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**Working Paper**

## Bringing an elementary agent-based model to the data: Estimation via GMM and an application to forecasting of asset price volatility

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# FinMaP-Working Paper No.38



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FINMAP –

**FINANCIAL DISTORTIONS AND MACROECONOMIC  
PERFORMANCE: EXPECTATIONS, CONSTRAINTS AND  
INTERACTION OF AGENTS**

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**TITLE**

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We apply our estimator to a sample of long records of returns of various stock and foreign exchange markets as well the price of gold. Using the estimated parameters to form the best linear forecasts for future volatility we find that the behavioral model generates sensible forecasts that get close to those of a standard GARCH (1,1) model in their overall performance, and often provide useful information on top of the information incorporated in the GARCH forecasts.

*Keywords: Sentiment dynamics, GMM estimation, volatility forecasting.*

*JEL classification: G12, C22, C53.*

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# Bringing an Elementary Agent-Based Model to the Data: Estimation via GMM and an Application to Forecasting of Asset Price Volatility\*

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## Abstract

We explore the issue of estimating a simple agent-based model of price formation in an asset market using the approach of Alfarano et al. (2008) as an example. Since we are able to derive various moment conditions for this model, we can apply generalized method of moments (GMM) estimation. We find that we can get relatively accurate parameter estimates with an appropriate choice of moment conditions and initialization of the iterative GMM estimates that reduce the biases arising from strong autocorrelations of the estimates of certain parameters. We apply our estimator to a sample of long records of returns of various stock and foreign exchange markets as well the price of gold. Using the estimated parameters to form the best linear forecasts for future volatility we find that the behavioral model generates sensible forecasts that get close to those of a standard GARCH(1,1) model in their overall performance, and often provide useful information on top of the information incorporated in the GARCH forecasts.

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# 1 Introduction

Asset markets have been known for a long time to be characterized by a set of ubiquitous ‘stylised facts’ that are hard to explain by any traditional approach to asset pricing. The best known of these are the fat tails of the unconditional distribution of returns and the volatility clustering characterizing their conditional distribution. Known since the 60s, but largely mysterious in terms of their behavioral origins, these salient features have also occasionally been classified as ‘anomalies’. However, the latter term appears odd in view of the fact that these are really the constants in the statistical analysis of time series of financial markets across time, countries and asset types, i.e. the imprint of their apparently ‘normal’ mode of operation.

Under the traditional ‘efficient market paradigm’ the time series (a)nomalies are interpreted as the mere reflection of the same (a)nomalies of the ‘fundamental factors’ of asset prices. However, the fundamentals consist of a conglomerate of diverse factors (macroeconomic, firm-specific, etc.) and as an ensemble they are not observable so that this aspect of the efficient market paradigm cannot be tested directly. While there do not even exist examples of fundamental factors whose time-variation would share the phenomenology of fat tails and clustered volatility, these features could also suggest a more behavioral explanation rather than the mere transmission of information from fundamental factors into changing prices. For instance, ‘fat tails’ are represented by an unusually high number of extremely large observations which resonates with the notion of excessive volatility of financial markets. Indeed, one of the most convincing components of the body of empirical evidence against the efficient market hypothesis is evidence for excessive volatility according to the test strategy developed first by Shiller (1981). In a similar vein, clustering of volatility could originate from waves of speculative behavior or overoptimism of market participants occasionally switching to overpessimism, ‘risk appetite’ changing over time, and similar descriptions of financial market turmoil.

The first attempts at explaining the stylized facts with behavioral models have come forth since about the beginning of the 1990s. Examples include Kirman (1991, 1993), de Grauwe et al. (1993), and Lux (1995), among others. While early contributions were targeting phenomena like excessive volatility and endogenous emergence of bubbles and crashes, the subsequent literature has also concentrated on reproducing time series from behavioral market models that could reproduce those of empirical records. Most of this research is conducted via simulation studies, since ‘stylized facts’ are statistically characterized by higher-order conditional and unconditional moments that for complex models of the market process with heterogeneous agents are hard to derive analytically. Meanwhile, a broad range of models exist that all get close to empirical market behavior or even generate synthetic data

that are hard to distinguish with statistical tests from ‘real’ ones, cf. the surveys by Hommes (2006), LeBaron (2006), Samanidou et al. (2007) and Lux (2008). It appears that the combination of stabilizing and destabilizing (centripetal and centrifugal) forces as represented by, e.g., trend following and chartist strategies, on the one hand, and arbitrage activities based on some perception of an underlying fundamental value by some agents, on the other hand, is generally sufficient to generate a process that with a bit of stochasticity added, is able to provide very realistic data for a broad class of models varying in their exact details. Models in this vein range from simple “zero intelligence” settings (Kirman, 1991; Cont and Bouchaud, 2000) over chartist-fundamentalist models (Lux and Marchesi, 1999; Brock and Hommes, 1998) to models in which agents continuously develop their strategies with some kind of artificial learning algorithm (LeBaron et al., 1999; Lux and Schornstein, 2005).

With this literature having reached a status of consolidation, one natural further research direction is the empirical estimation and validation of such models. This is an endeavor economists are not accustomed to as empirical implementation in an economic context has typically been concerned with (sets of) reduced form equations of behavioral characterizations of representative agent models (expressed, for instance, via the Euler equations characterizing the optimal path of economic activity of such a representative agent). In the present context one would rather have to estimate models with an ensemble of agents with more or less complex interactions. However, there is no fundamental problem involved in such an undertaking. Typical agent-based models can be represented as Markovian stochastic processes and, thus, often generically fulfill a number of ‘regularity conditions’ that are needed for the application of certain estimation strategies. Relatively simple models might also be amenable to some kind of reduced-form condensation which, in fact, will be the case for the model studied in the following chapters.

Previous attempts of estimation of behavioral or agent-based models of financial markets are sparse. Examples include: Amilon (2008) who estimated the model of Brock and Hommes (1998) by a maximum likelihood approach, Gilli and Winker (2003) who attempted to estimate the ‘ant’ model of Kirman (1993) via nonlinear optimization techniques, Alfarano et al. (2005) who estimated an extended version of the same model via an approximate likelihood approach, and a recent series of papers by Franke and Westerhoff (2011, 2012, 2014) in which a variety of extremely simplified versions of agent-based models are estimated via moment matching approaches. Jang (2013) also uses a simulated method of moments approach for a closely related model, and reports certain principal difficulties in estimating even a very basic agent-based model: He finds the surface of the objective function to be very flat over certain regions so that the chosen moments provide little

scope in differentiating between different parameter sets, and he highlights that a very rugged surface of the objective function also makes the search for a global optimum computationally difficult as the danger to get trapped in one out of many local minima of the objective function is hard to assess.

In the present study, we attempt to contribute to this literature by exploring more systematically the issues surrounding the estimation of a prototype agent-based behavioral market model. Our model is of the same class as investigated already by Gilli and Winker (2003), Alfarano et al. (2005) and Jang (2013). Among a number of closely related varieties we choose the specification of Alfarano et al. (2008). The latter has the advantage that the authors have derived already a set of moments that with a bit more of effort can be expanded into moment conditions of an estimable version of their model. These are the basic moments that characterize the stochastic dimension of the stylized facts, i.e. higher moments of returns and autocorrelations of such higher moments. We use these moments in a generalised method of moments (GMM) setting which, due to the absence of simulation in the estimation process, is more transparent than an SMM approach without any analytical input. By and large, we will see that the problems highlighted in previous literature can be overcome with a judicious choice of the moment conditions and the estimation strategy. An empirical application indicates that the present simple model already gets so close to the key moments characterising fat tails and clustered volatility that it can mostly not be rejected as the underlying data-generating process for these moments even for relatively long data sets extending over several decades. A forecasting exercise demonstrates that the behavioral model possesses significant forecast capacity for short- and medium-term volatility. While its forecast performance remains generally inferior to that of a baseline GARCH model, it is also mostly not ‘encompassed’ by the GARCH model, i.e. it adds valuable information on top of that extracted by the GARCH model.

The rest of this paper proceeds as follows. Section 2 introduces the agent-based model and its ‘reduced form’ representation in the form of a stochastic differential or Langevin equation. Section 3 provides an exposition of the GMM estimation and the moment conditions used. Section 4 provides Monte Carlo results for different specifications of the estimator. Section 5, then, contains the empirical application, and Section 6 concludes.

## 2 An elementary agent-based model of sentiment formation and asset price dynamics

The model investigated in Alfarano et al. (2008) basically extends Kirman’s (1993) seminal herding model into a simple asset-pricing model. The overall market consists of fundamental traders as well as those driven by *sentiment* and the time-variation of sentiment is formalized via the herding dynamics.

Referring to the second group of agents as noise traders, any one of these at any point in time might be labelled as an optimist or pessimist. The number of agents in an optimistic mood will be denoted by  $n$ . With the complete pool of noise traders consisting of  $N$  agents, the fraction of optimists at time  $t$ ,  $x_t = (n_t - (N - n_t))/N = 2n_t/N - 1$  can be denoted the current configuration of the sentiment-driven part of the market. Agents' mood might, of course, change over time, and so an optimistic noise trader might become pessimistic and vice versa. This happens with transition rates  $\pi_{\downarrow,t}$  and  $\pi_{\uparrow,t}$  in continuous time. Transitions consist of an autonomous part for idiosyncratic changes of opinion and an “interactive” part driven by interpersonal communication:

$$\pi_{\uparrow,t} = a + \frac{n_t}{N}b = a + bx_t, \quad (1)$$

$$\pi_{\downarrow,t} = a + \frac{N - n_t}{N}b = a + b(1 - x_t). \quad (2)$$

Here  $a$  is the propensity to change one's opinion due to idiosyncratic reasons (as a rate per time unit) and  $b$  is the propensity to change one's opinion under the influence of an agent of the other group (again as a rate per time unit). To meet one person with an antagonistic opinion happens with a probability proportional to  $n_t/N$  for a pessimist and  $(N - n_t)/N$  for an optimist which defines the second part of both equations ( $\uparrow$  denotes switch of a pessimist to optimist and the reverse for  $\downarrow$ ). Population transition rates ( $\omega_{\uparrow,t}$  and  $\omega_{\downarrow,t}$ ) are then simply obtained by multiplying the individual ones by the respective number of agents, and are, thus, obtained as:

$$\omega_{\uparrow,t} = (N - n_t) \left( a + b \frac{n_t}{N} \right) \quad (3)$$

$$\omega_{\downarrow,t} = n_t \left( a + b \frac{N - n_t}{N} \right). \quad (4)$$

Technically, this is a system of Poisson processes (one for each agent) with transition rates being linear and state-dependent. Such systems have also been denoted as jump Markov processes (cf. Aoki, 2002).

The configuration of sentiment,  $x_t$ , is embedded into an asset pricing framework by assuming that optimist (pessimist) agents will buy (sell) a fixed volume of shares  $T_c$ . In addition, there is another component of excess demand composed of fundamentalists who are sensitive to deviations between the log market price,  $p_t$ , and their assumed log fundamental value,  $p_{f,t}$ .

Price adjustments are assumed to occur in the usual Walrasian manner in the presence of excess demand. Combining these aspects we arrive at the price adjustment equation:

$$\frac{dp}{dt} = \beta (T_f (p_{f,t} - p_t) + NT_c x_t). \quad (5)$$



Assuming instantaneous adjustment to market equilibrium, i.e. letting the price adjustment speed  $\beta \rightarrow \infty$ , we obtain:

$$p_t = p_{f,t} + \frac{NT_c}{T_f} x_t \quad (6)$$

which shows how sentiment may trigger deviations from fundamental valuation.

Returns over finite time increments (e.g., daily returns) can be written as:

$$r_{t+1} = p_{t+1} - p_t = p_{f,t+1} - p_{f,t} + \frac{NT_c}{T_f} (x_{t+1} - x_t) \quad (7)$$

and, hence, are driven by innovations in the fundamentals and innovations in the sentiment dynamics. We assume that the (log) fundamental value follows Brownian motion with a variance  $\sigma_f^2$  so that the unit changes can be written as:

$$p_{f,t+1} - p_{f,t} = \sigma_f \varepsilon_t, \quad \varepsilon_t \sim N(0, 1). \quad (8)$$

So far, we have a macroscopic equation for the price combined with the result of the interaction of  $N$  agents,  $x_t$ . However, the microscopic dynamics of  $x_t$  can as well be transformed into a macroscopic process equation. In particular, it can be shown that the transient probability density of  $x_t$ ,  $p(x, t)$ , follows a Fokker-Planck or forward Kolmogorov equation:

$$\frac{\partial p(x, t)}{\partial t} = \frac{\partial}{\partial x} A(x) p(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (D(x) p(x, t)) \quad (9)$$

with drift  $A(x, t) = -2ax_t$  and diffusion  $D(x, t) = 2b(1 - x_t^2) + 4a/N$ . As a consequence, the macroscopic dynamic process can be approximately characterized by a continuous-time diffusion:

$$d\tilde{x}_t = A(\tilde{x}_t) + \sqrt{D(\tilde{x}_t)} dB_t \quad (10)$$

i.e. a diffusion process that ‘shares’ the same forward Kolmogorov equation with the agent-based process (cf. Lux, 2008, and Ethier and Kurtz, 1986, for more details). It is this diffusion process as the reduced-form of our agent-based model that together with the price process, eq. (6), will be used to derive a set of moment conditions to be used in our subsequent GMM estimation.

### 3 Estimation

We will assume throughout that we are given equidistant observations of market prices,  $p_{t_i}$ ,  $t_i = \Delta ti$ ,  $i = 0, \dots, T$ , making a sample of size  $(T + 1)$ . We calculate from equations (6), (7) and (8)

$$r_t = \sigma_f \varepsilon_t + x_{t+1} - x_t, \quad (11)$$

where  $\varepsilon_t$  is a standard normal variate that is independent from  $x_t$  at all lags. Thus, we observe a sequence of the log-returns  $r_t$  of size  $T$ . The parameter vector  $\theta$  to be estimated is  $\theta := (a, b, \sigma_f)'$ . Note that we do not estimate the parameters  $N$ ,  $T_c$ , and  $T_f$ . We assume that these are given by  $N = 100$ , and  $NT_c/T_f = 1$ . Given that these three parameters enter in a multiplicative way, we could at best hope to estimate  $N$  (which also enters the diffusion function of the reduced-form equation of the sentiment process), and  $T_c/T_f$ . However, preliminary investigations showed that when including these as free parameters, the problems of weak identification reported below for the remaining ones will become paramount. Essentially, with these parameters entering as pre-factors for  $z_t \equiv x_{t+1} - x_t$ , they are so close to colinear with  $a$ , the drift factor of the diffusion process, that with limited data, their separate estimation becomes impossible. Nevertheless, inspection of the moment conditions derived below immediately indicates that with a sufficient number of moment conditions used, also for a 5-parameter model  $\hat{\theta} = (a, b, \sigma_f, N, T_c/T_f)$  identification in the strict sense does hold, and so with an infinite supply of data one would be able to estimate these parameters.

### 3.1 The GMM procedure

We use  $\theta$  to denote an arbitrary element of the three dimensional parametric space  $\Theta$  and let  $\theta_0$  to be the true parameter vector. The GMM procedure due to Hansen (1982) requires to formulate a number  $q \geq 3$  moment conditions,  $m_t = [m_{1t}, \dots, m_{qt}]'$ . Here, we will be using simple unconditional and conditional moments like:

$$m_{1t} = r_t^2 \quad (12)$$

$$m_{2t} = r_t^4 \quad (13)$$

$$m_{3t} = r_t^2 r_{t-1}^2 \quad (14)$$

and higher lags of the autocovariances of squared returns. Their sample counterparts are  $M_T = [M_{1T}, \dots, M_{qT}]'$ , where

$$M_{iT} = \sum_{t=k+1}^T \frac{m_{it}}{(T-k)}$$

for all  $i$  from the set  $\{1, \dots, q\}$  and  $k$  defines the maximum lag between the variables that enter the sample moments.

Denote the corresponding  $q$ -dimensional vector of analytic moments,  $F(\theta) := [F_1(\theta), \dots, F_q(\theta)]'$  such that

$$\mathbb{E}[m_t(\theta)] = F(\theta).$$

It is customary to introduce a vector-valued function  $g_T(\theta) := M_T - F(\theta)$  which beyond  $\theta$  depends also on the  $T$ -dimensional data vector  $\mathbf{r}_T$  through  $M_T$ .

The GMM estimator,  $\hat{\theta}_T$ , minimizes the distance between  $F(\theta)$  and  $M_T$  over the parametric space  $\Theta$  in the following quadratic form,

$$Q(\theta)_T := g_T(\theta)' W_T g_T(\theta), \quad (15)$$

where  $W_T$  is a positive semi-definite matrix which may depend on the data but converges in probability to a positive definite matrix of constants (see Hall, 2005, p. 14). That is,

$$\hat{\theta}_T = \arg \min_{\theta \in \Theta} Q(\theta)_T. \quad (16)$$

The GMM theory does not give recommendations on the optimal selection of the moment conditions. In practice, this choice is up to the researcher. However, the theory provides information on the optimal choice of the weighting matrix in the sense that it leads to an estimator with the smallest asymptotic errors. In particular, if  $\Omega$  denotes the asymptotic covariance matrix of the moment conditions (to be specified below), then choosing  $W = \Omega^{-1}$  gives us the sought efficient weighting matrix. Let  $g_t(\theta) := m_t(\theta) - F(\theta)$  and consider  $D := \mathbb{E}[\partial g_t(\theta_0)/\partial \theta']$ . Then, under appropriate regularity conditions,  $\hat{\theta}_T$  is consistent and  $\sqrt{T}(\hat{\theta}_T - \theta_0)$  is asymptotically normal with mean zero and covariance matrix  $V := D' W D$ , that is,  $\sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{d} \mathcal{N}(0, V)$ .

Following most of the extant GMM literature, we have adopted the standard Newey-West estimation of the covariance matrix, and used an iterative GMM scheme, i.e. we computed a new estimate of the covariance matrix based upon the estimated set of parameters in each round and iterated the sequence of GMM estimations until convergence of the parameter estimates was obtained.

### 3.2 The moment conditions

The moment conditions that are available by appropriate extension of the results of Alfaro et al. (2008) are basically various powers of returns and their covariances. Among the unconditional powers, odd moments are uninteresting as they are mostly close to zero both in the empirical data and in the analysis of our model.<sup>1</sup> Flat or almost flat moments are, however, not informative and would presumably increase the uncertainty of parameter estimates and distort the size of the  $J$  test. Since higher powers also become increasingly noisy, the most practical choice is to use squared returns and the fourth moment. Theoretical approximations of autocovariances can be obtained for squared returns over arbitrary time lags while again, autocorrelations of odd powers would not be very informative and autocovariances

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<sup>1</sup>In simulations, we find the small autocorrelation of raw returns at lag 1 imposed by the bounded variation of the sentiment dynamics to be practically indistinguishable from zero, i.e. the sampling standard error exceeded the estimated autocorrelation by far, even for large samples.

of higher powers than the second would be very hard to obtain, and perhaps also very noisy. Thus, we are basically depending on the most elementary measurements of the stylized facts for our GMM implementation: Kurtosis and autoregressive dependency of squared returns. For univariate time series of asset prices, this is essentially the “portfolio” of moments for GMM or SMM estimation. Note that some authors (e.g. Franke and Westerhoff, 2012, 2014) have used more refined measures such as the tail index for the decay of the density of the unconditional distribution, or similar measures for the decay of the autocorrelation function. While GMM/SMM based on such more complicated objects derived from the data is perfectly legitimate, a certain backdrop is that an efficient estimate of the covariance matrix of these moment conditions is not easy to define. Hence, one has to sacrifice efficiency compared to a potential optimal estimator. We believe, however, that we can go some way towards a more robust description of the stylized facts by not just considering single autocovariances  $r_t r_{t-h}$  but rather sums of these:  $\sum_{h=t_1}^{T_1} r_t^2 r_{t-h}^2$  as these sums characterize the curvature over a certain range of lags and might, therefore, give less noisy information than single autocovariances. Such moments might actually convey structural information equivalent to the decay rate of the ACF. The use of such sums of moments is perfectly in line with the GMM framework and we will explore their effectiveness in the next section.

Now let us turn to the exact moment conditions. Starting with the second moment, we easily find that

$$\mathbb{E}[r_t^2] = \sigma_f^2 + \mathbb{E}[(x_{t+1} - x_t)^2] = \sigma_f^2 + \mathbb{E}[z_t^2] = \sigma_f^2 + 2\mathbb{E}[x_t^2](1 - e^{-2a}) \quad (17)$$

with  $z_t = x_{t+1} - x_t$  and

$$\mathbb{E}[x_t^2] = \frac{b}{b + 2a}. \quad (18)$$

Moving on to the fourth moment, it is also easy to see that:

$$\mathbb{E}[r_t^4] = 3\sigma_f^4 + 6\sigma_f^2\mathbb{E}[z_t^2] + \mathbb{E}[z_t^4]. \quad (19)$$

We can derive  $\mathbb{E}[z_t^4]$  as follows (cf. the Appendix):

$$\mathbb{E}[z_t^4] = 8\mathbb{E}[x_t^4] \left\{ \frac{a+b}{2a+b} - e^{-2a} + \frac{a}{2a+b} e^{-2(2a+b)} \right\} \quad (20)$$

and

$$\mathbb{E}[x_t^4] = \frac{3b}{2a+3b} \mathbb{E}[x_t^2]. \quad (21)$$

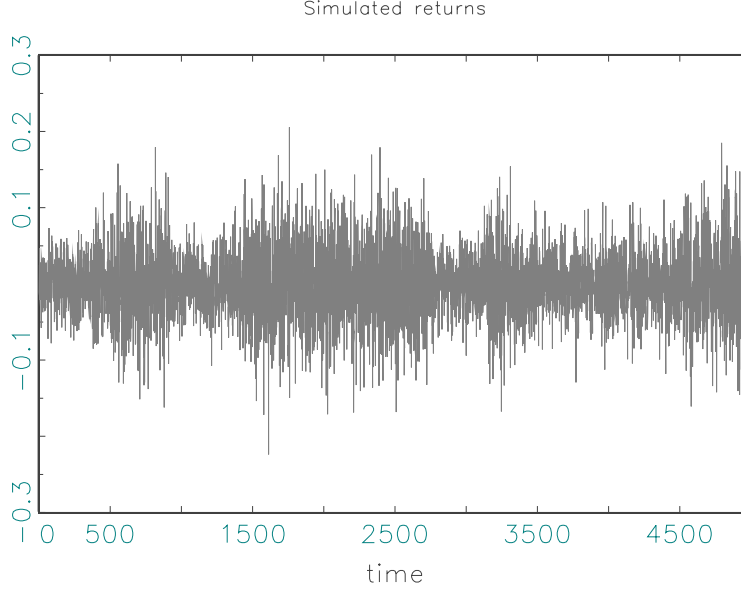
Finally, autocovariances of squared returns are defined as

$$\mathbb{E}[r_t^2 r_{t-h}^2] = \sigma_f^4 + 2\sigma_f^2 \mathbb{E}[z_t^2] + \mathbb{E}[z_t^2 z_{t-h}^2] \quad (22)$$

where we show again in the Appendix that

$$\mathbb{E}[z_t^2 z_{t-h}^2] = 4b^2(1 - 2(2a+b))^{h-1} (\mathbb{E}[x_t^4] - \mathbb{E}[x_t^2]^2). \quad (23)$$

Figure 1: Typical simulation with parameter set  $\theta$



## 4 Monte Carlo Results

In order to assess the quality of the proposed GMM estimator, we have conducted a series of Monte Carlo experiments.

In the baseline setting, we assume that the log returns follow Wiener Brownian motion without drift. The fundamental dynamics is, thus, simply characterized by its standard deviation  $\sigma_f$ . We choose the parameter set  $\theta = (a, b, \sigma_f) = (0.0003, 0.014, 0.03)$  together with the number of agents  $N = 100$ . This is a parameter set in the vicinity of typical empirical estimates, and it satisfies a number of conditions: (i) it generates data that is roughly in line with the empirical appearance of financial returns featuring fat tails and clustering of volatility, see Fig. 1, (ii) it has roughly equal contributions of the fundamental changes and the sentiment component to the overall variation of asset prices (which appears to be a good starting point), (iii) it enables us to approximate the underlying agent-based model by its associated Langevin equation to a reasonable degree of accuracy. We simulate the Langevin equation with time increments  $\Delta t < \frac{2}{N(2a+bN)}$  that guarantee that the sum of all transition probabilities remains smaller than 1. Since the parameter set  $\theta$  leads to a bimodal outcome of the sentiment process, we have also conducted Monte Carlo runs for a case with uni-modal dynamics given by the parameter set  $\theta' = (0.014, 0.0003, 0.03)$ . Baseline GMM results are documented as “GMM1” in Tables 1 and 2. The moments we have used

Table 1: Sensitivity of Moments to Parameter Changes.

Parameter set I								
	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m'_3$	$m'_4$
$a$	0.079	0.136	0.136	0.136	0.136	0.136	0.136	0.137
$b$	0.121	0.329	0.33	0.327	0.324	0.319	0.316	0.293
$\sigma_f$	0.412	0.889	0.889	0.889	0.889	0.888	0.888	0.889
Parameter set II								
$a$	0.007	0.014	0.014	0.014	0.014	0.014	0.014	0.014
$b$	0.094	0.197	0.197	0.197	0.197	0.197	0.197	0.197
$\sigma_f$	0.445	1.087	1.087	1.087	1.087	1.087	1.087	1.087

Notes: The table displays calculations of the sensitivity of the six moment conditions with respect to parameter variations. Changes by 25 percent of each one of the parameters have been considered. The moments we have used are in their order of appearance in the table: squared returns, the fourth moment and autocovariances of squared returns at lags 1, 5, 10 and 20, as well as the sums of autocovariances over lags 1 through 50 ( $m'_3$ ) and 51 through 100 ( $m'_4$ ), respectively.

are squared returns, as well as their fourth moment and autocovariances of squared returns at lags 1, 5, 10 and 20. Table 1 shows that our chosen moment conditions display sufficient variation when changing the parameters around  $\theta$  and  $\theta'$  so that the underlying parameters should be guaranteed to be identified. Note, however, that the autocovariances  $m_3$  through  $m_6$  are all sensitive mainly with respect to  $b$ , but show homogeneous variation with  $\sigma_f$  and  $a$ . This could be both an advantage or disadvantage for our estimator. Indeed, it may imply that it is easier to identify  $b$  than the remaining parameters,  $\sigma_f$  and  $a$ .

Since this is a nonlinear model with possibly multiple local minima of the objective function, and we would not have any clear ex-ante perception on the range of “realistic” parameter values in an empirical application, we have first conducted a grid search in the admissible parameter space and have subsequently initiated the GMM estimation from the 10 best out of  $9^3$  grid points. These grid points have been chosen equidistantly along all three dimensions: given the variance of each set of test data, we have chosen  $\sigma_f$  so that the fundamental variation would account for 0.1, 0.2, ..., 0.9 of the total sample variance. Given the grid value of  $\sigma_f$ ,  $a$  and  $b$  have been varied around the bifurcation value  $\epsilon_0 = a/b$  over 9 equidistant points each with center at  $\epsilon_0$  and four points to its right and left within the unimodal and bimodal regimes. The exact location of the grid points of  $a$  and  $b$  has also been chosen for each sample to bring them into rough agreement with the

Table 2: Monte Carlo results for GMM estimation of ALW Model.

True:	0.3	1.4	30	0.3	1.4	30	0.3	1.4	30
	GMM1			GMM2			GMM3		
	$a$	$b$	$\sigma_f$	$a$	$b$	$\sigma_f$	$a$	$b$	$\sigma_f$
T = 5,000									
Mean	0.465	1.197	28.050	0.411	1.253	27.691	0.410	1.328	28.197
FSSE	0.288	0.496	4.859	0.275	0.312	6.185	0.309	0.332	6.520
RMSE	0.331	0.535	5.230	0.296	0.345	6.595	0.326	0.338	6.733
T = 10,000									
Mean	0.419	1.134	28.399	0.352	1.306	28.655	0.352	1.431	29.351
FSSE	0.212	0.468	5.039	0.194	0.276	5.621	0.213	0.272	6.709
RMSE	0.243	0.538	5.281	0.201	0.292	5.773	0.219	0.273	6.707
T = 20,000									
Mean	0.429	1.150	27.874	0.307	1.460	30.347	0.344	1.503	30.067
FSSE	0.205	0.446	5.143	0.160	0.205	4.901	0.214	0.182	5.748
RMSE	0.242	0.511	5.559	0.160	0.214	4.907	0.217	0.209	5.719

True:	1.4	0.3	30	1.4	0.3	30	1.4	0.3	30
	GMM1			GMM2			GMM3		
	$a$	$b$	$\sigma_f$	$a$	$b$	$\sigma_f$	$a$	$b$	$\sigma_f$
T = 5,000									
Mean	1.787	0.441	27.092	0.935	0.692	27.390	1.322	0.434	27.298
FSSE	5.883	0.029	1.112	0.440	0.570	2.756	0.295	0.017	0.428
RMSE	5.888	0.144	3.113	0.640	0.691	3.793	0.304	0.136	2.735
T = 10,000									
Mean	2.054	0.441	27.018	1.014	0.693	27.107	1.524	0.433	27.207
FSSE	8.283	0.033	1.099	0.424	0.576	2.760	0.910	0.015	0.300
RMSE	8.298	0.145	3.177	0.573	0.696	3.996	0.914	0.134	2.809
T = 20,000									
Mean	2.006	0.440	26.997	1.118	0.633	26.707	1.529	0.430	27.206
FSSE	5.012	0.025	0.857	0.354	0.521	3.132	0.258	0.014	0.288
RMSE	5.042	0.142	3.123	0.452	0.618	4.542	0.287	0.130	2.809

Notes: The table shows the means, finite sample standard errors (FSSE) and root-mean squared errors (RMSE) of 400 replications of each scenario. Estimated parameters are multiplied by  $10^3$  for better readability. GMM1 stands for a standard GMM estimation, while in GMM2 the estimation has been initiated with the inverse of the variance-covariance matrix of the test data as the weighting matrix in the first step of the estimation. In GMM3, single autocovariances have been replaced by sums of autocovariances.

sample variance.

We have used sample sizes of 5000, 10000 and 20000 observations. While our estimates are centered in the vicinity of their ‘true’ values, and for  $a$  and  $b$  show roughly declining biases with increasing sample size, the overall quality of the estimates appears to show no significant improvement with higher sample sizes – their finite sample standard errors (FSSE) and root mean-squared errors (RMSEs) are about the same for all sample sizes. This behavior is puzzling as our process should meet all standard “regularity” conditions required for GMM estimation, and the chosen six moments should allow identification of the parameters. Note that our moment conditions are nonlinear in all parameters so that identification should be generic (cf. McManus, 1992). Upon closer inspection, we find that our parameter estimates suffer from a particularly high correlation between  $\sigma_f$  and  $a$  that leads to a certain number of outliers. The reason is that these variables are somewhat complementary in that they determine the variances of the fundamental and sentiment component (while the sentiment variance depends on both  $a$  and  $b$ , the behavior of the estimates actually shows that  $a$  is more relevant for the variance of the process). Thus, shifting between  $\sigma_f$  and  $a$  roughly leads the second moment intact. While this might effect autocovariances, changes of  $b$  might also bring those closer to the theoretical moments. As a consequence, we find a certain number of runs that get stuck in local minima.

Note also that in the grid search and initialization we have used unweighted moments (or weighted by the identity matrix – the default initialization of a routine GMM estimation). In the present setting, the second moment will be much larger in absolute value than the fourth moments and the autocovariances of squared returns. Hence, to not let the second moment exert an all-dominating influence in the first step and to bias to one side the overall results, it could be useful to already use some information on the precision of different moments in the first step. To this end, we have computed a weighting matrix that is the inverse of the variance-covariance matrix of the moments in the test data. Using this unconventional initialization (for both the grid search and the first iteration of the GMM) we indeed find a uniform improvement of our results in the more realistic parameter set  $\theta$ . The pertinent outcome is summarized under the heading “GMM2” in Table 2. As it turns out, the performance of this alternative estimator beats that of the conventional one in all respects under parameter set  $\theta$ : biases are uniformly decreasing with sample size, FSSEs and RMSEs are always smaller than with GMM1 (particularly so far  $b$  where the FSSEs and RMSEs are only about 50 percent of those of GMM1), and the precision of the estimates becomes recognizably better with increasing sample size. For  $\theta'$ , we find a strong improvement in the quality of the estimate for  $a$ , but also a sizable deterioration for  $b$ , and higher accuracy with increasing sample size is also confined to parameter  $a$ . Table 2 and Fig. 2 provide additional



illustrations of the distortive effects explained above which are still present (though less pronounced) in the results from GMM2.

Table 3: Correlation Matrices of GMM Estimates.

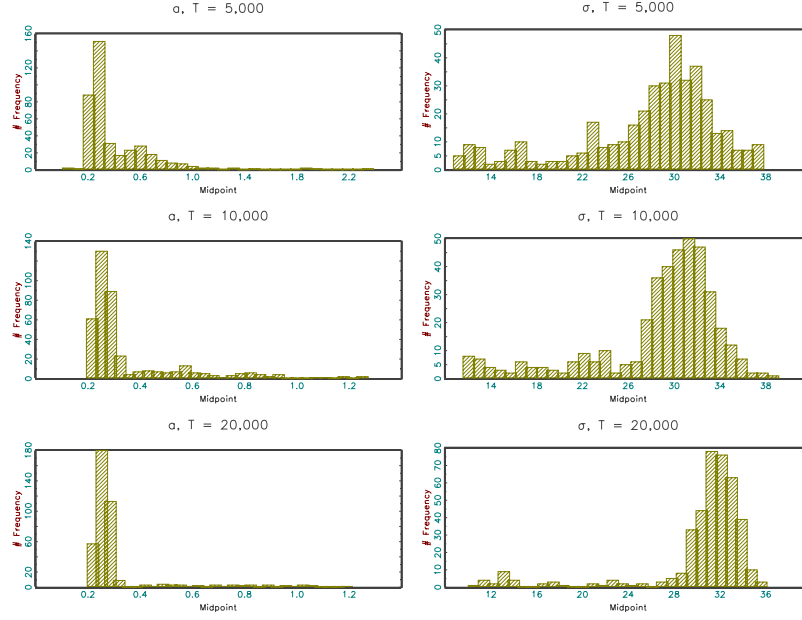
	$a$	$b$	$\sigma_f$
T = 5,000			
	1.000	-0.002	-0.647
	-0.002	1.000	0.417
	-0.647	0.417	1.000
T = 10,000			
	1.000	-0.013	-0.796
	-0.013	1.000	0.363
	-0.796	0.363	1.000
T = 20,000			
	1.000	-0.039	-0.888
	-0.039	1.000	0.320
	-0.888	0.320	1.000

Notes: The table shows the population correlation matrices of the parameter estimates from GMM2 of the 400 replications of each scenario summarized in Table 2.

Table 3 exhibits the population correlation matrices of the 400 Monte Carlo runs of Table 2. The dominating feature is the strong negative correlation between  $a$  and  $\sigma_f$  across all sample sizes. Note that this entry is even increasing with sample size which shows that with generally more precise estimates for higher  $T$  this distortive feature explains even more of the deviation from the ‘true’ values. In contrast, the two parameters of the sentiment process are virtually uncorrelated while  $b$  and  $\sigma_f$  are moderately positively correlated. Presumably, this shows that a higher estimate of the fundamental variance has to be compensated by more pronounced herding to match the volatility clustering measured by the autocovariances of second moments. Fig. 2 shows how this effect leads to a right-skewed distribution of the estimates of  $a$  and a left-skewed distribution for the estimates of  $\sigma_f$ . The distortion due to this near-colinearity trickles away with increasing sample size showing that indeed the parameters are well identified asymptotically.

Can we do any better with our limited information on these key unconditional and conditional statistics? Indeed we can, at least for certain sets of parameters. Table 2 lists as GMM3 a setting with four moments only, where, however, the conditional moments are sums of autocovariances, i.e.  $m'_3 = \sum_{i=1}^{50} r_t^2 r_{t-i}^2$ ,  $m'_4 = \sum_{i=51}^{100} r_t^2 r_{t-i}^2$ , while  $m_1$  and  $m_2$  again denote the second and fourth conditional moments. We see that this change greatly improves the quality of the estimates under the second unimodal set of pa-

Figure 2: Distribution of parameter estimates  $a$  and  $\sigma_f$  .

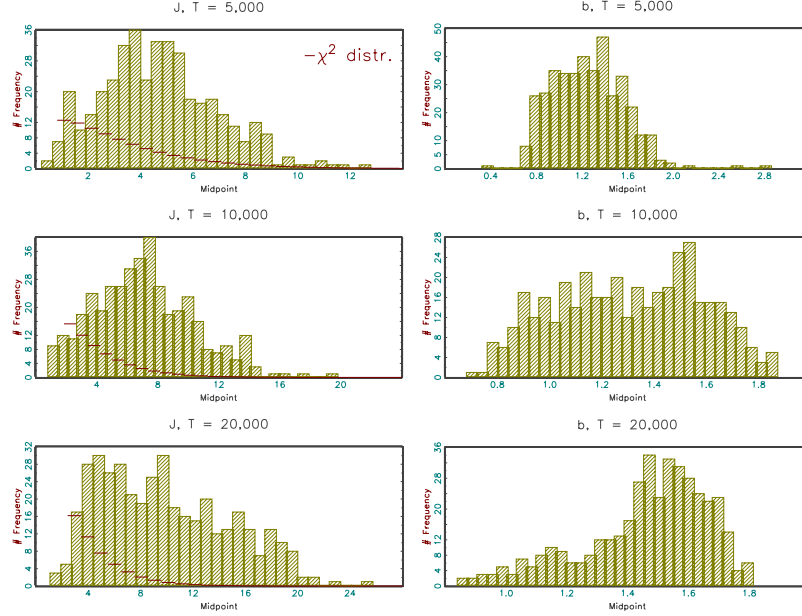


Note: The underlying data set is  $\theta = (0.03, 0.014, 0.0003)$ .

rameters, while it is more or less on par with GMM2 for the first parameter set. FSSEs and RMSEs are typically only half of those found for GMM2 and particularly for  $b$  the distribution of the estimates becomes very narrow. This more precise estimation might even be a bit surprising in view of the observation that  $m'_3$  and  $m'_4$  show almost the same variation like  $m_3$  through  $m_6$  in Table 1. Comparisons of the FSSE and RMSE suggest that at least for parameter set  $\theta'$  (lower part of Table 2) most of the remaining deviation from the ‘true’ value is due to a bias of the estimate, and not due to sampling variability. Of course, graphical representations of the Monte Carlo results from GMM3 look much ‘nicer’ than those of GMM2 depicted in Figs. 2 and 3.

A cumbersome feature one finds in Table 2 is that for parameter vector  $\theta'$  the Monte Carlo results show not the slightest indication of improving from sample sizes  $T = 5000$  to  $T = 10000$  or  $T = 20000$ . So while overall one obtains the most precise estimates in that setting, both these typical sample sizes do seem to be far away from the realm of the asymptotic distribution of the estimates, and more data within this range do not lead to significantly better estimates.

Figure 3: Distribution of parameter  $b$  and  $J$  test statistics



Note: The underlying Parameter set is  $\theta = (0.03, 0.014, 0.0003)$  The  $J$  test statistics is shown together with its asymptotic  $\chi^2$  distribution.

## 5 An Empirical Application

Given the acceptable performance of our modified GMM estimates, we turn to an empirical application. We have selected a number of important financial indices and other assets to explore the performance of our model. Table 4 exhibits estimation results for three stock market indices, three exchange rates and the price of gold. The stock market indices are: the S&P 500, the German DAX and the Japanese Nikkei. For these series and for the gold price, we have used daily data from the start of 1980 until the end of the year 2004, a total of exactly 25 years. The foreign exchange markets are represented by the exchange rates of the Euro against the U.S. dollar (USD/EUR) and against the Swiss franc (CHF/EUR) as well as the Japanese yen against the U.S. dollar (YEN/USD). Data for these series include the periods 01/01/1999 to 12/31/2009 (USD/EUR), 7/15/2003 to 12/31/2009 (CHF/EUR) and 1/1/1986 to 12/31/2004 (YEN/USD). Since due to the introduction of the Euro in 1999, the two exchange rates involving this currency have a shorter history, we have extended the pertinent samples by five years until the end of 2009. Table 4 shows results for both the GMM2 and GMM3 algorithms. Interestingly, we find that the results from both sets of moments are very close to each other with particularly

Table 4: Empirical Parameter Estimates.

Assets	GMM2					GMM3				
	$J$	$a$	$b$	$\sigma_f$	rel.var	$J$	$a$	$b$	$\sigma_f$	rel.var
DAX	3.962 (0.266 )	0.026 (0.009 )	0.315 (0.079 )	8.368 (1.642 )	0.56	5.913 (0.015 )	0.026 (0.020 )	0.315 (0.115 )	8.368 (3.539 )	0.56
S&P500	2.357 (0.502 )	0.016 (0.017 )	0.098 (0.029 )	6.597 (2.600 )	0.53	1.439 (0.230 )	0.016 (0.062 )	0.131 (0.177 )	6.597 (11.302 )	0.54
Nikkei	5.747 (0.125 )	0.024 (0.008 )	0.191 (0.034 )	7.983 (1.110 )	0.55	5.134 (0.023 )	0.046 (0.082 )	0.143 (0.013 )	5.645 (10.160 )	0.78
CHF/EUR	1.544 (0.672 )	0.001 (0.002 )	0.023 (0.009 )	1.701 (1.370 )	0.56	2.541 (0.111 )	0.001 (0.000 )	0.020 (0.007 )	1.701 (0.465 )	0.56
USD/EUR	14.315 (0.003 )	0.029 (0.053 )	0.012 (0.041 )	3.476 (8.618 )	0.62	3.748 (0.053 )	0.012 (0.011 )	0.024 (0.010 )	3.476 (2.117 )	0.67
YEN/USD	12.348 (0.006 )	0.000 (0.003 )	0.108 (0.831 )	6.428 (1.037 )	NA	6.574 (0.010 )	0.000 (0.000 )	0.093 (0.031 )	6.528 (0.238 )	NA
GOLD	4.111 (0.250)	0.021 (0.028)	0.332 (0.329)	5.263 (8.458)	0.73	6.172 (0.013)	0.021 (1.142 )	0.208 (5.049)	5.263 (276.290)	0.72

Notes: The table shows parameter estimates in-sample for selected stock market indices, foreign exchange rates, and the price of gold together with the value of Hansen's test of overidentification restrictions ( $J$ ). Standard errors and the p-value of the test are given in parentheses. The last column shows the relative contribution of the sentiment dynamics to the volatility of returns (measured by its variance).

the parameter  $\sigma_f$  often being undistinguishable. The later finding indicates that the level of fundamental noise is mainly extracted from the second and fourth unconditional moments (since these are represented in both estimations), while conditional moments are more exploited for the determination of  $a$  and  $b$ .

As the sentiment dynamics is responsible for volatility clustering, the estimations indeed sensibly decompose the relationships between parameters and moments. Inspecting the estimation results in Table 4 one observes that the estimates of  $a$ ,  $b$  and  $\sigma_f$  have about the same order of magnitudes, respectively, across the seven assets under consideration. The estimation, thus, typically converges to very similar configurations. In all cases except for the USD/EUR exchange rate under the GMM2 approach we find  $b > a$  indicating a bimodal distribution of the underlying sentiment dynamics.

The other exchange rates (CHF/EUR and YEN/USD) are somewhat unusual in that they are characterized by very small values of  $a$  indicating a very small noise factor in the sentiment process. The  $J$  statistics accepts the

model as a possible data generating process for the selected moments at any traditional level of significance except for the two exchange rates USD/EUR and YEN/USD under GMM2. It is more critical under GMM3 where we find non-rejection at the 5 percent confidence level only for S&P 500, USD/EUR and CHF/EUR. The last column of the Table exhibits the percentage of the return volatility that according to the estimated parameters would be attributed to sentiment changes. As we observe these numbers are relatively uniform varying mostly between 50 and 60 percent with gold as an outlier with a fraction of 73 percent as well as the Nikkei under GMM3 with 0.78 compared to 0.55 with GMM2 (for CHF/EUR this number cannot easily be obtained due to the boundary value  $\hat{a} = 0$ ).

To just estimate parameters of a model does, however, not yet provide much evidence on its closeness to the behavior of certain data. We, therefore, were interested to compare the ALW model to a standard econometric model for financial returns, the GARCH model of Bollerslev (1986). This is certainly the most common of all volatility models proposed in the literature, and it is well-known that it captures to a large extend the time-varying dynamics of financial volatility. While dozens if not hundreds of extensions exist that add features like asymmetry and long-term dependence we stick to the basic GARCH (1,1) model for comparison basically because it has the same number of parameters (3) like the agent-based model (denoted as ALW model in the following), and because we would not expect our simple model to be a coequal competitor for the most refined econometric models. We compare ALW and GARCH (1,1) by an out-of-sample forecasting exercise: To this end, we take the samples defined above as in-samples for parameter estimation, and use both models to generate out-of-sample forecasts of daily volatility for the remaining available data until the end of February, 2015 (i.e. data from either the beginning of 2005 or 2010 to the end of 2/2015). We forecast daily volatility (proxied by squared returns) over horizons of 1,5,10,20,30,40 and 50 days and evaluate the quality of these forecasts via their root mean squared errors (RMSEs). ALW forecasts are constructed as best linear forecasts (cf. Brockwell and Davis, 1991, chap. 5). In this approach, autoregressive forecasts are computed using a vector of optimal weights that can be obtained in an iterative fashion using the autocovariances of the process. Since we are forecasting squared returns as a proxy for volatility, we only need autocovariances of squared returns to implement this approach. These are luckily available in closed form from our moment conditions. For the GARCH model, we use conditional expectations from the ML estimates. Note that this puts our agent-based model on a disadvantage as due to its nonlinear nature, there should, in principle, be better forecasts available than the best *linear* ones. These are, however, not straightforward to compute.

Results are depicted in Table 5 where we show the RMSE for each model

Table 5: Out-of-Sample Forecast Comparison

Data	RMSE ALW	RMSE GARCH	p(DM)	$\lambda$	std. error	RMSE $f^*$
S&P500	0.871	0.764	0.991	-0.119	0.056	0.758
	0.823	0.767	0.984	-0.289	0.089	0.763
	0.857	0.817	0.899	-0.138	0.097	0.815
	0.910	0.883	0.812	-0.007	0.110	0.882
	0.957	0.941	0.828	0.133	0.120	0.939
	0.996	0.985	0.754	0.219	0.126	0.982
	1.020	1.003	0.780	0.120	0.126	1.001
DAX	0.896	0.866	0.824	0.148	0.068	0.844
	0.880	0.862	0.842	0.118	0.116	0.862
	0.903	0.884	0.928	0.031	0.126	0.884
	0.963	0.945	0.890	0.053	0.136	0.946
	0.993	0.963	0.936	-0.188	0.129	0.962
	1.028	0.992	0.952	-0.163	0.118	0.992
	1.050	1.007	0.964	-0.142	0.107	1.007
Nikkei	0.917	0.778	0.961	0.131	0.041	0.784
	0.908	0.880	0.674	0.346	0.053	0.858
	0.939	0.944	0.474	0.489	0.054	0.908
	0.997	1.108	0.158	0.945	0.057	0.996
	1.019	1.161	0.135	1.077	0.058	1.019
	1.036	1.204	0.127	1.194	0.058	1.031
	1.042	1.223	0.119	1.246	0.058	1.035
USD/EUR	0.953	0.935	0.968	0.144	0.152	0.931
	0.945	0.935	0.853	0.189	0.191	0.932
	0.947	0.940	0.736	0.237	0.195	0.936
	0.949	0.944	0.695	0.228	0.208	0.940
	0.951	0.949	0.600	0.311	0.220	0.945
	0.955	0.958	0.390	0.505	0.231	0.952
	0.956	0.958	0.431	0.461	0.241	0.953

Notes: The table show root-mean squared errors of volatility forecasts for selected assets on the base of the estimated ALW and a standard GARCH(1,1) model. p(DM) denotes the probability of the null hypothesis that both models have the same forecast accuracy against the alternative of better forecast accuracy of the ALW based predictions under the common Diebold-Mariano test.  $\lambda$  is the estimate of the encompassing test with its standard error following in the subsequent column.  $f^*$  denotes the relative RMSE of the optimal combination of GARCH and ALW.

Table 5 continued

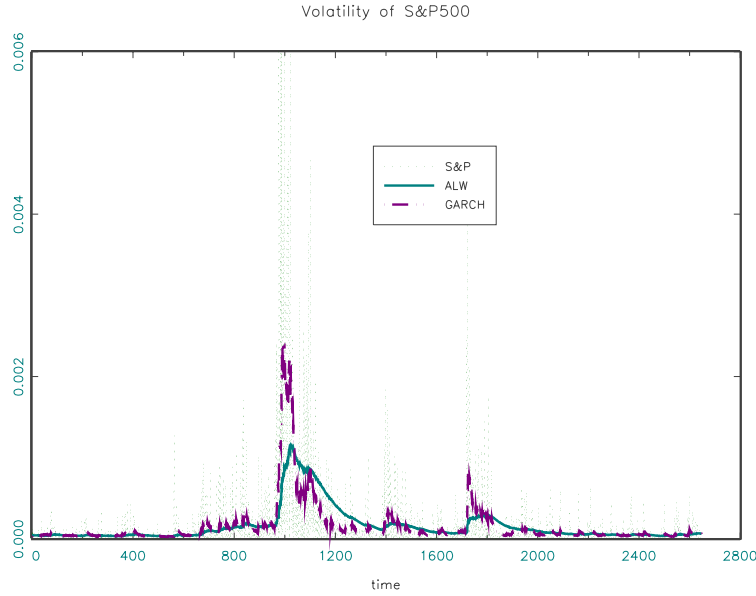
Data	RMSE ALW	RMSE GARCH	p(DM)	$\lambda$	std.error	RMSE $f^*$
YEN/USD						
	0.972	0.919	0.981	-0.138	0.092	0.919
	0.971	0.932	0.970	-0.113	0.105	0.932
	0.974	0.935	0.956	-0.216	0.114	0.934
	0.983	0.963	0.860	0.037	0.130	0.963
	0.989	0.977	0.838	0.183	0.138	0.976
	0.993	0.985	0.813	0.272	0.140	0.983
	0.996	0.989	0.783	0.301	0.139	0.987
CHF/EUR						
	0.995	1.003	0.404	0.513	0.107	0.981
	1.012	1.048	0.080	2.019	0.257	1.001
	1.015	1.046	0.134	2.261	0.304	1.002
	1.017	1.034	0.127	2.723	0.465	1.006
	1.002	1.004	0.259	0.270	0.943	1.002
	1.002	1.002	0.451	-0.268	0.886	1.001
	1.004	1.004	0.562	-0.452	0.788	1.002
Gold						
	0.944	0.926	0.886	0.392	0.135	0.937
	0.940	0.938	0.661	0.558	0.312	0.939
	0.948	0.946	0.669	0.515	0.313	0.947
	0.956	0.956	0.436	0.892	0.312	0.956
	0.973	0.972	0.530	0.760	0.311	0.972
	0.981	0.980	0.625	0.574	0.306	0.981
	0.996	0.994	0.578	0.616	0.300	0.995

Notes: The table show root-mean squared errors of volatility forecasts for selected assets on the base of the estimated ALW and a standard GARCH(1,1) model. p(DM) denotes the probability of the null hypothesis that both models have the same forecast accuracy against the alternative of better forecast accuracy of the ALW based predictions under the common Diebold-Mariano test.  $\lambda$  is the estimate of the encompassing test with its standard error following in the subsequent column.  $f^*$  denotes the relative RMSE of the optimal combination of GARCH and ALW.

and forecast horizon divided by the RMSE of a naive forecast using historical volatility (so that entries smaller than 1 indicate an improvement of the model-based forecast against a static one). Besides these relative RMSEs we also show the probability of the null hypothesis of equal predictive performance against the alternative of better performance of the ALW model under the Diebold-Mariano test statistics (denoted by p (DM)). Note that the complementary probabilities would give the results of the mirror-imaged tests of equal predictive performance against the alternative of better performance of the GARCH model. Overall, we see a surprisingly good performance of the ALW model that typically comes out only slightly worse than

the tailor-made GARCH (1,1) volatility forecast. For the stock market data, GARCH forecasts are only significantly better in 4 out of 21 cases at the 5 percent confidence level, mostly at the smallest horizon (for which GARCH has one particularly tailored parameter). For gold, we find again slightly larger RMSEs for ALW than GARCH but no significant differences at any forecast horizon. For the CHF/EUR, both models have often RMSEs above 1 and, thus, perform worse than historical volatility. For USD/EUR and YEN/USD results resemble those of the stock market: Both forecasts show improvements against historical volatility, but those of the GARCH model are more pronounced and significantly better at short horizons.

Figure 4: Forecasts for S&P 500 Volatility



Note: The figure shows squared returns of S&P 500 over the out-of-sample period together with 1-period forecasts from the GARCH and ALW models.

To give some visual impression of the typical out-of-sample performance of both models, Fig. 4 depicts the one-period ahead predictions for the S&P 500 squared returns together with the empirical series. As one can see, ALW shows more persistence of its predictions than GARCH, while the later shows stronger short-run variation to shocks. Forecasts from both models can differ to quite some extent at times so that it might be promising to explore their complementarities via forecast combinations. Motivated by this observation, we have explored the question whether GARCH and ALW could be combined to provide superior forecasts than those from single



models. The last three columns of Table 4 show the outcome of a standard test of forecast encompassing along the lines of Harvey et al. (1999). The fourth column of Table 4 exhibits the slope parameter  $\lambda$  in the forecast encompassing regression:

$$e_{1,t} = \lambda(e_{1,t} - e_{2,t}) + \epsilon_t \quad (24)$$

with  $e_{1,t}$  and  $e_{2,t}$  the errors of the forecasts from the GARCH and ALW model, respectively. One, thus, tests the null hypothesis  $H_0 : \lambda = 0$  which means that the GARCH model encompasses the ALW model. Rejection of  $H_0$ , i.e. estimates of  $\lambda$  significantly different from zero indicates that the ALW model contributes useful information to the forecast problem in question on top of what the GARCH model already contributes. The fifth column of Table 4 gives the standard error of  $\lambda$  (allowing to assess significance) and the last column shows the performance of optimally combined forecasts from GARCH and ALW ( $f_{1,t}$  and  $f_{2,t}$ ):

$$f_t^* = (1 - \lambda)f_{1,t} + \lambda f_{2,t}. \quad (25)$$

As it turns out, ALW adds value at all lags for the Nikkei, at the higher lags for USD/EUR, and YEN/USD, and the lower ones for the CHF/EUR and again at most lags for gold.

The higher persistence of the ALW forecasts underlines the proximity of this model to processes with long memory. As pointed out by Alfarano and Lux (2007) for a closely related model, the switching between different moods of market participants leads to a phenomenology of the resulting dynamics that mimics long-range dependence despite the Markovian nature of the process.

## 6 Conclusions

In this paper, we have explored the issues evolving around the estimation of agent-based asset pricing models as they have mushroomed over the last two decades. While we have concentrated on the particular example of the model by Alfarano et al. (2008), we believe that some of our findings would also be relevant for other models. Since we were able to derive analytical moment conditions, we believe that important features showed up more clearly than would have been under the additional complication of a simulated moment estimation.<sup>2</sup> First, we basically have available for the estimation of a univariate asset pricing model the moments that characterize the stylized facts of the empirical data: fat tails and volatility clustering. Since these features can be succinctly summarized by a small number of moments and additional

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<sup>2</sup>Chen and Lux (2015) compare SMM with GMM for the same model, and find that SMM is surprisingly inferior.

ones would presumably only add largely redundant information, not too many parameters could be identified in this way. Indeed, we find that even with a maximally stripped down model with only three parameters, patterns of high autocorrelation of parameter estimates and weak identification did show up.

However, at the same time, a judicious choice of moments and a careful design of the estimation algorithm are both able to increase the precision of the estimates. Applying our approach to a broad selection of asset markets we find structurally very similar results indicating a strong element of sentiment dynamics in the asset pricing process that, according to the model's decomposition, accounts for more than half of the overall variation of returns. We also used (presumably for the first time) an estimated behavioral model for forecasting of volatility. While we should certainly not expect too much success of a simple model like the present one in such an exercise, we find it interesting that this model indeed gets close in its performance to the seminal GARCH model, and often adds exploitable information on top of the one used by the GARCH model.

We believe that the present approach should be interesting to pursue further. In particular, it should be explored whether different specifications of agent-based models exhibit similar behavior in their empirical application, or whether we could discriminate between them on the base of their capacity to match moments and forecast volatility. Indeed, different specifications need not be equivalent and could, thus, possibly provide combined forecasts that improve upon single ones.

## Appendix

### A1. Derivation of moments in closed form

Unconditional even moments,  $\mathbb{E}[x_t^n]$ , for  $n \geq 2$  can be obtained from the compact formula available in Alfarano et al. (2008, p. 114)

$$\mathbb{E}[(1 - x^2)^k] = 2^{2k} \frac{\Gamma(2\epsilon)}{\Gamma(2\epsilon + 2k)} \left( \frac{\Gamma(\epsilon + k)}{\Gamma(\epsilon)} \right)^2 \quad (26)$$

by a suitable change of the parameter  $k$ . Here we preserve the original notation of Alfarano et al. (2008) which should not be mixed with our usage of  $k$  from Section 3. Recall,  $\epsilon = a/b$ . Thus, for  $k = 1$  we derive

$$\mathbb{E}[x_t^2] = \frac{1}{2\epsilon + 1} = \frac{b}{b + 2a}$$

and for  $k = 2$  we obtain

$$\mathbb{E}[x_t^4] = \frac{3}{(2\epsilon + 3)(2\epsilon + 1)} = \frac{3b}{2a + 3b} \mathbb{E}[x_t^2].$$

Additional details are available in Alfarano et al. (2008, p. 130).

In order to compute the moments  $\mathbb{E}[z_t^2]$ ,  $\mathbb{E}[z_t^4]$ , and  $\mathbb{E}[z_t^2 z_{t-h}^2]$ , we need to spend some more efforts. Recall,  $z_t = x_{t+1} - x_t$ . It is possible to obtain sensible approximations of these moments by basing calculations on the Eulerian approximation of the stochastic differential equation of  $\tilde{x}$  mentioned in Section 2:

$$d\tilde{x}_t = -2a\tilde{x}_t dt + \sqrt{2b(1 - \tilde{x}_t^2)} dB_t, \quad (27)$$

where we have suppressed the entry  $4a/N$  in the diffusion term as it will be negligible for sufficiently large  $N$ . The Eulerian approximation of (27) reads

$$\tilde{x}_{t+1} - \tilde{x}_t = -2a\tilde{x}_t \Delta t + \sqrt{2b(1 - \tilde{x}_t^2)} \Delta t \eta_t, \quad (28)$$

where  $\eta_t$  is a standard normal variate independent of  $\tilde{x}_t$  and in this paper we fix  $\Delta t \equiv 1$ .

Alfarano et al. (2008) make use of this approach and compute

$$\mathbb{E}[z_t^2] \simeq 2b(1 - \mathbb{E}[x_t^2]) \quad (29)$$

$$\mathbb{E}[z_t^4] \simeq (2b)^2 \mathbb{E}[(1 - 2x_t^2 + x_t^4)\eta_t^4] \quad (30)$$

$$= 3(2b)^2 \frac{4\epsilon(\epsilon + 1)}{(2\epsilon + 1)(2\epsilon + 3)} \quad (31)$$

$$\mathbb{E}[z_t^2 z_{t-h}^2] \simeq 4b^2(1 - 2(2a + b))^{h-1} (\mathbb{E}[x_t^4] - \mathbb{E}[x_t^2]^2). \quad (32)$$

Using these results it is then possible to compute the (approximate) auto-correlation function of the  $z_t^2$ -process at lag  $h$  as follows

$$C_{z^2}(h) = \frac{1}{4\epsilon^2 + 6\epsilon + 3} \exp(-2bh(2\epsilon + 1)), \quad (33)$$

cf. Alfarano et al. (2008, pp. 117, 131).

The second possibility to obtain the above moments is based directly on the Jacobi diffusion and its analytical tractability:

$$dx_t = -2ax_t dt + \sqrt{2b(1 - x_t^2)} dB_t. \quad (34)$$

The idea is to compute first the conditional expectations  $\mathbb{E}[x_{t+1}^n | x_t^n]$  for  $n = 1, 2, 3, 4$  in closed form by taking mathematical expectations on the both sides of the suitably Itô-transformed variables. In particular, first the Itô rule is applied to derive  $dx_t^n$  and then expectations are taken. Solutions to these conditional expectations are available in closed form, however, they are often very lengthy and for brevity are avoided in our presentation. Details are available upon request. One source of lengthy expressions is the dependence of the higher order conditional expectations on the solutions of the lower order ones.

After obtaining solutions to the mentioned conditional expectations, the next step is to consider the polynomial expansions of  $z_t^2$ ,  $z_t^4$ , and  $z_t^2 z_{t-h}^2$  in terms of (powers) of  $x_t$  and  $x_{t+1}$  and take mathematical expectations of the components. For instance, consider  $z_t^2 = x_{t+1}^2 - 2x_{t+1}x_t + x_t^2$  and take expectations from both sides. Note that the law of the iterated expectations applies to the middle term where we make use of the derived analytical conditional expectations. After very tedious calculations (details available upon request) we obtain the following results which we state in terms of more compact parameters  $\alpha \equiv 2a$  and  $\beta \equiv 2b$  and  $\Delta t \equiv 1$ :

$$\mathbb{E}[z_t^2] = 2\mathbb{E}[x_t^2] (1 - \exp[-\alpha\Delta t]) \quad (35)$$

$$\begin{aligned} \mathbb{E}[z_t^4] = 8\mathbb{E}[x_t^4] & \left( \frac{\alpha + \beta}{2\alpha + \beta} - \exp[-\alpha\Delta t] \right. \\ & \left. + \frac{\alpha}{2\alpha + \beta} \exp[-(2\alpha + \beta)\Delta t] \right) \end{aligned} \quad (36)$$

$$\mathbb{E}[z_t z_{t-j\Delta t}] = 2\mathbb{E}[x_t^2] (1 - \cosh[\alpha\Delta t]) \exp[-\alpha\Delta t j] \quad (37)$$

$$\mathbb{E}[z_t^2 z_{t-j\Delta t}^2] = c_1 + c_2 - 2c_3 \quad (38)$$

where

$$c_1 = x_{11} - 2x_{12} + x_{13} \quad (39)$$

$$c_2 = x_{21} - 2x_{22} + x_{23} \quad (40)$$

$$c_3 = x_{31} - 2x_{32} + x_{33} \quad (41)$$

and

$$\begin{aligned} x_{11} = & \exp(-(2\alpha + \beta)(j+1)\Delta t) \mathbb{E}[x_t^4] \\ & - \beta / [-(2\alpha + \beta)] (1 - \exp(-(2\alpha + \beta)(j+1)\Delta t)) \mathbb{E}[x_t^2] \end{aligned} \quad (42)$$

$$\begin{aligned} x_{12} = & \exp(-\alpha\Delta t) \exp(-(2\alpha + \beta)j\Delta t) \mathbb{E}[x_t^4] \\ & - \exp(-\alpha\Delta t) \beta / [-(2\alpha + \beta)] (1 - \exp(-(2\alpha + \beta)j\Delta t)) \mathbb{E}[x_t^2] \end{aligned} \quad (43)$$

$$\begin{aligned} x_{13} = & \exp(-(2\alpha + \beta)j\Delta t) \mathbb{E}[x_t^4] \\ & - \beta / [-(2\alpha + \beta)] (1 - \exp(-(2\alpha + \beta)j\Delta t)) \mathbb{E}[x_t^2] \end{aligned} \quad (44)$$

$$x_{21} = \exp(-(2\alpha + \beta)j\Delta t)\mathbb{E}[x_t^4] - \beta/[-(2\alpha + \beta)](1 - \exp(-(2\alpha + \beta)j\Delta t))\mathbb{E}[x_t^2] \quad (45)$$

$$x_{22} = \exp(-\alpha\Delta t)\exp(-(2\alpha + \beta)(j-1)\Delta t)\mathbb{E}[x_t^4] - \exp(-\alpha\Delta t)\beta/[-(2\alpha + \beta)](1 - \exp(-(2\alpha + \beta)(j-1)\Delta t))\mathbb{E}[x_t^2] \quad (46)$$

$$x_{23} = \exp(-(2\alpha + \beta)(j-1)\Delta t)\mathbb{E}[x_t^4] - \beta/[-(2\alpha + \beta)](1 - \exp(-(2\alpha + \beta)(j-1)\Delta t))\mathbb{E}[x_t^2] \quad (47)$$

$$x_{31} = \exp(-(2\alpha + \beta)j\Delta t)\exp(-3(\alpha + \beta)\Delta t)\mathbb{E}[x_t^4] + 3\beta/(-\alpha + 3(\alpha + \beta))\exp(-(2\alpha + \beta)j\Delta t)(\exp(-\alpha\Delta t) - \exp(-3(\alpha + \beta)\Delta t))\mathbb{E}[x_t^2] - \beta/[-(2\alpha + \beta)](1 - \exp(-(2\alpha + \beta)j\Delta t))\exp(-\alpha\Delta t)\mathbb{E}[x_t^2] \quad (48)$$

$$x_{33} = \exp(-(2\alpha + \beta)(j-1)\Delta t)\exp(-3(\alpha + \beta)\Delta t)\mathbb{E}[x_t^4] + 3\beta/(-\alpha + 3(\alpha + \beta))\exp(-(2\alpha + \beta)(j-1)\Delta t)(\exp(-\alpha\Delta t) - \exp(-3(\alpha + \beta)\Delta t))\mathbb{E}[x_t^2] - \beta/[-(2\alpha + \beta)](1 - \exp(-(2\alpha + \beta)(j-1)\Delta t))\exp(-\alpha\Delta t)\mathbb{E}[x_t^2] \quad (49)$$

$$x_{32} = \exp(-\alpha\Delta t)x_{33}. \quad (50)$$

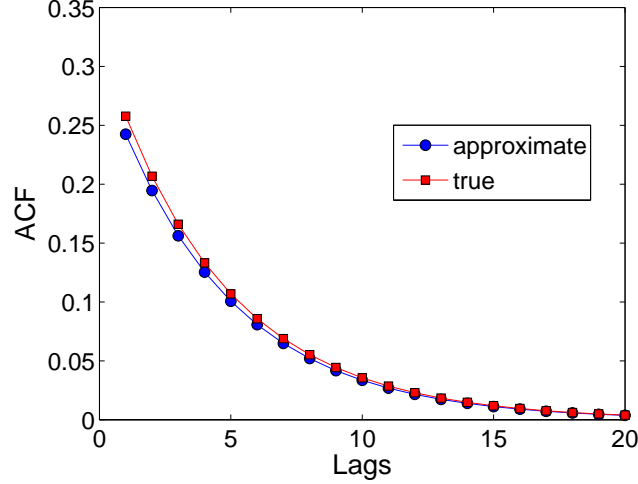
It is easy to see where the lengthy expressions come from. Simply, there are many polynomial components, unconditional expectations of which are based on very lengthy combinations of both lower-level conditional and unconditional expectations.

Using these results it is then possible, for instance, to compute the autocorrelation function of the  $z_t^2$ -process at lag  $h$  with  $\Delta t \equiv 1$  as follows:

$$\rho_{z^2}(h) = \frac{\mathbb{E}[z_t^2 z_{t-h}^2] - \mathbb{E}[z_t^2]^2}{\mathbb{E}[z_t^4] - \mathbb{E}[z_t^2]^2}. \quad (51)$$

Thus, we obtain all the necessary building blocks for deriving the expressions for  $\mathbb{E}[r_t^2]$ ,  $\mathbb{E}[r_t^4]$ , and  $\mathbb{E}[r_t^2 r_{t-h}^2]$  in terms of  $\mathbb{E}[x_t^2]$ ,  $\mathbb{E}[x_t^4]$ ,  $\mathbb{E}[z_t^2]$ ,  $\mathbb{E}[z_t^4]$ , and  $\mathbb{E}[z_t^2 z_{t-h}^2]$ . These moments would now be the exact counterpart of the ones obtained by Alfarano et al. (2008) via the Euler equation if (27) where the exact law of motion of the sentiment process. They are still approximate in the sense that the term  $4a/N$  has been neglected in the diffusion function and that eq. (27) is a diffusion approximation only to the ‘true’ underlying microscopic dynamics of agents’ sentiment formation.

Figure 5: ‘True’ and approximate autocorrelation functions of the  $z_t^2$ -process.



Notes: The expressions for the ‘true’,  $\rho_{z^2}(j)$ , and approximate,  $C_{z^2}(j)$ , autocorrelation functions can be found in equations (51) and (33), respectively. The parameters are set as  $\theta^* = (a, b) = (0.005, 0.1)$ . ‘True’ here means exact moments based on the Jacobi diffusion (27) which still is an approximation to the underlying agent-based process.

## A2. A comparative study of the approximating moments

From the perspective of the true agent-based model of this paper the derived moments following the two approaches are still approximations. However, among those two variants of moments the ones based on the Eulerian approximation can themselves be viewed as approximations to the ‘true’ moments of the Jacobi diffusion. To show their relative performance, we fix the parameters as in one calibration study of Alfarano et al. (2008, p. 119),  $\theta^* = (a, b) = (0.005, 0.1)$ , and compute the autocorrelations of the  $z_t^2$ -process at various lags for the two expressions in (33) and (51).

The results are visualized for 20 lags in Figure 5. Relative percentage deviations of the approximate ACF values from the ‘true’ ones for the first 10 lags are assessed in Table 6. Interestingly, visually both ACFs are in a very good agreement with each other, however, we observe virtually constant relative undervaluation at about six percent of the approximate ACF to the ‘true’ one. Nevertheless, the quality that is clearly obtained by the Euler approximation of the highly nonlinear Jacobi diffusion for the ACF is striking.

Another robust finding of our investigation is that approximate moments tend to consistently overestimate kurtosis, however, this overvaluation is

Table 6: True and approximate autocorrelation functions of the  $z_t^2$ -process.

Lag $h$	$\rho_{z^2}(h)$	$C_{z^2}(h)$	Relative % Difference
1	0.2576	0.2425	-5.8970
2	0.2068	0.1946	-5.8970
3	0.1659	0.1561	-5.8970
4	0.1332	0.1253	-5.8970
5	0.1069	0.1006	-5.8970
6	0.0858	0.0807	-5.8970
7	0.0688	0.0648	-5.8970
8	0.0552	0.0520	-5.8970
9	0.0443	0.0417	-5.8970
10	0.0356	0.0335	-5.8970

Notes: The expressions for the ‘true’,  $\rho_{z^2}(j)$ , and approximate,  $C_{z^2}(j)$ , autocorrelation functions can be found in equations (51) and (33), respectively. The parameters are set as  $\theta^* = (a, b) = (0.005, 0.1)$ . ‘True’ here means exact moments based on the Jacobi diffusion (27) which still is an approximation to the underlying agent-based process.

very small (almost zero) and, therefore, can be tolerated.

To summarize, we observe that both the simple Euler approximation and the much more involved derivation of exact moments from the diffusion (27) lead to almost the same numerical values. Small deviations are present but seem tolerable. Thus, the choice made in favor of the moment with a simpler expression in our empirical application seems to be justifiable.

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