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Misspecification Testing in GARCH-MIDAS Models

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Misspecification Testing in GARCH-MIDAS Models

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Abstract

We develop a misspecification test for the multiplicative two-component GARCH-MIDAS model suggested in Engle et al. (2013). In the GARCH-MIDAS model a short-term unit variance GARCH component fluctuates around a smoothly time-varying long-term component which is driven by the dynamics of a macroeconomic explanatory variable. We suggest a Lagrange Multiplier statistic for testing the null hypothesis that the macroeconomic variable has no explanatory power. Hence, under the null hypothesis the long-term component is constant and the GARCH-MIDAS reduces to the simple GARCH model. We provide asymptotic theory for our test statistic and investigate its finite sample properties by Monte Carlo simulation. Our test statistic can be considered as an extension of the Lundbergh and Teräsvirta (2002) ‘ARCH nested in GARCH’ test for evaluating GARCH models. We illustrate the usefulness of our procedure by an empirical application to S&P 500 return data.

Keywords: Volatility Component Models, LM test, Long-term Volatility.

JEL Classification: C53, C58, E32, G12

1 Introduction

During the recent financial crisis, macroeconomic determinants have been rediscovered as relevant factors for accurate prediction of stock market volatility. In particular, an increasing amount of empirical studies has employed the GARCH-MIDAS framework introduced by Engle et al. (2013) and identified economic variables that anticipate changes in long-term volatility (see, e.g., Asgharian et al., 2013, Conrad and Loch, 2014, Dorian, 2014, Opschoor et al., 2014). In a GARCH-MIDAS specification, the conditional variance consists of two multiplicative components, whereby macroeconomic conditions enter through the smooth long-term component around which a short-term unit variance GARCH component fluctuates. Besides the predictive evidence, however, it is still an open question if these macroeconomic effects are indeed significant. But standard test procedures for GARCH type specifications do not cover the case of macroeconomic explanatory variables. As most of them also require additive separability of the additional component under the alternative, they are also not straightforward to adapt to a general GARCH-MIDAS structure.

We develop a misspecification test for the multiplicative two-component GARCH-MIDAS model. In particular, we propose a Lagrange Multiplier (LM) statistic for testing the null hypothesis that the macroeconomic long-term component has no significant effect. Thus, under the null hypothesis, the long-term component is constant and the GARCH-MIDAS model reduces to the simple GARCH. Note that a simple t - or F -test cannot be employed in this context, as there exists no asymptotic theory yet for the general case of macroeconomic effects in the GARCH-MIDAS model. The most recent theoretical results available so far by Wang and Ghysels (2013), are specific to long-term components of realized volatility type and only hold in a restrictive parameter space which does not admit our null hypothesis. We derive the asymptotic properties for our test statistic. In a Monte Carlo simulation, we find good size and power properties in finite samples. Moreover, we illustrate the usefulness of our procedure by an empirical application to S&P 500 return data.

Our test statistic is closely related to the LM-test for evaluating GARCH models by Lundbergh and Teräsvirta (2002). In their ‘ARCH nested in GARCH’ setting, however, squared log-returns in the GARCH component are not rescaled by the the second multiplicative component which causes our test statistic to differ. In contrast to our set-up, log-returns standardized by the second “long-run” volatility component follow no longer

a GARCH process if the null hypothesis does not hold yielding different power properties of both tests. Considering general alternatives with exogenous variables, our work also complements the general GARCH specification tests with non-zero mean by Halunga and Orme (2009) and with multiplicative components by Amado and Teräsvirta (2014).

The plan of the paper is as follows. In Section 2, the GARCH-MIDAS model is introduced and the LM-test statistic is derived. In particular, we highlight how our statistic compares to existing GARCH-type specification tests of Lundbergh and Teräsvirta (2002) and Halunga and Orme (2009). Moreover, this section also contains the main asymptotic results. Section 3 provides some finite sample evidence in a Monte Carlo study. In Section 4, we illustrate how the test can contribute to modelling S&P 500 return data. Section 5 concludes. All proofs are contained in the Appendix.

2 Model and Test Statistic

In this section we first introduce the GARCH-MIDAS specification of Engle et al. (2013) and then discuss the null hypothesis of our test. The actual test statistic is closely related to the ‘ARCH nested in GARCH’ Lagrange multiplier (LM) test originally proposed in Lundbergh and Teräsvirta (2002). Moreover, Halunga and Orme (2009) suggest general misspecification tests for GARCH models. In particular, they focus on estimation effects from the correct specification of the conditional mean. While we simply assume that returns have mean zero and, therefore, do not deal with estimation effects from the mean, the proofs of our main results in Section 2.4 follow along similar lines as the proof of Theorem 2 in Halunga and Orme (2009).

2.1 The GARCH-MIDAS Model

We define the log-returns as given by

$$\varepsilon_t = \sigma_{0t} Z_t, \tag{1}$$

where Z_t is IID with mean zero and variance equal to one. σ_{0t}^2 is measurable with respect to the information set \mathcal{F}_{t-1} and denotes the conditional variance of the returns. We consider the following multiplicative decomposition of σ_{0t}^2 into a short-term and a long-term component:

$$\sigma_{0t}^2 = \tilde{h}_{0t} \tilde{\tau}_{0t} \tag{2}$$

The short-term component is specified as a unit variance GARCH(1,1):

$$\tilde{h}_{0t} = (1 - \alpha_0 - \beta_0) + \alpha_0 \frac{\varepsilon_{t-1}^2}{\tilde{\tau}_{0,t-1}} + \beta \tilde{h}_{0,t-1} \quad (3)$$

On the other hand, the long-term (MIDAS) component depends on the K lagged values of a (non-negative) explanatory variable x_t :

$$\tilde{\tau}_{0t} = \sigma_0^2 + \tilde{\pi}_0 \sum_{k=1}^K \psi_{0k}(\omega_{01}, \omega_{02}) x_{t-k} \quad (4)$$

with MIDAS weights $\psi_{0k}(\omega_{01}, \omega_{02}) \geq 0$ summing to one. A popular choice for determining the $\psi_{0k}(\cdot)$ is the Beta weighting scheme. The sign of the effect of x_t on long-term volatility can be inferred from the parameter $\tilde{\pi}_0$. Note that in equation (4) we consider a specification in which the explanatory variable and the returns are observed at the same frequency. Alternatively, one might also assume that the explanatory variable is observed at a lower frequency than the returns (see, e.g., Conrad and Loch, 2014). Nevertheless, our long-term component can be considered as a MIDAS specification in the sense that it parsimoniously models the dependence of $\tilde{\tau}_{0t}$ on (possibly) many lags of x_t in terms of only two parameters ω_{01} and ω_{02} via the flexible weighting scheme $\psi_{0k}(\cdot)$.

Following Conrad and Loch (2014), we denote the model with (exogenous) explanatory variables as GARCH-MIDAS-X. Engle et al. (2013) and Wang and Ghysels (2014) also consider a specification with the realized volatility, $RV_t = \sum_{j=0}^{N-1} \varepsilon_{t-j}^2$, of the last N days as the explanatory variable. In this case the long-term component can be rewritten as

$$\tilde{\tau}_{0t} = \sigma_0^2 + \tilde{\pi}_0 \sum_{l=1}^{N+K-1} c_{0l} \varepsilon_{t-l}^2$$

with $\sum_{l=1}^{N+K-1} c_{0l} = N \sum_{k=1}^K \psi_{0k}(\omega_{01}, \omega_{02}) = N$ (see Wang and Ghysels, 2014).¹ In other words, in this alternative representation the squared returns can be considered as the explanatory variables.

For the specific case of a GARCH-MIDAS-RV model, Wang and Ghysels (2014) provide conditions for the strict stationarity of ε_t and establish consistency and asymptotic normality of the QMLE. However, the proof of the asymptotic normality of the QMLE rests on the assumption that $\psi_{0k} > 0$ for $k = 1, \dots, K$ and, hence, their framework does not directly allow to test the null that all ψ_{0k} are equal to zero (see Assumption 4.3 in

¹Let $\nu_t = \sqrt{\tilde{h}_{0t}} Z_t$. Then $\varepsilon_t = \sqrt{\tilde{\tau}_{0t}} \nu_t$ can be interpreted as a semi-strong ARCH process with multiplicative GARCH error (see Wang and Ghysels, 2014).

Wang and Ghysels, 2014). In addition, there is no asymptotic theory for the general GARCH-MIDAS-X model yet. We will circumvent this problem by deriving an LM test for the hypothesis that x_t does not affect the long-term component, i.e. the long-term component is constant and the GARCH-MIDAS-X reduces to the simple GARCH(1,1). The LM test will require only the estimation of the model under the null. For deriving our test statistic we re-parameterize equation (2) as follows:

$$\sigma_{0t}^2 = \tilde{h}_{0t} \tilde{\tau}_{0t} = (\sigma_0^2 \tilde{h}_{0t}) \left(\frac{\tilde{\tau}_{0t}}{\sigma_0^2} \right) = \bar{h}_{0t} \tau_{0t}$$

The short-term component can then be expressed as

$$\bar{h}_{0t} = \omega_0 + \alpha_0 \frac{\varepsilon_{t-1}^2}{\tau_{0,t-1}} + \beta_0 \bar{h}_{0,t-1} \quad (5)$$

with $\omega_0 = \sigma_0^2(1 - \alpha_0 - \beta_0)$. We denote the vector of true parameters in the short-term component as $\boldsymbol{\eta}_0 = (\omega_0, \alpha_0, \beta_0)'$. Similarly, the long-term component can be rewritten as

$$\tau_{0t} = 1 + \sum_{k=1}^K \pi_{0k} x_{t-k} = 1 + \boldsymbol{\pi}'_0 \mathbf{x}_t \quad (6)$$

with $\pi_{0k} = \sigma_0^{-2} \tilde{\pi}_0 \psi_{0k}(\omega_{01}, \omega_{02})$, $\boldsymbol{\pi}_0 = (\pi_{01}, \dots, \pi_{0K})'$ and $\mathbf{x}_t = (x_{t-1}, \dots, x_{t-K})'$.

Using this notation, we are interested in testing $H_0 : \boldsymbol{\pi}_0 = \mathbf{0}$ against $H_1 : \boldsymbol{\pi}_0 \neq \mathbf{0}$. Under H_0 the long-term component is equal one and the GARCH-MIDAS-X reduces to the nested GARCH(1,1) with unconditional variance $\sigma_0^2 = \omega_0/(1 - \alpha_0 - \beta_0)$ (provided that $\alpha_0 + \beta_0 < 1$). Note that equation (5) is specified such that we can write

$$\bar{h}_{0t} = \omega_0 + (\alpha_0 Z_{t-1}^2 + \beta_0) \bar{h}_{0,t-1}$$

which means that $\varepsilon_t/\sqrt{\tau_{0t}} = \sqrt{\bar{h}_{0t}} Z_t$ follows a GARCH(1,1) both under the null and under the alternative.

We make the following assumptions about the true data generating process.

Assumption 1. $\boldsymbol{\eta}_0 \in \Theta$ where the parameter space is given by $\Theta = \{\boldsymbol{\eta} = (\omega, \alpha, \beta)' \in \mathbb{R}^{3 \times 1} | 0 < \omega < \bar{\omega}, 0 < \alpha, 0 < \beta, \alpha + \beta < 1\}$.

Assumption 2. Z_t^2 has a nondegenerate distribution, $\mathbf{E}[Z_t^2] = 1$ and $\kappa_Z = \mathbf{E}[Z_t^4] < \infty$.

Assumptions 1 and 2 imply that $\sqrt{\bar{h}_{0t}} Z_t$ is a covariance-stationary process with unconditional variance σ_0^2 . Further, by Jensen's inequality, they imply that $\mathbf{E}[\ln(\alpha_0 Z_t^2 + \beta_0)] < 0$ which ensures that under the null ε_t is strictly stationary and ergodic (see, e.g., Francq and Zakoian, 2004). Finally, the assumption on the existence of a fourth-order moment of Z_t will be necessary to ensure that the variance of the score vector exists.

2.2 Relation to Lundbergh and Teräsvirta (2002) LM Test

The LM test suggested by Lundbergh and Teräsvirta (2002) is based on the following specification $\varepsilon_t = \sqrt{h_{0t}}\xi_{0t} = \sqrt{h_{0t}\tau_{0t}}Z_t$, where

$$h_{0t} = \omega_0 + \alpha_0\varepsilon_{t-1}^2 + \beta_0h_{0,t-1} \quad (7)$$

and the long-term component τ_{0t} in equation (6) is based on the specific choice for the explanatory variable: $x_t = \xi_{0t}^2 = \varepsilon_t^2/h_{0t}$. Because under this assumption $\xi_{0t}^2 = \sqrt{\tau_{0t}}Z_t$ follows an ARCH(K), Lundbergh and Teräsvirta (2002) refer to this specification as ‘ARCH nested in GARCH’ and test the null hypothesis $H_0 : \boldsymbol{\pi}_0 = \mathbf{0}$. The LM test statistic in Lundbergh and Teräsvirta (2002) is based on

$$\left(\frac{\varepsilon_t^2}{\hat{h}_t} - 1 \right) \hat{\mathbf{r}}_t^{LT}, \quad (8)$$

where $\hat{\boldsymbol{\eta}}$ is the QMLE of $\boldsymbol{\eta}$ estimated under the null, $\hat{h}_t = \hat{\omega} + \hat{\alpha}\varepsilon_{t-1}^2 + \hat{\beta}\hat{h}_{t-1}$ and $\hat{\mathbf{r}}_t^{LT} = (\varepsilon_{t-1}^2/\hat{h}_{t-1}, \varepsilon_{t-2}^2/\hat{h}_{t-2}, \dots, \varepsilon_{t-K}^2/\hat{h}_{t-K})'$. Intuitively, equation (8) is used to test whether the standardized returns are still correlated.

The above specification is remarkably similar to the GARCH-MIDAS-X. However, there is also an important difference. In the specification of h_{0t} the ε_{t-1}^2 is *not* divided by τ_{0t} . Because of this, $\sqrt{h_{0t}}Z_t$ is a GARCH(1,1) process under the null but *not* under the alternative.² As a consequence, our test statistic – which is based on \bar{h}_{0t} – will rely on an appropriately modified version of equation (8).

2.3 Likelihood Function and Partial Derivatives

We denote the processes that can be constructed from the parameter vectors $\boldsymbol{\eta} = (\omega, \alpha, \beta)'$ and $\boldsymbol{\pi} = (\pi_1, \dots, \pi_K)'$ given initial observations for ε_t and x_t by \bar{h}_t and τ_t . We distinguish between the unobserved log-likelihood function based on $\bar{h}_t^\infty = \sum_{j=0}^{\infty} \beta^j (\omega + \alpha\varepsilon_{t-1-j}^2/\tau_{t-1-j})$ which depends on the infinite history of all past observations and the observed likelihood which is based on $\bar{h}_t = \sum_{j=0}^{t-1} \beta^j (\omega + \alpha\varepsilon_{t-1-j}^2/\tau_{t-1-j}) + \beta^t \bar{h}_0$. We write the unobserved log-likelihood function as

$$L_T^\infty(\boldsymbol{\eta}, \boldsymbol{\pi} | \varepsilon_T, x_T, \varepsilon_{T-1}, x_{T-1}, \dots) = \sum_{t=1}^T l_t^\infty \quad (9)$$

²The observation that h_{0t} does not follow a GARCH process under the alternative is closely related to the argument in Halunga and Orme (2009) that the alternative models considered in Lundbergh and Teräsvirta (2002) are not “recursive” in nature.

with

$$l_t^\infty = \ln(f(\varepsilon_t | \boldsymbol{\eta}, \boldsymbol{\pi})) = -\frac{1}{2} \left[\ln(\bar{h}_t^\infty) + \ln(\tau_t) - \frac{\varepsilon_t^2}{\bar{h}_t^\infty \tau_t} \right]. \quad (10)$$

Conditional on initial values $(\varepsilon_0, \bar{h}_0 = 0, \mathbf{x}_0)$ the observed log-likelihood can be written as

$$L_T(\boldsymbol{\eta}, \boldsymbol{\pi} | \varepsilon_T, x_T, \varepsilon_{T-1}, x_{T-1}, \dots, \varepsilon_1, x_1) = \sum_{t=1}^T l_t \quad (11)$$

with

$$l_t = \ln(f(\varepsilon_t | \boldsymbol{\eta}, \boldsymbol{\pi})) = -\frac{1}{2} \left[\ln(\bar{h}_t) + \ln(\tau_t) - \frac{\varepsilon_t^2}{\bar{h}_t \tau_t} \right]. \quad (12)$$

2.3.1 First derivatives

In the following, we consider the unobserved log-likelihood function. We define the average score vector evaluated under the null and at the true GARCH parameters as

$$\mathbf{D}^\infty(\boldsymbol{\eta}_0) = \begin{pmatrix} \mathbf{D}_\boldsymbol{\eta}^\infty(\boldsymbol{\eta}_0) \\ \mathbf{D}_\boldsymbol{\pi}^\infty(\boldsymbol{\eta}_0) \end{pmatrix} = \frac{1}{T} \sum_{t=1}^T \mathbf{d}_t^\infty(\boldsymbol{\eta}_0) = \frac{1}{T} \sum_{t=1}^T \begin{pmatrix} \mathbf{d}_{\boldsymbol{\eta},t}^\infty(\boldsymbol{\eta}_0) \\ \mathbf{d}_{\boldsymbol{\pi},t}^\infty(\boldsymbol{\eta}_0) \end{pmatrix}$$

where $\mathbf{d}_{\boldsymbol{\eta},t}^\infty(\boldsymbol{\eta}_0) = \partial l_t^\infty / \partial \boldsymbol{\eta} |_{\boldsymbol{\eta}_0, \boldsymbol{\pi}=\mathbf{0}}$ and $\mathbf{d}_{\boldsymbol{\pi},t}^\infty(\boldsymbol{\eta}_0) = \partial l_t^\infty / \partial \boldsymbol{\pi} |_{\boldsymbol{\eta}_0, \boldsymbol{\pi}=\mathbf{0}}$. Next, we derive expressions for $\mathbf{d}_{\boldsymbol{\eta},t}^\infty(\boldsymbol{\eta}_0)$ and $\mathbf{d}_{\boldsymbol{\pi},t}^\infty(\boldsymbol{\eta}_0)$. First, consider the partial derivative of the log-likelihood with respect to $\boldsymbol{\eta}$:

$$\frac{\partial l_t^\infty}{\partial \boldsymbol{\eta}} = \frac{1}{2} \left[\frac{\varepsilon_t^2}{\bar{h}_t^\infty \tau_t} - 1 \right] \left(\frac{1}{\bar{h}_t^\infty} \frac{\partial \bar{h}_t^\infty}{\partial \boldsymbol{\eta}} + \frac{1}{\tau_t} \frac{\partial \tau_t}{\partial \boldsymbol{\eta}} \right) \quad (13)$$

with $\partial \tau_t / \partial \boldsymbol{\eta} = (\partial \mathbf{x}_t / \partial \boldsymbol{\eta})' \boldsymbol{\pi}$. Under the null hypothesis, the long-term component reduces to $\tau_t = 1$ and the short term component simplifies to $\bar{h}_t^\infty = h_t^\infty$, i.e. the expression in equation (7). We then distinguish between

$$\mathbf{d}_{\boldsymbol{\eta},t}^\infty(\boldsymbol{\eta}) = \frac{\partial l_t^\infty}{\partial \boldsymbol{\eta}} \Big|_{\boldsymbol{\pi}=\mathbf{0}} = \frac{1}{2} \left[\frac{\varepsilon_t^2}{h_t^\infty} - 1 \right] \mathbf{y}_t^\infty \quad (14)$$

with

$$\mathbf{y}_t^\infty = \frac{1}{h_t^\infty} \frac{\partial \bar{h}_t^\infty}{\partial \boldsymbol{\eta}} \Big|_{\boldsymbol{\pi}=\mathbf{0}} = \frac{1}{h_t^\infty} \sum_{i=0}^{\infty} \beta^i \mathbf{s}_{t-i}^\infty, \quad (15)$$

where $\mathbf{s}_t^\infty = (1, \varepsilon_{t-1}^2, h_{t-1}^\infty)'$, and the corresponding quantity which is evaluated at $\boldsymbol{\eta}_0$:

$$\mathbf{d}_{\boldsymbol{\eta},t}^\infty(\boldsymbol{\eta}_0) = \frac{1}{2} \left[\frac{\varepsilon_t^2}{h_{0,t}^\infty} - 1 \right] \mathbf{y}_{0,t}^\infty, \quad (16)$$

where $h_{0,t}^\infty = \omega_0 + \alpha_0 \varepsilon_t^2 + \beta_0 h_{0,t-1}^\infty$ and $\mathbf{y}_{0,t}^\infty = (h_{0,t}^\infty)^{-1} \sum_{i=0}^{\infty} \beta_0^i \mathbf{s}_{0,t-i}^\infty$.

The partial derivative with respect to $\boldsymbol{\pi}$ leads to:

$$\frac{\partial l_t^\infty}{\partial \boldsymbol{\pi}} = \frac{1}{2} \left[\frac{\varepsilon_t^2}{\bar{h}_t^\infty \tau_t} - 1 \right] \left(\frac{1}{\bar{h}_t^\infty} \frac{\partial \bar{h}_t^\infty}{\partial \boldsymbol{\pi}} + \frac{1}{\tau_t} \frac{\partial \tau_t}{\partial \boldsymbol{\pi}} \right), \quad (17)$$

whereby the partial derivative of \bar{h}_t^∞ is given by

$$\frac{\partial \bar{h}_t^\infty}{\partial \boldsymbol{\pi}} = -\alpha \sum_{j=0}^{\infty} \beta^j \frac{\varepsilon_{t-1-j}^2}{\tau_{t-1-j}^2} \frac{\partial \tau_{t-1-j}}{\partial \boldsymbol{\pi}}. \quad (18)$$

Since $\partial \tau_t / \partial \boldsymbol{\pi} = \mathbf{x}_t + (\partial \mathbf{x}_t / \partial \boldsymbol{\pi})' \boldsymbol{\pi}$, we have $\partial \tau_t / \partial \boldsymbol{\pi} |_{\boldsymbol{\pi}=\mathbf{0}} = \mathbf{x}_t$ and, hence,

$$\mathbf{d}_{\boldsymbol{\pi},t}^\infty(\boldsymbol{\eta}) = \frac{\partial l_t^\infty}{\partial \boldsymbol{\pi}} \Big|_{\boldsymbol{\pi}=\mathbf{0}} = \frac{1}{2} \left[\frac{\varepsilon_t^2}{h_t^\infty} - 1 \right] \mathbf{r}_t^\infty \quad (19)$$

with

$$\mathbf{r}_t^\infty = \mathbf{x}_t - \alpha \frac{1}{h_t^\infty} \sum_{j=0}^{\infty} \beta^j \varepsilon_{t-1-j}^2 \mathbf{x}_{t-1-j}. \quad (20)$$

Similarly as before, the corresponding expression evaluated at $\boldsymbol{\eta}_0$ is given by:

$$\mathbf{d}_{\boldsymbol{\pi},t}^\infty(\boldsymbol{\eta}_0) = \frac{1}{2} \left[\frac{\varepsilon_t^2}{h_{0,t}^\infty} - 1 \right] \mathbf{r}_{0,t}^\infty \quad (21)$$

with

$$\mathbf{r}_{0,t}^\infty = \mathbf{x}_t - \alpha_0 \frac{1}{h_{0,t}^\infty} \sum_{j=0}^{\infty} \beta_0^j \varepsilon_{t-1-j}^2 \mathbf{x}_{t-1-j}. \quad (22)$$

Remark 1. Recall that in Lundbergh and Teräsvirta (2002) the log-likelihood function is not based on \bar{h}_t^∞ but on h_t^∞ . Since $\partial h_t^\infty / \partial \boldsymbol{\pi} = \mathbf{0}$, in their setting equation (22) reduces to $\mathbf{r}_{0,t}^\infty = \mathbf{x}_t$. Finally, for the specific choice of \mathbf{x}_t suggested in Lundbergh and Teräsvirta (2002), we obtain the version of $\mathbf{r}_{0,t}^\infty$ employed in equation (8). However, the form of equation (22) suggests that the properties of the two resulting test statistics might be similar if α_0 is small.

In summary, we have

$$\mathbf{D}^\infty(\boldsymbol{\eta}_0) = \frac{1}{T} \sum_{t=1}^T \mathbf{d}_t^\infty(\boldsymbol{\eta}_0) = \frac{1}{2T} \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{h_{0,t}^\infty} - 1 \right] \begin{pmatrix} \mathbf{y}_{0,t}^\infty \\ \mathbf{r}_{0,t}^\infty \end{pmatrix}.$$

Using that $\mathbf{E}[\varepsilon_t^2 / h_{0,t}^\infty] = \mathbf{E}[Z_t^2] = 1$, it follows that $\mathbf{E}[\mathbf{d}_t^\infty(\boldsymbol{\eta}_0) | \mathcal{F}_{t-1}] = \mathbf{0}$ and

$$\begin{aligned} \mathbf{Var}[\mathbf{d}_t^\infty(\boldsymbol{\eta}_0)] = \boldsymbol{\Omega} &= \begin{pmatrix} \boldsymbol{\Omega}_{\eta\eta} & \boldsymbol{\Omega}_{\eta\pi} \\ \boldsymbol{\Omega}_{\pi\eta} & \boldsymbol{\Omega}_{\pi\pi} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{E}[\mathbf{d}_{\eta,t}^\infty(\boldsymbol{\eta}_0) \mathbf{d}_{\eta,t}^\infty(\boldsymbol{\eta}_0)'] & \mathbf{E}[\mathbf{d}_{\eta,t}^\infty(\boldsymbol{\eta}_0) \mathbf{d}_{\pi,t}^\infty(\boldsymbol{\eta}_0)'] \\ \mathbf{E}[\mathbf{d}_{\pi,t}^\infty(\boldsymbol{\eta}_0) \mathbf{d}_{\eta,t}^\infty(\boldsymbol{\eta}_0)'] & \mathbf{E}[\mathbf{d}_{\pi,t}^\infty(\boldsymbol{\eta}_0) \mathbf{d}_{\pi,t}^\infty(\boldsymbol{\eta}_0)'] \end{pmatrix} \\ &= \frac{1}{4} (\kappa_Z - 1) \begin{pmatrix} \mathbf{E}[\mathbf{y}_{0,t}^\infty (\mathbf{y}_{0,t}^\infty)'] & \mathbf{E}[\mathbf{y}_{0,t}^\infty (\mathbf{r}_{0,t}^\infty)'] \\ \mathbf{E}[\mathbf{r}_{0,t}^\infty (\mathbf{y}_{0,t}^\infty)'] & \mathbf{E}[\mathbf{r}_{0,t}^\infty (\mathbf{r}_{0,t}^\infty)'] \end{pmatrix}. \end{aligned} \quad (23)$$

In the proof of Theorem 1 we will show that $\boldsymbol{\Omega}$ is finite and positive definite. This will allow us to apply a central limit theorem for martingale difference sequences to $\frac{1}{\sqrt{T}} \sum_{t=1}^T \mathbf{d}_t^\infty(\boldsymbol{\eta}_0)$.

2.3.2 Second derivatives

In the subsequent analysis we also make use of the following second derivatives:

$$\frac{\partial \mathbf{d}_{\boldsymbol{\eta},t}^\infty(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}'} = -\frac{1}{2} \frac{\varepsilon_t^2}{h_t^\infty} \mathbf{y}_t^\infty (\mathbf{y}_t^\infty)' + \frac{1}{2} \left[\frac{\varepsilon_t^2}{h_t^\infty} - 1 \right] \frac{\partial \mathbf{y}_t^\infty}{\partial \boldsymbol{\eta}'} \quad (24)$$

and

$$\frac{\partial \mathbf{d}_{\boldsymbol{\pi},t}^\infty(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}'} = -\frac{1}{2} \frac{\varepsilon_t^2}{h_t^\infty} \mathbf{r}_t^\infty (\mathbf{y}_t^\infty)' + \frac{1}{2} \left[\frac{\varepsilon_t^2}{h_t^\infty} - 1 \right] \frac{\partial \mathbf{r}_t^\infty}{\partial \boldsymbol{\eta}'} \quad (25)$$

We then define

$$\mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}} = -\mathbf{E} \left[\frac{\partial \mathbf{d}_{\boldsymbol{\eta},t}^\infty(\boldsymbol{\eta}_0)}{\partial \boldsymbol{\eta}'} \right] = \frac{1}{2} \mathbf{E}[\mathbf{y}_{0,t}^\infty (\mathbf{y}_{0,t}^\infty)'] \quad (26)$$

and

$$\mathbf{J}_{\boldsymbol{\pi}\boldsymbol{\eta}} = -\mathbf{E} \left[\frac{\partial \mathbf{d}_{\boldsymbol{\pi},t}^\infty(\boldsymbol{\eta}_0)}{\partial \boldsymbol{\eta}'} \right] = \frac{1}{2} \mathbf{E}[\mathbf{r}_{0,t}^\infty (\mathbf{y}_{0,t}^\infty)']. \quad (27)$$

Note that $\mathbf{d}_{\boldsymbol{\eta},t}^\infty(\boldsymbol{\eta}_0)$ corresponds to the score of observation t in a standard GARCH(1,1) model and $\partial \mathbf{d}_{\boldsymbol{\eta},t}^\infty(\boldsymbol{\eta}_0) / \partial \boldsymbol{\eta}'$ to the respective second derivative. It then directly follows from the results for the pure GARCH model in Francq and Zakoïan (2004) that $\mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}$ is finite and positive definite. Finally, note that $\boldsymbol{\Omega}_{\boldsymbol{\eta}\boldsymbol{\eta}} = \frac{1}{2}(\kappa_Z - 1)\mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}$ and $\boldsymbol{\Omega}_{\boldsymbol{\pi}\boldsymbol{\eta}} = \frac{1}{2}(\kappa_Z - 1)\mathbf{J}_{\boldsymbol{\pi}\boldsymbol{\eta}}$. If Z_t is normally distributed, then $\kappa_Z = 3$ and $\boldsymbol{\Omega}_{\boldsymbol{\eta}\boldsymbol{\eta}} = \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}$ and $\boldsymbol{\Omega}_{\boldsymbol{\pi}\boldsymbol{\eta}} = \mathbf{J}_{\boldsymbol{\pi}\boldsymbol{\eta}}$, respectively.

2.4 The LM Test Statistic

If Assumptions 1 and 2 hold and the model is estimated under the null, the QMLE of the GARCH(1,1) parameters will be consistent and asymptotically normal (see Francq and Zakoïan, 2004). More precisely, if Assumptions 1 and 2 hold and the model is estimated under the null:

$$\sqrt{T}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0) \xrightarrow{d} \mathcal{N}(\mathbf{0}, (\kappa_Z - 1)\mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1}). \quad (28)$$

In the following theorem we derive the asymptotic distribution of the average score evaluated at $\boldsymbol{\eta}_0$. In order to ensure the finiteness of the covariance matrix of the average score we assume that x_t has finite fourth moment.

Assumption 3. $x_t \geq 0$ is strictly stationary and ergodic with $\mathbf{E}[|x_t|^4] < \infty$.

For simplicity, we also assume that the explanatory variable takes only nonnegative values. This assumption is line with the GARCH-MIDAS-RV model with $x_t = \varepsilon_t^2$ or the specification of Lundbergh and Teräsvirta (2002) with $x_t = \varepsilon_t^2/h_{0t}$. For testing the GARCH-MIDAS-RV against the simple GARCH model, Assumption 3 requires that the observed process has finite eighth moment: $\mathbf{E}[|\varepsilon_t|^8] < \infty$. The corresponding constraints on the parameters of the GARCH(1,1) are provided in Francq and Zakoian (2010), equation (2.54). In general, nonnegative explanatory variables could be the unemployment rate, interest rates, the VIX, disagreement among forecasters or measures of political uncertainty. However, our results can be generalized for the case that $\tau_t = \exp(1 + \boldsymbol{\pi}'\mathbf{x}_t)$.

Theorem 1. *If Assumptions 1-3 hold, then*

$$\sqrt{T}\mathbf{D}^\infty(\boldsymbol{\eta}_0) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \boldsymbol{\Omega}). \quad (29)$$

Next, we consider the asymptotic distribution of the relevant lower part of the score vector evaluated at $\hat{\boldsymbol{\eta}}$. As an intermediate step, we show that $\mathbf{J}_{\boldsymbol{\pi}\boldsymbol{\eta}}$ can be consistently estimated by

$$-\frac{1}{T} \sum_{t=1}^T \frac{\partial \mathbf{d}_{\boldsymbol{\pi},t}^\infty(\tilde{\boldsymbol{\eta}})}{\partial \boldsymbol{\eta}'},$$

where $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta}_0 + o_P(1)$. The result is presented in Proposition 1 in the Appendix. For doing so the following Assumption 4 is required.

Assumption 4. $\mathbf{E}[|\varepsilon_t|^{4(1+s)}] < \infty$ for some $s > 0$.

Note that in general $\varepsilon_t^2 = \bar{h}_{0t}^\infty \tau_{0t} Z_t^2$, i.e. depends on $\boldsymbol{\eta}_0$ and $\boldsymbol{\pi}_0$. Under the null, $\varepsilon_t^2 = h_{0,t}^\infty Z_t^2$ depends on $\boldsymbol{\eta}_0$ only. In the proof of Proposition 1 we will use this insight to argue that $\mathbf{E}[\sup_{\boldsymbol{\eta}} |\varepsilon_t|^{4(1+s)}] = \mathbf{E}[|\varepsilon_t|^{4(1+s)}]$.

Theorem 2. *If Assumptions 1-4 hold, then*

$$\sqrt{T}\mathbf{D}_\pi^\infty(\hat{\boldsymbol{\eta}}) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}), \quad (30)$$

where

$$\begin{aligned} \boldsymbol{\Sigma} &= \boldsymbol{\Omega}_{\boldsymbol{\pi}\boldsymbol{\pi}} - \mathbf{J}_{\boldsymbol{\pi}\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} \boldsymbol{\Omega}'_{\boldsymbol{\pi}\boldsymbol{\eta}} \\ &= \frac{1}{4} (\kappa_Z - 1) \left(\mathbf{E}[\mathbf{r}_{0,t}^\infty (\mathbf{r}_{0,t}^\infty)'] - \mathbf{E}[\mathbf{r}_{0,t}^\infty (\mathbf{y}_{0,t}^\infty)'] (\mathbf{E}[\mathbf{y}_{0,t}^\infty (\mathbf{y}_{0,t}^\infty)'])^{-1} \mathbf{E}[\mathbf{y}_{0,t}^\infty (\mathbf{r}_{0,t}^\infty)'] \right). \end{aligned} \quad (31)$$

The covariance matrix $\boldsymbol{\Sigma}$ in equation (31) takes the same form as in Lundbergh and Teräsvirta (2002).

The actual test statistic will be based on the observed quantity $\mathbf{D}_\pi(\hat{\boldsymbol{\eta}})$. The following theorem states the test statistic and its asymptotic distribution. For the proof of the theorem, we show that $\sqrt{T}\mathbf{D}_\pi(\hat{\boldsymbol{\eta}})$ and $\sqrt{T}\mathbf{D}_\pi^\infty(\hat{\boldsymbol{\eta}})$ have the same asymptotic distribution.

Theorem 3. *If Assumptions 1-4 hold, then*

$$\begin{aligned} LM &= T\mathbf{D}_\pi(\hat{\boldsymbol{\eta}})' \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{D}_\pi(\hat{\boldsymbol{\eta}}) \\ &= \frac{1}{4T} \left(\sum_{t=1}^T \begin{bmatrix} \varepsilon_t^2 \\ \hat{h}_t \end{bmatrix} - 1 \right)' \hat{\boldsymbol{\Sigma}}^{-1} \left(\sum_{t=1}^T \begin{bmatrix} \varepsilon_t^2 \\ \hat{h}_t \end{bmatrix} - 1 \right) \underset{a}{\approx} \chi^2(K) \end{aligned} \quad (32)$$

where $\hat{\boldsymbol{\eta}} = (\hat{\omega}, \hat{\alpha}, \hat{\beta})'$ is the vector of parameter estimates from the model under the null, $\hat{h}_t = \hat{\omega} + \hat{\alpha}\varepsilon_{t-1}^2 + \hat{\beta}\hat{h}_{t-1}$, $\hat{\mathbf{r}}_t = \mathbf{x}_t - \hat{\alpha}/\hat{h}_t \sum_{j=0}^{t-1} \hat{\beta}^j \varepsilon_{t-1-j}^2 \mathbf{x}_{t-1-j}$ and $\hat{\boldsymbol{\Sigma}}$ is a consistent estimator of $\boldsymbol{\Sigma}$.

Following Lundbergh and Teräsvirta (2002), we also consider a regression version of the test. The corresponding test statistic is given by

$$\widetilde{LM} = T \frac{SSR_0 - SSR_1}{SSR_0} \quad (33)$$

where $SSR_0 = \sum_{t=1}^T (\varepsilon_t^2/\hat{h}_t - 1)^2$ and SSR_1 is the sum of squared residuals from a regression of $(\varepsilon_t^2/\hat{h}_t - 1)$ on $\hat{\mathbf{r}}_t'$ and $\hat{\mathbf{y}}_t'$. Hence, \widetilde{LM} is simply T times the uncentered R^2 of the regression.

Remark 2. *It is interesting to consider two special cases of our test. If there are no GARCH effects, i.e. $\alpha = \beta = 0$, then $\bar{h}_t = \omega$. In this case, the model under $H_0 : \boldsymbol{\pi}_0 = \mathbf{0}$ has constant conditional and unconditional variance equal to $\sigma_0^2 = \omega_0$. Under the alternative there is multiplicative conditional heteroskedasticity, i.e. $\mathbf{Var}[\varepsilon_t | \mathcal{F}_{t-1}] = \sigma^2 \tau_t$. Without GARCH effects the partial derivative of the log-likelihood under H_0 simplifies to*

$$\left. \frac{\partial l_t^\infty}{\partial \boldsymbol{\pi}} \right|_{\boldsymbol{\pi}=\mathbf{0}} = \frac{1}{2} \left[\frac{\varepsilon_t^2}{\sigma_0^2} - 1 \right] \frac{\mathbf{x}_t}{\sigma_0^2}. \quad (34)$$

The regression based test would be to regress $(\varepsilon_t^2/\sigma_0^2 - 1)$ on a constant and \mathbf{x}_t/σ_0^2 and to compute TR^2 which is $\chi^2(K)$. Thus, if there is a conditional mean function with explanatory variables \mathbf{x}_t , our LM test reduces to the Godfrey (1978) test for multiplicative heteroskedasticity. Finally, if we choose $x_{t-k} = \varepsilon_{t-k}^2$ our test coincides with the Engle (1982) test for ARCH effects.

3 Simulation

In this section, we examine the finite sample behavior of the proposed test in a Monte Carlo experiment. We simulate return series with $T = 1000$ observations and use $M = 1000$ Monte Carlo replications. Four alternative GARCH(1,1) specifications are considered. The first and second one can be described by medium (M1/M2) persistence ($\alpha_0 + \beta_0 = 0.95$), the third and fourth by high (H1/H2) persistence ($\alpha_0 + \beta_0 = 0.99$). The first and third specifications are also used in Halunga and Orme (2009). In order to illustrate the effect of α being low/high while holding the degree of persistence fixed, we additionally employ the second and fourth specification. ω_0 is always chosen such that under the null $\sigma_0^2 = 1$. The innovation Z_t is assumed to be either normally distributed or t -distributed with 5 degrees of freedom.

$$\begin{aligned} \text{M1:} \quad \bar{h}_t &= 0.05 + 0.05 \frac{\varepsilon_{t-1}^2}{\tau_{t-1}} + 0.90 \bar{h}_{t-1} \\ \text{M2:} \quad \bar{h}_t &= 0.05 + 0.10 \frac{\varepsilon_{t-1}^2}{\tau_{t-1}} + 0.85 \bar{h}_{t-1} \\ \text{H1:} \quad \bar{h}_t &= 0.01 + 0.09 \frac{\varepsilon_{t-1}^2}{\tau_{t-1}} + 0.90 \bar{h}_{t-1} \\ \text{H2:} \quad \bar{h}_t &= 0.01 + 0.15 \frac{\varepsilon_{t-1}^2}{\tau_{t-1}} + 0.84 \bar{h}_{t-1} \end{aligned}$$

We first consider the size properties of the test. Under null, we test for remaining ARCH effects by choosing $x_t = \varepsilon_t^2/h_t$. In Table 1 we report the empirical size of the LM test given in equation (32), the regression version of the test, \widetilde{LM} , and the Lundbergh and Teräsvirta (2002) test statistic \widetilde{LM}_{LT} . As the table shows, the empirical size of all three versions of the test statistic is very close to the nominal size when Z_t is normally distributed. Also, the size properties do not depend on the choice of K .

Under the alternative we consider the following ‘ARCH(1) nested in GARCH’ specification:

$$\tau_{0,t} = 1 + \pi_{0,1} \frac{\varepsilon_{t-1}^2}{h_{0,t-1}}$$

with $\pi_{0,1} \in \{0.1, 0.2\}$. In the test statistic we correctly specify $x_t = \varepsilon_t^2/h_t$. As Table 2 shows, all three versions of the test statistic lead to almost identical results for a given GARCH specification. However, the simulated power is much higher for GARCH specifications with high persistence than for GARCH specifications with low persistence. Also, the power of the tests strongly increases with the deviation from the null hypothesis. For example, for the process H1 and at the nominal size of 5% the simulated power of the LM

Table 1: Empirical size.

		$K = 1$				$K = 5$			
		M1	M2	H1	H2	M1	M2	H1	H2
LM	1%	0.9	0.9	1.3	1.1	1.3	1.6	1.2	1.7
	5%	4.6	5.0	5.4	5.2	5.1	5.0	5.3	5.4
	10%	9.2	10.5	10.1	10.7	9.2	9.4	9.7	9.5
\widetilde{LM}	1%	0.9	0.9	1.3	1.1	1.3	1.5	1.2	1.7
	5%	4.6	5.0	5.5	5.2	5.1	5.0	5.3	5.4
	10%	9.2	10.5	10.1	10.7	9.2	9.4	9.8	9.5
\widetilde{LM}_{LT}	1%	0.9	1.0	1.4	1.2	1.6	1.5	1.8	1.9
	5%	5.0	5.3	5.3	5.4	5.1	4.3	5.0	5.5
	10%	10.2	10.3	11.1	11.2	9.4	8.8	8.8	9.1

Notes: Entries are rejection rates in percent over the 1000 replications at the 1%, 5% and 10% nominal level.

statistic is 66.9 when $\pi_{0,1} = 0.1$ but increases to 94.0 when $\pi_{0,1} = 0.2$. Table 2 also shows that the power deteriorates if the lag length K is chosen too large in comparison to the true data generating process. Again, the loss in power is more severe if the underlying GARCH process is less persistent. Interestingly, when the persistence is low, the power is much higher for the process with high α_0 (M2) than for the process with low α_0 (M1). A comparison of the power of the LM and \widetilde{LM} test with the Lundbergh and Teräsvirta (2002) version, \widetilde{LM}_{LT} , of the test shows that the correct specification of $\hat{\mathbf{r}}_t$ leads to a modest increase in power when the data generating process is an ‘ARCH(1) nested in GARCH’.

Next, we consider two more realistic examples in which the long-term component is based actual data. We use continuously compounded daily stock returns on the S&P 500 for the period January 2000 to August 2014. Using five-minute intra-day data, we construct daily realized volatility, RV_t , as the sum of the squared five-minute returns over the day. In addition, we construct monthly and quarterly realized volatilities as $RV_t^{(N)} = \sum_{j=0}^{N-1} RV_{t-j}$ with $N = 22$ and $N = 65$. As a second explanatory variable we use the daily VIX index. As for the realized volatility, we also construct monthly and quarterly rolling window versions of the VIX as $VIX_t^{(N)} = \sum_{t=0}^{N-1} VIX_t$, with $N = 22$ and $N = 65$. Figure 1 shows the evolution of the VIX and the realized volatility over the

Table 2: Size-adjusted power.

		$K = 1$				$K = 5$			
		M1	M2	H1	H2	M1	M2	H1	H2
		$\pi_{0,1} = 0.1$							
LM	1%	16.0	27.5	46.0	44.2	7.6	13.2	24.3	22.5
	5%	31.0	49.0	66.9	69.2	17.2	27.4	41.8	40.9
	10%	42.3	62.5	77.2	77.7	26.0	38.9	53.3	51.3
\widetilde{LM}	1%	16.0	27.6	45.9	44.2	7.5	13.2	24.4	22.5
	5%	30.9	49.2	66.9	69.1	17.3	27.3	41.8	40.9
	10%	42.2	62.5	77.2	77.7	25.9	39.1	53.3	51.5
\widetilde{LM}_{LT}	1%	16.0	26.4	43.9	41.7	6.8	11.8	22.6	20.7
	5%	30.7	47.0	65.0	62.8	17.3	26.0	41.6	39.6
	10%	41.0	59.0	74.6	74.0	26.5	37.8	51.7	50.9
		$\pi_{0,1} = 0.2$							
LM	1%	19.0	51.8	86.9	92.6	11.5	29.3	68.4	71.2
	5%	29.9	69.6	94.0	97.5	22.2	50.5	84.7	87.6
	10%	38.7	78.2	96.1	98.6	31.5	62.8	88.9	92.9
\widetilde{LM}	1%	19.0	51.8	87.0	92.7	11.5	29.3	68.4	71.4
	5%	29.9	69.7	94.0	97.5	22.3	50.4	84.7	87.6
	10%	38.7	78.2	96.1	98.6	31.5	62.8	89.0	93.0
\widetilde{LM}_{LT}	1%	18.6	46.1	84.5	87.3	11.4	30.7	64.0	66.3
	5%	29.3	64.2	91.6	94.9	22.6	49.8	81.1	83.8
	10%	37.4	73.5	93.6	97.2	32.4	62.0	88.6	91.8

Notes: Entries are rejection rates in percent over the 1000 replications at the 1%, 5% and 10% nominal level.

sample period.

Using the rolling window versions of the realized volatility and the VIX we can create smooth long-term components. We specify the long-term component as

$$\tau_{0,t} = 1 + 0.6x_{t-1} + 0.3x_{t-2} + 0.07x_{t-3} + 0.03x_{t-4}. \quad (35)$$

As Table 3 shows, for the smooth long-term components that are based on actual data the difference in the power of the tests based on $\hat{\mathbf{r}}_t$ and the Lundbergh and Teräsvirta (2002) version based on $\hat{\mathbf{r}}_t^{LT}$ becomes much more pronounced.

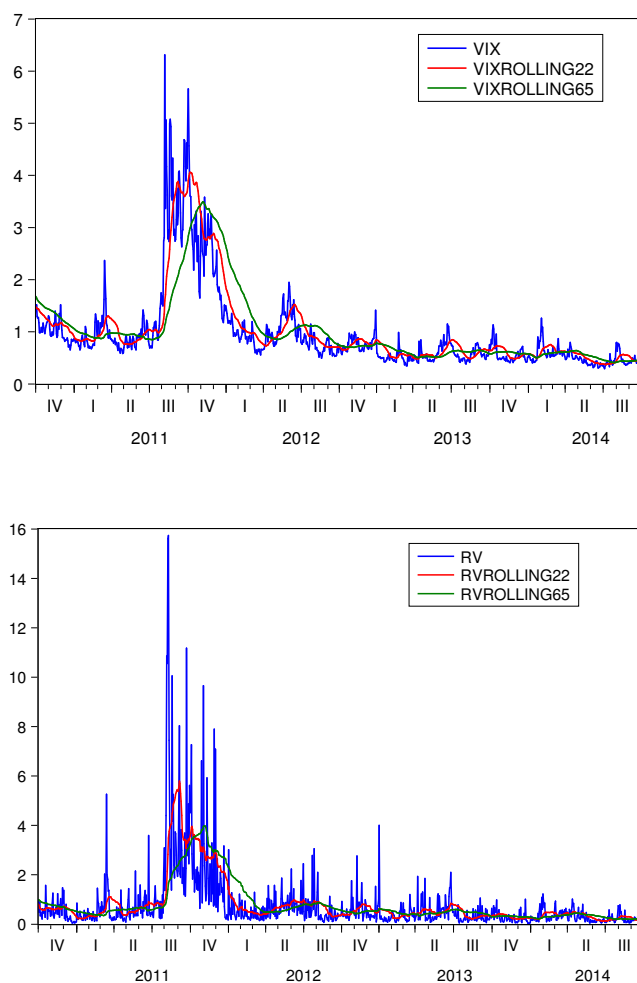


Figure 1: The upper and lower panel show the evolution of the VIX and the realized volatility (and the rolling window versions) over the period October 2010 to October 2014.

4 Conclusions

We develop a Lagrange-Multiplier test for the null hypothesis of a GARCH volatility against the alternative of a GARCH-MIDAS specification. The test provides a first solution to statistically evaluate if there is a separate long-term varying volatility component driven by a macroeconomic explanatory variable, besides the standard short-term GARCH part. We derive the asymptotic properties of our test and study its finite sample performance. In an application to S&P 500 returns, we find that the test provides useful guidance in model specification.

Table 3: Rejection rates for long-term components based on actual data.

		$K = 1$					
	x_t	$VIX_{t-1}^{(65)}$	$VIX_{t-1}^{(22)}$	VIX_{t-1}	$RV_t^{(65)}$	$RV_{t-1}^{(22)}$	RV_{t-1}
LM	1%	38.2	73.2	88.8	62.5	96.4	99.7
	5%	64.7	89.9	96.1	84.0	99.0	99.9
	10%	78.2	94.8	97.7	91.2	99.6	99.9
\widetilde{LM}	1%	38.2	73.0	88.6	62.1	96.3	99.7
	5%	64.4	89.8	96.1	83.8	99.0	99.9
	10%	78.0	94.7	97.7	91.2	99.6	99.9
\widetilde{LM}_{LT}	1%	19.1	34.3	65.5	29.1	57.5	100
	5%	45.1	62.2	85.8	56.2	84.2	100
	10%	62.5	75.3	92.7	70.6	92.6	100

Notes: See Table 1. In the GARCH component we choose specification H1.

Table 4: Rejection rates for long-term components based on actual data.

		$K = 4$					
	x_t	$VIX_{t-1}^{(65)}$	$VIX_{t-1}^{(22)}$	VIX_{t-1}	$RV_t^{(65)}$	$RV_{t-1}^{(22)}$	RV_{t-1}
LM	1%	25.0	54.7	83.0	43.2	86.1	99.9
	5%	47.8	77.7	93.6	66.7	96.7	100
	10%	61.5	85.8	96.6	78.4	98.3	100
\widetilde{LM}	1%	24.7	54.3	82.8	42.5	85.4	99.9
	5%	47.5	77.6	93.4	66.4	96.6	100
	10%	60.8	85.8	96.4	77.8	98.3	100
\widetilde{LM}_{LT}	1%	28.9	54.5	63.8	46.3	79.6	99.5
	5%	51.0	77.8	82.3	67.2	92.0	99.9
	10%	63.6	85.7	88.5	78.3	96.2	100

Notes: See Table 1. In the GARCH component we choose specification H1.

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A Proofs

Proof of Theorem 1. First, we show that $\mathbf{\Omega}$ is finite and positive definite. From Francq and Zakioan (2004) it follows that $\mathbf{\Omega}_{\eta\eta}$ is finite and positive definite. What remains to be shown is that $\mathbf{\Omega}_{\pi\pi}$ is finite and positive definite. If this is true then, by Cauchy-Schwarz, also the “off-diagonal matrices” will be finite and positive definite.

Finiteness of $\mathbf{\Omega}_{\pi\pi}$:

Recall from equation (23) that $\mathbf{\Omega}_{\pi\pi} = \mathbf{E}[\mathbf{d}_{\pi,t}^\infty(\boldsymbol{\eta}_0)\mathbf{d}_{\pi,t}^\infty(\boldsymbol{\eta}_0)'] = \frac{1}{4}(\kappa_Z - 1)\mathbf{E}[\mathbf{r}_{0,t}^\infty(\mathbf{r}_{0,t}^\infty)']$. It follows from Assumption 2 that $0 < \kappa_Z - 1 < \infty$. Hence, $\|\mathbf{E}[\mathbf{r}_{0,t}^\infty(\mathbf{r}_{0,t}^\infty)']\|$ is finite if $\mathbf{E}[\|\mathbf{r}_{0,t}^\infty(\mathbf{r}_{0,t}^\infty)'\|] < \infty$. A typical element of the $K \times 1$ vector $\mathbf{r}_{0,t}^\infty$ is given by

$$r_{0,kt}^\infty = x_{t-k} - \alpha_0 \frac{1}{h_{0,t}^\infty} \sum_{j=0}^{\infty} \beta_0^j \varepsilon_{t-1-j}^2 x_{t-1-k-j}. \quad (36)$$

First, $\mathbf{E}[|x_{t-k}|^2] < \infty$ by Assumption 3. Second,

$$\left(\mathbf{E} \left| \frac{\sum_{j=0}^{\infty} \alpha_0 \beta_0^j \varepsilon_{t-1-k-j}^2 x_{t-1-j}}{h_{0,t}^\infty} \right|^2 \right)^{1/2} \leq \left(\mathbf{E} \left| \sum_{j=0}^{\infty} \frac{\alpha_0 \beta_0^j \varepsilon_{t-1-j}^2}{(\omega_0 + \alpha_0 \beta_0^j \varepsilon_{t-1-j}^2)} x_{t-1-k-j} \right|^2 \right)^{1/2} \quad (37)$$

$$\leq \sum_{j=0}^{\infty} \left(\mathbf{E} \left| \frac{\alpha_0 \beta_0^j \varepsilon_{t-1-j}^2}{(\omega_0 + \alpha_0 \beta_0^j \varepsilon_{t-1-j}^2)} x_{t-1-k-j} \right|^2 \right)^{1/2} \quad (38)$$

$$\leq \sum_{j=0}^{\infty} \left(\mathbf{E} \left| \left(\frac{\alpha_0 \beta_0^j}{\omega_0} \varepsilon_{t-1-j}^2 \right)^{s/4} x_{t-1-k-j} \right|^2 \right)^{1/2} \quad (39)$$

$$\leq \frac{\alpha_0^{s/4}}{\omega_0^{s/4}} (\mathbf{E} [\varepsilon_{t-1-j}^{2s}])^{1/4} (\mathbf{E} [|x_{t-1-k-j}|^4])^{1/4}$$

$$\sum_{j=0}^{\infty} \beta_0^{js/4} < \infty$$

The arguments used above are similar to the ones in Francq and Zakioan (2004, Eq. (4.19), p.619). In particular, in equation (37) we use that $h_{0,t}^\infty \geq \omega_0 + \alpha_0 \beta_0^j \varepsilon_{t-1-j}^2$. In equation (38) we use Minkowski's inequality. Next, in equation (39) we use the fact that $w/(1+w) \leq w^s$ for all $w > 0$ and any $s \in (0, 1)$. Finally, Assumption 1 implies that there exists some $s > 0$ such that $\mathbf{E} [\varepsilon_{t-1-j}^{2s}] < \infty$ (see Proposition 1 in Francq and Zakioan, 2004, p.607). By Assumption 3, $\mathbf{E} [|x_{t-1-k-j}|^4] < \infty$.

This implies $\mathbf{E}[r_{0,kt}^\infty]^2 < \infty$ and $\mathbf{E}[r_{0,kt}^\infty r_{0,jt}^\infty] < \infty$ by Cauchy-Schwarz inequality which means that $\mathbf{\Omega}_{\pi\pi}$ is finite.

Positive definiteness of $\mathbf{\Omega}_{\pi\pi}$:

As $\kappa_Z - 1 > 0$, it remains to show that $\mathbf{c}'\mathbf{E}[\mathbf{r}_{0,t}^\infty(\mathbf{r}_{0,t}^\infty)']\mathbf{c} > 0$ for any non-zero $\mathbf{c} \in \mathbb{R}^{K \times 1}$. Assume the contrary, i.e., there exists a $\mathbf{c} \neq 0$ such that $\mathbf{c}'\mathbf{E}[\mathbf{r}_{0,t}^\infty(\mathbf{r}_{0,t}^\infty)']\mathbf{c} = 0$. This implies $\mathbf{E}[(\mathbf{c}'\mathbf{r}_{0,t}^\infty)^2] = 0$ and thus $\mathbf{c}'\mathbf{r}_{0,t}^\infty = 0$ a.s.. Hence, there exists a linear combination of $r_{0,1t}^\infty, \dots, r_{0,Kt}^\infty$ which equals zero a.s., i.e.,

$$0 = \sum_{k=1}^K c_k \left(x_{t-k} - \frac{\alpha_0}{h_{0,t}^\infty} \sum_{j=0}^{\infty} \beta_0^j \varepsilon_{t-1-j}^2 x_{t-1-k-j} \right) \quad \text{a.s.} \quad (40)$$

With rearranging, this requires

$$\mathbf{c}'\mathbf{x}_t = \left[\frac{\alpha_0}{h_{0,t}^\infty} (1 - \beta_0 L)^{-1} L \right] (\varepsilon_t^2 \mathbf{c}'\mathbf{x}_t) \quad \text{a.s.} \quad (41)$$

According to Assumption 1, $\beta_0 < 1$, i.e. the operator in square brackets cannot have an eigenvalue 1. Moreover, Assumption 2 imposes Z_t^2 and therefore also ε_t^2 to be non-degenerate. Hence the only way to fulfill the above equation is by $\mathbf{c}'\mathbf{x}_t = 0$ a.s.. As the x_t are positive, this implies that $c_1 = \dots = c_K = 0$ which is a contradiction to the initial assumption. Thus $\mathbf{\Omega}_{\pi\pi}$ must be invertible and hence positive definite.

Next, $\mathbf{E}[\mathbf{d}_t^\infty(\boldsymbol{\eta}_0) | \mathcal{F}_{t-1}] = \mathbf{0}$. From Francq and Zakoian (2004) and Assumptions 1-3 it then follows that $\mathbf{d}_t^\infty(\boldsymbol{\eta}_0)$ is a stationary and ergodic martingale difference sequence with finite second moment. Applying Billingsley's (1961) central limit theorem for martingale differences gives the result.

■

The following proposition will be used in the proof of Theorem 2.

Proposition 1. *Under Assumptions 1-4, we have that*

$$-\frac{1}{T} \sum_{t=1}^T \frac{\partial \mathbf{d}_{\pi,t}^\infty(\tilde{\boldsymbol{\eta}})}{\partial \boldsymbol{\eta}'} \xrightarrow{P} \mathbf{J}_{\pi\eta} = -\mathbf{E} \left[\frac{\partial \mathbf{d}_{\pi,t}^\infty(\boldsymbol{\eta}_0)}{\partial \boldsymbol{\eta}'} \right] \quad (42)$$

where $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta}_0 + o_P(1)$

Proof of Proposition 1. We obtain (42) in two steps. a) In a first step, we show that $\mathbf{J}_{\pi\eta}(\boldsymbol{\eta}) = -\mathbf{E} \left[\frac{\partial \mathbf{d}_{\pi,t}^\infty(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}'} \right]$ is finite with a uniform bound for all $\boldsymbol{\eta} \in \Theta$. Then a uniform weak law of large numbers (see, e.g., Theorem 3.1. in Ling and McAleer (2003)) implies

$$\sup_{\boldsymbol{\eta}} \left\| -\frac{1}{T} \sum_{t=1}^T \frac{\partial \mathbf{d}_{\pi,t}^\infty(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}'} - \mathbf{J}_{\pi\eta}(\boldsymbol{\eta}) \right\| = o_p(1).$$

Equation (42) then follows from the triangle inequality and the fact that $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta}_0 + o_P(1)$.

Using equation (25) we obtain

$$\begin{aligned} \left\| \frac{\partial \mathbf{d}_{\pi,t}^\infty(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}'} \right\| &\leq \frac{1}{2} \left(\left| \frac{\varepsilon_t^2}{h_t^\infty} \right| \cdot \|\mathbf{r}_t^\infty\| \cdot \|(\mathbf{y}_t^\infty)'\| + \left| \frac{\varepsilon_t^2}{h_t^\infty} - 1 \right| \cdot \left\| \frac{\partial \mathbf{r}_t^\infty}{\partial \boldsymbol{\eta}'} \right\| \right) \\ &\leq C|\varepsilon_t^2 + \omega| \left(\|\mathbf{r}_t^\infty\| \cdot \|(\mathbf{y}_t^\infty)'\| + \left\| \frac{\partial \mathbf{r}_t^\infty}{\partial \boldsymbol{\eta}'} \right\| \right). \end{aligned} \quad (43)$$

The last inequality follows with a generic constant $0 < C < \infty$ and $h_t^\infty \geq \omega > 0$.

First, consider the three elements of $\|(\mathbf{y}_t^\infty)'\|$. To simplify the notation note that $\frac{\partial \bar{h}_t^\infty}{\partial \boldsymbol{\eta}}|_{\boldsymbol{\pi}=\mathbf{0}} = \frac{\partial h_t^\infty}{\partial \boldsymbol{\eta}}$. Since $\frac{\partial h_t^\infty}{\partial \omega} = 1/(1-\beta)$, we have $|\frac{1}{h_t^\infty} \frac{\partial h_t^\infty}{\partial \omega}| \leq 1/(\omega(1-\beta)) < \infty$. Then $\alpha \frac{\partial h_t^\infty}{\partial \alpha} = \sum_{j=0}^{\infty} \alpha \beta^j \varepsilon_{t-1-j}^2 \leq h_t^\infty$ and, therefore, $|\frac{1}{h_t^\infty} \frac{\partial h_t^\infty}{\partial \alpha}| \leq 1/\alpha < \infty$. Finally, $\frac{\partial h_t^\infty}{\partial \beta} = \sum_{j=0}^{\infty} j \beta^{j-1} (\omega + \alpha \varepsilon_{t-1-j}^2)$. We then obtain

$$\begin{aligned} \left| \frac{1}{h_t^\infty} \frac{\partial h_t^\infty}{\partial \beta} \right| &\leq \left| \frac{1}{\beta} \sum_{j=0}^{\infty} \frac{j \beta^j (\omega + \alpha \varepsilon_{t-1-j}^2)}{\omega + \beta^j (\omega + \alpha \varepsilon_{t-1-j}^2)} \right| \\ &\leq \frac{1}{\beta \omega^s} \sum_{j=0}^{\infty} j |\beta^{js} (\omega + \alpha \varepsilon_{t-1-j}^2)^s|, \end{aligned} \quad (44)$$

where we again use the fact that $w/(1+w) \leq w^s$ for all $w > 0$ and any $s \in (0, 1)$. It follows that $\|(\mathbf{y}_t^\infty)'\| \leq K(1 + \sum_{j=0}^{\infty} j |\beta^{js} (\omega + \alpha \varepsilon_{t-1-j}^2)^s|)$ for some constant $K > 0$.

Hence, by Cauchy-Schwarz inequality, the first summand in equation (43), i.e.

$\mathbf{E} [\sup_{\boldsymbol{\eta}} |\varepsilon_t^2 + \omega| \cdot \|\mathbf{r}_t^\infty\| \cdot \|(\mathbf{y}_t^\infty)'\|]$, can be bounded from above by the terms

$$\sqrt{\mathbf{E}[\sup_{\boldsymbol{\eta}} |\varepsilon_t^2 + \omega|^2] \mathbf{E}[\sup_{\boldsymbol{\eta}} \|\mathbf{r}_t^\infty\|^2]} \quad (45)$$

and

$$\begin{aligned} &\sup_{\boldsymbol{\eta}} \sum_{j=0}^{\infty} j \beta^{js} \mathbf{E}[\sup_{\boldsymbol{\eta}} (\omega + \alpha \varepsilon_{t-1-j}^2)^s |\varepsilon_t^2 + \omega| \|\mathbf{r}_t^\infty\|] \leq \\ &\sup_{\boldsymbol{\eta}} \sum_{j=0}^{\infty} j \beta^{js} \sqrt{\mathbf{E}[\sup_{\boldsymbol{\eta}} (\omega + \alpha \varepsilon_{t-1-j}^2)^{2s} |\varepsilon_t^2 + \omega|^2] \mathbf{E}[\sup_{\boldsymbol{\eta}} \|\mathbf{r}_t^\infty\|^2]}. \end{aligned} \quad (46)$$

The finiteness of (45) follows from Assumption 4 and similar arguments as in the proof of Theorem 1. The finiteness of (46) follows by applying Hölder's inequality, since for the elements in the sum which involve expectations of the squared observations we have

$$\begin{aligned} &\mathbf{E}[\sup_{\boldsymbol{\eta}} (\omega + \alpha \varepsilon_{t-1-j}^2)^{2s} |\varepsilon_t^2 + \omega|^2] \leq \\ &(\mathbf{E}[\sup_{\boldsymbol{\eta}} (\omega + \alpha \varepsilon_{t-1-j}^2)^{2(1+s)}])^{s/(1+s)} (\mathbf{E}[\sup_{\boldsymbol{\eta}} |\varepsilon_t^2 + \omega|^{2(1+s)}])^{1/(1+s)} \end{aligned} \quad (47)$$

and again Assumption 4.

Using the Cauchy-Schwarz-Inequality for the two factors in the second term in (43), we are left with showing that $\mathbf{E} \left[\sup_{\boldsymbol{\eta}} \left\| \frac{\partial \mathbf{r}_t^\infty}{\partial \boldsymbol{\eta}'} \right\|^2 \right]$ is finite. This follows from

$$\begin{aligned} \frac{\partial \mathbf{r}_t^\infty}{\partial \boldsymbol{\eta}'} &= \frac{\partial}{\partial \boldsymbol{\eta}'} \mathbf{x}_t - \frac{\partial}{\partial \boldsymbol{\eta}'} \left(\frac{1}{h_t^\infty} \sum_{j=0}^{\infty} \alpha \beta^j \varepsilon_{t-1-j}^2 \mathbf{x}_{t-1-j} \right) \\ &= \frac{\partial}{\partial \boldsymbol{\eta}'} \mathbf{x}_t - \frac{1}{h_t^\infty} \left(\sum_{j=0}^{\infty} \alpha \beta^j \varepsilon_{t-1-j}^2 \frac{\partial}{\partial \boldsymbol{\eta}'} \mathbf{x}_{t-1-j} \right) \\ &\quad + \left(\frac{1}{h_t^\infty} \sum_{j=0}^{\infty} \alpha \beta^j \varepsilon_{t-1-j}^2 \mathbf{x}_{t-1-j} \right) (\mathbf{y}_t^\infty)' - \frac{1}{h_t^\infty} \sum_{j=0}^{\infty} \mathbf{x}_{t-1-j} \left(\frac{\partial}{\partial \boldsymbol{\eta}'} \alpha \beta^j \varepsilon_{t-1-j}^2 \right) \end{aligned} \quad (48)$$

The first two terms vanish in the GARCH-MIDAS-X with exogenous explanatory variable \mathbf{x}_t as $\frac{\partial \mathbf{x}_t}{\partial \boldsymbol{\eta}'} = 0$ or in the GARCH-MIDAS-RV with $x_{kt} = \varepsilon_{t-k}^2$.

Remark 3. Note that in the case of \mathbf{x}_t with elements $\mathbf{x}_{kt} = \frac{\varepsilon_{t-k}^2}{h_{t-k}^\infty}$ there also exists a bound for $\mathbf{E} \left[\sup_{\boldsymbol{\eta}} \left\| \frac{\partial \mathbf{r}_t^\infty}{\partial \boldsymbol{\eta}'} \right\|^2 \right]$. Here in the last two terms in (48) we have $\frac{\partial \mathbf{x}_{kt}}{\partial \boldsymbol{\eta}'} = -\frac{\varepsilon_{t-k}}{(h_{t-k}^\infty)^2} \frac{\partial h_{t-k}^\infty}{\partial \boldsymbol{\eta}'}$ and explicit bounds for terms of this type apply as before.

Boundedness of the norm of the third term follows for all $\boldsymbol{\eta}$ in expectation with a combination of the argument right above and the considerations in the proof of Theorem 1.

The fourth term can be written as:

$$\frac{1}{h_t^\infty} \begin{pmatrix} 0 & \sum_{j=0}^{\infty} \beta^j \varepsilon_{t-1-j}^2 x_{1,t-1-j} & \alpha \sum_{j=0}^{\infty} j \beta^{j-1} \varepsilon_{t-1-j}^2 x_{1,t-1-j} \\ 0 & \sum_{j=0}^{\infty} \beta^j \varepsilon_{t-1-j}^2 x_{2,t-1-j} & \alpha \sum_{j=0}^{\infty} j \beta^{j-1} \varepsilon_{t-1-j}^2 x_{2,t-1-j} \\ \vdots & \vdots & \vdots \\ 0 & \sum_{j=0}^{\infty} \beta^j \varepsilon_{t-1-j}^2 x_{K,t-1-j} & \alpha \sum_{j=0}^{\infty} j \beta^{j-1} \varepsilon_{t-1-j}^2 x_{K,t-1-j} \end{pmatrix} \quad (49)$$

Hence, for typical elements of the second and third column it follows that

$$\mathbf{E} \sup_{\boldsymbol{\eta}} \left| \frac{1}{h_t^\infty} \sum_{j=0}^{\infty} \beta^j \varepsilon_{t-1-j}^2 x_{k,t-1-j} \right|^2 < \infty$$

and

$$\mathbf{E} \sup_{\boldsymbol{\eta}} \left| \frac{1}{h_t^\infty} \alpha \sum_{j=0}^{\infty} j \beta^{j-1} \varepsilon_{t-1-j}^2 x_{K,t-1-j} \right|^2 < \infty$$

by similar arguments as used before.

■

Proof of Theorem 2. First, consider a mean value expansion of $\sqrt{T} \mathbf{D}_{\boldsymbol{\eta}}^\infty(\hat{\boldsymbol{\eta}})$ around the true value $\boldsymbol{\eta}_0$

$$\mathbf{0} = \sqrt{T} \mathbf{D}_{\boldsymbol{\eta}}^\infty(\hat{\boldsymbol{\eta}}) = \sqrt{T} \mathbf{D}_{\boldsymbol{\eta}}^\infty(\boldsymbol{\eta}_0) + \frac{1}{T} \sum_{t=1}^T \frac{\partial \mathbf{d}_{\boldsymbol{\eta},t}^\infty(\tilde{\boldsymbol{\eta}})}{\partial \boldsymbol{\eta}'} \sqrt{T}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0) \quad (50)$$

with $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta}_0 + o_P(1)$. Under Assumptions 1 and 2, Francq and Zakoian (2004) have shown that

$$-\frac{1}{T} \sum_{t=1}^T \frac{\partial \mathbf{d}_{\tilde{\boldsymbol{\eta}},t}^\infty}{\partial \boldsymbol{\eta}'} \xrightarrow{P} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}} = -\mathbf{E} \left[\frac{\partial \mathbf{d}_{\boldsymbol{\eta}_0,t}^\infty}{\partial \boldsymbol{\eta}'} \right] \quad (51)$$

and, hence, equation (50) can be written as

$$\sqrt{T}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0) = \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} \mathbf{D}_{\boldsymbol{\eta}}^\infty(\boldsymbol{\eta}_0) + o_p(1). \quad (52)$$

Similarly, a mean value expansion of $\sqrt{T} \mathbf{D}_\pi^\infty(\hat{\boldsymbol{\eta}})$ around the true value $\boldsymbol{\eta}_0$ leads to

$$\sqrt{T} \mathbf{D}_\pi^\infty(\hat{\boldsymbol{\eta}}) = \sqrt{T} \mathbf{D}_\pi^\infty(\boldsymbol{\eta}_0) + \frac{1}{T} \sum_{t=1}^T \frac{\partial \mathbf{d}_{\pi,t}^\infty(\tilde{\boldsymbol{\eta}})}{\partial \boldsymbol{\eta}'} \sqrt{T}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0). \quad (53)$$

Combining equation (52) and Proposition 1 leads to

$$\sqrt{T} \mathbf{D}_\pi^\infty(\hat{\boldsymbol{\eta}}) = \sqrt{T} \mathbf{D}_\pi^\infty(\boldsymbol{\eta}_0) - \mathbf{J}_{\pi\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} \sqrt{T} \mathbf{D}_\eta^\infty(\boldsymbol{\eta}_0) + o_P(1) \quad (54)$$

$$= [-\mathbf{J}_{\pi\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} : \mathbf{I}] \sqrt{T} \begin{pmatrix} \mathbf{D}_\eta^\infty(\boldsymbol{\eta}_0) \\ \mathbf{D}_\pi^\infty(\boldsymbol{\eta}_0) \end{pmatrix} + o_P(1) \quad (55)$$

$$= [-\mathbf{J}_{\pi\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} : \mathbf{I}] \sqrt{T} \mathbf{D}^\infty(\boldsymbol{\eta}_0) + o_P(1) \quad (56)$$

Applying Theorem 1 gives the asymptotic distribution

$$\sqrt{T} \mathbf{D}_\pi^\infty(\hat{\boldsymbol{\eta}}) \xrightarrow{d} \mathcal{N}(\mathbf{0}, [\mathbf{J}_{\pi\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} : \mathbf{I}] \boldsymbol{\Omega} [\mathbf{J}_{\pi\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} : \mathbf{I}]') \quad (57)$$

which has the form of $\mathbf{A}\boldsymbol{\Omega}\mathbf{A}'$ in Halunga and Orme (2009, p.372/373). The covariance matrix can be written as

$$\begin{aligned} \boldsymbol{\Sigma} &= [-\mathbf{J}_{\pi\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} : \mathbf{I}] \boldsymbol{\Omega} [-\mathbf{J}_{\pi\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} : \mathbf{I}]' \\ &= \boldsymbol{\Omega}_{\pi\pi} + \mathbf{J}_{\pi\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} \boldsymbol{\Omega}_{\boldsymbol{\eta}\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} \mathbf{J}'_{\pi\boldsymbol{\eta}} - \mathbf{J}_{\pi\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} \boldsymbol{\Omega}_{\boldsymbol{\eta}\pi} - \boldsymbol{\Omega}_{\pi\boldsymbol{\eta}} \mathbf{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1} \mathbf{J}'_{\pi\boldsymbol{\eta}} \end{aligned}$$

Finally, using equations (23), (26) and (27) the expression for $\boldsymbol{\Sigma}$ simplifies to:

$$\boldsymbol{\Sigma} = \frac{1}{4}(\kappa_Z - 1) \left(\mathbf{E}[\mathbf{r}_{0,t}^\infty (\mathbf{r}_{0,t}^\infty)'] - \mathbf{E}[\mathbf{r}_{0,t}^\infty (\mathbf{y}_{0,t}^\infty)'] (\mathbf{E}[\mathbf{y}_{0,t}^\infty (\mathbf{y}_{0,t}^\infty)'])^{-1} \mathbf{E}[\mathbf{y}_{0,t}^\infty (\mathbf{r}_{0,t}^\infty)'] \right). \quad (58)$$

■

Proof of Theorem 3. First, we show that

$$\sqrt{T} \mathbf{D}_\pi(\hat{\boldsymbol{\eta}}) = \sqrt{T} \mathbf{D}_\pi^\infty(\hat{\boldsymbol{\eta}}) + o_p(1). \quad (59)$$

Hence, $\sqrt{T} \mathbf{D}_\pi(\hat{\boldsymbol{\eta}})$ will have the same asymptotic distribution as $\sqrt{T} \mathbf{D}_\pi^\infty(\hat{\boldsymbol{\eta}})$. The asymptotic distribution of the test statistic follows then directly from Theorem 2. Standardization with the estimated consistent $\hat{\boldsymbol{\Sigma}}$ instead of the theoretical $\boldsymbol{\Sigma}$, has no effect on

the final χ^2 -distribution of the LM test statistic. This can be easily seen from similar considerations as the ones outlined above and below in detail.

Since

$$\sup_{\boldsymbol{\eta}} \|\sqrt{T}\mathbf{D}_{\boldsymbol{\pi}}^{\infty}(\boldsymbol{\eta}) - \sqrt{T}\mathbf{D}_{\boldsymbol{\pi}}(\boldsymbol{\eta})\| \leq \frac{1}{\sqrt{T}} \sum_{t=1}^T \sup_{\boldsymbol{\eta}} \|\mathbf{d}_{\boldsymbol{\pi},t}^{\infty}(\boldsymbol{\eta}) - \mathbf{d}_{\boldsymbol{\pi},t}(\boldsymbol{\eta})\| \quad (60)$$

we establish equation (59) by showing that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^T \sup_{\boldsymbol{\eta}} \|\mathbf{d}_{\boldsymbol{\pi},t}^{\infty}(\boldsymbol{\eta}) - \mathbf{d}_{\boldsymbol{\pi},t}(\boldsymbol{\eta})\| = o_p(1). \quad (61)$$

Consider the following decomposition:

$$\begin{aligned} 2(\mathbf{d}_{\boldsymbol{\pi},t}^{\infty}(\boldsymbol{\eta}) - \mathbf{d}_{\boldsymbol{\pi},t}(\boldsymbol{\eta})) &= \left(\frac{\varepsilon_t^2}{h_t^{\infty}} - 1\right) \mathbf{r}_t^{\infty} - \left(\frac{\varepsilon_t^2}{h_t} - 1\right) \mathbf{r}_t \\ &= \left(\frac{\varepsilon_t^2}{h_t^{\infty}} - 1\right) \mathbf{r}_t^{\infty} - \left(\frac{\varepsilon_t^2}{h_t} - 1\right) \mathbf{r}_t + \left[\left(\frac{\varepsilon_t^2}{h_t} - 1\right) \mathbf{r}_t^{\infty} - \left(\frac{\varepsilon_t^2}{h_t} - 1\right) \mathbf{r}_t^{\infty} \right] \\ &= \left(\frac{\varepsilon_t^2}{h_t^{\infty}} - \frac{\varepsilon_t^2}{h_t}\right) \mathbf{r}_t^{\infty} + \left(\frac{\varepsilon_t^2}{h_t} - 1\right) (\mathbf{r}_t^{\infty} - \mathbf{r}_t) \\ &= \varepsilon_t^2 \left(\frac{h_t - h_t^{\infty}}{h_t^{\infty} h_t}\right) \mathbf{r}_t^{\infty} + \left(\frac{\varepsilon_t^2}{h_t} - 1\right) (\mathbf{r}_t^{\infty} - \mathbf{r}_t) + \\ &\quad \left[\left(\frac{\varepsilon_t^2}{h_t^{\infty}} - 1\right) (\mathbf{r}_t^{\infty} - \mathbf{r}_t) - \left(\frac{\varepsilon_t^2}{h_t^{\infty}} - 1\right) (\mathbf{r}_t^{\infty} - \mathbf{r}_t) \right] \\ &= \varepsilon_t^2 \left(\frac{h_t - h_t^{\infty}}{h_t^{\infty} h_t}\right) \mathbf{r}_t^{\infty} + \varepsilon_t^2 \left(\frac{h_t - h_t^{\infty}}{h_t^{\infty} h_t}\right) (\mathbf{r}_t^{\infty} - \mathbf{r}_t) + \left(\frac{\varepsilon_t^2}{h_t^{\infty}} - 1\right) (\mathbf{r}_t^{\infty} - \mathbf{r}_t) \end{aligned}$$

Since $h_t \geq \omega > 0$ and $h_t^{\infty} \geq \omega > 0$ we have

$$\|\mathbf{d}_{\boldsymbol{\pi},t}^{\infty}(\boldsymbol{\theta}) - \mathbf{d}_{\boldsymbol{\pi},t}(\boldsymbol{\theta})\| \leq c \left\{ |\varepsilon_t^2 + \omega| \|\mathbf{r}_t^{\infty} - \mathbf{r}_t\| + \varepsilon_t^2 \|\mathbf{r}_t^{\infty}\| \left| \frac{h_t^{\infty} - h_t}{h_t^{\infty}} \right| + \varepsilon_t^2 \|\mathbf{r}_t^{\infty} - \mathbf{r}_t\| \left| \frac{h_t^{\infty} - h_t}{h_t^{\infty}} \right| \right\}$$

with $c = 1/\omega$.

First, note that

$$\mathbf{r}_t^{\infty} - \mathbf{r}_t = -\alpha \frac{1}{h_t^{\infty}} \sum_{j=t}^{\infty} \beta^j \varepsilon_{t-1-j}^2 \mathbf{x}_{t-1-j}. \quad (62)$$

Next, consider a typical element:

$$\begin{aligned}
(\mathbf{E}\sup_{\boldsymbol{\eta}} |r_{k,t}^\infty - r_{k,t}|^2)^{1/2} &= \left(\mathbf{E}\sup_{\boldsymbol{\eta}} \left| \alpha \frac{1}{h_t^\infty} \sum_{j=t}^{\infty} \beta^j \varepsilon_{t-1-j}^2 x_{t-1-k-j} \right|^2 \right)^{1/2} \\
&\leq \sum_{j=t}^{\infty} \left(\mathbf{E}\sup_{\boldsymbol{\eta}} \left| \frac{\alpha \beta^j \varepsilon_{t-1-j}^2}{\omega + \alpha \beta^j \varepsilon_{t-1-k-j}^2} x_{t-1-k-j} \right|^2 \right)^{1/2} \\
&\leq \sum_{j=t}^{\infty} \left(\mathbf{E}\sup_{\boldsymbol{\eta}} \left| \left(\frac{\alpha \beta^j}{\omega} \varepsilon_{t-1-j}^2 \right)^{s/4} x_{t-1-k-j} \right|^2 \right)^{1/2} \\
&\leq (\mathbf{E}[|\varepsilon_{t-1-j}|^{2s}])^{1/4} (\mathbf{E}[|x_{t-1-k-j}|^4])^{1/4} \\
&\quad \sup_{\boldsymbol{\eta}} \left(\frac{\alpha}{\omega} \right)^{s/4} \sum_{j=t}^{\infty} \beta^{js/4} \\
&= (\mathbf{E}[|\varepsilon_{t-1-j}|^{2s}])^{1/4} (\mathbf{E}[|x_{t-1-k-j}|^4])^{1/4} \\
&\quad \sup_{\boldsymbol{\eta}} \left(\frac{\alpha}{\omega} \right)^{s/4} \frac{(\beta^{s/4})^t}{1 - \beta^{s/4}}
\end{aligned} \tag{63}$$

which shows that $\mathbf{E}\sup_{\boldsymbol{\eta}} \|\mathbf{r}_{k,t}^\infty - \mathbf{r}_{k,t}\|^2 = O(\beta^{ts/2})$.

Hence,

$$\mathbf{E}\sup_{\boldsymbol{\eta}} |\varepsilon_t^2| \|\mathbf{r}_t^\infty - \mathbf{r}_t\| \leq \sqrt{\mathbf{E}\sup_{\boldsymbol{\eta}} |\varepsilon_t^4| \mathbf{E}\sup_{\boldsymbol{\eta}} \|\mathbf{r}_t^\infty - \mathbf{r}_t\|^2} = O(\rho^{ts/4})$$

by Assumption 1 and equation (63). Therefore, $\frac{1}{\sqrt{T}} \sum_{t=1}^T \mathbf{E}\sup_{\boldsymbol{\eta}} |\varepsilon_t^2| \|\mathbf{r}_t^\infty - \mathbf{r}_t\| = o(1)$ and, hence, by Markov's inequality $\frac{1}{\sqrt{T}} \sum_{t=1}^T \sup_{\boldsymbol{\eta}} |\varepsilon_t^2| \|\mathbf{r}_t^\infty - \mathbf{r}_t\| = o_P(1)$.

For the treatment of the second term we employ that

$$\left| \frac{h_t^\infty - h_t}{h_t^\infty} \right| \leq \frac{\alpha^s}{\omega^s} \sum_{j=t}^{\infty} (\beta^s)^j \varepsilon_{t-j}^{2s} \tag{64}$$

where again we use that $w/(1+w) \leq w^s$ for all $w > 0$ and any $s \in (0, 1)$. Then,

$$\begin{aligned}
\mathbf{E}\sup_{\boldsymbol{\eta}} \varepsilon_t^2 \|\mathbf{r}_t^\infty\| \left| \frac{h_t^\infty - h_t}{h_t^\infty} \right| &\leq \mathbf{E}\sup_{\boldsymbol{\eta}} \|\varepsilon_t^2 \mathbf{r}_t^\infty \varepsilon_{t-j}^{2s}\| \sup_{\boldsymbol{\eta}} \frac{\alpha^s}{\omega^s} \sum_{j=t}^{\infty} (\beta^s)^j \\
&\leq \sqrt{\mathbf{E}\sup_{\boldsymbol{\eta}} \|\mathbf{r}_t^\infty\|^2 \mathbf{E}|\varepsilon_t^4 \varepsilon_{t-j}^{4s}|} \sup_{\boldsymbol{\eta}} \frac{\alpha^s}{\omega^s} (\beta^s)^t \sum_{j=0}^{\infty} (\beta^s)^j \\
&= \sqrt{\mathbf{E}\sup_{\boldsymbol{\eta}} \|\mathbf{r}_t^\infty\|^2 \mathbf{E}|\varepsilon_t^4 \varepsilon_{t-j}^{4s}|} \sup_{\boldsymbol{\eta}} \frac{\alpha^s}{\omega^s (1 - \beta^s)} (\beta^s)^t \\
&= O((\beta^s)^t)
\end{aligned} \tag{65}$$

The last line follows because it can be shown by similar arguments as in the proof of Theorem 1 that $\mathbf{E}\sup_{\boldsymbol{\eta}} \|\mathbf{r}_t^\infty\|^2 < \infty$ and because Hölder's inequality and Assumption 4

imply that $\mathbf{E}|\varepsilon_t^4 \varepsilon_{t-j}^{4s}| \leq \left(\mathbf{E}|\varepsilon_t^{4(1+s)}|\right)^{1/(1+s)} \left(\mathbf{E}|\varepsilon_{t-j}^{4(1+s)}|\right)^{s/(1+s)} < \infty$. Equation (65) implies that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^T \mathbf{E} \sup_{\boldsymbol{\eta}} \varepsilon_t^2 \|\mathbf{r}_t^\infty\| \left| \frac{h_t^\infty - h_t}{h_t^\infty} \right| = o(1), \quad (66)$$

and, again, by Markov's inequality $\frac{1}{\sqrt{T}} \sum_{t=1}^T \sup_{\boldsymbol{\eta}} \varepsilon_t^2 \|\mathbf{r}_t^\infty\| |(h_t^\infty - h_t)/h_t^\infty| = o_P(1)$.

The third term can be treated as follows:

$$\begin{aligned} \frac{1}{\sqrt{T}} \sum_{t=1}^T \sup_{\boldsymbol{\eta}} \varepsilon_t^2 \|\mathbf{r}_t^\infty - \mathbf{r}_t\| \left| \frac{h_t^\infty - h_t}{h_t^\infty} \right| &\leq \sqrt{\frac{1}{T} \sum_{t=1}^T \sup_{\boldsymbol{\eta}} \varepsilon_t^4 \|\mathbf{r}_t^\infty - \mathbf{r}_t\|^2 \sum_{t=1}^T \sup_{\boldsymbol{\eta}} \left| \frac{h_t^\infty - h_t}{h_t^\infty} \right|^2} \\ &\leq \left\{ \frac{1}{\sqrt{T}} \sum_{t=1}^T \sup_{\boldsymbol{\eta}} \varepsilon_t^2 \|\mathbf{r}_t^\infty - \mathbf{r}_t\| \right\} \left\{ \sum_{t=1}^T \sup_{\boldsymbol{\eta}} \left| \frac{h_t^\infty - h_t}{h_t^\infty} \right| \right\}. \end{aligned}$$

because $\sum_{t=1}^T w_t^2 \leq \left\{ \sum_{t=1}^T z_t \right\}$ when $w_t \geq 0$ for all t . Above, we have already shown that $\sum_{t=1}^T \mathbf{E} \sup_{\boldsymbol{\eta}} \varepsilon_t^2 \|\mathbf{r}_t^\infty - \mathbf{r}_t\| = O(1)$ and $\mathbf{E} \sup_{\boldsymbol{\eta}} \left| \frac{h_t^\infty - h_t}{h_t^\infty} \right| = O(\beta^{ts})$.

■