Fadina, Tolulope; Herzberg, Frederik

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Hyperfinite Construction of $G$-expectation

Tolulope Fadina and Frederik Herzberg
Hyperfinite Construction of $G$-expectation

Tolulope Fadina† Frederik Herzberg‡

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Abstract

We prove a lifting theorem, in the sense of Robinsonian nonstandard analysis, for the $G$-expectation. Herein, we use an existing discretization theorem for the $G$-expectation by T. Fadina and F. Herzberg (Bielefeld University, Center for Mathematical Economics in its series Working Papers, 503, (2014)).

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1 Introduction

The hyperfinite $G$-expectation is a nonstandard discrete analogue of $G$-expectation (in the sense of Robinsonian nonstandard analysis) which is infinitely close to the continuous time $G$-expectation. We develop the basic theory for the hyperfinite $G$-expectation. We prove a lifting theorem for the $G$-expectation. For the proof of the lifting theorem, we use an existing discretization theorem for the $G$-expectation from Fadina and Herzberg [8, Theorem 6]. Very roughly speaking, we extend the discrete time analogue of the $G$-expectation to a hyperfinite time analogue. Then, we use the characterization of convergence in nonstandard analysis to prove that the hyperfinite discrete-time analogue of the $G$-expectation is infinitely close to the (standard) $G$-expectation.

Nonstandard analysis makes consistent use of infinitesimals in mathematical analysis based on techniques from mathematical logic. This approach is very promising because it also allows, for instance, to study

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†Faculty of Mathematics, Bielefeld University, D-33615 Bielefeld, Germany. Email: tfadina@math.uni-bielefeld.de
‡Center for Mathematical Economics (IMW), Bielefeld University, D-33615 Bielefeld, Germany. Email: fherzberg@uni-bielefeld.de
continuous-time stochastic processes as formally finite objects. Many authors have applied nonstandard analysis to problems in measure theory, probability theory and mathematical economics (see for example, Anderson and Raimondo [3] and the references therein or the contribution in Berg and Neves [4]), especially after Loeb [12] converted nonstandard measures (i.e. the images of standard measures under the nonstandard embedding \( ^* \)) into real-valued, countably additive measures, by means of the standard part operator and Carathéodory’s extension theorem. One of the main ideas behind these applications is the extension of the notion of a finite set known as hyperfinite set or more causally, a formally finite set. Very roughly speaking, hyperfinite sets are sets that can be formally enumerated with both standard and nonstandard natural numbers up to a (standard or nonstandard, i.e. unlimited) natural number.


Dolinsky et al. [7] and Fadina and Herzberg [8] showed the (standard) weak approximation of the \( G \)-expectation. Dolinsky et al. [7] introduced a notion of volatility uncertainty in discrete time and defined a discrete version of Peng’s \( G \)-expectation. In the continuous-time limit, it turns out that the resulting sublinear expectation converges weakly to the \( G \)-expectation. To allow for the hyperfinite construction of \( G \)-expectation which require a discretization of the state space, in Fadina and Herzberg [8] Theorem 6 we refine the discretization by Dolinsky et al. [7] and obtain a discretization where the martingale laws are defined on a finite lattice rather than the whole set of reals.

The aim of this paper is to give an alternative, combinatorially inspired construction of the \( G \)-expectation based on the aforementioned Theorem 6. We hope that this result may eventually become useful for applications in financial economics (especially existence of equilibrium on continuous-time financial markets with volatility uncertainty) and provides additional intuition for Peng’s \( G \)-stochastic calculus. We begin the nonstandard treat-
ment of the $G$-expectation by defining a notion of $S$-continuity, a standard part operator, and proving a corresponding lifting (and pushing down) theorem. Thereby, we show that our hyperfinite construction is the appropriate nonstandard analogue of the $G$-expectation. For details on nonstandard analysis, we refer the reader to Albeverio et al. [1], Cutland [5], Loeb and Wolff [13] and Stroyan and Luxemburg [16].

The rest of this paper is organised as follows: in Section 2 we introduce the $G$-expectation, the continuous-time setting of the sublinear expectation and the hyperfinite-time setting needed for our construction. In Section 3 we introduce the notion of $S$-continuity and also define the appropriate lifting notion needed for our construction. Finally, we prove that the hyperfinite $G$-expectation is infinitely close to the (standard) $G$-expectation.

2 Framework

The $G$-expectation $\xi \mapsto \mathcal{E}_G(\xi)$ is a sublinear function that takes random variables on the canonical space $\Omega$ to the real numbers. The symbol $G$ is a function $G : \mathbb{R} \to \mathbb{R}$ of the form

$$G(\gamma) := \frac{1}{2} \sup_{c \in D} c \gamma,$$

(1)

where $D = [r_D, R_D]$ and $0 \leq r_D \leq R_D < \infty$. Let $\mathcal{P}_G^\mathbb{D}$ be the set of probabilities on $\Omega$ such that for any $P \in \mathcal{P}_G^\mathbb{D}$, $B$ is a martingale with volatility $d \langle B \rangle_t / dt \in D$ in $P \otimes dt$ a.e. Then, the dual view of the $G$-expectation via volatility uncertainty (cf. Denis et al. [6]) can be denoted as

$$\mathcal{E}_G^G(\xi) = \sup_{P \in \mathcal{P}_G^\mathbb{D}} \mathbb{E}_P[\xi].$$

The canonical process $B$ under the $G$-expectation $\mathcal{E}_G^G(\xi)$ is called $G$-Brownian motion (cf. Peng [14]).

2.1 Continuous-time construction of sublinear expectation

Let $\Omega = \{\omega \in \mathcal{C}([0, T]; \mathbb{R}) : \omega_0 = 0\}$ be the canonical space of continuous paths on $[0, T]$ endowed with the maximum norm $\|\omega\|_\infty = \sup_{0 \leq t \leq T} |\omega_t|$, where $| \cdot |$ is the Euclidean norm on $\mathbb{R}$. $B$ is the canonical process defined by $B_t(\omega) = \omega_t$ and $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$ is the filtration generated by $B$. $\mathcal{P}_\mathbb{D}$ is the set of all martingale laws on $\Omega$ such that under any $P \in \mathcal{P}_\mathbb{D}$, the coordinate process $B$ is a martingale with respect to $\mathcal{F}_t$ with volatility $d \langle B \rangle_t / dt$ taking values in $D$, $P \otimes dt$ a.e., for $D = [r_D, R_D]$ and $0 \leq r_D \leq R_D < \infty$.

$$\mathcal{P}_\mathbb{D} = \{P \text{ martingale law on } \Omega; d \langle B \rangle_t / dt \in D, P \otimes dt \text{ a.e.}\}.$$
Thus, the sublinear expectation is given by
\[ \mathcal{E}_D(\xi) = \sup_{P \in \mathcal{P}_D} \mathbb{E}^P[\xi], \] (2)
for any \( \xi : \Omega \to \mathbb{R} \), \( \xi \) is \( \mathcal{F}_T \)-measurable and integrable for all \( P \in \mathcal{P}_D \). \( \mathbb{E}^P \) denotes the expectation under \( P \). It is important to note that the continuous-time sublinear expectation (2) can be considered as the classical \( G \)-expectation (for every \( \xi \in L^1_G \) where \( L^1_G \) is defined as the \( \mathbb{E}[|\cdot|] \)-norm completion of \( \mathcal{C}_b(\Omega; \mathbb{R}) \)) provided (1) is satisfied (cf. Dolinsky et al. [7]).

2.2 Hyperfinite-time setting

Here we present the nonstandard version of the discrete-time setting of the sublinear expectation and the strong formulation of volatility uncertainty on the hyperfinite timeline. For the (standard) strong formulation of volatility uncertainty in the discrete-time see Fadina and Herzberg [8], and for the continuous-time see Dolinsky et al. [7] and Fadina and Herzberg [8].

**Definition 2.1.** *\( \Omega \) is the *-image of \( \Omega \) endowed with the *-extension of the maximum norm *\( \| \cdot \|_{\infty} \). *\( \mathcal{D} = *[\mathcal{D}, \mathbb{R}] \) is the *-image of \( \mathcal{D} \), and as such it is internal. It is important to note that \( st : *\Omega \to \Omega \) is the standard part map, and \( st(\omega) \) will be referred to as the standard part of \( \omega \), for every \( \omega \in *\Omega \). °z denotes the standard part of a hyperreal \( z \).

**Definition 2.2.** For every \( \omega \in \Omega \), if there exists \( \tilde{\omega} \in *\Omega \) such that \( \| \tilde{\omega} - *\omega \|_{\infty} \simeq 0 \), then \( \tilde{\omega} \) is a nearstandard point in *\( \Omega \). This will be denoted as \( ns(\tilde{\omega}) \in *\Omega \).

For all hypernatural \( N \), let
\[ \mathcal{L}_N = \left\{ \frac{K}{N\sqrt{N}}, \ -N^2\sqrt{R_D} \leq K \leq N^2\sqrt{R_D}, \ K \in *\mathbb{Z} \right\} \] (3)
and the hyperfinite timeline
\[ T = \left\{ 0, \frac{T}{N}, \cdots, -\frac{T}{N} + T, T \right\}. \] (4)

We consider \( \mathcal{L}_N^T \) as the canonical space of paths on the hyperfinite timeline, and \( X^N = (X^N_k)_{k=0}^N \) as the canonical process denoted by \( X^N_k(\tilde{\omega}) = \tilde{\omega}_k \) for \( \tilde{\omega} \in \mathcal{L}_N^T \). \( \mathcal{F}_N \) is the internal filtration generated by \( X^N \). The linear interpolation operator can be written as
\[ \hat{\cdot} : \hat{\cdot} \circ i^{-1} \to *\Omega, \text{ for } \mathcal{L}_N^T \subseteq *\Omega, \]
where
\[ \bar{\omega}(t) := ([Nt/T] + 1 - Nt/T)\omega_{[Nt/T]} + (Nt/T - [Nt/T])\omega_{[Nt/T]+1}, \]
for \( \omega \in \mathcal{L}_N^{N+1} \) and for all \( t \in [0, T] \). \([y]\) denotes the greatest integer less than or equal to \( y \) and \( \iota : T \rightarrow \{0, \cdots, N\} \) for \( t : t \mapsto Nt/T \).

For the hyperfinite strong formulation of the volatility uncertainty, fix \( N \in {}^*\mathbb{N} \setminus \mathbb{N} \). Consider \( \{ \pm \frac{1}{\sqrt{N}} \}^T \), and let \( P_N \) be the uniform counting measure on \( \{ \pm \frac{1}{\sqrt{N}} \}^T \). \( P_N \) can also be seen as a measure on \( \mathcal{L}_N^T \), concentrated on \( \{ \pm \frac{1}{\sqrt{N}} \}^T \). Let \( \Omega_N = \{ \omega = (\omega_1, \cdots, \omega_N); \omega_i = \{\pm 1\}, i = 1, \cdots, N \} \), and let \( \Xi_1, \cdots, \Xi_N \) be a \( * \)-independent sequence of \( \{\pm 1\} \)-valued random variables on \( \Omega_N \) and the components of \( \Xi_k \) are orthonormal in \( L^2(P_N) \). We denote the hyperfinite random walk by
\[ X_t = \frac{1}{\sqrt{N}} \sum_{l=1}^{Nt/T} \Xi_l \quad \text{for all} \quad t \in T. \]
The hyperfinite-time stochastic integral of some \( F : T \times \mathcal{L}_N^T \rightarrow {}^*\mathbb{R} \) with respect to the hyperfinite random walk is given by
\[ \sum_{s=0}^{t} F(s, X(s)) \Delta X_s : \Omega_N \rightarrow {}^*\mathbb{R}, \quad \omega \in \Omega_N \mapsto \sum_{s=0}^{t} F(s, X(s)) \Delta X_s(\omega). \]
Thus, the hyperfinite set of martingale laws can be defined by
\[ \bar{\mathcal{Q}}_D^N = \{ P_N \circ (M^{F,X})^{-1}; F : T \times \mathcal{L}_N^T \rightarrow \sqrt{\mathcal{D}'_N} \} \]
where
\[ \mathcal{D}'_N = \mathcal{D} \cap \left( \frac{1}{N} {}^*\mathbb{N} \right)^2 \]
and
\[ M^{F,X} = \left( \sum_{s=0}^{t} F(s, X(s)) \Delta X_s \right)_{t \in \mathbb{T}}. \]

**Remark 2.3.** Up to scaling, \( \bar{\mathcal{Q}}_D^N = \mathcal{Q}_D^* \).

### 3 Results and proofs

**Definition 3.1** (Uniform lifting of \( \xi \)). Let \( \Xi : \mathcal{L}_N^T \rightarrow {}^*\mathbb{R} \) be an internal function, and let \( \xi : \Omega \rightarrow \mathbb{R} \) be a continuous function. \( \Xi \) is said to be a uniform lifting of \( \xi \) if and only if
\[ \forall \bar{\omega} \in \mathcal{L}_N^T \left( \bar{\omega} \in \text{ns}({}^*\Omega) \Rightarrow {}^0\Xi(\bar{\omega}) = \xi(st(\bar{\omega})) \right), \]
where \( st(\tilde{\omega}) \) is defined with respect to the topology of uniform convergence on \( \Omega \).

In order to construct the hyperfinite version of the \( G \)-expectation, we need to show that the *-image of \( \xi, \star \xi \), with respect to \( \tilde{\omega} \in \text{nsl}(\star \Omega) \), is the canonical lifting of \( \xi \) with respect to \( st(\tilde{\omega}) \in \Omega \). i.e., for every \( \tilde{\omega} \in \text{nsl}(\star \Omega), \end{equation}

\( \circ (\star \xi(\tilde{\omega})) = \xi(st(\tilde{\omega})) \). To do this, we need to show that \( \star \xi \) is \( S \)-continuous in every nearstandard point \( \tilde{\omega} \).

It is easy to prove that there are two equivalent characterizations of \( S \)-continuity on \( \star \Omega \).

**Remark 3.2.** The following are equivalent for an internal function \( \Phi : \star \Omega \to \star \mathbb{R} \):

1. \( \forall \omega' \in \star \Omega \left( \| \omega - \omega' \|_{\infty} \simeq 0 \Rightarrow \star |\Phi(\omega) - \Phi(\omega')| \simeq 0 \right) \).
2. \( \forall \varepsilon \gg 0, \exists \delta \gg 0 : \forall \omega' \in \star \Omega \left( \| \omega - \omega' \|_{\infty} < \delta \Rightarrow \star |\Phi(\omega) - \Phi(\omega')| < \varepsilon \right) \).

(The case of Remark 3.2 where \( \Omega = \mathbb{R} \) is well known and proved in Stroyan and Luxemburg [16, Theorem 5.1.1])

**Definition 3.3.** Let \( \Phi : \star \Omega \to \star \mathbb{R} \) be an internal function. We say \( \Phi \) is \( S \)-continuous in \( \omega \in \star \Omega \), if and only if it satisfies one of the two equivalent conditions of Remark 3.2.

**Proposition 3.4.** If \( \xi : \Omega \to \mathbb{R} \) is a continuous function satisfying \( |\xi(\omega)| \leq a(1 + \| \omega \|_{\infty})^b \), for \( a, b > 0 \), then, \( \Xi = \star \circ \xi \) is a uniform lifting of \( \xi \).

**Proof.** Fix \( \omega \in \Omega \). By definition, \( \xi \) is continuous on \( \Omega \). i.e., for all \( \omega \in \Omega \), and for every \( \varepsilon \gg 0 \), there is a \( \delta \gg 0 \), such that for every \( \omega' \in \Omega \), if

\[
\| \omega - \omega' \|_{\infty} < \delta, \text{ then } |\xi(\omega) - \xi(\omega')| < \varepsilon.
\]

(5)

By the Transfer Principle: For all \( \omega \in \Omega \), and for every \( \varepsilon \gg 0 \), there is a \( \delta \gg 0 \), such that for every \( \omega' \in \star \Omega \), (5) becomes,

\[
\star \| \omega - \omega' \|_{\infty} < \delta, \text{ and } \star |\xi(\omega) - \xi(\omega')| < \varepsilon.
\]

(6)

So, \( \star \xi \) is \( S \)-continuous in \( \star \omega \) for all \( \omega \in \Omega \). Applying the equivalent characterization of \( S \)-continuity, Remark 3.2 (6) can be written as

\[
\star \| \omega - \omega' \|_{\infty} \simeq 0, \text{ and } \star |\xi(\omega) - \xi(\omega')| \simeq 0.
\]

We assume \( \tilde{\omega} \) to be a nearstandard point. By Definition 2.2 this simply implies,

\[
\forall \tilde{\omega} \in \text{nsl}(\star \Omega), \exists \omega \in \Omega : \| \tilde{\omega} - \omega \|_{\infty} \simeq 0.
\]

(7)
Thus, by $S$-continuity of $\xi$ in $\omega$,

$$\ast|\ast\xi(\tilde{\omega}) - \ast\xi(\ast\omega)| \simeq 0.$$  

Using the triangle inequality, if $\omega' \in *\Omega$ with $\ast\|	ilde{\omega} - \omega'\|_{\infty} \simeq 0$,

$$\ast\|\ast\omega - \omega'\|_{\infty} \leq \ast\|\ast\omega - \tilde{\omega}\|_{\infty} + \ast\|\tilde{\omega} - \omega'\|_{\infty} \simeq 0$$

and therefore again by the $S$-continuity of $\ast\xi$ in $\ast\omega$,

$$\ast|\ast\xi(\ast\omega) - \ast\xi(\omega')| \simeq 0.$$  

And so,

$$\ast|\ast\xi(\tilde{\omega}) - \ast\xi(\omega')| \leq \ast|\ast\xi(\tilde{\omega}) - \ast\xi(\ast\omega)| + \ast|\ast\xi(\ast\omega) - \ast\xi(\omega')| \simeq 0.$$  

Thus, for all $\tilde{\omega} \in ns(\ast\Omega)$ and $\omega' \in *\Omega$, if $\ast\|\tilde{\omega} - \omega'\|_{\infty} \simeq 0$, then,

$$\ast|\ast\xi(\tilde{\omega}) - \ast\xi(\omega')| \simeq 0.$$  

Hence, $\ast\xi$ is $S$-continuous in $\tilde{\omega}$. Equation (7) also implies

$$\tilde{\omega} \in m(\omega) \left( m(\omega) = \bigcap \{^*\mathcal{O}; \mathcal{O} \text{ is an open neighbourhood of } \omega \} \right)$$

such that $\omega$ is unique, and in this case $st(\tilde{\omega}) = \omega$.

Therefore,

$$\circ\left(\ast\xi(\tilde{\omega})\right) = \xi(st(\tilde{\omega})).$$

\[ \square \]

**Definition 3.5.** Let $E : *\mathbb{R}^N \to *\mathbb{R}$. We say that $E$ lifts $E^G$ if and only if for every $\xi : \Omega \to \mathbb{R}$ that satisfies $|\xi(\omega)| \leq a(1 + \|\omega\|_{\infty})^b$ for some $a, b > 0$,

$$E(\ast\xi \circ \cdot) \simeq E^G(\xi).$$

**Theorem 3.6.**

$$\max_{\hat{Q} \in Q^n_{D^N}} \mathbb{E}\hat{Q}[\cdot] \text{ lifts } E^G(\xi). \quad (8)$$

**Proof.** From the standard approximation in Fadina and Herzberg [8, Theorem 6],

$$\max_{\hat{Q} \in Q^n_{D^N}} \mathbb{E}\hat{Q}[\xi(\hat{X}^n)] \to E^G(\xi), \quad \text{as } n \to \infty. \quad (9)$$

For all $N \in *\mathbb{N} \setminus \mathbb{N}$, we know that (9) holds if and only if

$$\max_{\hat{Q} \in *Q^n_{D^N}} \mathbb{E}\hat{Q}[\ast\xi(\hat{X}^N)] \simeq E^G(\xi), \quad (10)$$
(see Albeverio et al. [1], Proposition 1.3.1). Now, we want to express (10) in term of $\bar{Q}_N^D$, i.e., to show that

$$\max_{\bar{Q} \in \bar{Q}_N^D} \mathbb{E}^Q[\xi \circ \cdot] \simeq \mathcal{E}(\xi).$$

To do this, use

$$\mathbb{E}^Q[\xi \circ \cdot] = \mathbb{E}^Q[\xi \circ \cdot \circ t^{-1} \circ t]$$

and

$$\mathbb{E}^Q[\xi \circ \cdot \circ t^{-1} \circ t] = \mathbb{E}^Q[\xi \circ \cdot \circ t]$$

$$= \int_{\mathbb{R}^{N+1}} \mathcal{E}(Q \circ j),$$

$$= \mathbb{E}^{Q_j}[\xi \circ \cdot]$$

for $j : \mathbb{R}^T \to \mathbb{R}^{N+1}$, $(xt)_{t \in T} \mapsto (\frac{x_N}{T})_{t \in \mathbb{R}^{N+1}}$. Thus,

$$\bar{Q}_N^D = \{Q \circ j : Q \in \bar{Q}^N_D\}.$$

This implies,

$$\max_{\bar{Q} \in \bar{Q}_N^D} \mathbb{E}^Q[\xi \circ \cdot] = \max_{Q \in \bar{Q}^N_D} \mathbb{E}^Q[\xi \circ \cdot].$$

\[\square\]

References


Appendix

Proof of Remark 3.2. Let $\Phi$ be an internal function such that condition (1) holds. To show that (1) $\Rightarrow$ (2), fix $\varepsilon \gg 0$. We shall show there exists a $\delta$ for this $\varepsilon$ as in condition (2). Since $\Phi$ is internal, the set

$$I = \left\{ \delta \in *\mathbb{R}_{>0} : \forall \omega' \in *\Omega \ (\|\omega - \omega'\|_{\infty} < \delta \Rightarrow *|\Phi(\omega) - \Phi(\omega')| < \varepsilon) \right\},$$

is internal by the Internal Definition Principle and also contains every positive infinitesimal. By Overspill (cf. Albeverio et al. [1], Proposition 1.27]) \( I \) must then contain some positive \( \delta \in \mathbb{R} \).

Conversely, suppose condition (1) does not hold, that is, there exists some \( \omega' \in \ast \Omega \) such that

\[ \ast \| \omega - \omega' \|_\infty \simeq 0 \text{ and } \ast |\Phi(\omega) - \Phi(\omega')| \text{ is not infinitesimal.} \]

If \( \varepsilon = \min(1, \ast |\Phi(\omega) - \Phi(\omega')|/2) \), we know that for each standard \( \delta > 0 \), there is a point \( \omega' \) within \( \delta \) of \( \omega \) at which \( \Phi(\omega') \) is farther than \( \varepsilon \) from \( \Phi(\omega) \). This shows that condition (2) cannot hold either. \qed