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An Exact Solution Method for Binary Equilibrium Problems with Compensation and the Power Market Uplift Problem

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An exact solution method for binary equilibrium problems with compensation and the power market uplift problem

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Abstract

We propose a novel method to find Nash equilibria in games with binary decision variables by including compensation payments and incentive-compatibility constraints from non-cooperative game theory directly into an optimization framework in lieu of using first order conditions of a linearization, or relaxation of integrality conditions. The reformulation offers a new approach to obtain and interpret dual variables to binary constraints using the benefit or loss from deviation rather than marginal relaxations. The method endogenizes the trade-off between overall (societal) efficiency and compensation payments necessary to align incentives of individual players. We provide existence results and conditions under which this problem can be solved as a mixed-binary linear program.

We apply the solution approach to a stylized nodal power-market equilibrium problem with binary on-off decisions. This illustrative example shows that our approach yields an exact solution to the binary Nash game with compensation. We compare different implementations of actual market rules within our model, in particular constraints ensuring non-negative profits (no-loss rule) and restrictions on the compensation payments to non-dispatched generators. We discuss the resulting equilibria in terms of overall welfare, efficiency, and allocational equity.

Keywords: binary Nash game, non-cooperative equilibrium, compensation, incentive compatibility, electricity market, power market, uplift payments

JEL Codes: C72, C61, L13, L94

MSC Codes: 90C11, 90C46, 91B26

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1 Introduction

There are many environments where several players interact non-cooperatively and face binary decisions, such as electricity markets (on-off decision regarding a power plant), transportation (e.g., facility location models, Caunhye et al. 2012), engineering (Rao 1996), as well as agriculture and land-use planning (Tóth et al. 2011).

Modeling Nash equilibria between players which face both binary and continuous decisions is a challenging problem (Scarf 1990). Economists and game theorists usually apply brute-force methods by exploring all possible combinations and check every solution for deviation incentives of each player. When market-clearing prices to support a Nash equilibrium in the Walrasian sense do not exist, economists suggest to use multi-part pricing (Hotelling 1938) or deviate from marginal-cost pricing to a “second-best” market outcome, such that no player should lose money from participating (Baumol and Bradford 1970). However, a canonical approach to find Nash equilibria in binary games does not exist.

In many large-scale practical applications, exploring the entire solution space is not realistically possible. A common approach is to linearize the binary decision; the Nash equilibrium can then be computed by solving the system of first order optimality conditions, a.k.a. equilibrium modeling using mixed complementarity problems or variational inequalities, if certain assumptions on convexity of the linearized problem hold. Recent work seeks a trade-off between relaxation of the complementarity (slackness) conditions or the integrality of discrete constraints to obtain stationary points that are presumed to be equilibria of the original problem (Gabriel et al. 2012, 2013).

In this work, we focus on applications where a relaxation or linearization of the binary decision variable is either not practical or yields incorrect results.

There are many instances where there exists no set of strategies or a price vector that supports a Nash equilibrium; i.e., there is no outcome where the pay-offs to each stakeholder are such that no player has a profitable deviation. This is due to the non-convexity introduced by the binary decision variables and indivisibilities (O’Neill et al. 2005). We introduce the notion of a “quasi-equilibrium” to describe situations where no equilibrium exists, but where a market operator or regulator can assign compensation payments in order to obtain an incentive-compatible outcome. These payments align the incentives of individual players with the objectives of the overall system, such as cost minimization or welfare maximization. A regulator may also choose to intervene when an equilibrium exists but its outcome is inferior to the solution that a benevolent planner might achieve. That is, the market operator may seek to minimize the deviation from the system optimum (i.e., all decisions by one planner) caused by the non-cooperative game among a number of decision makers, each seeking to optimize competing objectives. Our solution approach allows to endogenously consider the trade-off between regulatory intervention to improve market efficiency and the distortions caused by said interventions.

Electricity markets are the real-world application of binary games which have received the most attention in the recent mathematical optimization literature (O’Neill et al. 2013, Liu and Hobbs 2013, Wogrin et al. 2013, Liu and Ferris 2013, Philpott et al. 2013, Bjørndal and Jörnsten 2008, Hu and Ralph 2007, Philpott and Schultz 2006, O’Neill et al. 2005). A challenging problem arises from the on-off decision of power plants, which usually incur substantial start-up or shut-down costs and, if operational, face minimum-generation constraints. Because power markets are usually based on marginal-cost, short-term pricing, the ramping costs are not necessarily covered by resulting market prices. Numerous other features of power systems further complicate the market, such as variable and stochastic feed-in of power from renewable energy sources, the necessity to maintain sufficient backup capacity to deal with contingencies, and switching of lines to optimize grid topology.

There are additional interesting aspects to consider: most electricity markets have
rules that generators must be “made whole” or have to be “in the money”; i.e., they receive compensation to make sure that they do not lose money (commonly referred to as a “no-loss rule”). However, this may not be required from a game-theoretic point of view, and thereby lead to higher-than-necessary compensation payments. At the same time, there might exist regulations that only power plants that are actually generating electricity can receive compensation – the rationale being that it may create perverse incentives for market participants to be paid to not do something. We will discuss and illustrate in a numerical example how such market rules can actually overly restrict operational efficiency and thereby reduce welfare.

The rules governing market operations obviously differ across various fields. A topic that has received far less attention from the applied Operations Research community is agriculture. Yet it is also a form of binary game, between independent farmers (or agro-commercial enterprises) which decide whether to plant a field or not. Interestingly, and in contrast to power markets, there are subsidies being paid in many countries to farmers that let some of their land lay idle. Here, there is a payment for doing nothing! The key difference may be that farming is seen as a market less prone to exertion of market power and strategic behavior, and that letting a field lay idle is still seen to provide a benefit beyond mere price support, for instance by maintaining the landscape or providing breeding grounds for wildlife.

The outline of this paper is as follows: in the next section, we summarize current approaches to solve binary Nash games and place our contribution in the context of methods applied to solve such problems in the power sector. In Section 3, we propose an exact solution method to solve binary equilibrium problems, which explicitly incorporates the trade-off between overall efficiency and compensation payments in cases where no equilibrium exists. Section 4 applies our method to a power market example from the literature to illustrate its advantages and flexibility to incorporate distinct market rules regarding uplift payments. Section 5 concludes with a discussion on methods, applications, and future work.

2 Current approaches to solve binary games

In this section we motivate our method by describing how current solution methods for binary games obtain equilibria, and we identify where our formulation can improve this process. While there exist brute-force methods that solve for an equilibrium considering all possible values of the binary variables and check ex-post for deviation incentives, we want to concentrate on mathematical programming techniques for obtaining equilibria. For large-scale applications such as those considered in this paper, computational efficiency proves a hurdle in these brute-force methods.

2.1 Optimization-based approaches to obtain equilibria

Game theory and equilibrium problems have been an integral part of the history of mathematical programming. Their formulations fit neatly in the application areas and computational techniques of optimization. Moreover, ideas and algorithms from equilibrium problems have benefited solution techniques for optimization as well. First-order optimality (Karush-Kuhn-Tucker, KKT) conditions, derived from each individual player’s optimization problem, can be solved simultaneously by stacking them to form an equilibrium problem. Interpretations from dual variables to constraints in a game theory analysis can prove very insightful; they provide essential information in equilibrium problems and are often interpreted as prices or marginal benefits for individual players (Facchinei and Pang 2003, Ferris and Pang 1997, Murphy et al. 1982).

However, this relationship between optimality conditions and equilibrium problems fails once you have a game in binary decision variables. The reason is that you cannot
directly derive the optimality conditions for a binary optimization problem. Thus, applied optimization researchers aim to relax these binary problems to be able to derive optimality conditions, or obtain duals to constraints of individual players’ optimization problems in other ways. Our main contribution is that we overcome both these hurdles to obtain exact equilibria of the binary games and dual variables to constraints that can be interpreted as prices or marginal benefits for a player.

Optimization formulations have been a natural fit for studying energy markets. The economics and the physics of power systems can, for the most part, be handled with equations that are amenable to current optimization algorithms. There are two notable exceptions: one is incorporating the physical characteristics of alternate-current power flow in economic models, while the other is properly including the on/off decisions for power plants. The former caveat is commonly tackled by using a linearization of the problem, namely the direct-current load flow (DCLF) formulation. The application presented in Section 4 focuses on the latter problem: solving the binary game among non-cooperative generators, which even in a DCLF model is computationally intensive and often does not have an equilibrium.

Solving a large number of equilibrium problems is not very elegant and suffers from a curse of dimensionality, because the number of equilibrium problems to be solved is $2^k$, where $k$ is the number of binary variables. Hence, mathematicians and Operations Researchers are looking for ways to apply advances in Variational Inequalities and Integer Programming to develop faster methods to solve such problems.

For games where the individual players’ optimization problems are non-convex with continuous variables, [Pang and Scutari (2013)] introduce the notion of a “quasi-Nash equilibrium” to describe solutions that are stationary points derived from relaxed constraint qualifications. Their paper provides analysis into these equilibria and their interpretation. In our paper, we are looking at a similar but distinct concept of an equilibrium. While Pang and Scutari obtain “quasi-Nash equilibria” where standard Nash equilibria exist but are computationally unattainable, we focus on cases where standard Nash equilibria do not exist, and obtain reasonable solution points instead. Moreover, the analysis by Pang and Scutari is focused on continuous games while we focus exclusively on binary games.

A method based on a trade-off between relaxing the integrality and the complementarity constraints is developed by [Gabriel et al. (2013)]. While relaxing integrality has been employed as a way to solve integer programs, relaxing complementarity – essentially the optimality conditions – was the novel idea of their contribution. By relaxing the optimality conditions, the authors were able to obtain solutions that were close to being stationary points and presumed to be equilibria. By relaxing both integrality and complementarity, the authors were also able to obtain duals to constraints, though interpreting them was not straightforward. The drawback to this approach is that there was no clear evidence that the solutions obtained were guaranteed to be equilibria or the duals of the constraints were guaranteed to be the marginal benefit of relaxing those constraints. In our work, we prove that the solutions from our formulation are in fact binary equilibria and provide duals which can be interpreted as marginal benefits when relaxing the constraints of the players.

One recent paper focuses on solving integral Nash-Cournot games [Todd (2014)] and provides an efficient algorithm to obtain equilibria. The method does not employ a relaxation but still converges to integer equilibria under plausible conditions. However, this method works very well for a specific integer game with no constraints, and the algorithm is not applicable to the broad class of binary games considered in this paper.

2.2 Dual variables in binary programs

As mentioned above, dual variables in constrained convex optimization contain useful information both for computational purposes and interpretation of the problem under
consideration. In principle, dual variables can be interpreted as multipliers and provide the marginal improvement of the objective function given a relaxation of the associated primal constraint. In particular, dual variables can be interpreted as prices supporting a Walrasian equilibrium in market models derived from convex optimization problems.

However, in mathematical programs with binary or discrete constraints, the interpretation as marginal relaxation is not valid any more because of the non-convex and disjoint feasible region. This is related to the difficulty of determining the value function of the original problem (Guzelsoy and Ralphs, 2007).

To overcome this caveat in practical applications and obtain dual variables from binary programs, the following approach is often used (cf. O’Neill et al., 2005). Consider the general constrained problem:

\[
\begin{align*}
\min_{x,y} & \quad f(x,y) \\
\text{s.t.} \quad & g(x,y) \leq 0 \\
& x \in \{0,1\}^n, y \in \mathbb{R}^m
\end{align*}
\]  

(1)

To obtain dual variables to the constraints \(g(x,y)\), such problems are commonly solved in a two-step procedure: first, the original problem (1) is solved using integer programming techniques; then, the binary variables \(x\) are linearized, and constraints are added to fix these variables at the level determined to be optimal, \(x^*\), in the first step:

\[
\begin{align*}
\min_{x,y} & \quad f(x,y) \\
\text{s.t.} \quad & g(x,y) \leq 0 \quad (\lambda) \\
& x = x^* \quad (\mu) \\
& (x, y) \in \mathbb{R}^{n+m}
\end{align*}
\]  

(2)

Now, solving the reformulated Problem (2) allows to interpret the dual variables \((\lambda, \mu)\) in the sense of multipliers or shadow values; offering these prices as contracts to market participants yields a Nash equilibrium. Note that while the dual variables \(\mu\) are not part of the original problem, they are obtained here and can be thought of as the "price […] representing the integral activity for (each) agent" (O’Neill et al., 2005, p. 279).

These duals are also important for binary programs, so that most numerical solvers automatically report these values when solving mixed-integer programs. However, one must be careful when using this approach in practical applications, as the duals from the linearized model cannot be readily interpreted as marginal relaxations of the original binary model – that is, the marginal value \(\lambda\) of the linearized fixed program cannot be interpreted as dual to the original, mixed-integer program (Problem [1]). This is, however, what many power markets are currently doing in practice: they use the dual variable to the energy balance constraint as locational marginal price and clear the market based on these pay-offs. The dual prices of the binary activities \(\mu\) are neglected. Instead, market operators assign compensation payments to make whole individual generators after the fact.

2.3 Uplifts, compensation, and equilibria in power markets

There already exists a substantial breadth of Operations Research literature with regard to electricity markets and pricing in non-convex problems, and binary games are a prevalent concern in this area. The current practice in many power markets is that, first, the welfare-optimal dispatch is computed by the Independent System Operator (ISO) and prices at each node in the network are determined using the two-step approach outlined above. Uplift payments (i.e., compensation to individual
players) are then calculated after market-clearance, ex-post, to ensure that each market player is made whole based on these prices. This is usually called bid cost recovery, though actual implementations and rules differ across markets.

This approach does not actually guarantee that the incentives of all players are aligned in the resulting market outcome, because the nature of the non-cooperative binary game between market participants is side-stepped. Generators that are not dispatched by the ISO may have an incentive to enter the market, if they would earn positive profits given resulting market prices, or to deviate from the announced schedule. Some markets allow self-scheduling, which gives generators the option to determine their dispatch individually rather than surrendering their generation decision to the ISO (cf. Sioshansi et al. 2010).

An alternative to the current approach is the minimum uplift or convex hull pricing method, which relies on a convex approximation of the lower bound of the aggregate cost function to derive prices and the minimal uplifts to support the market outcome (Gribik et al. 2007; Hogan and Ring 2003). This method acknowledges the compensation required to deter generators of following profitable deviations from the dispatch chosen by the ISO. Alas, using the convex hull relaxes the integrality of the underlying problem, and therefore also does not solve for the exact binary equilibrium.

Another important problem of the two-step approach arises from the fact that the budget for necessary compensation payments is not considered when determining the dispatch, but only computed ex-post. This neglects the potential trade-off between efficient market operation and minimizing the budget required for compensation payments, which is usually funded from fees or levies on market participants. These fees may in turn cause distortions in the market. It is easy to conceive of situations where accepting a slight reduction in market efficiency (i.e., reduction in welfare, higher costs for dispatch) allows to significantly reduce the compensation payments required. That is, there may exist opportunities for improvements, which the current two-step market operation fails to identify. The illustrative example in Section 4 shows such a situation.

The method developed in this work tackles these caveats of current approaches and proposes an exact solution method for games in binary variables. Our method offers an important practical advantage: it allows to directly balance efficient market operation based on an exact method for finding solutions to binary equilibrium problems with the amount of compensation payments to make these outcomes stable against deviation incentives by individual players.

2.4 Marginal relaxation vs. the loss from a binary deviation

There is a further caveat of using the duals of Problem (2) for algorithms and (economic) interpretation of results: this approach introduces the dual $\mu$ as the marginal relaxation of the constraint that fixes $x$ at its optimal value. However, it is more appropriate to ask not about a marginal relaxation, but a switch from one possible value of the binary variable to the other. We introduce the “switch value” $\kappa$ as the loss incurred by switching from the optimal value of the binary problem $f(x^*, y^*)$ to the optimal value of the objective function given that the binary variable takes the other value, $x^* = 1 - x^*$, where $y^*$ is chosen so as to minimize $f(x^*, y)$, i.e., $y^* = \arg\min_y f(x^*, y)$. Then, $\kappa$ can be determined by solving

$$f(x^*, y^*) = f(x^*, y^*) + \kappa.$$ 

When $x \in \{0, 1\}^n$ is a binary vector rather than a one-dimensional variable, the switch value can be computed component-wise, i.e., for element $j \in [1, \ldots, n]$ of the switch vector $\kappa \in \mathbb{R}_+^n$:

$$\kappa_j = f(x^*, y^*) - f(x_{1}^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*), y^*) \quad \forall j \in [1, \ldots, n].$$
where \( y^\times \) is computed accordingly for each element of \( \kappa_j \).

In the method proposed below, we use this notion of a switch value \( \kappa \) to choose between equilibria in binary games. This variable also serves as a selection mechanism in such cases where no binary equilibrium exists, because it can be used as a solution strategy to find an appropriate quasi-equilibrium.

We believe that this approach holds substantial promise with regard to algorithmic advances of binary and integer programming, as well as allow a better representation of real-world problems in economics, engineering, and beyond. Given that most large-scale applied integer programs are not solved to optimality, a branch-and-bound algorithm or stopping criterion based on the switch value \( \kappa \) may yield better results than current methods; these are usually based on satisfying a tolerance criterion between the best integer solution found so far and the optimal objective value of the linearized problem.

3 An exact solution for binary equilibrium problems

We now turn to our exact solution method to solve an equilibrium problem with binary variables. In the following section, we assume that each player has exactly one binary decision variable; this simplification is made only in the interest of a concise and simple exposition of our approach. The number of continuous decision variables and constraints is arbitrary. In the electricity market example presented in the following section, we show that our methodology can easily be extended to larger problems with multiple binary decision variables for each player.

The core idea for our approach is intuitive: for each player, we derive the first order optimality conditions with respect to the continuous decision variables for both values that the binary variable can take. In addition, we formulate an explicit incentive-compatibility constraint to ensure that each player chooses the state of the binary variable that is most beneficial to her.

The game is defined by a set of players \( i \in I = [1, \ldots, n] \), where each player seeks to minimize an objective function \( f_i(\cdot) \). In the following formulation, each player only has one binary decision variable \( x_i \in \{0, 1\} \), a (vector of) continuous decision variable(s) \( y_i \in \mathbb{R}^m \), and a set of \( k \) constraints \( g_i : \mathbb{R}^m \times \{0, 1\} \rightarrow \mathbb{R}^k \) with a vector of length \( k \) of associated dual variables \( \lambda_i \). As elaborated above, these dual variables are only meaningful for a fixed \( x_i \). The feasible region of each player is denoted by \( K_i = \{(x_i, y_i) \mid g_i(x_i, y_i) \leq 0\} \).

Each player’s optimization problem reads as follows:

\[
\min_{x_i \in \{0, 1\}, y_i \in \mathbb{R}^m} f_i(x_i, y_i, y_{-i}(x_{-i})) \quad (3a)
\]

\[
s.t. \quad g_i(x_i, y_i) \leq 0 \quad (\lambda_i) \quad (3b)
\]

The vector \( y_{-i} = (y_j)_{j \in I \setminus \{i\}} \) is the collection of all rivals’ decisions in continuous variables, and thus is of dimension \( m \times (n - 1) \). The set of feasible strategies by the rivals is denoted by \( K_{-i} = \Pi_{j \in I \setminus \{i\}} (y_j(x_j)) \). Because the continuous variables of the rivals depend on their binary decisions, \( K_{-i} \) is usually disjoint and not convex.

The formulation of Problem (3) implicitly assumes that each player’s payoff is only affected by the continuous decision variables of its rivals, but not directly affected by their binary variables. This is a simplification only for notational convenience and can easily be relaxed.

A Nash equilibrium to this game is a set of strategies such that no player has an incentive to unilaterally change her decision; there exists no profitable deviation.

Definition 1 (Nash equilibrium in a binary game). We define the binary game as a set of players \( i \in I \), each seeking to solve an optimization problem as given by
Problem (3). A Nash equilibrium to this game is a vector \((x_i^*, y_i^*) \in K_i \forall i \in I\) such that \(y_i^*\) is the optimal decision (i.e., best response) by player \(i\) given \(x_i^*\) and \(y_{-i}^*(x_{-i}^*)\),

\[
f_i(x_i^*, y_i^*, y_{-i}^*(x_{-i}^*)) \leq f_i((x_i^*, y_i, y_{-i}^*(x_{-i}^*))) \quad \forall y_i \in \{y_i \mid g_i(x_i^*, y_i) \leq 0\} \quad \forall i \in I,
\]

and such that there is no profitable deviation with regard to the binary variable,

\[
f_i(x_i^*, y_i^*, y_{-i}^*(x_{-i}^*)) \leq f_i((x_i^*, y_i, y_{-i}^*(x_{-i}^*))) \quad \forall i \in I,
\]

where \(x_i^*\) is the alternative value of \(x_i\), i.e., \(x_i^* = 1 - x_i^*\), and \(y_i^*\) is a best response of player \(i\) under the assumption that \(x_i = x_i^*\), i.e.,

\[
f_i((x_i^*, y_i^*, y_{-i}^*(x_{-i}^*))) \leq f_i((x_i^*, y_i, y_{-i}^*(x_{-i}^*))) \quad \forall y_i \in \{y_i \mid g_i(x_i^*, y_i) \leq 0\} \quad \forall i \in I.
\]

Because existence or uniqueness of equilibria cannot be guaranteed in binary games, we need to devise a method to select among several possible outcomes, or to arrive at a desired point which is “almost” an equilibrium (or a quasi-equilibrium). For this purpose, we introduce another player, which we call market operator, as a coordination agent and equilibrium selection mechanism. This entity is modeled as the upper-level player within a hierarchical, two-stage setup, where the lower-level constraints represent the binary equilibrium problem. The market operator guides the players towards a binary (quasi-) equilibrium and assigns compensation payments if necessary.

This upper-level player ensures that the optimality conditions and incentive compatibility (i.e., no profitable unilateral deviation from the market outcome) for each player are satisfied. If there exists more than one equilibrium, the market operator can choose which solution will materialize according to an adequate objective function.

The market operator can also assign compensation payments to individual players if no equilibrium is feasible without it, and objective function can include the trade-off between efficient operation and the required level of compensation disbursements.

We now introduce the term quasi-equilibrium for solutions to the binary game that are not Nash equilibria according to Definition [1] but which can be made incentive-compatible with appropriate compensation payments by the market operator.

**Definition 2** (Quasi-equilibrium in a binary game with compensation).

We define the binary game with compensation as a set of players \(i \in I\), each seeking to solve an optimization problem as given by Problem (3). A binary quasi-equilibrium to this game is a vector \((x_i^*, y_i^*) \in K_i \forall i \in I\) and a compensation vector \((\zeta_i \in \mathbb{R}_+) \forall i \in I\) such that for each player:

1. \(y_i^*\) is the optimal feasible decision (i.e., best response) by player \(i\) given \(x_i^*\) and \(y_{-i}^*(x_{-i}^*)\),

\[
f_i(x_i^*, y_i^*, y_{-i}^*(x_{-i}^*)) \leq f_i((x_i^*, y_i, y_{-i}^*(x_{-i}^*))) \quad \forall y_i \in \{y_i \mid g_i(x_i^*, y_i) \leq 0\} \quad \forall i \in I,
\]

2. no player can improve her own payoff by deviating from \(x_i^*\) by more than the compensation payment \(\zeta_i\); i.e., the compensation is at least as great as the benefit from deviation with regard to the binary variable. Hence, there is no profitable deviation with regard to the binary variable given the compensation payment,

\[
f_i(x_i^*, y_i^*, y_{-i}^*(x_{-i}^*)) - \zeta_i \leq f_i((x_i^*, y_i, y_{-i}^*(x_{-i}^*))) \quad \forall i \in I,
\]
where $x_i^x$ is the alternative value of $x_i$, i.e., $x_i^x = 1 - x_i^r$, and $y_i^x$ is a best response of player $i$ under the assumption that $x_i = x_i^x$, i.e.,

$$f_i((x_i^r, y_i^x, y_{-i}(x_i^r))) \leq f_i((x_i^r, y_i^r, y_{-i}(x_i^r)))$$

$$\forall y_i \in \{y_i | g_i(x_i^r, y_i) \leq 0\} \quad \forall i \in I. \tag{9}$$

3. the compensation payments are minimal, i.e., if a compensation payment is required for a player, then the incentive-compatibility condition holds with equality. That is,

$$\zeta_i = \min_{\zeta_i \in \mathbb{R}_+} \zeta_i \quad \text{s.t.} \quad f_i(x_i^r, y_i^x, y_{-i}(x_i^r)) - \zeta_i \leq f_i((x_i^r, y_i^r, y_{-i}(x_i^r))) \quad \forall i \in I. \tag{10}$$

Note that when $\sum_{i \in I} \zeta_i = 0$, the binary quasi-equilibrium is also a Nash equilibrium. In the definition of the quasi-equilibrium, we directly incorporate the notion that the compensation payments should be minimal. This is helpful because it eliminates those incentive-compatible solutions where the market operator “over-compensates” some players, and it allows to focus on a smaller set of candidate solutions to the binary game.

3.1 Determining each player’s best response

In the definitions above, we have simply stated that the continuous decision variables $y_i^r$ are the optimal decision of player $i$ given the binary variable and the rivals’ decisions. In order to efficiently compute the best response of each player given the decisions of the rivals, we use first order optimality conditions with regard to the continuous decision variables. Hence, we need to make sure these conditions are necessary and sufficient so that we can capture the entire equilibrium set. An assumption on compactness is also needed for the selection of certain parameters of our method.

A1 We assume that for each player $i \in I$, Problem 3 is such that the first order optimality (KKT) conditions are necessary and sufficient with respect to the variables $y_i$, and the feasible region defined by the constraints $g_i(x_i, y_i)$ is compact and non-empty, for any fixed realization of $x_i$ and for any fixed feasible strategy by the rivals $y_{-i} \in K_{-i}$.

As an example, the KKT conditions are necessary and sufficient for Problem 3 if $f_i(x_i, y_i)$ are convex and $g_i(x_i, y_i)$ affine for any fixed value $x_i \in \{0, 1\}$ and any fixed vector $y_{-i} \in K_{-i}$. Let the vector $(x_i^r, y_i^r)$ denote the best response for each player within the overall problem, given the decision vector $(y_{-i}(x_i^r))_{i \in I}$ by all rivals, and let $\tilde{y}_i(x_i)$ denote the best response of player $i$ for a fixed $x_i = x_i \in \{0, 1\}$. Then, the objective value $f_i(x_i, \tilde{y}_i(x_i), y_{-i}(x_i))$ is the best pay-off that a player can do given $x_i$ and the rivals’ strategies.

Under Assumption A1, if the value of $x_i$ is fixed at $x_i$, the best response $\tilde{y}_i(x_i)$ can be found by solving the respective first order optimality conditions:

$$0 = \nabla y_i f_i \left( x_i, \tilde{y}_i(x_i), y_{-i}(x_i) \right) + \lambda_i(x_i) \nabla g_i \left( x_i, \tilde{y}_i(x_i) \right), \quad \tilde{y}_i(x_i) \text{ (free)} \tag{11a}$$

$$0 \geq g_i \left( x_i, \tilde{y}_i(x_i) \right), \quad \lambda_i(x_i) \geq 0 \tag{11b}$$

Player $i$ will choose the binary variable $x_i$ such that its objective value is minimal given the decisions of the rivals $y_{-i}(x_i)$. Mathematically, the best response of player $i$
regarding her binary variable $x_i$ can be written as follows:

$$f_i(1, \tilde{y}_i^{(1)}, y_{-i}(x_{-i})) < f_i(0, \tilde{y}_i^{(0)}, y_{-i}(x_{-i})) \quad \Rightarrow \quad x_i^* = 1 \quad (12a)$$

$$f_i(1, \tilde{y}_i^{(1)}, y_{-i}(x_{-i})) > f_i(0, \tilde{y}_i^{(0)}, y_{-i}(x_{-i})) \quad \Rightarrow \quad x_i^* = 0 \quad (12b)$$

$$f_i(1, \tilde{y}_i^{(1)}, y_{-i}(x_{-i})) = f_i(0, \tilde{y}_i^{(0)}, y_{-i}(x_{-i})) \quad \Rightarrow \quad x_i^* = \{0, 1\} \quad (12c)$$

The logic of Conditions 12 is similar to the notion of incentive compatibility in game theory, i.e., there exists no profitable deviation given the decisions of all rivals. Hence, a vector $(x_i^*, y_i^*(x_i^*))$ (or $\tilde{x}_i, \tilde{y}_i$) that satisfies the incentive-compatibility constraints in Definition 1 for each player constitutes a Nash equilibrium. If the incentive-compatibility condition is not satisfied for any feasible strategy, it may be necessary to financially compensate a player to ensure that she doesn’t deviate, as stated in Definition 2.

### 3.2 An efficient formulation for incentive compatibility

Now, we propose a mathematically equivalent formulation to represent the incentive-compatibility logic by introducing four non-negative variables $(\kappa_i^{(1)}, \kappa_i^{(0)}, \zeta_i^{(1)}, \zeta_i^{(0)})$ for each player, and a sufficiently large scalar (or vector of scalars), $\tilde{K}$. In applied work, one can of course make the parameter $\tilde{K}$ specific to each player and constraint to improve computation efficiency. We omit this specification for notational convenience. The vector $\kappa_i^{(\tilde{K})}$ is the switch value introduced in Section 2.4; it can be interpreted as the loss the player would incur by switching from its optimal decision to the alternative with regard to the binary variables.

The vector $\zeta_i^{(\tilde{K})}$ denotes compensation payments to guarantee incentive compatibility; that is, in cases where the market operator requires a player to act against her own objective with regard to the binary variable so that a binary quasi-equilibrium can be obtained, a remuneration must be paid as compensation and to ensure that the player does not have a profitable deviation.

The vectors $\kappa_i^{(1)}$ and $\zeta_i^{(\tilde{K})}$ are not dual variables in the original sense, but they do contain similar information regarding the solution. Hence, they are analogous in interpretation to a dual – but in terms of a binary deviation, not in the sense of a marginal relaxation. Alas, the term “shadow price” often used in economics as synonymous for dual variables could also be applied here.

We can now replace the incentive compatibility conditions (Equations 12) by a more efficient formulation:

$$f_i(1, \tilde{y}_i^{(1)}, y_{-i}) + \kappa_i^{(1)} - \zeta_i^{(1)} - \kappa_i^{(0)} + \zeta_i^{(0)} = f_i(0, \tilde{y}_i^{(0)}, y_{-i}) \quad (13a)$$

$$\kappa_i^{(1)} + \zeta_i^{(1)} \leq x_i \tilde{K} \quad (13b)$$

$$\kappa_i^{(0)} + \zeta_i^{(0)} \leq (1 - x_i) \tilde{K} \quad (13c)$$

The market operator selects a solution such that the first order conditions and the incentive-compatibility constraints are satisfied for all players. In line with the definition of the quasi-equilibrium as the minimal compensation payment for each player $i$, the variables $\kappa_i^{(\tilde{K})}$ and $\zeta_i^{(\tilde{K})}$ cannot both be strictly greater than zero at a solution; this will be formally discussed when we introduce the problem of the market operator.

First, let us illustrate and discuss the interpretation of the variables $\kappa_i^{(\tilde{K})}$ and $\zeta_i^{(\tilde{K})}$ in more detail. The question is whether the solution for the overall equilibrium problem chosen by the market operator is aligned with the best response of each player. By this, we mean whether a player’s individually optimal decision coincides with the (quasi-)equilibrium chosen by the market operator.
Table 1: Incentive alignment between a player’s individually optimal decision (her best response) and the (quasi-) equilibrium chosen by the market operator

There are five possible outcomes regarding the incentive alignment of an individual player and the market operator; the cases are illustrated in Table 1. In cases I and II, the incentives are aligned, as the player would incur losses (a strictly worse payoff) by deviating from the outcome decided by the market operator. The respective switch value $\kappa_i(x_i)$ is strictly positive. There may be situations where the player has a strict preference regarding her binary decision, but the market operator is actually indifferent. In this case, we assume that the market operator will choose the solution so that it coincides with the player’s preference; otherwise, the market operator would have to disburse compensation payments.

In cases III and IV, the solution chosen by the market operator is not in line with the player’s individual best response; only by disbursing compensation payments can the market operator convince the player not to deviate to the individually optimal decision. As a consequence, the respective compensation payment $\zeta_i(x_i)$ is strictly positive, and a quasi-equilibrium is realized.

In the last case (no. V), the player is indifferent between her options, so the market operator is not restricted in selecting either outcome. The player does not have a positive switch value in either direction, and no compensation is required.

### 3.3 Translating each player’s best response into the overall game

From Equations (11a), we have obtained two optimal decision vectors, $y^i(x_i)$, for each player for both values that the variable $x_i$ can take. We now need to translate which of these two decision variables is “seen” by the rivals in their own optimization problem, contingent on the equilibrium decision $x_i$. We do this by adding the following constraints:

\[
\begin{align*}
\tilde{y}_i^{(0)} - x_i \bar{K} &\leq y_i \leq \tilde{y}_i^{(0)} + x_i \bar{K} \\
\tilde{y}_i^{(1)} - (1 - x_i) \bar{K} &\leq y_i \leq \tilde{y}_i^{(1)} + (1 - x_i) \bar{K}
\end{align*}
\]  

(14a)  

(14b)

The logic of Constraints (14) is straightforward: the decision vector $y_i$, as it is considered by the rivals and the market operator in their optimization problems, must be equal to the optimal decision $\tilde{y}_i^{(x_i)}$ for whichever value of $x_i$ is the solution in the quasi-equilibrium, as stated by equations (12) and reformulated using equations (13), i.e., $x_i = 0 \Rightarrow y_i = \tilde{y}_i^{0}$ and $x_i = 1 \Rightarrow y_i = \tilde{y}_i^{1}$. The parameter $\bar{K}$ must be chosen suitably large so as not to constrain the continuous decision variable(s). Since this is a primal variable, it should be straightforward to find a suitable bound in any practical application.

We can now combine the incentive-compatibility constraints (13) with the equilibrium conditions (11a) for the continuous decision variables into one large set of constraints with binary variables. The non-linearity of the complementarity condi-
3.4 A multi-objective program subject to binary (quasi-) equilibria

So far, we have only replaced a large number of equilibrium problems (for each possible combination of binary variables) by a large set of integer constraints that exactly represent all binary (quasi-) equilibria. Next, we can apply multi-objective programming to direct the game towards desired solutions. To this end, we introduce the market operator, and we assume that she seeks to minimize an objective function consisting of two terms: a function $F(\cdot)$, which only depends on the actual market outcome (efficiency of the solution); cost-minimization or welfare-maximization may be a natural choice for this term; and a function $G(\cdot)$, which serves as a regularizer; in economic applications, it can be interpreted as a penalty term that seeks to minimize the compensation payments required to ensure incentive compatibility of the market solution.

As is common in multi-objective programming, one can easily explore the range of equilibria by solving the reformulated problem using different integer forms. Furthermore, one can sort binary (quasi-) equilibria from a "best" to a "worst" outcome by solving the model consecutively and in each step excluding the previous solutions; here, "best" is to be understood as given by the objective function $F(\cdot) + G(\cdot)$.

The overall problem then reads as follows, where $x_i \in \{0,1\}$; all constraints and variables contingent on $x_i$ have to be included twice in the mathematical program, once for each value that $x_i$ can take.

\[
\begin{align*}
\min_{x_i, y_i, \gamma_i, \lambda_i, \kappa_i, \zeta_i} & \quad F \left( (x_i, y_i) \right) + G \left( (\gamma_i) \right) \\
\text{s.t.} & \quad \nabla y_i f_i \left( x_i, \gamma_i \right) + \left( \lambda_i \right)^T \nabla y_i g_i \left( x_i, \gamma_i \right) = 0 \\
& \quad 0 \leq -g_i \left( x_i, \gamma_i \right) \perp \lambda_i \geq 0 \\
& \quad x_i \left( 1, y_i \right) + \kappa_i \left( 1 \right) - \zeta_i \left( 0 \right) = f_i \left( 0, y_i \right) \\
& \quad \kappa_i \left( 1 \right) + \zeta_i \left( 1 \right) \leq x_i K \\
& \quad \kappa_i \left( 0 \right) + \zeta_i \left( 0 \right) \leq \left( 1 - x_i \right) K \\
& \quad y_i \left( 0 \right) - x_i K \leq y_i \left( 0 \right) + x_i K \\
& \quad \zeta_i \left( 1 \right) - \left( 1 - x_i \right) K \leq y_i \left( 1 \right) + \left( 1 - x_i \right) K \\
& \quad x_i \in \{0,1\}, \ (y_i, \gamma_i) \in \mathbb{R}^{m_1}, (\lambda_i, \kappa_i, \zeta_i) \in \mathbb{R}^{2k+4}.
\end{align*}
\]

It is important to note that the binary variable has an additional role in this formulation; it also controls which of the two potential states with regard to the continuous variables are active and “visible” to the other players (Constraints $15g$ and $15d$). Furthermore, it ensures that the correct switch values and compensation payments are active (Constraints $15a$ and $15b$), in line with Table 1.

**Theorem 1** (Exact solutions of the binary Nash game). Under Assumption $A1$, any vector $(x_i, y_i) \in I$ is a solution to the binary game as stated in Definition 2 if and only if there exists a vector $(\gamma_i, \lambda_i, \kappa_i, \zeta_i) \in I$, with $\zeta_i = 0 \forall i \in I$, such that $(x_i, y_i, \gamma_i, \lambda_i, \kappa_i, \zeta_i) \in I$ is a feasible point to Problem $15$.\[15a\]

**Proof.** First, assume $(x_i, y_i, \gamma_i, \lambda_i, \kappa_i, \zeta_i) \in I$, with $\zeta_i = 0 \forall i \in I$ is a feasible point to Problem $15$. Then, by Assumption $A1$, we know that second order sufficiency conditions hold for each player’s optimization problem. Hence, the point $(x_i, y_i) \in I$ is an optimal
solution for each player given fixed values of $x_i$ and $y_{-i}$ $\forall \ i \in I$. This satisfies the first part of Definition 1. Furthermore, we know that $\zeta_i(x_i) = 0 \ \forall \ i \in I$, and $\{\kappa^*(x_i)\}_{i \in I}$ will be selected according to the constraints of Problem 15. By these constraints, we know that $f_i(x_i, y_{-i}(x_{-i})) \leq f_i(x_i^*, y_{-i}(x_{-i})) \ \forall \ i \in I$, where $x_i^*$ is the alternative value of $x_i$ (i.e., $x_i^* = 1 - x_i$) and $y_i^*$ is a best response of player $i$, i.e.,

$$f_i(x_i^*, y_{-i}(x_{-i})) \leq f_i((x_i^*, y_{-i}(x_{-i})) \ \forall \ y_i \in \{y_i \mid g_i(x_i^*, y_i) \leq 0\} \ \forall \ i \in I.$$  

This satisfies the second part of Definition 1 and thus we have shown that $(x_i, y_i)_{i \in I}$ is a solution to the binary game defined by Definition 1.

Now, we assume that $(x_i, y_i)_{i \in I}$ is a solution to the binary game defined by Definition 1. Choose $\bar{K}$ large enough so that it is greater than the difference between the minimum and maximum value of $f_i(\cdot) \ \forall \ i \in I$. Such a value exists since by Assumption A1 the feasible set is compact and $f_i(\cdot)$ is continuous, so the maximum and minimum must exist. Then, by Definition 1, for any fixed value of $x_i$ and $y_{-i}$, $y_i$ is an optimal solution to the individual player’s optimization problem. Thus, you can find $(\tilde{y}_i^{(x_i)}, \tilde{x}_i^{(x_i)}, \tilde{\lambda}(x_i), \tilde{\kappa}(x_i), \tilde{\zeta}(x_i))_{i \in I}$ that satisfies the Constraints 15b and 15c. Take $\zeta(x_i) = 0 \ \forall \ i \in I$, and choose $\kappa(x_i)$ according to Constraint 15a. Thus, for any solution to the binary game in Definition 1, given by $(x_i, y_i)_{i \in I}$, we have shown that there exists a vector $(\tilde{y}_i^{(x_i)}, \tilde{x}_i^{(x_i)}, \tilde{\lambda}(x_i), \tilde{\kappa}(x_i), \tilde{\zeta}(x_i))_{i \in I}$ with $\zeta(x_i) = 0 \ \forall \ i \in I$, such that $(x_i, y_i, \tilde{y}_i^{(x_i)}, \tilde{x}_i^{(x_i)}, \tilde{\lambda}(x_i), \tilde{\kappa}(x_i), \tilde{\zeta}(x_i))_{i \in I}$ is a feasible point to Problem 15.

Note that if any vector is a global minimum to the objective function 15a, it is the equilibrium with the lowest objective function value $F(\cdot)$, while $G(\cdot)$ is constant because $\zeta(x_i) = 0 \ \forall \ i \in I$. An efficient technique to choose $\bar{K}$ is to linearize the binary variables in the individual optimization problems and minimize and maximize over $f_i(\cdot)$ to find the largest difference possible. The next theorem shows that the method can also be applied to obtain a quasi-equilibrium. Note that we need an assumption on the objective function before we can prove that our method can obtain a quasi-equilibrium.

**A2** Assume that $F(\cdot)$ and $G(\cdot)$ are convex quadratic or linear functions for every fixed binary variable $x_i$ and that $\partial G(\cdot)/\partial \zeta_i > 0 \ \forall \ i \in N$.

**Theorem 2** (Exact solutions of the binary Nash game with compensation). Under Assumptions A1 and A2, if there is a vector $(x_i, y_i, \tilde{y}_i^{(x_i)}, \tilde{x}_i^{(x_i)}, \tilde{\lambda}(x_i), \tilde{\kappa}(x_i), \tilde{\zeta}(x_i))_{i \in I}$ that is an optimal solution to Problem 15, then the vector $(x_i, y_i)$ with compensation $(\zeta_i)_{i \in I}$ is a solution to the binary game as stated in Definition 3. Following the term introduced in Definition 3, we refer to this as a binary quasi-equilibrium.

Furthermore, under Assumptions A1 and A2, if $(x_i, y_i)_{i \in I}$ is a solution to the binary game with compensation $(\zeta_i)_{i \in I}$ as stated in Definition 3, then there exists a vector $(\tilde{y}_i^{(x_i)}, \tilde{x}_i^{(x_i)}, \tilde{\lambda}(x_i), \tilde{\kappa}(x_i), \tilde{\zeta}(x_i))_{i \in I}$, such that $(x_i, y_i, \tilde{y}_i^{(x_i)}, \tilde{x}_i^{(x_i)}, \tilde{\lambda}(x_i), \tilde{\kappa}(x_i), \tilde{\zeta}(x_i))_{i \in I}$ is a feasible point to Problem 15.

**Proof.** First, assume $(x_i, y_i, \tilde{y}_i^{(x_i)}, \tilde{x}_i^{(x_i)}, \tilde{\lambda}(x_i), \tilde{\kappa}(x_i), \tilde{\zeta}(x_i))_{i \in I}$ is an optimal solution to Problem 15. Then, by A1, we know that second order sufficiency conditions hold for each players optimization problem. Hence, the point $(x_i, y_i)_{i \in I}$ is an optimal solution for each player given fixed values of $x_i$ and $y_{-i}$. This satisfies the first part of Definition 2. Furthermore, we know that $\kappa_i(x_i) \ \forall \ i \in I$ is selected according to the constraints of 15. By these constraints, we know that $f_i(x_i, y_{-i}(x_{-i})) - \zeta_i \leq f_i((x_i^*, y_{-i}(x_{-i})) \ \forall \ i \in I$, where $x_i^*$ is the alternative value of $x_i$ (i.e., $x_i^* = 1 - x_i$) and $y_i$ is a best response of player $i$ with fixed $x_i^*$, i.e.,

$$f_i((x_i^*, y_{-i}(x_{-i})) \leq f_i((x_i^*, y_{-i}(x_{-i})) \ \forall \ y_i \in \{y_i \mid g_i(x_i^*, y_i) \leq 0\} \ \forall \ i \in I.$$
This satisfies the second part of Definition 2.

By Assumption $\textbf{A2}$ we know that $\partial G(\cdot)/\partial \zeta_i > 0 \quad \forall \ i \in I$ and, hence, for any optimal solution, for each player, $(\zeta_i)_{i \in I}$ is minimal. This satisfies the third part of Definition 2 and hence we have shown that if $(x_i, y_i, \overline{y}_i, \overline{x}_i, \kappa_i, \zeta_i)_{i \in I}$ is an optimal solution to Problem (15), then $(x_i, y_i)_{i \in I}$ with compensation payments $(\zeta_i)_{i \in I}$ is a solution to the binary game defined by Definition 2.

Now, we assume that $(x_i, y_i)_{i \in I}$ is an optimal solution to the binary game with compensation defined by Definition 2. Choose $\kappa$ large enough so that it is greater than the difference between the maximum and minimum value of $f_i \forall i \in I$. Such a value exists since by Assumption $\textbf{A1}$ the feasible set is compact and $f_i$ is continuous, so the maximum and minimum must exist. Then, by Definition 2 for any fixed value of $x_i$ and $y_{-i}$, $y_i$ is an optimal solution to the individual player’s optimization problem. Thus, you can find $(\overline{y}_i, \overline{x}_i)_{i \in I} \quad \forall \ i \in I$ such that $(x_i, y_i, \overline{y}_i, \overline{x}_i)_{i \in I}$ satisfies the first two constraints in Problem (15). Calculate $\zeta_i \forall i \in I$ and choose $\kappa_i$ according to the third constraint in Problem (15). Thus, for any solution to the binary game in Definition 2 given by $(x_i, y_i)_{i \in I}$ and compensation payment $(\zeta_i)_{i \in I}$, we have shown that there exists a vector $(\overline{y}_i, \overline{x}_i, \kappa_i, \zeta_i)_{i \in I}$ such that $(x_i, y_i, \overline{y}_i, \overline{x}_i, \kappa_i, \zeta_i)_{i \in I}$ is a feasible solution to Problem (15).

**Corollary 1.** Under Assumptions $\textbf{A1}$ and $\textbf{A2}$, any vector $(x_i, y_i)_{i \in I}$ is a binary quasi-equilibrium for the Nash game in binary variables with compensation $(\zeta_i)_{i \in I}$ as defined in Definition 2 if there exists a vector $(\overline{y}_i, \overline{x}_i, \kappa_i, \zeta_i)_{i \in I}$, such that $(x_i, y_i, \overline{y}_i, \overline{x}_i, \kappa_i, \zeta_i)_{i \in I}$ is a feasible solution to Problem (15) and $\zeta_i$ is minimal as defined in Definition 2.

**Proof.** By the arguments in Theorem 2 for any point $(x_i, y_i)_{i \in I}$ that is feasible to Problem (15), the vectors $(x_i, y_i)_{i \in I}$ and $(\zeta_i)_{i \in I}$ satisfy the first two conditions of Definition 2. If, in addition, Condition 3 of Definition 2 is satisfied, i.e., $\zeta_i$ is minimal for each player $i \in I$, then $(x_i, y_i)_{i \in I}$ is a binary quasi-equilibrium with compensation $(\zeta_i)_{i \in I}$.

If a vector is a global minimum to the objective function (15), this is the binary quasi-equilibrium with the lowest objective function value $F(\cdot) + G(\cdot)$. The following lemma and theorem provide conditions for the existence of a binary quasi-equilibrium that can be supported by appropriate compensation payments.

**Lemma 1** (Existence of a Nash equilibrium for a game with fixed binary variables). If for a fixed vector $(\pi_i)_{i \in I}$, the objective function $f_i(\pi_i, y_{-i}(x_{-i}))$ of every player $i \in I$ is continuous with regard to $y_i$ and $y_{-i}(x_{-i})$, and quasi-convex in $y_i$, and the feasible region defined by the constraints $g_i(\pi_i, y_i)$ is compact, convex and non-empty, then the resulting continuous game has a solution.

**Proof.** The existence follows readily from Kakutani’s fixed-point theorem (Glicksberg 1952).

Relaxations of these conditions for the existence of a Nash equilibrium in continuous decision variables are also discussed in the literature (Facchinei and Pang 2003; Nishimura and Friedman 1981). The existence result for Nash equilibria in continuous games given fixed binary variables in Lemma 1 can be combined with Theorem 3 to provide reasonable conditions for the existence of binary quasi-equilibria.
Theorem 3 (Existence of a binary quasi-equilibrium). Under Assumptions $A1$ and $A2$ if for any fixed vector $(\tau_i)_{i \in I}$, the resulting continuous game has a solution, then a corresponding binary quasi-equilibrium exists for the Nash game in binary variables.

Proof. For any $(y_i)_{i \in I}$ that is a Nash equilibrium given the fixed vector $(\tau_i)_{i \in I}$, we can find vectors $(\tilde{y}_i^{(\tau_i)}, \tilde{\lambda}_i^{(\tau_i)})_{i \in I}$ such that $(\tau_i, y_i, \tilde{y}_i^{(\tau_i)}, \tilde{\lambda}_i^{(\tau_i)})_{i \in I}$ is a feasible point to Constraints (15b) and (15c). Recall that our notation implies $\tilde{\tau}_i = (\tau_i, 1 - \tau_i)_{i \in I}$. Choose $\tilde{K}$ as in the proof for Theorem 2.

We can then find values for $(\kappa_i^{(\tau_i)}, \zeta_i^{(\tau_i)})_{i \in I}$ such that $(\tilde{y}_i^{(\tau_i)}, \tilde{\lambda}_i^{(\tau_i)}, \kappa_i^{(\tau_i)}, \zeta_i^{(\tau_i)})_{i \in I}$ satisfy Constraints (15d), and where either $\kappa_i^{(\tau_i)} = 0$ or $\zeta_i^{(\tau_i)} = 0$ for every player $i \in I$. By Equation (15d), this implies that $\zeta_i^{(\tau_i)}$ is minimal. By Corollary 1, $(\tau_i, y_i)_{i \in I}$ with compensation payments $(\zeta_i)_{i \in I}$ is a binary quasi-equilibrium.

Theorem 3 implies that, if a solution to the continuous game exists for every fixed realization of the binary variables, then this solution can be supported as a quasi-equilibrium with appropriate compensation payments.

The reformulated multi-objective program subject to a binary quasi-equilibrium (Problem 15) is a mixed-integer problem. However, the incentive-compatibility constraint (Condition 15d) can still cause numerical difficulties, because the players’ objective function are often not linear and not even convex in terms of all variables, even when they are linear from the point of view of the player itself.

We will now introduce a special case, which allows to reduce Problem 15 to a linear or quadratic convex mixed-integer program with linear constraints.

A3 Assume that the each lower-level player’s objective function $f_i(x_i, y_i, y_{-i}(x_{-i}))$ can be separated into two functions, where the first part is linear with respect to $x_i$ and $y_{-i}$, and the second part is linear only with respect to $y_i$,

$$f_i(x_i, y_i, y_{-i}(x_{-i})) = f_i^{(x)}(x_i, y_{-i}(x_{-i})) + f_i^{(y)}(y_i | y_{-i}(x_{-i})),$$

and the partial derivative of the objective function with regard to the continuous variable $y_i$, $\nabla_{y_i} f_i^{(y)}(y_i^{(\tau_i)} | y_{-i}(x_{-i}))$, is linear in all variables.

Furthermore, assume that all constraints $g_i(x_i, y_i)$ are affine and can therefore be written as:

$$g_i(x_i, y_i) \leq 0 \Rightarrow a_i x_i + A_i y_i \leq b_i$$

where $a_i$, $b_i$ are vectors and $A_i$ a matrix of suitable dimensions.

Note that the function $f_i^{(y)}(y_i | y_{-i}(x_{-i}))$ need not be linear with respect to all variables, only with regard to the player’s own continuous decision variable $y_i$ given a fixed value of $y_{-i}$.

Theorem 4 (Exact reformulation as a mixed-integer linear/quadratic program with linear constraints). Under Assumptions $A1$, $A2$ and $A3$, the multi-objective program subject to a binary quasi-equilibrium (Problem 15) can be reformulated as a quadratic integer program with linear constraints. Theorems 1 and 2 remain valid.

Proof. By Assumptions $A1$, $A2$ and $A3$, the objective function is linear or convex quadratic, and the first order conditions and the players’ constraints are linear. The complementarity conditions (15e) can be reformulated using disjunctive constraints [Fortuny-Amat and McCarl 1981].

The incentive-compatibility constraint (15d) can be reformulated as follows: since the function $f_i^{(y)}(y_i^{(\tau_i)} | y_{-i}(x_{-i}))$ is linear with respect to $y_i^{(\tau_i)}$, we know that

$$f_i^{(y)}(y_i^{(\tau_i)} | y_{-i}(x_{-i})) = (\nabla_{y_i} f_i^{(y)}(y_i^{(\tau_i)} | y_{-i}(x_{-i}))^T y_i^{(\tau_i)}).$$

15
By first order optimality, \( \left( \nabla_{y_i} f_i^{(y)} (y_i^{(x)}) | y_i - (x_i) \right) = - \left( (\lambda_i^{(x_i)})^T \nabla_{y_i} g_i (x_i, y_i^{(x)}) \right) \).

Then, applying the definition of \( g_i(\cdot) \) in Assumption \( A.3 \), \( \nabla_{y_i} g_i (x_i, y_i^{(x_i)}) = A_i \), and using the complementarity condition of constraint (15), \( (a_i x_i + A_i y_i - b_i)^T \lambda_i = 0 \), the reformulation proceeds as follows:

\[
\begin{align*}
  f_i(x_i, y_i, y_{-i}(x_{-i})) &= f_i^{(x)}(x_i, y_{-i}(x_{-i})) + f_i^{(y)}(y_i^{(x)}) | y_{-i}(x_{-i})) \\
  &= f_i^{(x)}(x_i, y_{-i}(x_{-i})) + \left( \nabla_{y_i} f_i^{(y)} (y_i^{(x)}) | y_{-i}(x_{-i})) \right)^T y_i^{(x)} \\
  &= f_i^{(x)}(x_i, y_{-i}(x_{-i})) - \left( (\lambda_i^{(x_i)})^T \nabla_{y_i} g_i (x_i, y_i^{(x_i)}) \right)^T y_i^{(x)} \\
  &= f_i^{(x)}(x_i, y_{-i}(x_{-i})) - (\lambda_i^{(x_i)})^T (b_i - a_i) \\
\end{align*}
\]

Therefore, Problem (15) can be reformulated as a linear/quadratic convex integer program with linear constraints.

With this theorem, we show that our approach can be applied to a large number of problem classes and still be solved as a mixed-integer linear program. These include operational constraints such as capacity constraints or minimum generation constraints, and market forms including linear inverse demand functions. We discuss some impacts on the computational aspects of the problem in Section 4.5.

### 3.5 Computational complexity of the binary quasi-equilibrium

For illustration, we now recast, along the notation used in the derivation of binary equilibrium, the current practice in power markets as explained in Section 2.3: first, solve the welfare-optimal dispatch, derive prices from the linearized model [O’Neill et al. 2005], and then compute compensation payments. This method is, in a way, a lexicographic solution approach to the over-all problem; mathematically, it can be seen as a hierarchical min-min problem.

\[
\begin{align*}
  \min_{\zeta_i} G \left( (\zeta_i)_{i \in I} \right) + \left\{ \min_{x_i, y_i} F \left( (x_i, y_i) \right) \right. \\
  &\left. \text{s.t. } g_i(x_i, y_i) \leq 0 \quad \forall i \in I \right\} \\
  \text{s.t. } f_i(x_i, y_i, y_{-i}(x_{-i})) - \zeta_i \leq 0 \\
\end{align*}
\]

We now compare the mathematical complexity of the current approach to our method in a linear problem setting, i.e., Assumption \( A.3 \) holds and the upper-level objective function is linear. Here, we only focus on the number of binary variables in each approach, because the number of continuous variables, linear constraints, and the subsequent linear program in the two-stage approach are significantly less computationally challenging.

The two-stage approach following [O’Neill et al. 2005] requires to solve a linear mixed-binary optimization problem with \( mn \) binary variables, where \( n \) is the number of players and \( m \) is the number of binary decision variables of each player. In contrast, the multi-objective program subject to a binary quasi-equilibrium (Problem 15) requires to solve a mixed-binary optimization problem with \( (2^m + 2k) n \) binary variables, where \( k \) is the number of inequality constraints. This is due to the disjunctive constraints reformulation to replace the associated complementarity conditions. Nevertheless, as will be seen in the numerical application presented in the following section, the number of binary variables can be significantly reduced depending on the actual underlying problem. In the power market uplift problem, the number of binary variables to obtain an exact solution is only \( (m + k) n + l \), where \( l \) is the number of binary...
variables required for the ISO; this is, in principle, not a substantial increase in computational complexity. In short, our approach scales well in the number of players, but not necessarily in the number of binary variables of each player.

The standard approach in power market operation, as illustrated above, neglects the nature of the binary, non-cooperative game between generators; because generators are usually only compensated for actual costs, they may have profitable deviations with regard to the binary decisions. We illustrate such a case in the application in Section 4. Convex-hull pricing, as introduced by Gribik et al. (2007), considers the deviation incentives of individual players, but this comes at the cost of relaxing integrality of the underlying problem. Neither approach directly considers the trade-off between market efficiency and compensation payments.

4 A binary game: The power market uplift problem

In order to illustrate that our approach is advantageous relative to other methods proposed in the literature, we adapt the nodal power market example from Gabriel et al. (2013) and solve it for an exact solution under various market settings. We model a nodal-pricing electricity market, in which an Independent System Operator (ISO) collects bids and offers from all players and seeks to clear the market in a welfare-optimal manner.

4.1 The power market model

In contrast to the notation introduced earlier, the ISO takes two roles in this example: first, taking dispatch as given, she seeks to maximize short-run market efficiency. She assigns locational prices, which the generators consider in their individual optimization problems (cf. Hobbs, 2001). Second, she disburses compensation payments to align the incentives of generators with the overall (societally most beneficial) outcome. In the parlance of sequential games, the first function takes place at the same hierarchical level as the generators’ decisions; the second function is equivalent to the market operator introduced in Section 3.4.

After formulating the model, we will compare numerical results under different sets of market rules regarding whether players are allowed to deviate from the schedule announced by the ISO, which players may receive compensation, and whether stakeholders may incur losses. Table 2 provides a summary of the notation used in the example.

The generators

Each generator $i \in I$ seeks to maximize her profits from generating and selling electricity over the time horizon $t \in T$. For consistency with the previous chapter, we write the generator’s optimization problem in minimization form:

$$\begin{align*}
\min_{x_{ti}, y_{ti}, z_{on}^{i}, z_{off}^{i}} & - p_{tn(i)} y_{ti} + c^{G}_{i} y_{ti} + d_{on}^{i} z_{on}^{i} + d_{off}^{i} z_{off}^{i} \\
\text{s.t.} & x_{ti} g_{min} \leq y_{ti} \leq x_{ti} g_{max} \quad (\alpha_{on}^{i}, \beta_{on}^{i}) \\
& x_{ti} = x_{(t-1)i} + z_{on}^{i} - z_{off}^{i} = 0 \\
& x_{ti} \in \{0, 1\}, \quad y_{ti}, z_{on}^{i}, z_{off}^{i} \in \mathbb{R}^{+}
\end{align*}$$

(17a) (17b) (17c)

where the linear generation costs are given by $c^{G}_{i}$, the (binary) start-up costs are given by $d_{on}^{i}$, the (binary) shut-down costs are given by $d_{off}^{i}$. The operation schedule is denoted by $x_{ti}$, the decision how much electricity to generate and sell to the grid is $y_{ti}$. The ramping decisions in a particular period are denoted by $z_{on}^{i}$ and $z_{off}^{i}$, respectively.
The status of a power plant at the beginning of the model horizon (i.e., $x_{0i}$) is given by the parameter $x_{init}^i$.

The first set of constraints (17b) concerns the maximum generation capacity and the minimum activity level ($g_{min}^i$, $g_{max}^i$), if the power plant is operating ($x_{ti} = 1$). The shadow variables ($\alpha_{on}^i$, $\beta_{on}^i$) are only meaningful given a fixed operation schedule $x_{ti}$, and we only compute them if the power plant is operational. If the power plant is not switched on, generation is equal to zero; however, due to costs incurred by shutting down the plant (assuming it was operational in the previous period or at the beginning of the model horizon), total profits may be negative even when the plant is not generating electricity.

### Sets & Mappings

| $n, m \in N$ | ... nodes |
| $t \in T$ | ... time slices, hours |
| $i \in I$ | ... generators, power plant units |
| $j \in J$ | ... load, demand units |
| $l \in L$ | ... power lines |
| $z \in Z$ | ... set of dispatch options for each generator |
| $i \in I$, $j \in J_n$ | ... generator/load unit mapping to node $n$ |
| $n(i), n(j)$ | ... node mapping to generator $i$/load unit $j$ |
| $t \in T_z$ | ... hours in which a generator is active in dispatch option $z$ |

### Primal variables

- $x_{ti}$ ... on/off decision for generator $i$ in hour $t$
- $z_{on}^{in}, z_{off}^{in}$ ... inter-temporal start-up/shut-down decision
- $y_{on}^{in}$ ... actual generation by generator $i$ in hour $t$
- $y_{on}^{in}$ ... generation if binary variable is fixed at $x_{ti}$
- $d_{ij}$ ... demand by unit $j$ in hour $t$
- $\delta_{tn}$ ... voltage angle

### Dual variables

- $\alpha_{on}^i$, $\beta_{on}^i$ ... dual to minimum activity/maximum generation capacity
- $p_{tn}$ ... locational marginal price
- $\nu_{ij}$ ... dual to maximum load constraint
- $\mu_{ij}^{+}, \mu_{ij}^{-}$ ... dual to voltage angle band constraints
- $\xi_{lm}^{+}, \xi_{lm}^{-}$ ... dual to thermal line capacity constraints
- $\gamma_t$ ... dual to slack bus constraints

### Switch and compensation variables

- $\kappa_{on}^i$, $\kappa_{off}^i$ ... switch value
- $\zeta_i$ ... compensation payment

### Parameters

- $c_i^G$ ... linear generation costs
- $d_{on}^{in}, d_{off}^{in}$ ... start-up/shut-down costs
- $d_{on}^{in}$ ... total ramping costs in dispatch option $z$
- $g_{min}^i$ ... minimum activity level if power plant is online
- $g_{max}^i$ ... maximum generation capacity
- $x_{init}^i$ ... power plant status at start of model horizon ($t = 0$)
- $u_{ij}^+$ ... utility of demand unit $j$ for using electricity
- $d_{max}^j$ ... maximum load of unit $j$
- $f_{l}^{max}$ ... thermal capacity of power line $l$
- $B_{nk}, H_{lk}$ ... line/node impedance/susceptance matrices

Table 2: Notation for the nodal power market problem
The second constraint (17c) concerns the inter-temporal consideration, i.e., the decision in which time periods the power plant is operational: while the start-up and shut-down variables $z_{on}^{ti}$ and $z_{off}^{ti}$ are binary in nature, they can be relaxed to positive real numbers without loss of information. Integrality on these variables is automatically enforced by the on/off variables $x_{ti}$.

The market clearing price $p_{tn}$ is the vector of the locational marginal prices over time, where $n(i)$ denotes the node at which generator $i$ is located; the set $n, m \in N$ denotes the nodes in the network. The price arises out of the ISO’s welfare maximization problem, and each generator takes the price at her node as given.

Assuming that the power plant $i$ is switched on in period $t$, the optimal amount of power generated $y_{on}^{ti}$ and the dual variables associated with the constraints can be determined by solving the generator’s first order optimality (KKT) conditions:

$$0 = c_i^G - p_{tn}(i) + \beta_{on}^{ti} - \alpha_{on}^{ti}, \quad y_{on}^{ti} \text{ (free)} \quad (18a)$$

$$0 \leq -g_i^m + y_{on}^{ti} \quad \alpha_{on}^{ti} \geq 0 \quad (18b)$$

$$0 \leq g_i^m - y_{on}^{ti} \quad \beta_{on}^{ti} \geq 0 \quad (18c)$$

Otherwise, the amount generated is zero, and we do not require the dual variables in that case. Hence, in contrast to the general formulation in Section 3, we can omit the KKT conditions in this example for the case that the power plant is not operating in period $t$.

The Independent System Operator

The spot market is cleared by an ISO; this player assigns nodal prices $p_{tn}$, which the generators consider in their optimization problems. The ISO seeks to maximize welfare of demand and the transmission system; she assigns locational prices and load such that resulting flows on the network are feasible and the market clears. At this stage, she takes the unit commitment at time $t$ and the dispatch of each power plant as given, written here as $y_{ti}$.

There are a set of units that demand electricity $j \in J$ (load), each located at a specific node $n(j)$. The sets $I_n$ and $J_n$ are the generators and load units located at node $n$, respectively. There are a set of power lines $l \in L$ connecting the nodes; the direct-current load flow (DCLF) characteristics are captured using the susceptance matrices $B_{nm}$ (node-to-node) and $H_{nl}$ (node-to-line mapping, cf. Leuthold et al., 2012). This formulation is equivalent to using a power transfer distribution factor (PTDF) matrix. In line with the previous notation, we write the objective function as a minimization problem:

$$\min \sum_{d_{ij} \in I_n} -d_{ij} \sum_{j \in J_n} y_{ti} + \sum_{m \in N} B_{nm} \delta_{lm} = 0 \quad (p_{tn}) \quad (19a)$$

s.t. $d_{ij} = \sum_{i \in I_n} y_{ti} + \sum_{m \in N} B_{nm} \delta_{lm} = 0 \quad (p_{tn}) \quad (19b)$

$$d_{ij}^{max} - d_{ij} \geq 0 \quad (\nu_{ij}) \quad (19c)$$

$$f_{ij}^{max} - \sum_{n \in N} H_{on} \delta_{in} \geq 0 \quad (\mu_{ij}^+) \quad (19d)$$

$$f_{ij}^{max} + \sum_{n \in N} H_{on} \delta_{in} \geq 0 \quad (\mu_{ij}^-) \quad (19e)$$

$$\pi - \delta_{in} \geq 0 \quad (\zeta_{in}^+) \quad (19f)$$

$$\pi + \delta_{in} \geq 0 \quad (\zeta_{in}^-) \quad (19g)$$

$$\delta_{in} = 0 \quad (\gamma_{in}) \quad (19h)$$
The ISO maximizes the utility of demand from using electricity $d_{tj}$, where the per-unit utility is given by $u_{D_{tj}}$. The first constraint is the energy balance constraint (19b), balancing load, generation and net injections into the network at each node $n$. The second constraint is the maximum demand, since a load unit cannot use more than $d_{tj}^{max}$ units.

The next constraints (19d,19e) ensure that network flows are feasible and the thermal capacity $f_{l}^{max}$ of each power line is observed. Constraints (19f,19g) guarantee that the voltage angle $\delta_{tn}$ is within the range $[-\pi, \pi]$. The B-H-formulation requires to define one arbitrary node as slack bus $\hat{n}$, at which the voltage angle $\delta_{t\hat{n}}$ is zero by assumption (constraint 19h).

Because the decision variables of the ISO are continuous, this problem can be solved simultaneously with the generators’ problems using first order optimality conditions:

$$
0 \leq -u_{D_{tj}} + p_{tn(i)} + \nu_{tj} \perp d_{tj} \geq 0 \quad (20a)
$$

$$
0 = \sum_{m \in N_{B}} B_{mn}p_{tn} + \sum_{l \in L} H_{ln}(\mu_{l} + \mu_{l}^\perp) + \xi_{tn}^+ - \xi_{tn}^- = \begin{cases} \gamma_{t} & \text{if } n = \hat{n} \\ 0 & \text{else} \end{cases}, \quad \delta_{tn} \text{ (free)} \quad (20b)
$$

$$
0 = \sum_{j \in J_{n}} d_{tj} - \sum_{i \in I_{n}} y_{ti} + \sum_{m \in N} B_{nm}\delta_{tn} - p_{tn} \text{ (free)} \quad (20c)
$$

$$
0 \leq d_{tj}^{max} - d_{tj} \perp \nu_{tj} \geq 0 \quad (20d)
$$

$$
0 \leq f_{l}^{max} - \sum_{n \in N_{H}} H_{ln}\delta_{tn} \perp \mu_{l} \geq 0 \quad (20e)
$$

$$
0 \leq f_{l}^{max} + \sum_{n \in N_{H}} H_{ln}\delta_{tn} \perp \mu_{l}^\perp \geq 0 \quad (20f)
$$

$$
0 \leq \pi - \delta_{tn} \perp \xi_{tn}^+ \geq 0 \quad (20g)
$$

$$
0 \leq \pi + \delta_{tn} \perp \xi_{tn}^- \geq 0 \quad (20h)
$$

$$
0 = \delta_{\hat{n}}, \quad \gamma_{t} \text{ (free)} \quad (20i)
$$

The market operator

As stated earlier, we distinguish between the short-term role of the ISO, which clears the market taking the dispatch of the generators as given, and the market operator as introduced in Section 3, which takes the role of an equilibrium selection mechanism and assigns compensation payments to align the incentives of market participants and ensure that no player has a profitable deviation. Mathematically, this player forms the upper level of a two-stage, hierarchical game; the lower level is the binary quasi-equilibrium between the generators and the ISO, with compensation payments if necessary.

It is obvious from the upper-level player’s objective function that this is closely related to the ISO’s and generators’ objective, but not identical – the market operator does not only consider short-term market efficiency, but includes the welfare loss from the disbursement of compensation payments $\zeta_{t}$:

$$
\min_{t \in T} \left\{ \sum_{i \in I} c_{ti}^G y_{ti} + d_{on_{ti}} z_{on_{ti}} + d_{of_{ti}} z_{of_{ti}} - \sum_{j \in J} u_{ij}d_{tj} \right\} + \sum_{i \in I} \zeta_{t} \quad (21)
$$

s.t. KKT conditions of the ISO (Equations 20)

KKT conditions of the generators (Equations 18)

binary equilibrium between generators (Equations 22) specified below

The first part of the objective function is the sum of the generators’ incurred costs and the utility of load units from using electricity; this is equivalent to $F(\cdot)$ in the
theoretical formulation (Problem 15). The second part is the regularizer $G(\cdot)$, although it has a distinct interpretation in this example: because the compensation payments to generators have to be funded through fees or levies on market participants or from general taxation, they usually involve some efficiency loss from market distortions. Here, we assume that every dollar paid in compensation is a 100% loss to overall welfare. One could assume that the loss is even greater because of the aforementioned distortions; alternatively, one could argue that compensation does not incur a loss at all, as it is just a redistribution of rents between different stakeholders. We leave it to future, more policy-applied work to tackle this question.

Let us now turn to the equations necessary to guarantee the binary equilibrium between the generators. When $x_{ti} = 0$ (i.e., the power plant of generator $i$ is switched off in period $t$), the first order conditions can be omitted altogether; the short-term profits in this case are zero, and the fixed costs from starting up or shutting down will be included in the incentive-compatibility constraint. This leaves the KKT conditions of the generators (Equations 18) to determine the optimal, short-term dispatch in the case that the generator is operating.

In addition, the inter-temporal constraint of the power plant operation status and the incentive-compatibility constraint have to be considered. We directly use the linearization proposed in Theorem 4; however, because we now have multiple time periods, we replace the incentive-compatibility constraint from the theoretical formulation (Equation 15d) with two constraints. Thereby, we separate the short-term (per-period) deviation incentives from the compensation payments ($\zeta_i$) that ensure deviation alignment over the entire time horizon. Because of this discrepancy of the time horizon, the switch values are not necessarily non-negative, in contrast to the overview in Table 1. This can be interpreted such that there may be a loss in one period, which the generator is willing to suffer in order to gain larger profits in a different period, hence $(\kappa_{on}^{\text{on}}, \kappa_{off}^{\text{off}}) \in \mathbb{R}$.

\begin{align}
\sum_{t \in T} \left[ \kappa_{on}^{\text{on}} - \alpha_{on}^{\text{on}} y_{ti}^{\text{on}} - \kappa_{on}^{\text{off}} + \kappa_{off}^{\text{off}} \right] = 0
\end{align}

The last constraint (Equation 22e) guarantees that the profits for each generator (short-term revenue from generating less the ramping costs) in the actual market outcome plus the compensation (left-hand side) are greater than the profits which that player could earn in any other dispatch schedule $z$ (right-hand side). The revenues for each dispatch schedule can easily be computed from the duals to the maximum generation and minimum activity constraints, summing over the periods in which the generator is operational in schedule $z$; these periods are denoted by the set $T_z \subseteq T$.

The total ramping costs for each schedule are denoted by $d_{zi}$. The final set of constraints of the binary quasi-equilibrium “translates” the optimal generation decision for both states of the binary variable into the generation level which is actually realized in equilibrium.

\begin{align}
0 \leq y_{ti} \leq x_{ti} g_{ti}^{\text{max}}
\end{align}

The logic of these constraints is straightforward: if $x_{ti} = 1$, the actual dispatch $y_{ti}$ is equal to the generation level $y_{ti}^{\text{on}}$; otherwise, it is zero.
As stated in the previous section, the binary variable has an additional role in this formulation relative to a standard unit-commitment model: rather than simply stating whether a plant is operating or not, it controls which of the two potential states with regard to the continuous variables are active. Furthermore, it ensures that the switch variables $\kappa_i^{on}$ are correctly assigned.

In this power market uplift payment example, Assumptions $A1$, $A2$, and $A3$ hold, hence Theorem 4 shows that the over-all problem is a mixed-binary linear program.

### 4.2 Myopic behavior vs. consistent deviation incentives

There is one subtle difference between the general binary Nash game formulation in Section 3 and the present market power example: in the general formulation of the incentive-compatibility constraint, each player considers the impact of its own binary decision on the market outcome, comparing the respective pay-offs $f_i(1, \tilde{y}_i(y_i), y_i - x_i)$ and $f_i(0, \tilde{y}_i(0), y_i - x_i)$, as specified in Equations (12). In our problem formulation, each generator takes the price as given, irrespective of her own decision. One could say that the generators are assumed to be myopic; i.e., they do not consider that their decision to deviate will impact the resulting market price. That is in line with Scarf’s definition that introducing a new activity must be profitable at the “old” prices (Scarf, 1990).

From a game-theoretic point of view, this is inconsistent; however, from the point of view of market operation in the electricity context, our formulation is valid. Of course, this argument does not hold in general, because a market operator (or coordination agent) in our sense usually does not exist for commodities markets or other real-world applications of binary games. To model such situations, each player’s own impact on prices contingent on its entry or exit should be captured. The method proposed here allows to do this, although the power market model would have to be formulated differently.

Even in the electricity context, it may be appropriate to compute compensation payments based on the alternative prices (i.e., prices resulting after a deviation), rather than those in the actual market outcome. It is not a priori obvious whether compensation payments to guarantee a binary quasi-equilibrium would be larger or smaller in either formulation. We leave this analysis for future research.

### 4.3 Alternative rules for compensation payments

The incentive-compatibility constraint as stated above (Equation 22c) is the direct extension of Constraint (13) in a multi-period setting. The short-term profits or deviation incentives are succinctly captured by the vector $(\kappa_i^{on}, \kappa_i^{off}) \in \mathbb{R}$, and the start-up and shut-down costs are linear terms. As a consequence, this method is flexible and allows to easily implement a wide range of market rules regarding compensation disbursements. To illustrate different settings, we formulate two alternative versions of the model.

First, we implement a no-loss rule to replace the incentive-compatibility constraints: no generator may earn negative profits (i.e., loose money out of pocket):

$$\sum_{t \in T} \left[ \kappa_i^{on} z_i^{on} - d_i^{on} z_i^{on} - d_i^{off} z_i^{off} \right] + \zeta_i \geq 0 \quad (22c')$$

In this setting, there is no constraint stating that the dispatch selected by the ISO has to be incentive-compatible for each generator. Instead, every market participant is forced to follow the schedule selected by the market operator. Hence, this setting omits the game-theoretic considerations. Furthermore, the power plant is compensated even
if it would occur losses irrespective of the selected dispatch, so there may be over-compensation. This can happen in our setting because there are shut-down costs and some generators are operational at the beginning of the model horizon.

The second rule stipulates that only power plants can receive compensation payments if they were active at least once over the model horizon:

\[ \zeta_i \leq \sum_{t \in T} x_{ti} \tilde{K} \]  

(22c")

This is to reflect the potential concern that no generator should receive compensation for doing nothing.

4.4 Illustrative results

The power system adapted from Gabriel et al. (2013) consists of 6 nodes, with 9 generators and 4 load units (see Figure 1). Each generator has a maximum generation capacity of 100 MW and a minimum generation level, if operating, of 50 MW. Because we assume that one time period \( t \) lasts for one hour, capacity (MW) and energy (MWh) are equivalent.

Generators \( g_3 \) to \( g_6 \) are operational at the beginning of the model horizon and the power plants differ with regard to ramping and marginal generation costs. Demand for electricity varies over time, with a high utility for energy (or large willingness-to-pay, WTP) in the first hour and lower WTP in the second hour. All data for generators and load units are provided in Table 3. Regarding the multi-objective function representing the market operator, we assume that each monetary unit paid in compensation is a one-for-one loss of welfare (sum of consumer utility, generator profits, and congestion rents).

All lines have a thermal capacity of 300 MW, except for the two inter-connector lines \( n_2 - n_4 \) and \( n_3 - n_6 \), which have a reduced thermal capacity of 20 MW. Due to these bottlenecks, the standard, welfare-optimal unit commitment model yields losses for some generators.

We compare three different market rule cases; the game-theoretic considerations (constraints) concerning losses and compensations are repeated here for clarity:

**Game-theoretic:** This is the binary quasi-equilibrium solution, as selected by the market operator; every generator receives compensation such that she has no profitable deviation.

\[ \sum_{t \in T} \left[ \kappa_{on} - d_{on} z_{on} - d_{off} z_{off} \right] + \zeta_i \geq \sum_{t \in T} \left[ \bar{g}_{on} - \alpha_{on} g_{on} - \beta_{on} g_{on} \right] - d_{i} \]

\[ \forall \ i \in I, z \in Z \ (\text{cf. } 22e) \]

**No-loss rule:** We solve the multi-objective program (Problem 21) subject to the constraint that no player earns negative profits (instead of Constraint 22e).

Players may have profitable deviations, for which they are not compensated.

\[ \sum_{t \in T} \left[ \kappa_{on} - d_{on} z_{on} - d_{off} z_{off} \right] + \zeta_i \geq 0 \ \forall \ i \in I \]  

(cf. 22e’)

Incidentally, this case yields identical results as the standard approach, in which the power market model is solved according to the two-stage procedure following O’Neill et al. (2005), and pay-offs are calculated based on the dual to the energy-balance constraint from the relaxed problem. However, the observation that the two-stage procedure and the integrated multi-objective yield an identical result is specific to this stylized example, and not a general property of the multi-objective program under a no-loss rule.
No-loss & active: We add the constraint to the previous case No-loss rule stating that only active generators can receive compensation.

\[
\zeta_i \leq \sum_{t \in T} x_{ti} \tilde{K} \quad \forall \ i \in I
\]

The rationale for such a rule may be to prevent gaming, or in response to the public perception that there should not be payments for not providing a service.

Table 4 summarizes the rents earned by the generators in the three market rule cases, as well as the required compensation. Table 5 shows the resulting nodal prices as well as the amount generated and consumed by each unit. Generators \( g_1 \) and \( g_2 \) are not operational at the beginning of the time horizon and are not switched on in any of these cases; therefore, they are omitted from the tables wherever all entries are zero.

For reasons of illustration, we first discuss the results for the no-loss case; as stated before, this yields the same outcome as the (O’Neill et al., 2005) method in this example. Generator \( g_3 \) is switched off immediately and therefore incurs shut-down costs of $300, while generator \( g_4 \) is operating, but the nodal price at node \( n_2 \) is below her marginal costs. Both of these generators are made whole such that they do not incur losses. The deviation incentives of each player are shown in Table 6; here, one

<table>
<thead>
<tr>
<th>( g_i )</th>
<th>( d_{i}^{on} )</th>
<th>( d_{i}^{off} )</th>
<th>( x^{init} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_1 )</td>
<td>24</td>
<td>100</td>
<td>500</td>
</tr>
<tr>
<td>( g_2 )</td>
<td>22</td>
<td>140</td>
<td>350</td>
</tr>
<tr>
<td>( g_3 )</td>
<td>20</td>
<td>180</td>
<td>300</td>
</tr>
<tr>
<td>( g_4 )</td>
<td>18</td>
<td>220</td>
<td>250</td>
</tr>
<tr>
<td>( g_5 )</td>
<td>16</td>
<td>250</td>
<td>220</td>
</tr>
<tr>
<td>( g_6 )</td>
<td>14</td>
<td>300</td>
<td>180</td>
</tr>
<tr>
<td>( g_7 )</td>
<td>12</td>
<td>350</td>
<td>140</td>
</tr>
<tr>
<td>( g_8 )</td>
<td>10</td>
<td>500</td>
<td>100</td>
</tr>
<tr>
<td>( g_9 )</td>
<td>14</td>
<td>105</td>
<td>100</td>
</tr>
</tbody>
</table>

(a) Cost structure (in $/MWh and $, respectively) and initial operational status by generator

<table>
<thead>
<tr>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( d_1^{max} )</th>
<th>( d_2^{max} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_1 )</td>
<td>25</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>( d_2 )</td>
<td>26</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>( d_3 )</td>
<td>26</td>
<td>21</td>
<td>100</td>
</tr>
<tr>
<td>( d_4 )</td>
<td>17</td>
<td>21</td>
<td>100</td>
</tr>
</tbody>
</table>

(b) Utility and maximum demand by load unit and time period (in $/MWh and MW, respectively)

Table 3: Data for generators and load
can see that generator $g_9$ has a profitable deviation given market prices in the no-loss case. That means, this generator has an incentive to switch on the power plant in spite of the schedule announced by the ISO, as she would earn positive profits given the prevailing market prices. Hence, if self-scheduling is an option for the generator, this outcome would not be a Nash equilibrium and the solution would not be stable against deviations.

In the game-theoretic case, the market operator could choose to also compensate generator $g_9$ to counteract self-scheduling by that player, and then obtain the same dispatch as in the no-loss case. This would require to compensate generator $g_9$ to the tune of $95 and generator $g_3$ with $40; generator $g_3$ does not have a profitable deviation and does not receive compensation. The objective value of the market operator would be $2965.

However, because of the integrated consideration of market efficiency and compensation payments, the market operator realizes that it is preferable to dispatch generator $g_9$ and instead shuts down generator $g_4$, realizing an objective value of $2975. Generator $g_9$ now incurs losses, because the resulting locational marginal prices given the new dispatch are lower than her marginal costs; for these losses, she is compensated by the market operator to prevent her from leaving the market. Overall, market efficiency is slightly reduced, but the compensation required to maintain incentive compatibility in this binary quasi-equilibrium is significantly lower than the payments necessary to guarantee incentive compatibility of the welfare-optimal solution.

The last case, No-loss & active, illustrates how strict market rules can hamper efficient market operation, even when they are intended to mitigate strategic behavior or increase public acceptance. Generator $g_3$ would incur losses from shutting down at the beginning of the period, but cannot receive compensation if she doesn’t generate at least in one period; therefore, the market operator dispatches here throughout the model horizon. Because this would result in infeasible flows on the network in the second, low-demand period, the market operator shuts down generator $g_4$ at the end of the first period. Now, this generator incurs losses from generating at a nodal price below marginal costs in the first period and the shutting-down costs, and she is compensated accordingly.

In this case, welfare is reduced by 3% compared to the no-loss case, while the required compensation payments are almost twice as high. Assuming that every dollar

<table>
<thead>
<tr>
<th>$x_i^{\text{init}}$</th>
<th>No-loss rule $(x_{t_i})$</th>
<th>Game-theoretic $(x_{t_i})$</th>
<th>No-loss &amp; active $(x_{t_i})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_3$ 1</td>
<td>$(0, 0)$ $-300$ $-300$</td>
<td>$(0, 0)$ $-300$ $15$</td>
<td>$(1, 1)$ $-525$ $525$</td>
</tr>
<tr>
<td>$g_4$ 1</td>
<td>$(1, 1)$ $-160$ $160$</td>
<td>$(0, 0)$ $-250$ $65$</td>
<td>$(1, 0)$ $-335$ $335$</td>
</tr>
<tr>
<td>$g_5$ 1</td>
<td>$(1, 1)$ $50$</td>
<td>$(1, 1)$ $-50$</td>
<td>$(1, 1)$ $-50$ $50$</td>
</tr>
<tr>
<td>$g_6$ 1</td>
<td>$(1, 1)$ $200$</td>
<td>$(1, 1)$ $100$</td>
<td>$(1, 1)$ $100$</td>
</tr>
<tr>
<td>$g_7$ 0</td>
<td>$(1, 1)$ $690$</td>
<td>$(1, 1)$ $715$</td>
<td>$(1, 1)$ $690$</td>
</tr>
<tr>
<td>$g_8$ 0</td>
<td>$(1, 1)$ $680$</td>
<td>$(1, 1)$ $730$</td>
<td>$(1, 1)$ $550$</td>
</tr>
<tr>
<td>$g_9$ 0</td>
<td>$(0, 0)$</td>
<td>$(1, 1)$ $-5$ $5$</td>
<td>$(0, 0)$</td>
</tr>
</tbody>
</table>

Table 4: Profits by generator ($\pi_i$) before compensation is disbursed, and rents by stakeholder group for each market rule case; on-off status and initial operational status by generator

Generator profit | 1160 460 | 940 85 | 435 910 |
Consumer surplus | 1380 | 1480 | 1830 |
Congestion rent | 560 | 640 | 740 |
Total welfare | 3100 460 | 3060 85 | 3005 910 |
of welfare, the overall utility and profits in the system is 30% lower than in the best-possible outcome, the game-theoretic case. At the same time, it is interesting to note that in the No-loss & active case, consumer surplus and congestion rents are substantially larger than in the other cases – before accounting for the transfers necessary to finance the compensation payments.

Of course, the question is why some generators were already operational at the beginning of the model horizon. If this is because the market operator/ISO dispatched them on the previous day, and then allows them to incur losses on the following day, this may not seem equitable. If, on the other hand, this was a decision taken by the generators themselves based on faulty projections or bad planning, it could be argued that letting them lose money from such myopic decisions is a good penalty to incentivize better planning. We do not venture further in this discussion here; our intention here is only to illustrate that a no-loss rule and compensation may not always be necessary from a game-theoretic point of view.

### 4.5 Numerical implementation and a note on computation

Reformulating the complementarity conditions of the ISO (Equations 20) and the generators (Equations 18b and 18c) using disjunctive constraints yields a mixed-integer linear program (Fortuny-Amat and McCarl, 1981). This approach to determine a binary quasi-equilibrium requires 3|T| |I| = 54 binary variables for the generators and 2|T|(|J| + |L| + |N| − 1) = 68 binary variables for the disjunctive-constraints reformulation of the ISO. The total number of binary variables is therefore |T| (2(|I| + |J| + |L| + |N| − 1) + |I|) = 122. The large scalars $\tilde{K}$ for the disjunctive constraints reformulation and the constraints on assigning the correct values $\kappa_i^t(x)$ (Equations 22c and 22c) were set to 1000.
<table>
<thead>
<tr>
<th></th>
<th>No-loss rule</th>
<th>Game-theoretic</th>
<th>No-loss &amp; active</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(0, 0) (1, 0) (0, 1) (1, 1) ζ_i</td>
<td>(0, 0) (1, 0) (0, 1) (1, 1) ζ_i</td>
<td>(0, 0) (1, 0) (0, 1) (1, 1) ζ_i</td>
</tr>
<tr>
<td>g_1</td>
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<td>0 -787 -380 -567</td>
<td>0 -862 -380 -642</td>
</tr>
<tr>
<td>g_2</td>
<td>0 -590 -370 -470</td>
<td>0 -627 -370 -507</td>
<td>0 -702 -370 -582</td>
</tr>
<tr>
<td>g_3</td>
<td>-300 -350 -690 -260 300</td>
<td>-300 -375 -690 -285 15</td>
<td>-300 -525 -690 -435 435</td>
</tr>
<tr>
<td>g_4</td>
<td>-250 -250 -630 -160 160</td>
<td>-250 -275 -630 -185 65</td>
<td>-250 -425 -630 -335 425</td>
</tr>
<tr>
<td>g_5</td>
<td>-220 -120 -520 500</td>
<td>-220 -220 -520 -50</td>
<td>-220 -220 -520 -50 50</td>
</tr>
<tr>
<td>g_6</td>
<td>-180 20 -480 200</td>
<td>-180 -80 -480 100</td>
<td>-180 -80 -480 100</td>
</tr>
<tr>
<td>g_7</td>
<td>210 -10 690</td>
<td>235 -10 715</td>
<td>210 -10 690</td>
</tr>
<tr>
<td>g_8</td>
<td>200 -120 680</td>
<td>250 -120 730</td>
<td>75 -120 555</td>
</tr>
<tr>
<td>g_9</td>
<td>0 -5 -105 95</td>
<td>-105 -105 -5 5</td>
<td>0 -105 -105</td>
</tr>
</tbody>
</table>

Table 6: Profits by generator for each on/off option, and compensation (in $); market outcome in bold
This compares to $|T| |I| = 18$ binary variables to compute the welfare-optimal dispatch and integer pricing following the approach proposed by O’Neill et al. (2005). Our method is therefore more computationally expensive, but the number of binary variables increases only linearly in the number of time periods, and sublinear in the number of generators, load units, nodes and lines. Solving the resulting equilibrium problem for all permutations of binary variables and checking for incentive compatibility and profitable deviations ex-post (i.e., the brute-force approach) grows exponentially in complexity and requires solving $2^{|T| |N|} > 266,000$ (linear) equilibrium problems.

The numerical model presented in this section is implemented in GAMS and solved using the CPLEX solver. The code includes the possibility of multiple bidding blocks for each generator and demand unit, as in the model proposed by Gabriel et al. (2013), even though this option is not used here. The GAMS code is available on request from the authors. An algorithm for enumerating all permutations and checking for deviation incentives ex-post was also implemented to verify the accuracy of our methodology.

5 Conclusions and outlook

Equilibrium problems with binary decision variables are often encountered in real-world applications, from engineering to economics. It is well known that equilibria in such problems do not necessarily exist, and even if they do exist, finding them is mathematically challenging. The most frequently studied example is the power market uplift problem, which seeks to reconcile the difficulty of finding market-clearing prices – based on the short-term, efficient dispatch – with obtaining incentive-compatible outcomes in decentralized, non-cooperative markets. Alas, to date, no approach to exactly solve such games exists.

In this work, we propose an exact solution method for binary equilibrium problems based on computing optimal responses for each player for both values of the binary variable (or vector), rather than assuming a linearization as a starting point. We then add an explicit incentive-compatibility constraint to ensure that no player has a profitable deviation. We define the notion of a binary quasi-equilibrium to describe situations where no equilibrium exists for the original problem, but in which compensation payments can align the incentives of players such that no player has a profitable deviation and a stable equilibrium is realized.

To this end, we introduce a market operator that acts as an equilibrium selection mechanism according to an upper-level objective function. By recasting the binary equilibrium problem as a hierarchical multi-objective program subject to a binary quasi-equilibrium, our method allows to explicitly incorporate the trade-off between market efficiency and the budget required for compensation payments to obtain an incentive-compatible market outcome. With regard to the power market, this can be interpreted as striking a balance between maximizing short-run welfare (consumer utility less generation costs) and the amount of uplift payments, which are usually funded through taxation, usage fees, or price mark-ups.

Rather than focusing on linearization or relaxation of the binary variables, we compute the best responses for each player for both states of the binary variables (or all states of a binary vector), and compare the respective objective values. This yields a “switch value,” which is the loss a player would incur if it were to deviate from her individually optimal decision. The switch value can also be readily interpreted as the compensation payment a player should receive if the market operator requires her to deviate from the individually optimal strategy. As elaborated earlier, we also believe that the switch value can be used for algorithmic improvements and new approaches to solve binary problems (e.g., act as a stopping criterion, guide a branch-and-bound algorithm).

Most importantly, we show that this method can be reformulated and solved as a
mixed-binary linear program under very general conditions. Hence, the approach can be applied to a wide range of real-world problems, including Nash equilibria in energy and natural resource markets. The approach allows to include a broad range of market regulations, such as “no-loss” rules common in power markets. These regulations can be formulated as linear constraints and therefore do not increase the mathematical complexity of obtaining a binary quasi-equilibrium.

The solution method for binary equilibrium problems proposed in this work can be extended to include Generalized Nash games (cf. [Harker, 1991]) or games with individual joint constraints ([Nabetani et al., 2011], as well as equilibrium problems in discrete rather than binary variables. Furthermore, more general non-cooperative games can be solved in this framework, such as games based on conjectural variations ([Wogrin et al., 2013]). In economic applications, the binary variables can be interpreted as on/off decisions, or as market-entrance or investment decisions in a dynamic, two-stage setting. Extending the mathematical approach to stochastic applications is also straightforward.

Because most power market operations are based on a welfare-optimal dispatch with ex-post compensation payments, there may exist numerous gaming opportunities for large utilities or other market participants in the current setting; the settlement in 2013 between the Federal Energy Regulatory Commission (FERC) and JP Morgan regarding manipulative bidding strategies is a case in point[1]. It will be the subject of future research to analyze whether our solution method for binary equilibrium problems will be more or less prone to market power exertion and strategic behavior. Our method explicitly incorporates game-theoretic considerations (i.e., deviation incentives) rather than simplistic no-loss rules, and it includes the trade-off between maximizing market efficiency and minimizing compensation payments. As a consequence, we believe that our approach has significant potential to improve the current practice in power market operation. In particular, our approach allows to include the incentives of non-active players to enter the market; hence, the behavior of non-dispatched generators entering the market through self-commitment (or “self-scheduling”) can be more effectively addressed (cf. [Sioshansi et al., 2010]).

As a numerical application, we solve the power market uplift problem analyzed by [Gabriel et al., 2013]. We illustrate that the current practice in power market operation easily leads to situations where players have profitable deviations. In particular, when considering the nature of the non-cooperative game in our stylized example, the market operator prefers to deviate from the welfare-optimal dispatch, because a slight reduction in market efficiency is traded for a strong reduction in compensation payments necessary to maintain incentive compatibility of the market outcome.

To illustrate the flexibility of our approach, we also solve the model under a hypothetical market regulation stating that a) no generator may loose money, and b) only active generators may receive compensation. This yields a welfare loss of 3% relative to the optimal solution. At the same time, compensation payments are almost twice as high as in the currently used approach, so that compensation payments eat up almost a third of total welfare in the market. We take this as a warning that market rules may have rather counter-intuitive effects, even when they are implemented with the aim of preventing strategic behavior or mitigating other inefficiencies.

As a next step, we will apply our method to examples beyond the power market. It remains to be seen whether this method is advantageous from the point of view of market operation, as well as whether it is numerically practicable in large-scale problems. The number of binary variables may quickly grow beyond the bounds of numerical tractability; at the same time, as we show in the numerical application in Section 4, the number of binary variables can be significantly reduced depending on the actual problem under consideration.

References


