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## Dynamic consistent alpha-maxim expected utility

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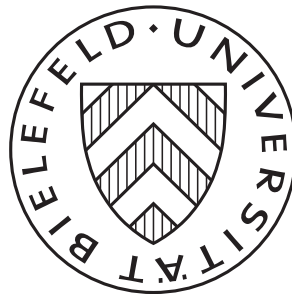
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January 2015

## Dynamic Consistent $\alpha$ -Maxim Expected Utility

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Patrick Beißner and Qian Lin



# Dynamic Consistent $\alpha$ -Maxmin Expected Utility\*

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## Abstract

We establish a class of fully nonlinear conditional expectations. Similarly to the usage of linear expectations when a probabilistic description of uncertainty is present, we observe analogue quantitative and qualitative properties. The type of nonlinearity captures the agents sentiments of optimism and pessimism in an ambiguous environment.

We then introduce an expected utility under a nonlinear expectation, and show monotonicity and continuity of utility. Risk aversion is characterized, and the properties of the certainty equivalent are discussed. Finally, we derive an Arrow-Pratt approximation of the static certainty equivalent and investigate the dynamic version via recursive equations.

*Key words:* Nonlinear expectation, Knightian Uncertainty, time consistency, risk aversion, certainty equivalent, optimism and pessimism

*JEL subject classification:* C60, D81, D90

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# 1 Introduction

In decision theory under uncertainty, the probabilistic expectation is invariably utilized as a representation of beliefs and results in a utility under a (linear) expectation as in Von Neumann and Morgenstern (1953) and Savage (1954).

Based on the standard approach, this (mathematical) expectation is in a one-to-one correspondence with a probabilistic description of uncertainty, and leads to the pleasant linearity of expectations. Implications within models in economics or finance is that, having such a description of beliefs, can only create conclusions that are in the scope of the supposed linearity.

We advance the concept of expectations, and discuss some implications if the belief system is no longer linear.

Our formalization of a nonlinear conditional expectation is based on a set of possible probabilistic scenarios  $\mathcal{P}$ , and allows for continuous time models. The description of the resulting expectation operator is determined by a stochastic weighting scheme that takes value in the unit interval and captures the time- and state-dependent degree of optimism and pessimism within the set  $\mathcal{P}$ . As such, the present expectation model is able to capture sentiments of an agent or investor in a concrete behavioral manner.

Our approach follows the axiomatization of Ghirardato, Maccheroni, and Marinacci (2004), Chateauneuf, Eichberger, and Grant (2007) and Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011), where the attitudes towards ambiguity, due to presence of multiple priors, are captured by a similar but static representation. These preferences allow for a functional form being consistent with a “generalized Hurwicz” criterion. As indicated in Proposition 2, the present class of preferences proposes a dynamic and time-consistent version of Monotonic Bernoullian Archimedean (MBA) Preferences.

Our starting point is the construction of a family of conditional nonlinear expectations, given an increasing system of information indexed by a finite time interval. Information, as usual and there is little reason to move on this assumption, is represented by an increasing sequence of  $\sigma$ -fields or partitions. The generalization of the conditional expectation relies on the same filtration and the same space of random variables (with finite variance).

As an initial and basic result of the paper (Theorem 1), we establish the unique existence of a consistent family of conditional expectations  $(\mathcal{E}_t)$ . Apart from the fundamental additivity, the present class of conditional expectations satisfies all the fundamental properties. For instance the law of iterated expectation holds true, which is an equivalent formulation of time consistency, (see Epstein and Schneider (2003) and Appendix B in Riedel (2009)).

Given a nonlinear expectation, we introduce a nonlinear expected utility with some utility index  $u : \mathbb{R} \rightarrow \mathbb{R}$  and show that under standard conditions, standard properties hold for the nonlinear expected utility, such as continuity and (strict) monotonicity. Although the nonlinear expected utility is no longer

concave, we are still able to characterize, as in the linear case, risk aversion in terms of concavity of the utility index.

Following, we discuss risk preferences via a modified certainty equivalent and present further properties such as continuity and monotonicity. We compute a generalized Arrow-Pratt approximation, where the classical measure of absolute risk aversion is still present, in contrast to Izhakian and Benninga (2011) and Maccheroni, Marinacci, and Ruffino (2013), where an ambiguity premium is derived from the utility representation axiomatized by Klibanoff, Marinacci, and Mukerji (2005). The premium of the present utility specification is directly encoded in the nonlinear expectation (see equation (6) and Figure 4). Depending on the type and degree of pessimism, the nonlinear expectation determines the sign of the *sentiment premium*.

Setting	Static	Continuous Time
<b>Maxmin</b>	$U(c) = \min_{P \in \mathcal{P}} E^P u(c) = \underline{\mathbb{E}}u(c)$ <i>Gilboa and Schmeidler (1989)</i>	$U_t(c) = \underline{\mathbb{E}} \int_t^T u(c_t) dt$ <i>Chen and Epstein (2002)</i>
$\alpha$ - <b>Maxmin</b>	$U(c) = \alpha \underline{\mathbb{E}}u(c) + (1 - \alpha) \bar{\mathbb{E}}u(c)$ <i>Ghirardato et al. (2004)</i>	$U_t(c) = \mathcal{E}_t u(c)$ <b>This Paper</b>
<b>Smooth</b>	$U(c) = \mathbb{E}^\mu[\phi(E^P u(c))]$ <i>Klibanoff et al. (2005)</i>	$dU = \mu dt + \Sigma dB$ <i>Skiadas (2013)</i>
<b>Variational</b>	$U(c) = \min_{P \in \mathcal{P}} E^P u(c) + c(P)$ <i>Maccheroni et al. (2006)</i>	$U_t(c) = \min_{P \in \mathcal{P}} E_t^P u(c) + c(P)$ <i>Cheng and Riedel (2013)</i>

Table 1: Functional forms of utility under risk and ambiguity

Likewise, for the formulation of nonlinear expectations and conditional certainty equivalents, we describe the objects in the language of backward stochastic differential equations (BSDE)<sup>1</sup>. In the theory of continuous-time stochastic processes this is a well-developed concept of recursive integral equations. For instance, the description of stochastic differential utility of Duffie and Epstein (1992) is formulated via the solution of a BSDE. From a conceptual view, one can consider a BSDE as a backward induction principle in continuous time, such that tools from stochastic analysis become available. Moreover, Chen and Epstein (2002) apply this approach to model the continuous-time version of Gilboa and Schmeidler (1989)-type preferences,

<sup>1</sup>This recursive theory of stochastic processes is introduced by Pardoux and Peng (1990) and applied to finance and recursive utility in the overview article by El Karoui, Peng, and Quenez (1997), where, instead of the initial condition, a terminal condition serves as input data. Recently, this approach is applied to convex optimal stopping problems under ambiguity in Cheng and Riedel (2013). By virtue of BSDEs, Chiu and Wong (2011) study the mean-variance portfolio selection problem, and Oksendal, Sandal, and Uboe (2013) study stochastic Stackelberg equilibria.

whose description of the uncertainty model is adopted here. In Table 1, we list some well-known functional forms of utility. Interestingly, all continuous time extensions of the static setup are described in the language of BSDE's. Proposition 1 clarifies that this observation is by far no coincidence.

In the final part, we study the dynamic certainty equivalent via BSDE's, and obtain time consistency of certainty equivalents. A potential application refers to the indifference-pricing principle. In such a setting, it is possible to capture the agents sentiment within the pricing.<sup>2</sup>

The paper is organized as follows. Section 2 starts with the nonlinear expectation. In Sections 3 and 4, we introduce and analyze the utility and certainty equivalent. Section 5 concludes and the proofs are collected in the Appendix.

## 2 Nonlinear Expectations

Fix a finite time interval  $[0, T]$ . Before we introduce the nonlinear expectation, we recap the linear case and the construction of priors.

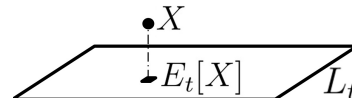
**Linear Expectations** Let us render more precisely the underlying structure of a linear benchmark expectation to illustrate how the properties of the mathematical expectation operators are exploited. These properties are of importance since conclusions usually depending upon them.

Based on a given probabilistic model  $(\Omega, \mathcal{F}, \mathbb{P})$ , we fix the information structure of increasing sub  $\sigma$ -fields  $\mathcal{F}_t \subset \mathcal{F}_T = \mathcal{F}$ ,  $t \in [0, T]$ . The formal definition of a conditional expectation reads as follows

$$E_t : L_T \rightarrow L_t,$$

where  $L_t = L^2(\Omega, \mathcal{F}_t, \mathbb{P})$  denotes the usual Hilbert space of square  $\mathbb{P}$ -integrable and  $\mathcal{F}_t$ -measurable random variables or payoffs. Existence and uniqueness of this family of conditioning operations is a classical result in probability theory, (see Figure 1 for an illustration of the underlying orthogonality principle, induced by the underlying scalar product  $\langle X, Y \rangle = E_0[X \cdot Y]$ ). Among linearity and the law of iterated expectation, many other properties hold true.

Figure 1: Orthogonality– a linear conditional expectation in the least square sense




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<sup>2</sup>see Zimper (2012); Ludwig and Zimper (2014)

**Priors and the Uncertainty Model** Let us now model the priors of the decision theoretic setup. We follow a construction in terms of Girsanov kernels as in Chen and Epstein (2002) (for details see therein Section 2.4.). Let  $B$  be a Brownian motion on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and  $(\mathcal{F}_t)_{t \in [0, T]}$  is the filtration generated by  $B$ , and completed by  $\mathbb{P}$ -null sets. We define a density in terms of an  $\mathbb{R}$ -valued process  $\theta = (\theta_t)$  as follows:

$$z_t^\theta \equiv \exp \left( - \int_0^t \theta_s dB_s - \int_0^t \theta_s^2 ds \right),$$

where  $\theta$  satisfies the Novikov condition  $E^\mathbb{P}[\exp(\frac{1}{2} \int_0^T \theta_s^2 ds)] < \infty$ . Such a density generator  $\theta$  determines a new prior  $P^\theta$  via a change of measure with density

$$\frac{dP^\theta}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = z_t^\theta. \quad (1)$$

The density generator  $\theta$  contains all information of  $P^\theta$  given  $\mathbb{P}$ . From this construction we may move to a set of density generators  $\Theta$ . The corresponding set of priors is defined as follows:

$$\mathcal{P} = \left\{ P^\theta : \theta \in \Theta, P^\theta \text{ is defined by (1)} \right\}.$$

**Lemma 1** *Let  $\Theta : \Omega \times [0, T] \Rightarrow \mathbb{R}$  be an adapted set-valued process. For every  $(\omega, t)$  the set  $\Theta_t(\omega)$  is convex and compact valued subset of  $\mathbb{R}$ . Then the induced set of prior  $\mathcal{P}$  is weakly compact and convex.*

The lemma follows from Theorem 3.1 in Chen and Epstein (2002).

## 2.1 A Class of Nonlinear Conditional Expectations

We introduce a novel notion of (behavioral) expectation. The exposition remains close to the connection between an expectation value and its usage to define the cardinal (expected) utility. Consider a random variable  $X \in L_T$  as a possible contingent claims or payoff,  $\|\cdot\|$  denotes the usual  $L^2$ -norm. The  $\|\cdot\|$ -closed subspace of  $\mathcal{F}_t$  measurable elements is again denoted by  $L_t$ .

Formally we aim to introduce the following nonlinear expectation operator  $\mathcal{E} : L_T \rightarrow \mathbb{R}$  based on  $\mathcal{P}$ . The minimal requirement to remain in the scope of  $\mathcal{P}$  is ensured by

$$\mathcal{E}[X] = \alpha^X \min_{P \in \mathcal{P}} E^P[X] + (1 - \alpha^X) \max_{P \in \mathcal{P}} E^P[X], \quad (2)$$

where the positive fraction  $\alpha^X \in [0, 1]$  depends on the evaluated  $X \in L_T$ . From this perspective,  $\alpha^X$  is the endogenous outcome of a more flexible and time-dependent weighting that is formulated in the following.

**Definition 1** Fix a set of priors  $\mathcal{P}$ , via (1). A specification of a nonlinear expectation is determined by an adapted weighting process  $(\alpha_t)$ , taking values in  $[0, 1]$ . All such processes we denote by  $\llbracket 0, 1 \rrbracket$ .

We interpret  $\alpha_t(\omega)$  as the degree of relative pessimism at time  $t$  in state  $\omega$ . In this regard, there is little reason to assume a constant weight, when the environment is uncertain and the arrival of new information makes it reasonable to adjust this degree in an accurate way.

**Remark 1** Definition 1 and the dynamic weighting process therein build the key primitives in our model. For instance, the constant weight  $\alpha_t \equiv \frac{1}{2}$  results into a completely different expectation as  $\alpha_t = \frac{1}{2} \cdot (1 + \sin(t))$ ,  $t \in [0, T]$ , where  $T$  is a multiple of  $2\pi$ . In Example 2, we explain that a constant specification may lead to a sole pessimistic specification with a reduced set of relevant priors.

In comparison with the axiomatization for the generalized  $\alpha$ -MEU in Section 6 of Ghirardato, Maccheroni, and Marinacci (2004), the dependence of  $\alpha^X$  with respect to an evaluated  $X$  is again present. We aim to present a mechanism for generating  $\alpha^X$  in (2) that is based on an initial value  $\mathcal{E}_0[X]$  of a family of time-consistent conditional expectations  $(\mathcal{E}_t)$ , induced by  $\mathcal{P}$  and  $\alpha \in \llbracket 0, 1 \rrbracket$ . Although there is no general consensus how to update in a multiple prior setting, we follow the convincing Bayesian updating rule, such that time consistency holds, as in Epstein and Schneider (2003). For instance, a well-behaved conditional expectation, satisfying especially the iterated law of conditional expectation, is indispensable when the reasoning (especially in continuous time) is based on a martingale concept of fair games against nature.

In the light of (2), the weight  $\alpha^X \in [0, 1]$  appears as a rather endogenous quantity of the given conditional expectation with specification  $[\mathcal{P}, \alpha]$ , being evaluated by  $X \in L_T$  at the initial time  $t = 0$ .

**Remark 2** One may ask if the case of constant  $\alpha$ -maxmin is plausible in the present set-up. As shown in Ghirardato, Maccheroni, and Marinacci (2004) this is strongly related to the issue that all information is contained in the certainty equivalent (see also Chateauneuf, Eichberger, and Grant (2007)).

In Example 2 below, we examine this issue in the case of  $\kappa$ -ambiguity. In Section 4, we discuss the conditional certainty equivalent in more detail.

Aiming for a dynamic prospect of utility theory, we introduce the conditioning of the nonlinear expectation, given the information  $\mathcal{F}_t$ . For the construction of a meaningful conditional expectation  $\mathcal{E}_t : L_T \rightarrow L_t$  the following bounds are needed. As a dynamic version of (2), they are given in the following way,

$$\underline{\mathcal{E}}_t[X] := \min_{P \in \mathcal{P}} E_t^P[X] \leq \mathcal{E}_t[X] \leq \max_{P \in \mathcal{P}} E_t^P[X] =: \overline{\mathcal{E}}_t[X]. \quad (3)$$

This representation is intuitively reasonable in terms of an implicit stochastic weighting  $\alpha_t^X$  between the conditional worst and best case. In the light of



Theorem 1, the functional dependency with the remaining weighting  $(\alpha_s)_{s \in [t, T]}$  persists as the dynamic analog of  $\alpha^X$  in (2).

Fortunately, we have an alternative formulation of the nonlinear conditional expectation. Theorem 1 collects properties and establishes unique existence. Apart from the linearity, all pleasant properties still hold.

**Theorem 1** *Let  $\mathcal{P}$  satisfy the condition of Lemma 1. For every  $\alpha \in \llbracket 0, 1 \rrbracket$  there is a unique nonlinear expectation  $\mathcal{E}$  that satisfies (3) and: For every  $X \in L_T$ , the pair  $(\mathcal{E}_t[X], \sigma_t^X)$  uniquely solves of the following BSDE*

$$d\mathcal{E}_t[X] = \mathbf{e}(t, \sigma_t^X)dt + \sigma_t^X dB_t, \quad \mathcal{E}_T[X] = X, \quad (4)$$

where  $t \mapsto \mathbf{e}(t, \sigma_t)$  is an adapted integrable process and only depends on  $[\mathcal{P}, \alpha]$ .<sup>3</sup> We have the following properties: For every  $s, t \in [0, T]$ ,  $X, Y \in L_T$ .

- (i) Monotonicity: *If  $X \geq Y$ , then  $\mathcal{E}_t[X] \geq \mathcal{E}_t[Y]$ .  
Moreover, if  $\mathbb{P}(X > Y) > 0$ , then  $\mathcal{E}_0[X] > \mathcal{E}_0[Y]$ .*
- (ii) Constant-preserving:  *$\mathcal{E}_t[\eta] = \eta$ , if  $\eta \in L_t$  and  $\mathcal{E}_t[c] = c$ , for all  $c \in \mathbb{R}$ .*
- (iii) Time consistency:  *$\mathcal{E}_s[X] = \mathcal{E}_s[\mathcal{E}_t[X]]$ , for all  $s \leq t$ .*
- (iv) Conditional linearity:  *$\mathcal{E}_t[X + \eta] = \mathcal{E}_t[X] + \eta$ , for every  $\eta \in L_t$ .*
- (v) Zero-one law: *For any  $A \in \mathcal{F}_t$ , we have  $\mathcal{E}_t[X \mathbf{1}_A] = \mathcal{E}_t[X] \mathbf{1}_A$ .*
- (vi) Positive homogeneity:  *$\mathcal{E}_t[\eta X] = \eta \mathcal{E}_t[X]$ , for all  $\eta \geq 0$ .*
- (vii) Contraction property: *There is a constant  $c > 0$  such that,*  
$$\sup_t E^{\mathbb{P}} [|\mathcal{E}_t[X] - \mathcal{E}_t[Y]|^2] \leq c E^{\mathbb{P}} [|X - Y|^2].$$
- (viii) Jensen inequality: *If  $u \in C^2(\mathbb{R})$  is increasing and concave, then*  
$$\mathcal{E}_t[u(X)] \leq u(\mathcal{E}_t[X]).$$

Part (iv) still allows for linearity, when the summand  $\eta$  is known, given information at time  $t$ . (i), (vi) and (viii) can be directly employed for the properties of the expected utility in Section 3. Clearly, the minimal requirement of (2) and the dynamic analogue (3) are satisfied by the expectation operator in Theorem 1, which is given by the initial value

$$\mathcal{E}[X] := \mathcal{E}_0[X].$$

**Remark 3** *Apart from the linearity, the consistent family of nonlinear expectations of Theorem 1 captures most properties of the classical linear case. Moreover, further properties are present. From (vii) we can get the  $\|\cdot\|$ -continuity, i.e., if  $X_n \rightarrow X$  in  $\|\cdot\|$ , then  $\lim_{n \rightarrow \infty} \mathcal{E}_t[X_n] = \mathcal{E}_t[X]$ .*

<sup>3</sup>Specifically, we can write the aggregator of the BSDE which defines the nonlinear expectation via  $\mathbf{e}(t, \sigma_t) = \alpha_t \max_{\theta \in \Theta} \theta_t \sigma_t + (1 - \alpha_t) \min_{\theta \in \Theta} \theta_t \sigma_t$ .

## 2.2 A Converse Result of Theorem 1

In this section, we start with the opposite point of view and reason that under some very intuitive properties for some arbitrary expectation on  $L_T$ , we immediately end up in the class of expectation that can be formulated in terms of BSDE's.

**Definition 2** *Let  $\mathbf{E} : L_T \rightarrow \mathbb{R}$  be a nonlinear expectation, satisfying (i) and (ii) in Theorem 1.  $\mathbf{E}$  is called a dynamic nonlinear expectation if for each  $X \in L_T$  and  $t \in [0, T]$ , there exists a random variable  $\eta \in L_t$ , such that*

$$\mathbf{E}[X \mathbf{1}_A] = \mathbf{E}[\eta \mathbf{1}_A], \quad \text{for any } A \in \mathcal{F}_t.$$

The indicator functions, parametrized for each event in  $\mathcal{F}_t$  can be considered as an adequate collection of test-random variable that control the coherence between  $X$  and  $\eta$  on  $\mathcal{F}_t$ . From the strict monotonicity of the nonlinear expectation  $\mathbf{E}$ , we know that  $\eta$  is unique, and we denote it by  $\eta = \mathbf{E}_t[X]$ .

$\mathbf{E}_t[X]$  is called the conditional nonlinear expectation of  $X$  under  $\mathcal{F}_t$ . Additionally, we suppose that  $\mathbf{E}$  satisfies the following property: for  $\mu > 0$ ,

$$\mathbf{E}[X + Y] - \mathbf{E}[Y] \leq \mathcal{E}_0^\mu[X], \quad \text{for any } X, Y \in L_T. \quad (5)$$

where  $(\mathcal{E}_t^\mu[X], \sigma_t)$  solves the BSDE in (4) with  $\mathbf{e}(t, \sigma) = -\mu|\sigma|$ . Given these preparatory notations, we are interested in the generality of Theorem 1. The following result clarifies that under mild conditions for  $\mathbf{E}$ , as a model of a belief system, a kind of converse result in comparison to Theorem 1 holds true.

**Proposition 1** *Let  $\mathbf{E}$  be a dynamic nonlinear expectation satisfying (5), conditional linearity and positive homogeneity, i.e., property (iv) and (vi) of Theorem 1. Then there exists an adapted generator  $(s, \omega) \mapsto g(s, \omega, \cdot)$ , such that*

$$\mathbf{E}[X] = \mathcal{E}_0^g[X], \quad \text{for any } X \in L_T.$$

where  $(\mathcal{E}_s^g[X], \sigma_s)$  is the unique solution of the BSDE in (4) with  $\mathbf{e}(s, \sigma) = g(s, \sigma)$ . Moreover,  $g(s, \cdot)$  is Lipschitz continuous,  $g(\cdot, 0) = 0$  and  $g(s, \eta \times \cdot) = \eta g(s, \cdot)$ , for all  $s \in [0, T]$ .

From this opposite starting point, Theorem 1 refers to a special case. The refinement via the generator  $\mathbf{e}(s, \sigma)$  allows for an interpretation of sentiments as the following examples aim to illustrate. The direct connection between a specification  $\alpha \in \llbracket 0, 1 \rrbracket$  and a family of nonlinear conditional expectations facilitates assigning a meaning to each nonlinear expectation.

## 2.3 Examples

The manifestation of the conditioning strongly depends on the specification  $(\mathcal{P}, \alpha)$ . In the example below, we illustrate some special specifications.

**Example 1** *If  $\Theta = \{\theta\}$  is a singleton, then  $\alpha$  is irrelevant and the expectation boils down to a linear one  $\mathcal{E}[X] = E^{P^\theta}[X]$ , where the subjective prior  $P^\theta$  is again given by (1). If  $\alpha \equiv 1$ , i.e., maximal pessimism, then  $\mathcal{E}[X] = \min_{P \in \mathcal{P}} E^P[X]$  results in the setting given by Chen and Epstein (2002).*

Apart from the unique existence of the this nonlinear expectation we have a list of plausible and important properties. Next we try to give some intuition for the new (mathematical) expectation operator. Departing from (2), for every  $X \in L_T$ , we may find an effective prior  $P_\alpha^X \in \mathcal{P}$ , since

$$\begin{aligned} \mathcal{E}[X] &= \alpha^X \min_{P \in \mathcal{P}} E^P[X] + (1 - \alpha^X) \max_{P \in \mathcal{P}} E^P[X] \\ &= \alpha^X E^{P_\alpha^X \min}[X] + (1 - \alpha^X) E^{P_\alpha^X \max}[X] = E^{P_\alpha^X}[X], \end{aligned}$$

by the convexity and compactness of  $\mathcal{P}$  and the linearity of  $P \mapsto E^P X$ . This linearization depends on  $X$  and also holds for the conditional expectation as well. For an illustration consider Figure 2 departing from Figure 1, where

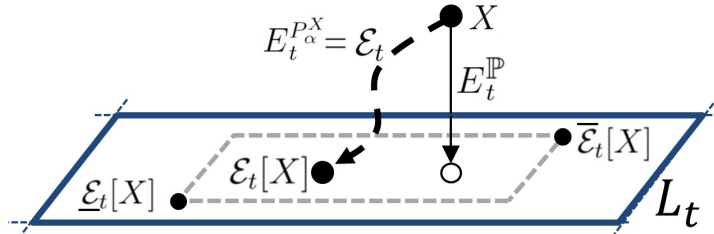


Figure 2: *Illustration of nonlinear conditional expectations*

the classical conditional expectation is portrayed. The nonlinear conditional expectation is described as a deformed least square principle. The following example illustrates in which sense a constant permanent weight  $\alpha$  may cause only sub-/super-linear expectations. As formulated in Remark 2 and Example 2, the case of extreme optimism/pessimism is still possible and serves as another argument to rely on stochastic weights as specified in Definition 1.

**Example 2** *In this example, we argue that in the present dynamic setup, constant weights  $\alpha$  may lead to extreme pessimism or optimism.*

*Consider only a constant weighting  $\alpha \in \llbracket 0, 1 \rrbracket$  and the easiest and well-known case of  $\kappa$ -ignorance, i.e.,  $\Theta = \llbracket -\kappa, \kappa \rrbracket$  and  $\kappa \in \mathbb{R}_+$ . By similar arguments as in Chen and Epstein (2002), we derive*

$$\begin{aligned} e(t, \sigma_t) &= \alpha \max_{\theta \in \Theta} \theta_t \sigma_t + (1 - \alpha) \min_{\theta \in \Theta} \theta_t \sigma_t \\ &= \alpha \kappa |\sigma_t| + (1 - \alpha)(-\kappa) |\sigma_t| = (1 - 2\alpha\kappa) \cdot |\sigma_t| \end{aligned}$$

We get two regions for  $\alpha$  namely  $\frac{1}{2} > \alpha\kappa$  and  $\frac{1}{2} < \alpha\kappa$ .

In the first case  $\mathbf{e}$  is convex, which yields a sublinear expectation given by

$$\underline{\mathcal{E}}[X] = \min_{P \in \mathcal{P}^{\underline{\Theta}}} E^P[X], \quad \text{where } \underline{\Theta} = [-1 + 2\alpha\kappa, 1 - 2\alpha\kappa].$$

In the second region ( $\frac{1}{2} < \alpha\kappa$ ) the nonlinear expectation becomes a superlinear one  $\bar{\mathcal{E}}[X] = \max_{P \in \mathcal{P}^{\bar{\Theta}}} E^P[X]$ , where  $\bar{\Theta} = -\underline{\Theta}$ .

At this stage, since the argument only depends on the concavity (convexity) of  $\sigma \mapsto \mathbf{e}(t, \sigma) = (1 - 2\alpha\kappa) \cdot |\sigma|$ , the derivations lead to a sub (super)-linear expectation even if  $\alpha_t(\omega)$  is stochastic but remains in the sub (super)-linear region. In Figure 3, we consider a stochastic and time-dependent weight  $\alpha_t(\omega)$ ,

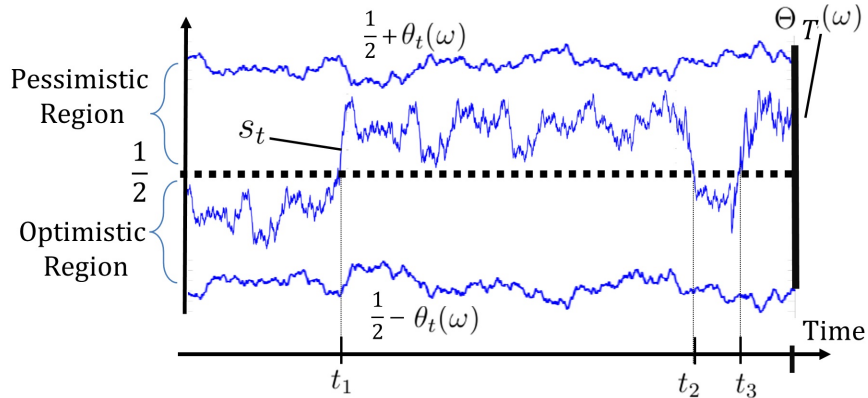


Figure 3: *Illustration of  $\alpha$ -regions*: The upper and lower paths are the boundaries of  $\Theta(\cdot, \omega)$ , given path  $\omega \in \Omega$ . For this  $\omega$ , at time  $t_1, t_2, t_3$ , the sentiment switches.

where it is possible that the expectation changes between the sublinear (optimistic) and superlinear (pessimistic) region, given  $\omega \in \Omega$ . Consider now the set of (symmetric) density generators  $\Theta = [\frac{1}{2} - \theta, \frac{1}{2} + \theta]$ , with  $\theta_t \geq 0$  for all  $t$ . The sentiment process  $s_t = \frac{1}{2} - \theta_t(1 - 2\alpha_t)$  detects the degree of optimism/pessimism.

### 3 Nonlinear Expected Utility

In this section we give a formal definition of a nonlinear expected utility functional on the space of gambles  $L_T$ . Fix a family of nonlinear conditional expectations  $(\mathcal{E}_t)$  induced by a specification  $[\mathcal{P}, \alpha]$  representing the belief system of an agent. A further primitive is a utility index  $u : \mathbb{R} \rightarrow \mathbb{R}$ . In order to concentrate on the implication of the nonlinear expectation, we omit the generalization of state or time depending consumption streams.<sup>4</sup>

<sup>4</sup>Although it is a natural question, we also ignore a possible recursive structure in the light of Duffie and Epstein (1992) and Chen and Epstein (2002).

**Definition 3** Fix  $\mathcal{P}$  and  $\alpha \in \llbracket 0, 1 \rrbracket$ . The Nonlinear Expected Utility  $U : L_T \rightarrow \mathbb{R}$  of a payoff  $X \in L_T$  under  $\mathcal{E}$  is given by

$$U(X) = \mathcal{E}[u(X)],$$

the resulting utility process  $U_t^X = \mathcal{E}_t[u(X)]$  refers to the predicted utility, given information  $\mathcal{F}_t$  at time  $t \in [0, T]$ .

In the light of Example 1 we compare the associated utility specifications. When  $\Theta = \{\theta\}$  is a singleton the nonlinear expected utility boils down to a linear subjective expected utility specification in the sense of Savage (1954). While in the case of  $\alpha_t \equiv 1$  we receive the Maxmin utility of Gilboa and Schmeidler (1989). The permanent pessimism builds a corner specification and results to the extreme case of the highest ambiguity aversion, while  $\alpha = 0$  would capture an extreme optimism. This is consistent with Definition 1, where  $\alpha$  determines the degree of relative pessimism.

Many properties which hold for the standard expected utility remain true for the case of nonlinear expected utility. Here, transitivity and completeness follows immediately. We have the following results.

**Proposition 2** Let  $u$  be continuous and strictly increasing. Then the nonlinear expected-utility functional  $U : L_T \rightarrow \mathbb{R}$  is

- (i) norm continuous: if  $X_n \rightarrow X$  in  $L_T$ , then  $U(X_n) \rightarrow U(X)$ .
- (ii) order continuous: If  $X_n \searrow X$ ,  $\mathbb{P}$ -a.s., then  $U(X_n) \rightarrow U(X)$ .
- (iii) (strictly) monotone: if  $X \geq Y$  then  $U_t^X \geq U_t^Y$ , for all  $t \in [0, T]$ . If also  $\mathbb{P}(X > Y) > 0$  and  $u$  is strictly increasing, then  $U(X) > U(Y)$ .

In the following subsection, we discuss the implication of the utility specification  $(\mathcal{P}, \alpha, u)$ . A first result clarifies, that risk aversion in the absolute sense is still encoded in the shape of  $u$ . Nevertheless, as we see in Example 3 and Figure 4 of Section 4, the specification of  $\alpha$  can influence the comparative risk aversion considerably.

**Risk Aversion** One of the first arising questions refers to the structural properties of risk preference. Recall the classical von Neumann Morgenstern (under the linear expectation  $E$ ) utility functional. Risk aversion of the utility specification  $X \mapsto E[u(X)]$  corresponds to  $E[u(X)] \leq u(E[X])$  for every gamble  $X$ . As usual let the utility index  $u$  be concave, then by the Jensen's inequality risk aversion follows. This is true under a linear expectation, but also, by Theorem 1 (viii), under a nonlinear expectation.

In the present setting, we repeat the same step for the nonlinear expectation. We follow the fundamental principle to define risk aversion in terms of a certainty principle.

**Definition 4** *An agent is conditionally  $\mathcal{E}$ -risk averse on  $L_T$  if*

$$U_t^X \leq u(\mathcal{E}_t[X]), \quad \text{for all } X \in L_T, t \in [0, T].$$

This modified definition precisely catches up the idea of risk aversion to the present setting. However, the novelty of the stated risk aversion concept refers to the dependence of the primitive nonlinear expectation. In the present setup, preferences for risk depends on the structure of the expectation operator  $\mathcal{E}$ , so that  $\alpha$  represents preferences for ambiguity.

The following simple but important result clarifies the analogy to the linear theory of expected utility. Specifically, risk aversion is still explicable in terms of the certainty equivalent, risk premium and related concepts. Nevertheless the premia changes in compare to the linear case and heavily depends on the structure of  $\mathcal{E}$ , see Figure 3. The following result states the announced characterization of risk aversion.

**Proposition 3** *Let  $u \in C^2(\mathbb{R})$  be increasing. The agent is conditionally  $\mathcal{E}$ -risk averse if and only if  $u$  is concave.*

Clearly, the “if” part follows directly from the nonlinear version of the Jensen inequality, stated in Theorem 1. The reverse direction is technically more demanding and relies on recent results of Peng’s  $g$ -expectation based on partial differential equations. In sharp contrast to a von Neumann-Morgenstern utility (linear) or Maxmin utility (superlinear) of Gilboa and Schmeidler (1989) type the concavity of  $u$  does in general not imply the concavity of the nonlinear expected utility, which is a direct consequence of the expectation’s nonlinear shape.

## 4 Certainty Equivalent

To get into the way of analyzing uncertainty preference of a nonlinear expected utility specification we continue with a discussion of the certainty equivalent. As mentioned in Remark 2, it is especially important to justify the endogenous weights  $\alpha^X$  within the present nonlinear expectation.

### 4.1 Static Case

We follow the usual argument within a static setting and say that the certainty equivalent  $\mathcal{C}^X \in \mathbb{R}$  at  $X \in L_T$  is the solution of the following equation

$$u(\mathcal{C}^X) = \mathcal{E}[u(X)].$$

Accomplishing the Arrow-Pratt approximation, we illustrate the quantitative difference with the linear case. The relation between the certainty equivalent

and absolute risk aversion  $A(x) = -\frac{u''(x)}{u'(x)}$  of Pratt (1964) and Arrow (1970) is still present. Consider a given wealth  $w \in \mathbb{R}$ . A Taylor expansion on both sides gives us the following approximative equality<sup>5</sup>

$$\begin{aligned} w - \mathcal{C}^{w+X} &\approx \mathcal{E} \left[ X - \frac{1}{2}A(w)X^2 \right] \\ &= \frac{1}{2}A(w)\mathbf{var}(X) - \mathbf{co}(X, -A(w)X^2), \end{aligned} \quad (6)$$

where  $X$  with  $\mathcal{E}[X] = 0$  denotes a centered risk distortion. For such centered random variables, the expression  $\mathbf{var}(X) = \mathcal{E}[X^2]$ , denotes the variance under the nonlinear expectation. The expression  $\mathbf{co}(X, Y) = \mathcal{E}[X + Y] - \mathcal{E}[X] - \mathcal{E}[Y]$  refers to the so-called *coexpectation*, which can be seen as the compensation term for the nonlinearity of the expectation under consideration. The polar case  $\alpha = 1$ , yields a super linear expectation, hence  $\mathbf{co}(\cdot, \cdot) \geq 0$ .

The formulation in (6) has an intuitive appeal in compare to the linear case, where the term in the bracket boils down to the usual variance factor  $\frac{1}{2}A(w)\mathbf{var}^P(X)$  under a given prior  $P$ . The additional coexpectation captures the nonlinearity of the expectation, may take positive and negative values and heavily depends on  $A(w)$ . On the other hand, if  $X \in \{X \in L_T : \mathcal{E}(X) = -\mathcal{E}(-X)\}$ , then the coexpectation in (6) vanishes.

In comparison with the approximation of the (second order) smooth ambiguity certainty equivalent in Maccheroni, Marinacci, and Ruffino (2013), the present certainty equivalent reveals the nonlinear structure of the expectation. In our approximation this is exposed by the coexpectation.

The following example discusses the implication of the new certainty equivalent and shows differences with the standard expected utility model. To keep the illustration simple, there are two states of the world and dynamic aspects are excluded.

**Example 3** *Suppose that there are only two states of the world (good) and (bad) and the nonlinear expected utility is given by the following specification  $(\mathcal{P}, \alpha)$ , where  $\mathcal{P} = \{P \in \Delta_2 : P(h) \in [\frac{2}{5}, \frac{3}{5}]\}$  and  $\alpha = (\frac{1}{3}, \frac{2}{3})$ . The utility index is given by  $u(x) = \ln(x)$ . Consider the gamble  $(X_{\text{bad}}, X_{\text{good}})$ .*

*Figure 4 compares the effect of the  $\mathcal{E}$ -certainty equivalence with the usual certainty equivalence of an expected utility under the fixed prior  $P$  such that  $P(\text{bad}) = \frac{1}{2}$ . We denote by  $R(X)$  the risk premium and by  $\mathcal{R}(X)$  the uncer-*

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<sup>5</sup>We consider the second-order Taylor expansion around  $w$  for  $u(w)$

$$\begin{aligned} \mathcal{E}[u(X + w)] &\approx u(w) + \mathcal{E}[u'(w)X + \frac{1}{2}u''(w)X^2] = u(w) + u'(w)\mathcal{E}[X - \frac{1}{2}A(w)X^2] \\ &= u(w) + u'(w)\left(-\frac{1}{2}A(w)\mathcal{E}[X^2] + \mathbf{co}(X, -\frac{1}{2}A(w)X^2)\right), \end{aligned}$$

where we applied the concavity of  $u$  via  $u' \geq 0, u'' \leq 0$  and Theorem 1. Using the first-order Taylor expansion for  $u(\mathcal{C}^{w+X})$  around  $w$ :  $u(\mathcal{C}^{w+X}) \approx u(w) + u'(w)(\mathcal{C}^{w+X} - w)$ . The above equalities yield the desired result.

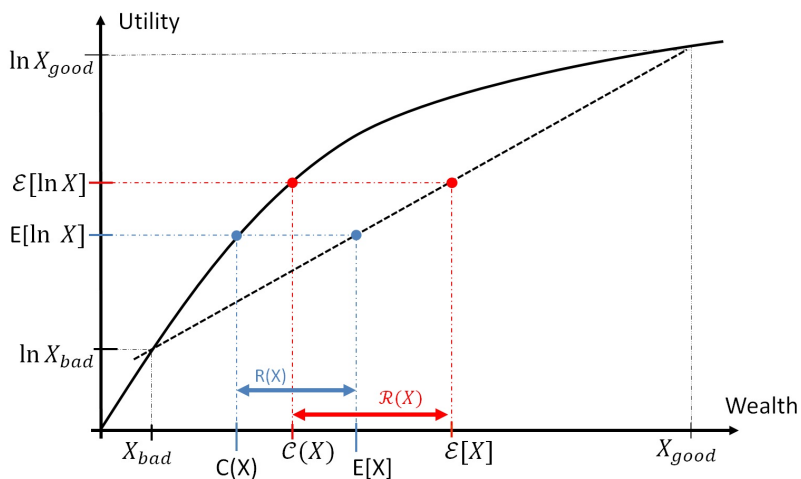


Figure 4: Illustration of risk premia and certainty equivalents under linear and nonlinear expectation.

tainty premium, respectively given by the difference between expected value and the corresponding certainty equivalent. As we emphasize in Proposition 2, nonlinear expected utility does not induce a convex preference relation. However, as stated in Proposition 3, this does not affect the risk aversion, as illustrated in Figure 4.

The modified risk premium captures a nonlinear component. Let  $P \in \mathcal{P}$  be considered as a reference measure. From this we may consider  $R(X) - \mathcal{R}(X) \in \mathbb{R}$  as a new uncertainty factor depending on  $\alpha$ .

For a small distortion, the example points out that under a nonlinear expectation the risk premium  $u^{-1}(\mathcal{E}[u(X)]) - \mathcal{E}[X]$  may vary considerably in comparison to the linear case. This is consistent with the derivations in (6), where the coexpectation  $\mathfrak{co} : L_T \times L_T \rightarrow \mathbb{R}$  controls this issue. Implicitly the ambiguity premium, caused by preferences for ambiguity, is manifested in the nonlinear behavior of the risk premium. As illustrated in Example 3 and Figure 4 this is determined by the ambiguity averse (pessimism) and ambiguity loving (optimism) fractions  $\alpha$  of the specification  $[\mathcal{P}, \alpha]$  of the decision maker.

## 4.2 Dynamic Case

This subsection is concerned with a discussion of the updating behavior of the certainty equivalent. As a revealing of preferences about uncertainty, we may check the rationality in the updating behavior. In the dynamic context, time consistency of the certainty equivalent is a natural property, as it requires that the certainty equivalent can be indifferently evaluated directly or using an intermediate time.



**Conditional Certainty Equivalent** In the light of Definition 2, we are especially interested in the dynamics of the certainty equivalent. The proposition below states a complete description of the conditional certainty equivalent. Moreover, we describe the local decomposition of the risk premium and its residual compensation due to the nonlinearity of the expectation.

**Proposition 4** *Let  $u \in C^2(\mathbb{R})$  be strictly increasing. Then the conditional certainty equivalent,  $\mathcal{C}_t^X = u^{-1}(\mathcal{E}_t[u(X)])$ , satisfies  $\mathcal{C}_t^X \leq \mathcal{E}_t[X]$  and is a first component of the solution of the following BSDE*

$$d\mathcal{C}_t^X = \left( \frac{\sigma_t^C}{\sigma_t^u} \mathbf{e}(t, \sigma_t^u) + \frac{1}{2} (\sigma_t^C)^2 A(\mathcal{C}_t^X) \right) dt + \sigma_t^C dB_t, \quad \mathcal{C}_T^X = X, \quad (7)$$

where  $\sigma_t^C = \frac{\sigma_t^u}{u'(\mathcal{C}_t^X)}$  and  $\sigma^u$  is the second component in the solution of

$$d\mathcal{E}_t[u(X)] = \mathbf{e}(t, \sigma_t^u) dt + \sigma_t^u dB_t, \quad \mathcal{E}_T[u(X)] = u(X).$$

In general, the differential equation (7) can have many different solutions. In the light of Remark 2, by selecting one possible conditional certainty equivalent, this selection could not fully capture the underlying utility process.

**Dynamic Certainty Equivalent** Concerning the notion of time-consistency of the nonlinear conditional expectation, we may rephrase this issue as a semi-group property, i.e.,  $\mathcal{E}_s \circ \mathcal{E}_t = \mathcal{E}_s$ , for  $s \leq t$ . In a similar form, homogeneous Markov processes satisfy a related semi-group property. However, the class of time non-homogeneous Markov processes, where the kernel of the transition probability from time  $s$  to times  $t$  does not depend on  $t - s > 0$ , but on  $(s, t)$ , satisfies a more general semi-group property. In such an analogy, time consistency of the dynamic certainty equivalent corresponds to the semi-group property for non-homogeneous Markov processes.

To formulate the time consistency of certainty equivalents, we consider the following backward stochastic differential equation, for fixed  $0 \leq t < \infty$ ,

$$d\mathcal{E}_{s,t}[X] = \mathbf{e}(s, \sigma_s) ds + \sigma_s dB_s, \quad s \in [0, t], \quad \mathcal{E}_{t,t}[X] = X. \quad (8)$$

In comparison to the last subsection, we fixed  $t = T$ . For given  $X \in L_t$ , the above equation has a unique solution by the same arguments. As mentioned, we introduce two time parameters, so that (8) corresponds to Theorem 1 via  $\mathcal{E}_{t,T} = \mathcal{E}_t$ .

**Definition 5** *Let  $u \in C^2(\mathbb{R})$  be strictly increasing and concave and  $X \in L_t$ . The dynamic certainty equivalent  $C_{s,t} : L_t \rightarrow L_s$  at  $X$  with  $s \in [0, t]$  is defined by  $u(C_{s,t}(Y)) = \mathcal{E}_{s,t}[u(Y)]$ , where and  $(\mathcal{E}_{s,t}[u(X)], \sigma_s)_{s \leq t}$  is the unique solution of the following BSDE on  $[0, t]$ :*

$$d\mathcal{E}_{s,t}[u(X)] = \mathbf{e}(s, \sigma_s) ds + \sigma_s dB_s, \quad \mathcal{E}_{t,t}[u(X)] = u(X).$$

The dynamic certainty equivalent has the following properties. Again, in compare to Proposition 3, we have  $C_{t,T} = C_t$ .

**Proposition 5** *For  $0 \leq r \leq s \leq t < \infty$ , and  $X, Y \in L_t$ , the following properties hold:*

- (i) Constant-preserving:  $C_{t,t}(X) = X$ .
- (ii) Time consistency:  $C_{r,t}(X) = C_{r,s}(C_{s,t}(X))$ .
- (iii) Monotonicity: *If  $C_{s,t}(X) \leq C_{s,t}(Y)$ , then  $C_{r,t}(X) \leq C_{r,t}(Y)$ . In particular, if  $X \leq Y$ , then  $C_{r,t}(X) \leq C_{r,t}(Y)$ .*
- (iv) Zero-one law: *for any  $A \in \mathcal{F}_s$ , we have  $C_{s,t}(X \mathbf{1}_A) = C_{s,t}(X) \mathbf{1}_A$  and*

$$C_{s,t}(X \mathbf{1}_A + Y \mathbf{1}_{A^c}) = C_{s,t}(X) \mathbf{1}_A + C_{s,t}(Y) \mathbf{1}_{A^c}.$$

- (v) Dominance:  $C_{r,t}(X) \leq \mathcal{E}_{r,s}[C_{s,t}(X)]$ . *In particular,  $C_{r,t}(X) \leq \mathcal{E}_{r,t}[X]$ .*

Figure 5 illustrates the time consistency of the dynamic certainty equivalent for  $X$  on  $[r, t]$  can be obtained directly or we can first get it for  $X$  on  $[s, t]$  and

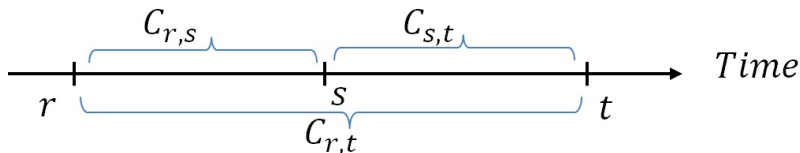


Figure 5: Time consistency of dynamic certainty equivalent

then take  $C_{s,t}(X)$  on  $[r, s]$ .

**Remark 4** *For  $0 < r < s < t < \infty$ , if  $C_{s,t}(X) = C_{s,t}(Y)$ , then from (iii) we know that  $C_{r,t}(X) = C_{r,t}(Y)$ . This precisely means that the indifference of the certainty equivalent of  $X$  and  $Y$  at time  $s \in (r, t)$  can carry over to any earlier time  $r < s$ , that is, when less information is available.*

*In (iv),  $C_{s,t}(X \mathbf{1}_A) = C_{s,t}(X) \mathbf{1}_A$  means that, at time  $s$ , we know whether  $\mathbf{1}_A$  is equal to 1 or 0. If it is equal to 0, then it is obvious that  $C_{s,t}(X \mathbf{1}_A) = C_{s,t}(0) = 0$ .*

## 5 Conclusion

Conditional expectations build a useful tool in many branches of economics and finance. This paper introduces a new functional form of sentiment-based belief systems that captures waves of optimism and pessimism.

The resulting utility functionals employ a Hurwicz-type criterion and allow for a dynamic consistent extension. We presented an approach in continuous time and characterized risk aversion via the same criterion as for linear expected utility.

Furthermore, it is possible to describe the (dynamic) certainty equivalent in a quite explicit way, so that the classical Arrow–Pratt approximation reveals a sentiment premium by means of the specification of relative pessimism within in the belief system.

## A Backward stochastic differential equation

In this section we recall some results of BSDEs in order to give readers some convenience. Pardoux and Peng (1990) introduced the following equation, called backward stochastic differential equation (BSDE):

$$dy_t = f(t, y_t, \sigma_t)dt + \sigma_t dB_t, \quad t \in [0, T], \quad y_T = X,$$

where the terminal condition  $X \in L_T$  is given and  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , so that  $f(\cdot, y, \sigma)$  is an adapted process for every  $y, \sigma \in \mathbb{R}$ .

A pair of adapted processes  $(y, \sigma) \in \mathbb{R} \times \mathbb{R}$  is called a solution of the above equation, if  $E^{\mathbb{P}}[\sup_t |y_t|^2] < \infty$ ,  $E^{\mathbb{P}}[\int_0^T |\sigma_t|^2 dt] < \infty$  and  $(Y, \sigma)$  satisfies the above equation.  $y$  is called the first component of the solution and  $\sigma$  is called the second component of the solution.

Pardoux and Peng (1990) obtained the following existence and uniqueness of the solution of the above equation.

**Lemma 2** *If  $f(t, \cdot, \cdot)$  is Lipschitz continuous on  $\mathbb{R} \times \mathbb{R}$ , then the above BSDE has a unique adapted solution  $(y, \sigma)$ .*

## B Proofs

**Proof of Theorem 1** *We start with the uniqueness and existence of the solution of the BSDE. For all  $x, y \in \mathbb{R}$ , we have*

$$\begin{aligned} & |e(t, x) - e(t, y)| \\ &= |\alpha_t \max_{\theta \in \Theta} \theta_t x + (1 - \alpha_t) \min_{\theta \in \Theta} \theta_t x - \alpha_t \max_{\theta \in \Theta} \theta_t y - (1 - \alpha_t) \min_{\theta \in \Theta} \theta_t y| \\ &\leq |\alpha_t| |\max_{\theta \in \Theta} \theta_t x - \max_{\theta \in \Theta} \theta_t y| + |(1 - \alpha_t)| |\min_{\theta \in \Theta} \theta_t x - \min_{\theta \in \Theta} \theta_t y| \\ &\leq \max_{\theta \in \Theta} |\theta_t(x - y)| + \max_{\theta \in \Theta} |\theta_t(x - y)|. \end{aligned}$$

*Since  $\Theta$  is compact, then there exists a positive constant  $C$  such that*

$$|e(t, x) - e(t, y)| \leq C|x - y|.$$

*Therefore,  $e(t, \cdot)$  is uniformly Lipschitz and  $e(t, 0) = 0$ , then from Lemma 2 in Appendix A, equation (4) has a unique solution, and we can get (i)-(v) from Lemma 36.6 and Theorem 37.3 in Peng (1997).*

For all  $x \in \mathbb{R}, \beta > 0$ , we have

$$\begin{aligned} \mathbf{e}(t, \beta x) &= \alpha_t \max_{\theta \in \Theta} \theta_t(\beta x) + (1 - \alpha_t) \min_{\theta \in \Theta} \theta_t(\beta x) \\ &= \beta \mathbf{e}(t, x), \end{aligned}$$

from which we get the positive homogeneity of  $\mathbf{e}(t, \cdot)$ . It follows from Example 10 in Peng (1997) to get (vi), and (vii) is derived from Lemma 36.9 in Peng (1997). Since  $u \in C^2(\mathbb{R})$  is increasing and concave, we can get (viii) from Theorem 1 in Jia and Peng (2010).

**Proof of Proposition 1** We can get the first part from Coquet, Hu, Mémin, and Peng (2002) directly. For  $\eta > 0$ , we define

$$\bar{g}(s, z) = \eta g(s, \frac{z}{\eta}), \text{ for } s \in [0, T].$$

Let  $(\mathcal{E}_s^{\bar{g}}[X], \bar{\sigma}_s)_{s \in [0, T]}$  be the unique solution of the following BSDE:

$$d\mathcal{E}_s^{\bar{g}}[X] = \bar{g}(s, \bar{\sigma}_s)ds + \bar{\sigma}_s dB_s, \quad s \in [0, T], \quad \mathcal{E}_T^{\bar{g}}[X] = X.$$

Then using the uniqueness of BSDE, we have

$$\mathcal{E}_0^{\bar{g}}[\eta X] = \eta \mathcal{E}_0^g[X], \text{ for all } X \in L_T.$$

From the positive homogeneity it follows that

$$\mathcal{E}_0^{\bar{g}}[X] = \mathcal{E}_0^g[X], \text{ for all } X \in L_T.$$

Using the similar arguments as in Jiang (2004, 2008), we obtain

$$\bar{g}^\eta(s, \cdot) = g(s, \cdot), \text{ for } s \in [0, T].$$

Consequently,

$$g(s, \eta \times \cdot) = \eta g(s, \cdot), \text{ for } s \in [0, T].$$

**Proof of Proposition 2** (i) The continuity follows from the presence of a dominating sublinear expectation, which implies norm-continuity as stated in Remark 3.

(ii) We just give the proof when  $\{X_n\}_{n \geq 1}$  is decreasing. From the monotonicity of the nonlinear expectation  $\mathcal{E}$ , we know that  $\{U(X_n)\}_{n \geq 1}$  is decreasing.

Since  $\{X_n\}_{n \geq 1}$  is decreasing and  $\lim_{n \rightarrow \infty} X_n = X$ ,  $\mathbb{P}$ -a.s., we get that  $|u(X_n) - u(X)| \leq |u(X_n)| + |u(X)| \in L_T$ , and  $\lim_{n \rightarrow \infty} |u(X_n) - u(X)| = 0$ ,  $\mathbb{P}$ -a.s. Then by virtue of the dominated convergence theorem we have,  $\lim_{n \rightarrow \infty} E^{\mathbb{P}} |u(X_n) -$

$u(X)|^2 = 0$ . From (vii) in Theorem 1 we know that, there is a constant  $C > 0$  such that,

$$|U(X_n) - U(X)|^2 \leq CE^{\mathbb{P}} [|u(X_n) - u(X)|^2],$$

from which we can get  $\lim_{n \rightarrow \infty} U(X_n) = U(X)$ .

(iii) Since  $X \geq Y$ ,  $\mathbb{P}$ -a.s., and  $u$  is increasing, then we have  $u(X) \geq u(Y)$ ,  $\mathbb{P}$ -a.s. From (i) in Theorem 1 it follows that

$$U_t^X = \mathcal{E}_t[u(X)] \geq \mathcal{E}_t[u(X)] = U_t^Y.$$

Moreover, if  $\mathbb{P}(X > Y) > 0$  and  $u$  is strictly increasing, then  $\mathbb{P}(u(X) > u(Y)) > 0$ . Using (i) in Theorem 1 again

$$U(X) = \mathcal{E}[u(X)] > \mathcal{E}[u(Y)] = U(Y).$$

**Proof of Proposition 3** Since  $e(t, \sigma)$  is a convex combination of an inf and sup operation,  $\mathbf{e}(t, \sigma)$  is positive homogeneous in  $z$ . By an application of Theorem 3.2 in Jia and Peng (2010) to  $\mathbf{e}(t, \sigma)$ , which is independent of  $\mathcal{E}_t[X]$ , the conditional  $\mathcal{E}$ -concavity, i.e.,  $u(\mathcal{E}_t[X]) \leq \mathcal{E}_t[u(X)]$ , can be characterized as follow

$$\frac{1}{2}u''(x)|\sigma|^2 + \mathbf{e}(t, u'(x)z) - u'(x)\mathbf{e}(t, \sigma) \leq 0$$

By the positive homogeneity of  $\mathbf{e}(t, \cdot)$  this is equivalent to  $u''(x) \leq 0$ , being equivalent to concavity.

**Proof of Proposition 4** Let  $(\mathcal{E}_t[u(X)], \sigma_t^u)_{t \in [0, T]}$  be the unique solution of the following equation:

$$\begin{aligned} d\mathcal{E}_t[u(X)] &= \mathbf{e}(t, \sigma_t^u)dt + \sigma_t^u dB_t, \quad t \in [0, T], \\ \mathcal{E}_T[u(X)] &= u(X). \end{aligned}$$

Then from Itô Lemma with respect to  $u^{-1}(\mathcal{E}_t[u(X)])$  it follows that

$$d\mathcal{C}_t(X) = \left( \frac{\mathbf{e}(t, \sigma_t^X)}{u'(\mathcal{C}_t(X))} - \frac{1}{2} \frac{u''(\mathcal{C}_t(X))}{u'(\mathcal{C}_t(X))^3} (\sigma_t^u)^2 \right) dt + \frac{\sigma_t^u}{u'(\mathcal{C}_t(X))} dB_t.$$

We denote  $\sigma_t^{\mathcal{C}} = \frac{\sigma_t^u}{u'(\mathcal{C}_t(X))}$ , then

$$\begin{aligned} d\mathcal{C}_t(X) &= \left( \frac{\sigma_t^{\mathcal{C}}}{\sigma_t^u} \mathbf{e}(t, \sigma_t^X) - \frac{1}{2} \frac{u''(\mathcal{C}_t(X))}{u'(\mathcal{C}_t(X))} (\sigma_t^{\mathcal{C}})^2 \right) dt + \sigma_t^{\mathcal{C}} dB_t \\ &= \frac{\sigma_t^{\mathcal{C}}}{\sigma_t^u} \mathbf{e}(t, \sigma_t^U) + \frac{1}{2} (\sigma_t^{\mathcal{C}})^2 A(\mathcal{C}_t^X) dt + \sigma_t^{\mathcal{C}} dB_t. \end{aligned}$$

**Proof of Proposition 5** (i) By (vi) in Theorem 1 and the definition of the dynamic certainty equivalent, we have

$$C_t(X) = u^{-1}(\mathcal{E}_{t,t}[u(X)]) = u^{-1}(u(X)) = X.$$

(ii) By (iv) in Theorem 1 and the definition of the dynamic certainty equivalent, we have

$$\begin{aligned} C_{r,t}(X) &= u^{-1}(\mathcal{E}_{r,t}[u(X)]) \\ &= u^{-1}(\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[u(X)]]) \\ &= u^{-1}(\mathcal{E}_{r,s}[u(C_{s,t}(X))]) \\ &= u^{-1}(u(C_{r,s}(C_{s,t}(X)))) \\ &= C_{r,s}(C_{s,t}(X)). \end{aligned}$$

(iii) From (ii) it follows that

$$C_{r,t}(X) = C_{r,s}(C_{s,t}(X)) \leq C_{r,s}(C_{s,t}(Y)) = C_{r,t}(Y).$$

In particular, when we take  $v = t$  and from (i) it follows that, if  $X \leq Y$ , then  $C_{r,t}(X) \leq C_{r,t}(Y)$ .

(iv) Since  $u(X\mathbf{1}_A + Y\mathbf{1}_{A^c}) = u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}$ , we have

$$\begin{aligned} C_{s,t}(X\mathbf{1}_A + Y\mathbf{1}_{A^c}) &= u^{-1}(\mathcal{E}_{s,t}[u(X\mathbf{1}_A + Y\mathbf{1}_{A^c})]) \\ &= u^{-1}(\mathcal{E}_{s,t}[u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}]). \end{aligned} \quad (9)$$

Let us consider the following two BSDEs

$$d\mathcal{E}_{s,t}[u(X)] = \hat{e}(\sigma_s^1)ds + \sigma_s^1 dB_s, \quad \mathcal{E}_{t,t}[u(X)] = u(X), \quad (10)$$

and

$$d\mathcal{E}_{s,t}[u(Y)] = \hat{e}(\sigma_s^2)ds + \sigma_s^2 dB_s, \quad \mathcal{E}_{t,t}[u(Y)] = u(Y). \quad (11)$$

Then (10)  $\times \mathbf{1}_A + (11) \times \mathbf{1}_{A^c}$  yields

$$\begin{aligned} &d(\mathcal{E}_{s,t}[u(X)]\mathbf{1}_A + \mathcal{E}_{s,t}[u(Y)]\mathbf{1}_{A^c}) \\ &= [\hat{e}(\sigma_s^1)\mathbf{1}_A + \hat{e}(\sigma_s^2)\mathbf{1}_{A^c}]ds + (\sigma_s^1\mathbf{1}_A + \sigma_s^2\mathbf{1}_{A^c})dB_s \\ &= \hat{e}(\sigma_s^1\mathbf{1}_A + \sigma_s^2\mathbf{1}_{A^c})ds + (\sigma_s^1\mathbf{1}_A + \sigma_s^2\mathbf{1}_{A^c})dB_s, \end{aligned}$$

with the terminal condition

$$\mathcal{E}_{t,t}[u(X)]\mathbf{1}_A + \mathcal{E}_{t,t}[u(Y)]\mathbf{1}_{A^c} = u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}.$$

Recall the following BSDE

$$\begin{aligned} d\mathcal{E}_{s,t}[u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}] &= \hat{e}(\sigma_s)ds + \sigma_s dB_s, \\ \mathcal{E}_{t,t}[u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}] &= u(X)\mathbf{1}_A + u(Y)\mathbf{1}_{A^c}. \end{aligned}$$

From the uniqueness of the solution of the above equations, we have

$$\mathcal{E}_{s,t}[u(X)]\mathbf{1}_A + \mathcal{E}_{s,t}[u(Y)]\mathbf{1}_{A^c} = \mathcal{E}_{s,t}[u(X)]\mathbf{1}_A + \mathcal{E}_{s,t}[u(Y)]\mathbf{1}_{A^c}.$$

Therefore, from (9) we have

$$\begin{aligned} C_{s,t}(X\mathbf{1}_A + Y\mathbf{1}_{A^c}) &= u^{-1}(\mathcal{E}_{s,t}[u(X)]\mathbf{1}_A + \mathcal{E}_{s,t}[u(Y)]\mathbf{1}_{A^c}) \\ &= u^{-1}(\mathcal{E}_{s,t}[u(X)])\mathbf{1}_A + u^{-1}(\mathcal{E}_{s,t}[u(Y)])\mathbf{1}_{A^c} \\ &= C_{s,t}(X)\mathbf{1}_A + C_{s,t}(Y)\mathbf{1}_{A^c}. \end{aligned}$$

Let  $Y = 0$ , Then by  $\mathcal{E}_{s,t}[0] = 0$  we have  $C_{s,t}(X\mathbf{1}_A) = C_{s,t}(X)\mathbf{1}_A$ .

(v) From (ii) we have

$$C_{r,t}(X) = C_{r,s}(C_{s,t}(X)) = u^{-1}(\mathcal{E}_{r,s}[u(C_{s,t}(X))]).$$

Therefore, by Jensen inequality in Theorem 1 we get

$$\mathcal{E}_{r,s}[u(C_{s,t}(X))] \leq u(\mathcal{E}_{r,s}[C_{s,t}(X)]).$$

From the above inequalities it follows that

$$C_{r,t}(X) \leq \mathcal{E}_{r,s}[C_{s,t}(X)].$$

In particular, by using (i) we get  $C_{r,t}(X) \leq \mathcal{E}_{r,t}[X]$ .

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