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Unique Equilibrium in Rent-Seeking Contests with a Continuum of Types

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Abstract  It is shown that rent-seeking contests with continuous and independent type distributions possess a unique pure-strategy Nash equilibrium.

Keywords  Rent-seeking · Private information · Pure-strategy Nash equilibrium · Existence · Uniqueness

JEL Classification  C7, D7, D8

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1 Introduction

While rent-seeking contests with continuous and independent type distributions are quite interesting, basic issues such as existence and uniqueness of a pure-strategy Nash equilibrium (PSNE) have been addressed only partially. Indeed, previous work on the issue of existence focused either on symmetric contests (Fey, 2008; Ryvkin, 2010) or on the case of a continuous technology (Wasser, 2013a, 2013b). Moreover, little general was known about the uniqueness of the equilibrium.

Below, it is shown that in any rent-seeking contest with independent and continuous types, there exists a unique PSNE. The result holds even when the contest is ex-ante asymmetric, so that the equilibrium may entail inactive types. Moreover, no restriction is imposed on the shape of the type distributions. Generally, existence ensures consistency of a model, whereas uniqueness strengthens numerical analyses, theoretical results, and experimental findings.

The rest of the paper is structured as follows. Section 2 describes the set-up. Existence is dealt with in Section 3. Section 4 discusses uniqueness. A numerical illustration can be found in Section 5. Section 6 concludes. An Appendix contains technical lemmas.

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1 Uniqueness means here that for any given player, any two PSNE strategies differ at most on a null set. This corresponds to the strongest form of uniqueness for PSNE.

2 Asymmetry may be reflected, e.g., in heterogeneous distributions of marginal costs or in heterogeneous economies of scale.

3 Wärneryd (2003) explicitly allows for inactive types in a common-value setting.
There are $N \geq 2$ players. Each player $i = 1, \ldots, N$ observes a signal (or type) $c_i$, drawn from an interval $D_i = [c_i, \bar{c}_i]$, where $0 < c_i < \bar{c}_i$. Signals are independent across players. Moreover, player $i$ does not observe the signal $c_j$ of any other player $j \neq i$. The distribution function of player $i$'s signal is denoted by $F_i = F_i(c_i)$. Each player $i$ chooses a level of activity $y_i \geq 0$ at cost $g_i(y_i)$. It is assumed that $g_i(0) = 0$, and that $g_i$ is twice continuously differentiable on $\mathbb{R}_+$, with $g'_i > 0$ on $\mathbb{R}_+$, and $g''_i \geq 0$. Player $i$'s payoff is
\begin{equation}
\Pi_i(y_i, y_{-i}, c_i) = p_i(y_i, y_{-i}) - c_i g_i(y_i), \quad \text{where } p_i(y_i, y_{-i}) = y_i/(y_i + \sum_{j \neq i} y_j) \text{ if } y_i + \sum_{j \neq i} y_j > 0, \text{ and } p_i(y_i, y_{-i}) = 1/N \text{ otherwise.} \tag{4}
\end{equation}

A strategy for player $i$ is a (measurable) mapping $\sigma_i : D_i \to \mathbb{R}_+$. Denote by $S_i$ the set of strategies for player $i$. For a profile $\sigma_{-i} = \{\sigma_j\}_{j \neq i} \in S_{-i} = \prod_{j \neq i} S_j$, and a type $c_i \in D_i$, player $i$'s interim expected payoff is given by
\begin{equation}
\bar{\Pi}_i(y_i, \sigma_{-i}, c_i) = \int_{D_{-i}} \Pi_i(y_i, \sigma_{-i}(c_{-i}), c_i) dF_{-i}(c_{-i}), \quad \text{where } D_{-i} = \prod_{j \neq i} D_j,
\end{equation}
\begin{equation}
\sigma_{-i}(c_{-i}) = \{\sigma_j(c_j)\}_{j \neq i}, \text{ and } dF_{-i}(c_{-i}) = \prod_{j \neq i} dF_j(c_j). \quad \text{A Bayesian Nash equilibrium (BNE) is a profile } \sigma^* = \{\sigma_i^*\}_{i=1}^N \in S = \prod_{i=1}^N S_i \text{ such that } \Pi_i(\sigma^*_i(c_i), \sigma^*_{-i}, c_i) \geq \Pi_i(y_i, \sigma^*_{-i}, c_i) \text{ for any } i = 1, \ldots, N, \text{ any } c_i \in D_i, \text{ and any } y_i \geq 0. \tag{5}
\end{equation}

A pure-strategy Nash equilibrium (PSNE) is a profile $\sigma^* \in S$ such that for any $i = 1, \ldots, N$, and for almost any $c_i \in D_i$, the inequality
\begin{equation}
\Pi_i(\sigma^*_i(c_i), \sigma^*_{-i}, c_i) \geq \Pi_i(y_i, \sigma^*_{-i}, c_i) \text{ holds for any } y_i \geq 0. \tag{6}
\end{equation}

\footnote{As usual, a simple change of variables allows to capture other types of contest success functions and other forms of uncertainty, e.g., about valuations. Cf. Ryvkin (2010).}

\footnote{As shown in the Appendix, this amounts to the standard definition.}
3 Existence

This section builds on prior work by Fey (2008), Ryvkin (2010), and Wasser (2013a). Existence is shown first for the $\varepsilon$-constrained contest, for $\varepsilon > 0$, in which each player $i = 1, \ldots, N$ may use only strategies with values in $[\varepsilon, \infty)$.

Lemma 3.1 There is a level of activity $E > 0$ such that, for any sufficiently small $\varepsilon > 0$, there exists a BNE $\sigma^\varepsilon$ in the $\varepsilon$-constrained contest such that each player $i$’s strategy $\sigma_i^\varepsilon$ is continuous, monotone, and bounded by $E$.

Proof. Since costs are strictly increasing and convex, there is an $E > 0$ such that any $y_i > E$ is suboptimal. Moreover, $\Pi_i$ exhibits decreasing differences in $y_i$ and $c_i$. Hence, existence of a monotone PSNE $\bar{\sigma}^\varepsilon$ in the $\varepsilon$-constrained contest follows from Athey (2001, Cor. 2.1). Note now that type $c_i$’s $\varepsilon$-constrained problem, $\max_{y_i \geq \varepsilon} \Pi_i(y_i, \bar{\sigma}^\varepsilon_{-i}, c_i)$, has a unique solution $y_i = \sigma_i^\varepsilon(c_i)$. Indeed, if $\bar{\sigma}^\varepsilon_{-i}(c_{-i}) \neq 0$ with positive probability, then $\Pi_i(\cdot, \bar{\sigma}^\varepsilon_{-i}, c_i)$ is strictly concave on $[\varepsilon, E]$, while otherwise, the unique solution is $y_i = \varepsilon$. Hence, $\sigma_i^\varepsilon(c_i) = \bar{\sigma}^\varepsilon_i(c_i)$ with probability one, for any $i = 1, \ldots, N$. This implies that $\sigma_i^\varepsilon(c_i)$ is also type $c_i$’s best response to $\sigma^\varepsilon_{-i}$, for any $i = 1, \ldots, N$, and any $c_i \in D_i$. Thus, $\sigma^\varepsilon = (\sigma_1^\varepsilon, \ldots, \sigma_N^\varepsilon)$ is a BNE in the $\varepsilon$-constrained contest. Clearly, each $\sigma_i^\varepsilon$ is monotone. Finally, continuity of $\sigma_i^\varepsilon$ follows from Berge’s Theorem, as $\Pi_i(\cdot, \sigma^\varepsilon_{-i}, \cdot)$ is continuous on the compact set $[\varepsilon, E] \times D_i$. \qed

Consider now a sequence $\{\varepsilon_m\}_{m=1}^\infty$ such that $\varepsilon_m \searrow 0$, and select a BNE $\sigma_m$ in the $\varepsilon_m$-constrained contest for each $m \in \mathbb{N}$, with the properties specified in the previous lemma.

Lemma 3.2 The sequence $\{\sigma^m\}_{m=1}^\infty$ has a uniformly converging subse-
Proof. In view of Lemma 3.1 and the Theorem of Arzelà-Ascoli, it suffices to find a \( \lambda > 0 \) such that \( \sigma_i^m \) has everywhere a slope exceeding \(-\lambda\) for any \( m \in \mathbb{N} \) and any \( i \). In terms of the transformed choice variable \( y_i^\lambda = y_i + \lambda c_i \), a type \( c_i \)'s expected payoff in \( \sigma^m \) may be written as

\[
\Pi_i(y_i^\lambda, \sigma_i^m, c_i) = \int_{D_{-i}} \frac{(y_i^\lambda - \lambda c_i) dF_{-i}(c_{-i})}{y_i^\lambda - \lambda c_i + \sum_{j \neq i} \sigma_j^m(c_j)} - c_i g_i(y_i^\lambda - \lambda c_i),
\]

provided that \( y_i^\lambda - \lambda c_i = y_i > 0 \). Hence, for \( \lambda \) sufficiently large, the cross-partial

\[
\frac{\partial^2 \Pi_i^\lambda}{\partial y_i^\lambda \partial c_i} = \int_{D_{-i}} \frac{2 \lambda \sum_{j \neq i} \sigma_j^m(c_j) dF_{-i}(c_{-i})}{(y_i + \sum_{j \neq i} \sigma_j^m(c_j))^3} - g_i'(y_i) - c_i \lambda g_i''(y_i)\geq 0
\]

\[
\geq \frac{2 \lambda}{NE} \int_{D_{-i}} \frac{\sum_{j \neq i} \sigma_j^m(c_j) dF_{-i}(c_{-i})}{(y_i + \sum_{j \neq i} \sigma_j^m(c_j))^2} - g_i'(y_i)
\]

\[
\geq \frac{2 \lambda c_i}{NE} \frac{1}{1} g_i'(y_i)
\]

is seen to be positive in the range of \( c_i \) where \( y_i = \sigma_i^m(c_i) > 0 \). Thus, for \( \lambda \) large, \( y_i^\lambda \) is weakly increasing in \( c_i \), which proves the claim. \( \square \)

By Lemma 3.2, one may assume that \( \{\sigma^m\}_{m=1}^\infty \) converges uniformly to some \( \sigma^* \in S \). Next, it is shown that in \( \sigma^* \), at least one player is active with probability one.

**Lemma 3.3** There is some player \( i \) such that \( \sigma_i^*(c_i) > 0 \) with probability one.
Proof. Suppose that for each $i$, there is a set $D_i \subseteq D_i$ of positive measure such that $\sigma^*_i(c_i) = 0$ for all $c_i \in D_i$. Then, by uniform convergence, there exists, for any $\varepsilon > 0$, an $m_0 = m_0(\varepsilon)$ such that $\sigma^m_i(c_i) < \varepsilon$ for any $i$, any $c_i \in D_i$, and any $m \geq m_0$. But, from the Kuhn-Tucker condition for type $c_i$ in the $\varepsilon_m$-constrained contest,

$$0 \geq \int_{D_{-i}} \frac{\sum_{j \neq i} \sigma^m_j(c_j) dF_{-i}(c_{-i})}{\left(\sigma^m_i(c_i) + \sum_{j \neq i} \sigma^m_j(c_j)\right)^2} - c_i g'_i(E),$$

(5)

where $D_{-i} = \prod_{j \neq i} D_j$. Integrating over $D_i$, and subsequently summing over $i = 1, \ldots, N$, one obtains

$$0 \geq \int_{D} \frac{(N-1)dF(c)}{\sum_{i=1}^{N} \sigma^m_i(c_i)} - \sum_{i=1}^{N} g'_i(E) \int_{D_i} c_i dF_i(c_i),$$

(6)

where $D = \prod_{i=1}^{N} D_i$ and $dF(c) = \prod_{i=1}^{N} dF_i(c_i)$. For $\varepsilon$ small, however, this is impossible. \qed

The following is the first main result of this paper.

**Theorem 3.4** In the unconstrained contest, $\sigma^*$ is a PSNE in continuous and monotone strategies.

Proof. Fix a player $i \in \{1, \ldots, N\}$. For any $m \in \mathbb{N}$, since $\sigma^m$ is a BNE in the $\varepsilon_m$-constrained contest, $\Pi_i(\sigma^m_i(c_i), \sigma^m_{-i}, c_i) \geq \Pi_i(y_i, \sigma^m_{-i}, c_i)$ for any $c_i \in D_i$ and any $y_i \geq \varepsilon_m$. Therefore, if the event $\sigma^*_m(c_{-i}) = 0$ is null, letting $m \to \infty$ implies $\Pi_i(\sigma^*_i(c_i), \sigma^*_m_{-i}, c_i) \geq \Pi_i(y_i, \sigma^*_m_{-i}, c_i)$ for any $c_i \in D_i$ and any $y_i > 0$. Suppose next that $\sigma^*_m(c_{-i}) = 0$ with positive probability. Then, by Lemma 3.3, $\sigma^*_i(c_i) > 0$ with probability one. Let $c_i \in D_i$ with
\( \sigma^*_i(c_i) > 0 \). If \( y_i > 0 \), then the argument proceeds as above. To complete the proof, note that \( \Pi_i(\cdot, \sigma^*_i, c_i) \) is l.s.c., so that \( y_i = 0 \) cannot be the only profitable deviation for \( c_i \). \( \square \)

4 Uniqueness

Consider two PSNE \( \sigma^* \) and \( \sigma^{**} \) such that, for some player \( i \), the event \( \sigma^*_i(c_i) \neq \sigma^{**}_i(c_i) \) has positive probability. Then, as noted below, \( \sigma^* \) and \( \sigma^{**} \) must differ in an essential way for at least two players.

**Lemma 4.1** There are players \( i \neq j \) such that each of the independent events \( \sigma^*_i(c_i) \neq \sigma^{**}_i(c_i) \) and \( \sigma^*_j(c_j) \neq \sigma^{**}_j(c_j) \) has positive probability.

**Proof.** Suppose there is some \( i \) such that \( \sigma^*_i(c_i) = \sigma^{**}_i(c_i) \) with probability one. Then, \( \Pi_i(\cdot, \sigma^*_i, c_i) = \Pi_i(\cdot, \sigma^{**}_i, c_i) \) for any \( c_i \in D_i \). Thus, \( \sigma^*_i(c_i) = \sigma^{**}_i(c_i) \) with probability one, which is a contradiction. \( \square \)

The following is the second main result of this paper.

**Theorem 4.2** The PSNE in the unconstrained contest is unique.

**Proof.** Following Rosen (1965), write \( \sigma^{s*} = (1-s)\sigma^* + s\sigma^{**} \) for \( 0 \leq s \leq 1 \), and consider

\[
\Phi_s = \sum_{i=1}^{N} \int_{D_i} \bar{\pi}_i(\sigma^{s*}, c_i) (\sigma^{**}_i(c_i) - \sigma^*_i(c_i)) dF_i(c_i) \tag{7}
\]

for \( s = 0, 1 \), where \( \pi_i(\sigma, c_i) = \partial \Pi_i(\sigma(c_i), \sigma_{-i}, c_i)/\partial y_i \) denotes type \( c_i \)'s marginal expected payoff at a profile \( \sigma \in S \).\(^6\) From the Kuhn-Tucker con-

\(^6\)It is shown in the Appendix that \( \Phi_0 \) and \( \Phi_1 \) are well-defined.
ditions, \( \pi_i(\sigma^*, c_i) \leq 0 \) for almost any \( c_i \in D_i \); moreover, \( \sigma^*_i(c_i) = 0 \) if \( \pi_i(\sigma^*, c_i) < 0 \). It follows that \( \Phi_0 \leq 0 \), and similarly, \( \Phi_1 \geq 0 \). To provoke a contradiction, it will be shown now that \( \Phi_1 - \Phi_0 < 0 \). Denote by 
\[
\pi_i(\sigma, c_i, c_{-i}) = \partial \Pi_i(\sigma_i(c_i), \sigma_{-i}(c_{-i}), c_i)/\partial y_i \text{ type } c_i \text{'s marginal ex-post payoff at } \sigma \in S, \text{ when facing } c_{-i} \in D_{-i}.
\]
Then, by Lemma A.2 in the Appendix,
\[
\Phi_1 - \Phi_0 = \int_D \sum_{i=1}^N (\pi_i(\sigma^{**}, c_i, c_{-i}) - \pi_i(\sigma^*, c_i, c_{-i})) z_i(c_i) dF(c) \quad (8)
\]
\[
= \int_D \sum_{i=1}^N \left\{ \int_0^1 \frac{\partial \pi_i(\sigma^{**}, c_i, c_{-i})}{\partial s} z_i(c_i) ds \right\} dF(c), \quad (9)
\]
where \( z_i(c_i) = \sigma_i^{**}(c_i) - \sigma_i^*(c_i) \). An application of the chain rule delivers
\[
\frac{\partial \pi_i(\sigma^{**}, c_i, c_{-i})}{\partial s} = \sum_{j=1}^N \frac{\partial^2 p_i(\sigma_i^{**}(c_i), \sigma_{-i}^{**}(c_{-i}))}{\partial y_i \partial y_j} z_j(c_j) - c_i g_i''(\sigma_i^{**}(c_i)) z_i(c_i), \quad (10)
\]
for any \( i \), any \( c_i \in D_i \), and any \( c_{-i} \in D_{-i} \). It follows that
\[
\Phi_1 - \Phi_0 \leq \int_D \left( \int_0^1 \left( \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 p_i(\sigma_i^{**}(c_i), \sigma_{-i}^{**}(c_{-i}))}{\partial y_i \partial y_j} z_i(c_i) z_j(c_j) \right) ds \right) dF(c). \quad (11)
\]
One can verify, however, that
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 p_i(y_i, y_{-i})}{\partial y_i \partial y_j} z_i z_j
\]
\[
= - \sum_{i=1}^{N} \frac{2Y_{-i}}{Y^3} z_i^2 + \sum_{i=1}^{N} \sum_{j \neq i} \frac{Y - 2Y_{-i}}{Y^3} z_i z_j
\]
\[
= - \frac{2}{Y^3} \sum_{i=1}^{N} Y_{-i} z_i^2 - \frac{2}{Y^3} \sum_{i=1}^{N} \sum_{j > i} \sum_{k \neq i, j} y_k z_i z_j
\]
\[
= - \frac{1}{Y^3} \sum_{i=1}^{N} Y_{-i} z_i^2 - \frac{1}{Y^3} \sum_{i=1}^{N} \sum_{j = 1}^{N} \sum_{k \neq i, j} y_k z_i z_j
\]
\[
= - \frac{1}{Y^3} \sum_{i=1}^{N} (z_i^2 Y_{-i} + y_i Z_{-i}) \leq 0
\]
for any \((y_1, ..., y_N) \in \mathbb{R}_+^N \setminus \{0\}\) and any \((z_1, ..., z_N) \in \mathbb{R}^N\), where \(Y = \sum_{i=1}^{N} y_i\), \(Y_{-i} = \sum_{j \neq i} y_j\), and \(Z_{-i} = \sum_{j \neq i} z_j\). Moreover, \(z_i^2 Y_{-i} = z_i(c_i)^2 \frac{2}{Y^3} \sum_{j \neq i} \sigma_j^*(c_j)\) is positive for any \(s \in (0, 1)\) if \(\sigma_i^*(c_i) \neq \sigma_i^*(c_j)\) and \(\sigma_j^*(c_j) \neq \sigma_j^*(c_j)\) for some \(j \neq i\). Thus, by Lemma 4.1, \(\Phi_1 - \Phi_0 < 0\).

5 Numerical illustration

Figure 1 shows PSNE strategies in a two-player lottery contest, where types are distributed uniformly on \(D_1 = [0.01, 1.01]\) and \(D_2 = [0.51, 5.51]\), respectively. Note that player 2 remains inactive for \(c_2 > c_2^* \approx 4.21\).
Figure 1: An equilibrium involving inactive types

6 Concluding remark

While this paper has focused on the existence and uniqueness of a PSNE in asymmetric rent-seeking contests, it follows from the proofs that also any of the BNE studied by Fey (2008) and Ryvkin (2010) is unique.

7 Appendix: Technical lemmas

Lemma A.1 A profile $\sigma^* \in S$ is a PSNE in the unconstrained contest if and only if $\int_D \Pi_i(\sigma^*_i(c_i), \sigma^*_{-i}(c_{-i}), c_i)dF(c) \geq \int_D \Pi_i(\tilde{\sigma}_i(c_i), \sigma^*_{-i}(c_{-i}), c_i)dF(c)$ for any $i = 1, ..., N$, and any $\tilde{\sigma}_i \in S_i$.

Proof. Let $\sigma^*$ be a PSNE, and consider a deviation $\tilde{\sigma}_i \in S_i$ for some player $i$. Then, $\Pi_i(\sigma^*_i(c_i), \sigma^*_{-i}, y_i) \geq \Pi_i(\tilde{\sigma}_i(c_i), \sigma^*_{-i}, c_i)$ for almost any $c_i \in D_i$. Integrating over $D_i$, the assertion follows via Fubini’s theorem. Conversely,
suppose that $\sigma^*$ is not a PSNE. Then, there is a player $i$ and a set $D_i \subseteq D_i$ of positive measure such that $\sigma^*_i(c_i)$ is not a best response to $\sigma^*_{-i}$ for $c_i$, for any $c_i \in D_i$. Define $\tilde{\sigma}_i(c_i)$ as $c_i$’s best response to $\sigma^*_{-i}$ if it exists; otherwise as $\sigma^*_i(c_i)/2$ if $\sigma^*_i(c_i) > 0$, and as $\Pr\{\sigma^*_{-i}(c_{-i}) = 0\}/(2\mathcal{g}_i(E))$ if $\sigma^*_i(c_i) = 0$. Then $\tilde{\sigma}_i$ is a profitable deviation. \(\square\)

**Lemma A.2** Let $\sigma^* \in S$ be a PSNE in the unconstrained contest. Then, for almost any $c_i \in D_i$, the function $\pi_i(\sigma^*, c_i, \cdot)$ is integrable, with $\pi_i(\sigma^*, c_i) = \int_{D_{-i}} \pi_i(\sigma^*, c_i, c_{-i})dF_{-i}(c_{-i})$. Moreover, $\pi_i(\sigma^*, \cdot)$ is integrable.

**Proof.** The first claim is obvious if $\sigma^*_i(c_i) > 0$ for almost any $c_i \in D_i$. Suppose that $\sigma^*_i(c_i) = 0$ with positive probability. Then, by Lemma 3.3, the event $\sigma^*_{-i}(c_{-i}) = 0$ is null. Take some $c_{-i} \in D_{-i}$ with $\sigma^*_{-i}(c_{-i}) \neq 0$. Then, for any $c_i \in D_i$, by concavity, the difference quotient $\Pi_i(y_i, \sigma^*_{-i}(c_{-i}), c_i)/y_i$ is monotone increasing as $y_i \searrow 0$, with limit $\pi_i(\sigma^*, c_i, c_{-i})$. Since also $\Pi_i(y_i, \sigma^*_{-i}(c_{-i}), c_i)/y_i \geq -\mathcal{g}_i(E)$, the first claim follows from Levi’s theorem. The second claim follows from Lebesgue’s theorem, because $\pi_i(\sigma^*, \cdot) \leq 0$ from the Kuhn-Tucker conditions, and because $\pi_i(\sigma^*, \cdot) \geq -\mathcal{g}_i(E)$, as above. \(\square\)

**References**


