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Working Paper

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Working Paper, No. 1208

Provided in Cooperation with:

Koç University - TÜSİAD Economic Research Forum, Istanbul

Suggested Citation: Gerratana, Emanuele; Koçkesen, Levent (2012) : Renegotiation-Proof Third-Party Contracts under Asymmetric Information, Working Paper, No. 1208, Koç University-TÜSİAD Economic Research Forum (ERF), Istanbul

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KOÇ UNIVERSITY-TÜSİAD ECONOMIC RESEARCH FORUM
WORKING PAPER SERIES

**RENEGOTIATION-PROOF THIRD-PARTY
CONTRACTS UNDER ASYMMETRIC
INFORMATION**

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Working Paper 1208
April 2012

Renegotiation-Proof Third-Party Contracts under Asymmetric Information^{*}

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April 2012

Abstract

This paper characterizes the equilibrium outcomes of two-stage games in which the second mover has private information and can sign renegotiable contracts with a neutral third-party. Our aim is to understand whether renegotiation-proof third-party contracts can confer a strategic advantage on the second mover. We first analyze *non-renegotiable* contracts and show that a “folk theorem” holds: Any outcome in which the second mover best responds to the first mover’s action and the first mover obtains a payoff at least as large as his “individually rational payoff” can be supported. *Renegotiation-proofness* imposes some restrictions, which is most transparent in games with externalities, i.e., games in which the first mover’s payoff increases (or decreases) in the second mover’s action. In such games, a similar folk theorem holds with renegotiation-proof contracts as well, but the first mover’s individually rational payoff is in general higher.

JEL Classification: C72, D80, L13.

Keywords: Third-Party Contracts, Strategic Delegation, Renegotiation, Asymmetric Information, Renegotiation-Proofness, Durability.

^{*}This research has been supported by TÜBİTAK Grant No. 106K317.

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1 Introduction

Could an incumbent firm deter entry by contracting with third parties, such as a bank or a labor union? Could a buyer get a better price from a seller if she is under a credit contract with a bank? More generally can contracts with third parties change the outcome of a game to the advantage of the contracting player? When contracts are observable and non-renegotiable, the answer to this question is in general yes.¹ In fact, there are several “folk theorem” type results for different classes of games with *observable and non-renegotiable* third-party contracts.² The effects of *unobservable and non-renegotiable* third-party contracts are also well-understood: Nash equilibrium outcomes of a game with and without third-party contracts are identical (Katz (1991)). In fact, all (and only) Nash equilibrium outcomes of the original game can be supported as a sequential equilibrium outcome of the game with unobservable and non-renegotiable contracts (Koçkesen and Ok (2004) and Koçkesen (2007)). In this paper we seek an answer to this question for *unobservable and renegotiable* contracts.³

More precisely we consider the effects of renegotiation-proof third-party contracts in two-player two-stage games where the second mover (player 2) has some payoff relevant private information. In what we call the *original game*, Nature moves first and determines the state of the world θ . After that, player 1 chooses an action a_1 without observing θ . Player 2 observes both θ and a_1 , chooses a_2 , and the game ends. Many models in economics such as the entry game, the Stackelberg game, and monopolistic screening belong to this class of games.

In the *game with contracts* we let player 2 sign a contract with a neutral third-party before the original game starts. A contract specifies transfers between player 2 and the third-party as a function of the contractible outcomes, which we assume to be the action choices of the two players, (a_1, a_2) . The underlying and crucial assumption is that the private information of player 2 is not observable by any other player, including the third-party, and thus non-contractible. We also assume that the contract is never observed by player 1 and analyze the perfect Bayesian equilibria of the resulting game with contracts.

Since contracts cannot depend on θ , in order to get a handle on the type of strategies that are incentive compatible, we assume that player 2’s payoff function exhibits increasing differences in (θ, a_2) . We first characterize outcomes that can be supported with non-renegotiable contracts, both as a logical step in the analysis and to better isolate the effects of renegotiation (Section 3.1). We show that any Bayesian Nash equilibrium of the original game in which player 2’s strategy is increasing can be supported with third-party contracts. In fact, we prove a folk theorem type result: any outcome $(a_1^*, a_2^*(\theta))$ of the original game in which $a_2^*(\theta)$ is a best response to a_1^* for each θ and player 1’s payoff is at least as large as his “individually rational” payoff, can be supported. Definition of individually rational payoff is different from the standard one in that player 2, in minimizing player 1’s payoff, is restricted to using increasing strategies.

We next consider renegotiable contracts: After player 1 moves, player 2 can make a renegotiation

¹See, among many others, Vickers (1985), Fershtman and Judd (1987), Sklivas (1987), Koçkesen et.al. (2000), Brander and Lewis (1986), Bolton and Scharfstein (1990), Snyder (1996), Spencer and Brander (1983), Brander and Spencer (1985), and Eaton and Grossman (1986).

²See Fershtman, Judd, and Kalai (1991), Polo and Tedeschi (2000), and Katz (2006).

³Prat and Rustichini (2003) and Jackson and Wilkie (2005) analyze related models in which players can write action contingent contracts before the game is played. Unlike the current paper, in these papers contractual relationships are not exclusive and the focus is on the efficiency properties of the equilibrium set. Also related is Bhaskar (2009), in which players need to pay a price to a supplier in order to play certain actions that are controlled by this supplier.

offer to the third-party, who knows a_1 , but not θ , and can either accept the offer or reject it. We define renegotiation-proof equilibrium as a perfect Bayesian equilibrium in which the equilibrium contract is not renegotiated after any (θ, a_1) and characterize the renegotiation-proof contracts and strategies (Section 3.2).⁴

We obtain a complete characterization for games with externalities, i.e., games in which player 1's payoff is increasing or decreasing in Player 2's action (Section 4). We show that, in this class of games, a similar folk theorem holds with renegotiation-proof contracts as well. The difference from the result for non-renegotiable contracts is in the definition of individually rational payoff: With renegotiation-proof contracts the strategy of the highest type of player 2 must be a best response to each a_1 , whereas no such restriction exists for non-renegotiable contracts.⁵

The class of games with externalities is large and contains many economic models. In Section 5 we apply our result to some that we find interesting. The canonical example, of course, is the Stackelberg competition. We show that in this game, the follower firm indeed benefits from renegotiation-proof third-party contracts. This game can also be construed as an entry deterrence game, in which case we show that entry can always be deterred with non-renegotiable contracts but only under certain conditions with renegotiation-proof contracts, i.e., renegotiation has a real bite in these games. We also analyze a standard monopolistic screening game in which the seller offers a menu of quality-price pairs and a privately informed buyer chooses one item in the menu (or none). We show that the buyer benefits from renegotiation-proof contracts but not as much as she would from non-renegotiable contracts.

The closest paper to ours is Dewatripont (1988), which analyzes an entry-deterrence game in which the incumbent signs a contract with a labor union before the game begins. A potential entrant observes the contract and then decides whether to enter or not. Renegotiation takes place after the entry decision is made, during which the union offers a new contract to the incumbent. The crucial assumption is that the incumbent has some payoff relevant private information during the renegotiation process. Dewatripont (1988) shows that commitment effects exist in such a model and may deter entry when contracts are publicly observable. We show that this commitment effect exists for unobservable contracts as well. Also, we analyze arbitrary two-stage games and hence can gauge the effects of third-party contracts in other interesting settings, for example in oligopoly models with price competition or in monopolistic screening. Lastly, in our renegotiation protocol, the informed party makes the new contract offer, whereas in Dewatripont's, it is the uninformed party who makes the offer. This turns out to make a difference as we discuss in some detail in Section 6.2.

In a related paper, Gerrata and Koçkesen (2011) also study the effects of renegotiation-proof third-party contracts in two-stage games. However, that paper assumes that the original game is with perfect information whereas the current one assumes it is a game with incomplete information. This difference is crucial, because in the current paper renegotiation takes place under asymmetric information about the type of player 2, whereas in Gerrata and Koçkesen (2011) the asymmetry stems from the inability of the third-party to observe player 1's move. Although, some aspects of the

⁴Our assumption that the third-party cannot observe θ during renegotiation is crucial. Otherwise, the result is trivial: One can only support the perfect Bayesian equilibria of the original game. This is because, if both a_1 and θ are common knowledge, then player 2 and the third-party would renegotiate away any strategy of player 2 that does not maximize the joint surplus, i.e., player 2's payoff in the original game.

⁵This is true when player 1's payoff is increasing in player 2's action. If his payoff is decreasing, then player 2's strategy must be a best response for the lowest type.

analyses of these two models are similar and use similar tools, namely theorems of the alternative, the games to which they can be applied are completely different. This becomes most transparent in Section 4 where we apply our results to games with externalities. Obtaining similarly sharp results in Gerratana and Koçkesen (2011) has been possible in a different class of games and the results are quite different.

Another related paper is Caillaud et al. (1995), which analyzes a game between two principal-agent hierarchies. In the first stage of their game each principal decides whether to publicly offer a contract to the agent; in the second stage each principal offers a secret contract to the agent, which, if accepted, overwrites the public contract that might have been offered in stage 1; in the third stage each agent receives payoff relevant information, decides whether to quit, and if he does not quit, he plays a normal form game with the other agent. Their main question is whether there exist equilibria of this game in which the principals choose not to offer a public contract in stage 1. If the answer to this question is no, then the interpretation is that contracts have commitment value. They show that contracts have commitment value if the market game stage is of Cournot type, but not if it is of Bertrand type. The crucial difference between Caillaud et al. (1995) and our model is that they allow initial public commitment to a contract and allow renegotiation only before the game begins, whereas in our setting there is no possibility of public commitment and renegotiation can happen both before and after the game begins.⁶

Finally, Bensaid and Gary-Bobo (1993) analyze a model in which the original game is a two-stage game and the initial contract can be renegotiated after player 1 chooses an action. However, in their model utility is not transferable between player 2 and the third-party. They show that, in a certain class of games, contracts with third parties have a commitment effect, even when they are renegotiable. We analyze a model with transferable utility and show that commitment effects still exist.

2 The Model

Our aim is to understand the effects of renegotiation-proof third-party contracts in extensive form games. We will do this in a particularly simple environment: two-stage games with private information, which we call the *original game*. We then allow one of the players to sign a contract with a third-party before the game begins and call this new game the *game with third-party contracts*. The contracts specify a transfer between the player and the third-party as a function of the contractible outcomes of the original game. The crucial aspect of our model is the presence of asymmetric information between this player and the third-party during the renegotiation phase.

More precisely, we define the *original game*, denoted G , as follows: Nature chooses $\theta \in \Theta$ according to probability distribution $p \in \Delta(\Theta)$. After the move of Nature, player 1, without observing θ , chooses $a_1 \in A_1$. Lastly, player 2 observes (θ, a_1) and chooses $a_2 \in A_2$. We assume that A_1 , A_2 , and Θ are finite and let $p(\theta)$ denote the probability of Nature choosing θ . Payoff function of player $i \in \{1, 2\}$ is given by $u_i : A \times \Theta \rightarrow \mathbb{R}$, where $A = A_1 \times A_2$.

We will first analyze the *game with non-renegotiable third-party contracts*, denoted $\Gamma(G)$, which is a three player extensive form game described by the following sequence of events:

Stage I. Player 2 offers a contract $f : A \rightarrow \mathbb{R}$ to a third-party.

⁶This is also related to the fact that they assume the agents play a simultaneous move game whereas we focus on sequential move games.

Stage II. The third-party accepts (denoted y) or rejects (denoted n) the contract.

1. In case of rejection the game ends, the third-party receives a fixed payoff of $\delta \in \mathbb{R}$, and player 1 and 2 receive $-\infty$.
2. In case of acceptance, the game goes to Stage III.

Stage III. Nature chooses $\theta \in \Theta$ according to p .

Stage IV. Player 1 chooses $a_1 \in A_1$ (without observing the contract or θ).

Stage V. Player 2 observes (θ, a_1) .

Stage VI. Player 2 chooses $a_2 \in A_2$.

It is easy to see that the contract offer is accepted in all equilibria, since offering a contract that is rejected yields player 2 a very small payoff. Therefore, we can omit the third-party's acceptance decision from histories and represent an outcome of the game as (f, θ, a_1, a_2) . The payoff functions in $\Gamma(G)$ are given by

$$\begin{aligned} \nu_1(f, a_1, a_2, \theta) &= u_1(a_1, a_2, \theta) \\ \nu_2(f, a_1, a_2, \theta) &= u_2(a_1, a_2, \theta) - f(a_1, a_2) \\ \nu_3(f, a_1, a_2, \theta) &= f(a_1, a_2) \end{aligned}$$

where ν_3 is the payoff function of the third-party.

The game is with renegotiable contracts if the contracting parties can renegotiate the contract after Stage V and before Stage VI. The following sequence of events describe the renegotiation process after any history (f, θ, a_1) .

Stage V(i). Player 2 either offers a new contract $g : A \rightarrow \mathbb{R}$ to the third-party or chooses an action a_2 .

In the latter case the game ends and the outcome is (f, θ, a_1, a_2) .

Stage V(ii). If player 2 offers a new contract, the third-party observes a_1 but not θ , and either accepts (denoted y) or rejects (denoted n) the offer.

If the third-party rejects the renegotiation offer g , then player 2 chooses $a_2 \in A_2$ and the outcome is payoff equivalent to (f, θ, a_1, a_2) . If he accepts, then player 2 chooses $a_2 \in A_2$ and the outcome is payoff equivalent to (g, θ, a_1, a_2) . This completes the description of the *game with renegotiable contracts*, which we denote as $\Gamma_R(G)$.

A *behavior strategy* for player $i \in \{1, 2, 3\}$ is defined as a set of probability measures $\beta_i \equiv \{\beta_i[I] : I \in \mathcal{I}_i\}$, where \mathcal{I}_i is the set of information sets of player i and $\beta_i[I]$ is defined on the set of actions available at information set I . One may write $\beta_i[h]$ for $\beta_i[I]$ for any history $h \in I$. By a *system of beliefs*, we mean a set $\mu \equiv \{\mu[I] : I \in \mathcal{I}_i \text{ for some } i\}$, where $\mu[I]$ is a probability measure on I . A pair (β, μ) is called an *assessment*. An assessment (β, μ) is said to be a *perfect Bayesian equilibrium* (PBE) if (1) each player's strategy is optimal at every information set given her beliefs and the other players' strategies; and (2) beliefs at every information set are consistent with observed histories and strategies.⁷

⁷See Fudenberg and Tirole (1991) for a precise definition of perfect Bayesian equilibrium.

We will limit our analysis to pure behavior strategies, and hence a strategy profile of the original game G is given by $(b_1, b_2) \in A_1 \times A_2^{A_1 \times \Theta}$. For any behavior strategy profile (b_1, b_2) in G , we say that an assessment (β, μ) in $\Gamma(G)$ (or $\Gamma_R(G)$) *induces* (b_1, b_2) if in $\Gamma(G)$ ($\Gamma_R(G)$) player 1 plays according to b_1 and, after the equilibrium contract, player 2 plays according to b_2 .⁸

Our ultimate aim is to characterize renegotiation-proof equilibria, in which the equilibrium contract is not renegotiated after any history.⁹ More precisely,

Definition 1 (Renegotiation-Proof Equilibrium). A perfect Bayesian equilibrium (β^*, μ^*) of $\Gamma_R(G)$ is *renegotiation-proof* if the equilibrium contract is not renegotiated after any $a_1 \in A_1$ and $\theta \in \Theta$.

We say that a strategy profile (b_1, b_2) of the original game G can be *supported with non-renegotiable contracts* if there exists a perfect Bayesian equilibrium of $\Gamma(G)$ that induces (b_1, b_2) . Similarly, a strategy profile (b_1, b_2) of the original game G can be *supported with renegotiation-proof contracts* if there exists a renegotiation-proof perfect Bayesian equilibrium of $\Gamma_R(G)$ that induces (b_1, b_2) .

An easy backward induction argument shows that there exists a pure strategy perfect Bayesian equilibrium of the original game G . It is also not difficult to see that there exists a pure strategy renegotiation-proof perfect Bayesian equilibrium of $\Gamma_R(G)$. Indeed, let (b_1, b_2) be a pure strategy perfect Bayesian equilibrium of the original game G and consider the following assessment (described only partially): Player 2 offers the constant contract $f(a_1, a_2) = \delta$ for all (a_1, a_2) , player 1 believes that this is the contract and plays according to b_1 , player 2 plays according to b_2 after (f, a_1, θ) for all (a_1, θ) . This, of course, induces the perfect Bayesian equilibrium of the original game. The interesting question is whether there are other outcomes of the original game that can be supported with renegotiation-proof contracts. The next section shows that the answer, in general, is yes.

3 Main Results

There may be legal or technological constraints that might render contracts non-renegotiable and therefore outcomes that can be supported by non-renegotiable contracts are of interest on their own. Furthermore, understanding non-renegotiable contracts will help place our results within the literature and allow us to isolate the effects of renegotiation. Therefore, in this section we will first analyze the game with unobservable and non-renegotiable third-party contracts before moving on to the analysis of renegotiable contracts.

3.1 Non-Renegotiable Contracts

Let G be an arbitrary original game and for any behavioral strategy profile $(b_1, b_2) \in A_1 \times A_2^{A_1 \times \Theta}$ of G , define the expected payoff of player $i = 1, 2$ as

$$U_i(b_1, b_2) = \sum_{\theta \in \Theta} p(\theta) u_i(b_1, b_2(b_1, \theta), \theta).$$

⁸Note that in $\Gamma_R(G)$, player 2 may choose an action $a_2 \in A_2$ either without renegotiating the initial contract or after attempting renegotiation.

⁹We follow the previous literature in our definition of renegotiation-proof equilibrium. See, for example, Maskin and Tirole (1992) and Beaudry and Poitevin (1995).

Also define the best response correspondences as

$$BR_1(b_2) = \operatorname{argmax}_{a_1 \in A_1} U_1(a_1, b_2) \text{ for all } b_2 \in A_2^{A_1 \times \Theta}$$

$$BR_2(a_1, \theta) = \operatorname{argmax}_{a_2 \in A_2} u_2(a_1, a_2, \theta) \text{ for all } (a_1, \theta) \in A_1 \times \Theta.$$

We say that a strategy profile (b_1^*, b_2^*) is a *Bayesian Nash equilibrium* of G if $b_1^* \in BR_1(b_2^*)$ and $b_2^*(b_1^*, \theta) \in BR_2(b_1^*, \theta)$ for all θ . The difference between a perfect Bayesian equilibrium and a Bayesian Nash equilibrium, of course, is that the former requires player 2 to best respond to every action of player 1 whereas the latter requires best response to only the equilibrium action. Therefore, every perfect Bayesian equilibrium is a Bayesian Nash equilibrium but not conversely.

Now let $\Gamma(G)$ be the game with non-renegotiable third-party contracts and for any strategy profile (b_1, b_2) of G define the expected transfer from player 2 to the third-party as

$$F(b_1, b_2) = \sum_{\theta \in \Theta} p(\theta) f(b_1, b_2(b_1, \theta)).$$

We first prove the following.

Proposition 1. *A strategy profile (b_1^*, b_2^*) of G can be supported with non-renegotiable contracts if and only if*

1. (b_1^*, b_2^*) is a Bayesian Nash equilibrium of G
and there exists a contract $f : A \rightarrow \mathbb{R}$ such that
2. $F(b_1^*, b_2^*) = \delta$,
3. $u_2(a_1, b_2^*(a_1, \theta), \theta) - f(a_1, b_2^*(a_1, \theta)) \geq u_2(a_1, b_2^*(a_1, \theta'), \theta) - f(a_1, b_2^*(a_1, \theta'))$, for all $a_1 \in A_1$ and all $\theta, \theta' \in \Theta$.

Proposition 1 provides necessary and sufficient conditions for an outcome of an arbitrary original game to be supported with non-renegotiable contracts. Condition 1 states that only Bayesian Nash equilibrium outcomes can be supported, which follows from sequential rationality of players 1 and 2 in $\Gamma(G)$. Condition 2 simply states that the third-party does not receive any rent in equilibrium, whereas condition 3 is the incentive compatibility constraint.

We can obtain a sharper characterization if we impose an order structure on Θ and A_2 and assume that u_2 exhibits increasing differences. Let \succsim_θ be a linear order on Θ and \succsim_2 a linear order on A_2 , and denote their asymmetric parts by \succ_θ and \succ_2 , respectively.

Definition 2 (Increasing Differences). $u_2 : A_1 \times A_2 \times \Theta \rightarrow \mathbb{R}$ is said to have *increasing differences* in $(\succsim_\theta, \succsim_2)$ if $\theta \succsim_\theta \theta'$ and $a_2 \succsim_2 a'_2$ imply that $u_2(a_1, a_2, \theta) - u_2(a_1, a_2, \theta') \geq u_2(a_1, a'_2, \theta) - u_2(a_1, a'_2, \theta')$. It is said to have *strictly increasing differences* if $\theta \succ_\theta \theta'$ and $a_2 \succ_2 a'_2$ imply that $u_2(a_1, a_2, \theta) - u_2(a_1, a_2, \theta') > u_2(a_1, a'_2, \theta) - u_2(a_1, a'_2, \theta')$.

Definition 3 (Increasing Strategies). $b_2 : A_1 \times \Theta \rightarrow A_2$ is called *increasing* in $(\succsim_\theta, \succsim_2)$ if for all $a_1 \in A_1$, $\theta \succsim_\theta \theta'$ implies that $b_2(a_1, \theta) \succsim_2 b_2(a_1, \theta')$. Denote the set of all increasing b_2 by \mathcal{B}_2 .

For the rest of the paper, we restrict attention to games G in which there exist a linear order on Θ and a linear order on A_2 such that u_2 has strictly increasing differences in $(\succsim_\theta, \succsim_2)$. We then have the following result.

Proposition 2. *A strategy profile (b_1^*, b_2^*) of the original game G can be supported with non-renegotiable contracts if and only if (b_1^*, b_2^*) is a Bayesian Nash equilibrium of G and b_2^* is increasing in $(\succsim_\theta, \succsim_2)$.*

This result completely characterizes the strategy profiles that can be supported when contracts are non-renegotiable (but still unobservable) in environments with asymmetric information. First, it shows that third-party contracts potentially enlarges the set of outcomes that can arise in equilibrium. Second, while earlier papers showed that, when there is no asymmetric information, any Nash equilibrium of the original game can be supported with unobservable contracts, this result shows that only the subset of Bayesian Nash equilibria in which the second player plays an increasing strategy can be supported if, instead, there is asymmetric information. The reason why only increasing strategies of the second player can be supported is very similar to the reason why only increasing strategies can be supported in standard adverse selection models: If the payoff function of player 2 exhibits increasing differences, then incentive compatibility is equivalent to increasing strategies.

This result has an immediate corollary in terms of the outcomes that can be supported. For any strategy profile $(b_1, b_2) \in A_1 \times A_2^{A_1 \times \Theta}$, we define an *outcome* $(a_1, a_2) \in A_1 \times A_2^\Theta$ of G as $a_1 = b_1$ and $a_2(\theta) = b_2(b_1, \theta)$. Define the individually rational payoff of player 1 as

$$\underline{U}_1 = \max_{a_1 \in A_1} \min_{b_2 \in \mathcal{B}_2} U_1(a_1, b_2). \quad (1)$$

This is the best payoff player 1 can guarantee for herself in game G , given that player 2 plays an increasing strategy.¹⁰ The following easily follows from Proposition 2.

Theorem 1. *An outcome (a_1^*, a_2^*) of the original game G can be supported with non-renegotiable contracts if and only if (1) $a_2^*(\theta) \in BR_2(a_1^*, \theta)$ for all θ and (2) $U_1(a_1^*, a_2^*) \geq \underline{U}_1$.*

Again, note that, in general, outcomes that are not perfect Bayesian equilibrium outcomes of the original game can also be supported. This can be achieved by writing a contract that leads player 2 to punish player 1 when he deviates from his equilibrium action. Since contracts cannot be conditioned on θ and u_2 has increasing differences, player 2 can only use punishment strategies that are increasing in θ . The best that player 1 can do by deviating is therefore given by \underline{U}_1 , and his equilibrium payoff cannot be smaller than this payoff. This is condition (2). Condition (1), on the other hand, simply follows from the requirement that only Bayesian Nash equilibrium outcomes can be supported, and hence, player 2 must be best responding along the equilibrium path.

Note that if θ were contractible as well, we would not need to limit the punishment strategies to be increasing. In this case, condition (2) would have the individually rational payoff defined as

$$\max_{a_1 \in A_1} \min_{b_2 \in A_2^{A_1 \times \Theta}} U_1(a_1, b_2).$$

In that case, the result would be the exact analog of those in models without asymmetric information, i.e., Koçkesen and Ok (2004) and Koçkesen (2007).

We should also note that there are interesting environments in which non-contractibility of θ does not restrict the set of outcomes that can be supported with non-renegotiable contracts. For example

¹⁰We should also note that this is different from the definition of individually rational payoff used in the repeated games literature, which is the minmax payoff rather than the maxmin payoff. The maxmin payoff is at most equal to the minmax payoff.

if player 1's payoff does not depend on θ , then the punishment does not have to depend on θ either. Therefore, one can simply use a constant punishment after each deviation, which would be increasing by construction. A second environment is *games with externalities*, in which u_1 is increasing (or decreasing) in a_2 . In this case, after any a_1 , the harshest punishment is the lowest (or highest) a_2 , which is constant and hence increasing.

3.2 Renegotiable Contracts

In this section we will provide results that help identify the set of outcomes of any original game G that can be supported by renegotiation-proof perfect Bayesian equilibria of the game with renegotiable contracts $\Gamma_R(G)$. In the next section, we will use these results to obtain a complete characterization in a special environment, namely, games with externalities.

In order to decide whether to accept a new contract offer in the renegotiation phase of $\Gamma_R(G)$, the third-party forms beliefs regarding player 2's strategy under the new contract and compares his payoffs from the old and the new contracts. In equilibrium, these beliefs must be such that player 2's strategies are incentive compatible under the new contract. For easy reference, we define incentive compatibility as a property of any contract-strategy pair $(f, b_2) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$.

Definition 4 (Incentive Compatibility). $(f, b_2) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ is *incentive compatible* if

$$u_2(a_1, b_2(a_1, \theta), \theta) - f(a_1, b_2(a_1, \theta)) \geq u_2(a_1, b_2(a_1, \theta'), \theta) - f(a_1, b_2(a_1, \theta')) \text{ for all } a_1 \in A_1 \text{ and } \theta, \theta' \in \Theta.$$

We next define our renegotiation-proofness concept, which follows from the definition of renegotiation-proof perfect Bayesian equilibrium (Definition 1).

Definition 5 (Renegotiation-Proofness). We say that $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ is *renegotiation-proof* if for all $a_1 \in A_1$ and $\theta \in \Theta$ for which there exists an incentive compatible (g, b_2) such that

$$u_2(a_1, b_2(a_1, \theta), \theta) - g(a_1, b_2(a_1, \theta)) > u_2(a_1, b_2^*(a_1, \theta), \theta) - f(b_2^*(a_1, \theta)) \quad (2)$$

there exists a $\theta' \in \Theta$ such that

$$f(a_1, b_2^*(a_1, \theta')) \geq g(a_1, b_2(a_1, \theta')) \quad (3)$$

In words, if, for some (θ, a_1) , there is a contract g and an incentive compatible continuation play b_2 such that player 2 prefers g over f (i.e., (2) holds), there must exist a belief of the third-party (over θ) under which it is optimal to reject g , which is implied by (3).¹¹

Finally, we define a renegotiation-proof strategy as,

Definition 6 (Renegotiation-Proof Strategy). A strategy $b_2 \in A_2^{A_1 \times \Theta}$ is *renegotiation-proof* if there exists an $f \in \mathcal{C}$ such that (f, b_2) is incentive compatible and renegotiation-proof.

The following result proves that Definitions 5 and 6 are indeed the correct definitions to work with, in the sense that they identify the conditions that any contract f and strategy b_2 must satisfy to be part of a renegotiation-proof perfect Bayesian equilibrium of $\Gamma_R(G)$.

¹¹This definition allows beliefs to be arbitrary following an off-the-equilibrium renegotiation offer. An alternative definition would be to require the beliefs to satisfy intuitive criterion. In Section 6.1 we show that our results go through with minor modifications when we adopt this stronger version.

Proposition 3. *A strategy profile (b_1^*, b_2^*) of the original game G can be supported with renegotiation-proof contracts if and only if (b_1^*, b_2^*) is a Bayesian Nash equilibrium of G and b_2^* is increasing and renegotiation-proof.*

However, it is not straightforward to apply these definitions directly to an arbitrary game. The next set of results we present in this section use theorems of the alternative to obtain conditions in terms of the primitives of the original game.

Fix an arbitrary $a_1 \in A_1$, let the number of elements of Θ be n , and order its elements so that $\theta^n \succsim_\theta \theta^{n-1} \succsim_\theta \dots \succsim_\theta \theta^2 \succsim_\theta \theta^1$. For any contract-strategy pair (f, b_2) , define $f(a_1)_j = f(a_1, b_2(a_1, \theta^j))$, $j = 1, \dots, n$, and let, with an abuse of notation, $f(a_1) \in \mathbb{R}^n$ be the vector whose j -th component is given by $f(a_1)_j$.

First, note that, under increasing differences, incentive compatibility of contract-strategy pair (g, b_2) is equivalent to b_2 being increasing. Second, it is immediate to realize that condition (3) in Definition 5 is satisfied trivially if the strategy b_2 does not lead to a higher surplus for the contract-ing parties after (a_1, θ) . In other words, for each a_1 and $i = 1, \dots, n$, we need to check renegotiation-proofness of (f, b_2^*) only against strategies belonging to the following set:

$$\mathfrak{B}(a_1, i, b_2^*) = \{b_2 \in A_2^{A_1 \times \Theta} : b_2 \text{ is increasing and } u_2(a_1, b_2(a_1, \theta^i), \theta^i) > u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)\}. \quad (4)$$

Third, by Definition 5, (f, b_2^*) is *not* renegotiation-proof if and only if there exist $a_1 \in A_1$, $i = 1, \dots, n$, and incentive compatible (g, b_2) such that $u_2(a_1, b_2(a_1, \theta^i), \theta^i) - g(a_1)_i > u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) - f(a_1)_i$ and $g(a_1)_j > f(a_1)_j$ for all $j = 1, \dots, n$. When u_2 has increasing differences, incentive compatibility of (g, b_2) is equivalent to the local upward and downward constraints:

$$\begin{aligned} g(a_1)_j - g(a_1)_{j+1} &\leq u_2(a_1, b_2(a_1, \theta^j), \theta^j) - u_2(a_1, b_2(a_1, \theta^{j+1}), \theta^j), \quad j = 1, \dots, n-1 \\ -g(a_1)_{j-1} + g(a_1)_j &\leq u_2(a_1, b_2(a_1, \theta^j), \theta^j) - u_2(a_1, b_2(a_1, \theta^{j-1}), \theta^j), \quad j = 2, \dots, n \end{aligned}$$

For any $a_1 \in A_1$, we can write these inequalities in matrix form as $Dg(a_1) \leq U(a_1, b_2)$, where D is a matrix of coefficients and $U(a_1, b_2)$ a column vector with $2(n-1)$ components, whose component $2j-1$ is given by

$$U(a_1, b_2)_{2j-1} = u_2(a_1, b_2(a_1, \theta^j), \theta^j) - u_2(a_1, b_2(a_1, \theta^{j+1}), \theta^j)$$

and component $2j$ is given by

$$U(a_1, b_2)_{2j} = u_2(a_1, b_2(a_1, \theta^{j+1}), \theta^{j+1}) - u_2(a_1, b_2(a_1, \theta^j), \theta^{j+1})$$

Therefore, (f, b_2^*) is not renegotiation-proof if and only if there exist a_1, i, b_2 and $\varepsilon \in \mathbb{R}^n$ such that

$$D(f(a_1) + \varepsilon) \leq U(a_1, b_2), \quad \varepsilon_i < u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i), \quad \varepsilon \gg 0$$

These conditions can be written as $[Ax \gg 0, Cx \geq 0 \text{ has a solution } x]$, once the vector x and matrices A and C are appropriately defined. Motzkin's theorem of the alternative (stated as Lemma 3 in section 8) then implies that the necessary and sufficient condition for being renegotiation-proof is $[A'y_1 + C'y_2 = 0, y_1 > 0, y_2 \geq 0 \text{ has a solution } y_1, y_2]$ (See Lemma 4 in section 8). The fact that u_2 has

increasing differences can then be used to prove the equivalence of this condition to the one stated in the following theorem.

Theorem 2. (f, b_2^*) is renegotiation-proof if and only if for any $a_1 \in A_1$, $i \in \{1, 2, \dots, n\}$, and $b_2 \in \mathfrak{B}(a_1, i, b_2^*)$ there exists a $k \in \{1, 2, \dots, i-1\}$ such that

$$u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) + \sum_{j=k}^{i-1} U(a_1, b_2)_{2j-1} \leq f(a_1)_k - f(a_1)_i \quad (5)$$

or there exists an $l \in \{i+1, i+2, \dots, n\}$ such that

$$u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) + \sum_{j=i+1}^l U(a_1, b_2)_{2(j-1)} \leq f(a_1)_l - f(a_1)_i \quad (6)$$

Theorem 2 characterizes the conditions for which (f, b_2^*) is renegotiation-proof. Our next step is to find conditions for a strategy b_2^* to be supported with renegotiation-proof contracts. The following definition facilitates the exposition.

Definition 7. For any a_1 , $i = 1, \dots, n$ and $b_2 \in \mathfrak{B}(a_1, i, b_2^*)$ we say that $m(b_2) \in \{1, 2, \dots, n\}$ is a *blocking type* if

$$u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) \leq \sum_{j=m(b_2)}^{i-1} [U(a_1, b_2^*)_{2j-1} - U(a_1, b_2)_{2j-1}] \quad (7)$$

or

$$u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) \leq \sum_{j=i+1}^{m(b_2)} [U(a_1, b_2^*)_{2(j-1)} - U(a_1, b_2)_{2(j-1)}] \quad (8)$$

We obtain the following necessary conditions for a strategy b_2^* to be renegotiation-proof.

Proposition 4. A strategy $b_2^* \in A_2^{A_1 \times \Theta}$ is renegotiation-proof only if for any $a_1 \in A_1$, $i \in \{1, 2, \dots, n\}$, and $b_2 \in \mathfrak{B}(a_1, i, b_2^*)$ there is a blocking type.

The above condition becomes also sufficient for renegotiation-proofness with an additional requirement about the relation of blocking types for different renegotiation opportunities.

Proposition 5. A strategy $b_2^* \in A_2^{A_1 \times \Theta}$ is renegotiation-proof if for any $a_1 \in A_1$, $i \in \{1, 2, \dots, n\}$, and $b_2 \in \mathfrak{B}(a_1, i, b_2^*)$ there is a blocking type $m(b_2^i)$ such that $k < l$, $m(b_2^k) > k$, and $m(b_2^l) < l$ imply $m(b_2^k) \leq m(b_2^l)$.

The conditions given in Proposition (4) and (5) coincide when player 2 has only two types. Therefore, Proposition (4) is a full characterization result for such games. Although, they fall short of providing a full characterization in games with more than two types, they help us do so in an environment that is common to many economic models, namely, games with negative or positive externalities.

4 Games with Externalities

Theorem 2 characterizes renegotiation-proof contract-strategy pairs for very general environments. However, its implications are not quite immediate in applications. In this section, we present a result

that is particularly easy to apply. The only additional condition we need to impose on the original game G is that player 1's payoff is monotonically increasing or decreasing in player 2's action, i.e., for any a_1 and θ , $a'_2 \succsim_2 a_2$ implies either $u_1(a_1, a'_2, \theta) \geq u_1(a_1, a_2, \theta)$ or $u_1(a_1, a'_2, \theta) \leq u_1(a_1, a_2, \theta)$. Such positive or negative externalities are very common in economic models. Indeed, the class of games that satisfy these conditions includes Stackelberg and entry games, sequential Bertrand games with differentiated products, monopolistic screening, and ultimatum bargaining, among others.

Let us first assume that player 1's payoff function is increasing in player 2's action. Consider any outcome $(a_1^*, a_2^*) \in A_1 \times A_2^\Theta$ of the original game G and let the expected payoff of player 1 at that outcome be $U_1(a_1^*, a_2^*) = \sum_{i=1}^n p(\theta^i) u_1(a_1^*, a_2^*(\theta^i), \theta^i)$. In order to support this outcome with renegotiation-proof contracts, a_1^* must be a best response to player 2's strategy. In other words, if player 1 plays another action, player 2's strategy must "punish" him sufficiently so that he does not receive a payoff higher than $U_1(a_1^*, a_2^*)$. Since player 1's payoff is increasing in a_2 , the harshest punishment is to play the smallest possible a_2 . However, such a punishment strategy is not renegotiation proof, as Lemma 7 in Section 8 shows.¹² In fact, the only way to make it so, while keeping the punishment as severe as possible, is to make the highest type of player 2 play a best response, while the other types play the smallest a_2 (see again Lemma 6 and 7 in the Section 8). Let us define such a punishment strategy more precisely. Let \underline{a}_2 be the smallest element of A_2 , and for any $a_1 \in A_1$, let $\hat{b}_2(a_1) \in \operatorname{argmin}_{a_2 \in BR_2(a_1, \theta^n)} u_1(a_1, a_2, \theta^n)$. Define the punishment strategy as

$$b_2^+(a_1, \theta) = \begin{cases} \underline{a}_2, & \theta < \theta^n \\ \hat{b}_2(a_1), & \theta = \theta^n \end{cases} \quad (9)$$

The best payoff that player 1 can achieve against this strategy is

$$\underline{U}_1^+ = \max_{a_1} U_1(a_1, b_2^+) \quad (10)$$

We can now state our result more formally.

Theorem 3. *Suppose u_1 is increasing in a_2 . Then, an outcome $(a_1^*, a_2^*) \in A_1 \times A_2^\Theta$ of the original game G can be supported with renegotiation-proof contracts if and only if (1) $a_2^*(\theta) \in BR_2(a_1^*, \theta)$ for all $\theta \in \Theta$ and (2) $U_1(a_1^*, a_2^*) \geq \underline{U}_1^+$.*

Condition (1) of Theorem 3 simply states that player 2's strategy must be sequentially rational along the equilibrium path, which is a direct consequence of Proposition 3. Condition (2) just recapitulates our previous discussion: Following a deviation, the harshest renegotiation-proof punishment player 2 can inflict upon player 1 is to make type θ^n best respond, while all the other types play the smallest possible action. Player 1's expected payoff cannot be smaller than the highest payoff he can get by deviating and getting punished in this manner by player 2.

Comparing Theorem 1 and 3 makes the effect of renegotiation in this environment very clear. If the contracts are non-renegotiable, then any outcome (a_1, a_2) in which player 2 best responds on the equilibrium path and player 1 receives his individually rational payoff $\max_{a_1} U(a_1, \underline{a}_2)$ can be supported.¹³ With renegotiation-proof contracts only the definition of the individually rational payoff

¹²More precisely, this lemma shows that renegotiation-proofness of a strategy $b_2 \in A_2^{A_1 \times \Theta}$ implies that the highest (lowest, resp.) type does not have a profitable deviation to a higher (lower, resp.) action.

¹³Remember from the discussion at the end of Section 3.1 that this would also be the set of outcomes that can be sup-

changes: All types of player 2 punishes a deviation by player 1 in the harshest possible way by playing \underline{a}_2 , except the highest type, who must play a best response to make her strategy renegotiation-proof.

Theorem 3 presents the result for the case of positive externalities. The negative externality case is symmetric. If player 1's payoff function is decreasing in player 2's action, the harshest punishment that player 2 can inflict upon player 1 is to play the largest possible a_2 . Again, such a punishment is not renegotiation-proof and the renegotiation-proof strategy that delivers the harshest punishment is to make the smallest type best respond and the other types play the largest a_2 , denoted \bar{a}_2 . More precisely, for any $a_1 \in A_1$ and letting $\hat{b}_2(a_1) \in \operatorname{argmin}_{a_2 \in BR_2(a_1, \theta^1)} u_1(a_1, a_2, \theta_1)$, the harshest renegotiation-proof punishment strategy is defined as

$$b_2^-(a_1, \theta) = \begin{cases} \hat{b}_2(a_1), & \theta = \theta^1 \\ \bar{a}_2, & \theta > \theta^1 \end{cases} \quad (11)$$

The result for negative externality is as in Theorem 3 except the condition (2) becomes $U_1(a_1^*, a_2^*) \geq U_1^+$, where U_1^+ is defined as

$$U_1^-(b_2) = \max_{a_1} U_1(a_1, b_2^-). \quad (12)$$

We should also note that a result similar to Theorem 3 holds even if player 1's payoff function is decreasing in a_2 at some a_1 while it is increasing at other a_1 . Condition (2) would then become $U_1(a_1^*, a_2^*) \geq \max\{U_1^+, U_1^-\}$.

5 Applications

In this section, we illustrate the results of our paper, in particular Theorem 1 and Theorem 3, within the context of two commonly used models. The first one is a (Stackelberg) quantity competition model with private information and the second, a standard monopolistic screening model.

In both of these games, player 2's payoff function has increasing differences in (a_2, θ) and player 1's payoff function is increasing in a_2 , which imply that we can apply Theorem 1 and 3. As we will show shortly, in all these models, renegotiation-proof third-party contracts can change the outcome of the game, but the extent to which player 2 can benefit from this is limited and depends on the specifics of the environment.

APPLICATION I: QUANTITY COMPETITION AND ENTRY-DETERRENCE

Consider a Stackelberg game in which firm 1 moves first by choosing an output level $q_1 \in Q_1$ and firm 2, after observing q_1 , chooses its own output level $q_2 \in Q_2$. Inverse demand function is given by $P(q_1, q_2) = \max\{0, \alpha - q_1 - q_2\}$, where $\alpha > 0$, and we assume Q_i a rich enough finite subset of \mathbb{R}_+ whose largest element is α .¹⁴ Cost function of firm 1 is $C_1(q_1) = cq_1$, where c is common knowledge, whereas the cost function of firm 2 is $C_2(q_2) = \theta q_2$. We assume that $\theta \in \{\theta^1, \theta^2, \dots, \theta^n\}$ is private information of firm 2 and $\theta^1 < \theta^2 < \dots < \theta^n$. Firm 1 believes that the probability of θ^i is given by $p(\theta^i)$ and for ease of exposition we assume that expected value of θ is equal to c . The profit function of firm i is given by $\pi_i(q_1, q_2, \theta) = P(q_1, q_2)q_i - C_i(q_i)$ and we assume that both firms are profit maximizers.

ported when θ is contractible.

¹⁴We introduce this assumption so that player 2 can choose a high enough output level to drive the price to zero.

To ensure positive output levels in equilibrium we assume that $\alpha > 2\theta^n - c$, in which case the (Stackelberg) equilibrium outcome of this game is given by

$$(q_1^s, q_2^s(\theta)) = \left(\frac{\alpha - c}{2}, \frac{\alpha - 2\theta + c}{4} \right)$$

Define the game G as follows: Let $A_1 = Q_1$ and $A_2 = \{-q_2 : q_2 \in Q_2\}$ and define \succsim_i on A_i as $a_i \succsim_i a'_i \Leftrightarrow a_i \geq a'_i$ and \succsim_θ as $\theta \succsim_\theta \theta' \Leftrightarrow \theta \geq \theta'$. Let the payoff function of player i be given by $u_i(a_1, a_2, \theta) = \pi_i(a_1, -a_2, \theta)$, for any $(a_1, a_2) \in A_1 \times A_2$. The game G is strategically equivalent to the Stackelberg game defined in the previous paragraph.

It is easy to show that u_2 has strictly increasing differences in (a_2, θ) and u_1 is increasing in a_2 , and hence we can apply Theorem 1 and 3 to characterize all the outcomes that can be supported with non-renegotiable as well as renegotiation-proof third-party contracts. In order to apply Theorem 1, we need to calculate the individually rational payoff of player 1, i.e., \underline{U}_1 as defined in equation (1). The harshest punishment firm 2 can inflict is to drive the price down to zero by producing α for any type θ . Since this is a constant (and hence an increasing) strategy, it follows that $\underline{U}_1 = 0$. In other words, any outcome $(a_1^*, a_2^*(\theta))$ such that firm 2 best responds to a_1^* and firm 1 gets at least zero profit can be supported with non-renegotiable contracts. In particular, entry can be deterred with non-renegotiable contracts.

Can entry be deterred with renegotiation-proof contracts? In order to apply Theorem 3, we need to first calculate player 1's payoff when the highest type of player 2 best responds while the other types choose the lowest a_2 , i.e., $a_2 = -\alpha$. This is given by $\frac{1}{2}p(\theta^n)(\alpha + \theta^n - a_1)a_1 - ca_1$, and its maximum, i.e., player 1's individually rational payoff, is equal to

$$\underline{U}_1^+ = \begin{cases} 0, & p(\theta^n)(\alpha + \theta^n) - 2c \leq 0 \\ \frac{(p(\theta^n)(\alpha + \theta^n) - 2c)^2}{8p(\theta^n)}, & p(\theta^n)(\alpha + \theta^n) - 2c > 0 \end{cases}$$

Condition (1) of Theorem 3 requires that $a_2^*(\theta) = \frac{a_1^* + \theta - c}{2}$ for all θ , and hence $U_1(a_1^*, a_2^*) = \frac{1}{2}(\alpha - c - a_1^*)a_1^*$. Therefore, by condition (2), any outcome such that $\frac{1}{2}(\alpha - c - a_1^*)a_1^* \geq \underline{U}_1^+$ can be supported.

Also note that if $p(\theta^n)(\alpha + \theta^n) - 2c > 0$, then \underline{U}_1^+ is strictly positive. This implies that entry cannot be deterred if $p(\theta^n)(\alpha + \theta^n) - 2c > 0$. Therefore, we have the following result:

Corollary 1. *Entry can be deterred with non-renegotiable contracts. It can be deterred with renegotiation-proof contracts if and only if $p(\theta^n)(\alpha + \theta^n) - 2c \leq 0$.*

Dewatripont (1988) has also analyzed a similar entry game and showed that entry can be deterred with renegotiation-proof contracts under certain conditions. His conditions are stronger than ours because he uses a different renegotiation-proofness concept, namely durability, first introduced by Holmstrom and Myerson (1983). We comment on durability in general and in the context of the entry game in more detail in Section 6.2.

APPLICATION II: ADVERSE SELECTION

We will analyze a simple but standard adverse selection model. For the sake of concreteness we will phrase the model as a monopolistic screening model in which a seller (player 1) offers the same product in two different qualities: high (H) and low (L), at possibly different prices. For simplicity we assume that the cost of producing either quality is zero and hence the seller maximizes his expected

revenue. A buyer (player 2) observes the prices and chooses to purchase H or L or does not buy at all. Her willingness to pay, θ , is private information and if she buys quality q at price t her payoff is $\theta q - t$, while if she does not buy at all her payoff is zero.

Define the original game G as follows: $\Theta = \{\theta_l, \theta_h\}$, $\theta_h > \theta_l > 0$, $\text{prob}(\theta_l) = p \in (0, 1)$, A_1 is a finite but rich enough subset of $\{(t_l, t_h) \in \mathbb{R}_+^2 : t_h \geq t_l\}$, $A_2 = \{0, L, H\}$, $H > L > 0$, the orders on A_2 and Θ are the natural orders and the payoff functions are given as follows:

$$u_1(a_1, a_2, \theta) = \begin{cases} 0, & a_2 = 0 \\ t_l, & a_2 = L \\ t_h, & a_2 = H \end{cases}, \quad u_2(a_1, a_2, \theta) = \begin{cases} 0, & a_2 = 0 \\ \theta L - t_l, & a_2 = L \\ \theta H - t_h, & a_2 = H \end{cases}$$

We assume that $\theta_l > (1 - p)\theta_h$, in which case the standard analysis shows that the optimal prices are $t_l = \theta_l L$, $t_h = \theta_h H - (\theta_h - \theta_l)L$ and the high type buys the high quality whereas the low type buys the low quality good. Expected payoff of the seller is $p\theta_l L + (1 - p)[\theta_h H - (\theta_h - \theta_l)L]$ and that of the buyer is $(1 - p)(\theta_h - \theta_l)L$.

An interesting question is whether the buyer can do better by signing a contract with a third-party, e.g., a financial contract with a bank. To answer this question first note that u_2 has strictly increasing differences in (a_2, θ) and u_1 is increasing in a_2 .¹⁵ Therefore, we can apply Theorem 1 and 3 and identify the best equilibria for the buyer.

The harshest punishment player 2 can inflict upon player 1 after some a_1 is not to buy, i.e., choose zero for any θ . Since this is increasing in θ , the individually rational payoff of player 1 is zero, i.e., in Theorem 1 we have $\underline{U}_1 = 0$. This implies that any offer (t_l, t_h) can be supported with non-renegotiable contracts. The best equilibrium for the buyer therefore is the one with $(t_l, t_h) = (0, 0)$, which gives the full surplus $(p\theta_l + (1 - p)\theta_h)H$ to the buyer.

How about if the contracts are renegotiable? In order to apply Theorem 3, we have to calculate the best payoff that the seller can get when the high type is best responding while the low type buys nothing. This implies that the best offer for the seller is to set $t_h = \theta_h H$, which yields him an expected payoff of $(1 - p)\theta_h H$. Condition (2) of Theorem 3 implies that this is the worst the seller can get in any equilibrium. In other words, and in contrast to the non-renegotiable contracts case, the buyer cannot get the full surplus with renegotiation-proof contracts.

What is the best that the buyer can obtain? If in equilibrium both types buy the high quality, then the previous paragraph implies that the price of the high quality good must be at least $(1 - p)\theta_h H$. The following outcome satisfies the conditions of Theorem 3 at the smallest such price: $(t_l^*, t_h^*) = (\theta_h H, (1 - p)\theta_h H)$ and $a_2^*(\theta) = H$ for all θ . Therefore, the best payoff the buyer can get in any equilibrium in which both types buy high quality is $p\theta_l H$.

Similarly, we can show that the best payoff of the buyer in any equilibrium in which both types buy low quality is $(p\theta_l + (1 - p)\theta_h)L - (1 - p)\theta_h H$, which is smaller than $p\theta_l H$. Lastly we need to consider equilibria in which low type buys low quality and the high type buys high quality. Condition (2) of Theorem 3 implies that $p t_l + (1 - p) t_h \geq (1 - p)\theta_h H$. In other words, the best expected payoff for the buyer in any such equilibrium is $p\theta_l L + p\theta_h H - (1 - p)\theta_h H = p\theta_l L$, which is, again, smaller than

¹⁵Definition of A_1 is the only non-standard part of the model and ensures that u_1 is increasing in a_2 . However, this is otherwise inconsequential since in equilibrium the seller never offers to sell the high quality at a lower price than the low quality product.

$p\theta_l H$.

Therefore, the highest expected payoff of the buyer with renegotiation-proof contracts is $p\theta_l H$, which is strictly smaller than what she can get with non-renegotiable contracts: $(p\theta_l + (1-p)\theta_h)H$. The difference is $(1-p)\theta_h H$, which is exactly the expected surplus from the high type.

6 Alternative Definitions of Renegotiation-Proofness

As Proposition 3 shows, our definition of renegotiation-proofness follows directly from the assumed game form for the renegotiation procedure, i.e., player 2, who is the informed party, makes a new contract offer and the third-party, who is uninformed, accepts or rejects. In a renegotiation-proof equilibrium, the contract is never renegotiated, and therefore any renegotiation offer is an out-of-equilibrium event. This allows us to specify the beliefs of the third-party freely after a new contract offer. This may be found unreasonable and a more plausible alternative could be to require beliefs satisfy the conditions specified in the intuitive criterion as defined by Cho and Kreps (1987). The next section provides an analysis of the implications of intuitive criterion and the section after that provides a discussion of another definition of renegotiation-proofness, namely durability, introduced by Holmstrom and Myerson (1983). Since durability is also the renegotiation-proofness concept used by Dewatripont (1988), this will allow us to compare the implications of alternative concepts of renegotiation-proofness within the context of the entry-deterrence game.

6.1 Strong Renegotiation-Proofness

In our setting, intuitive criterion requires that, given an equilibrium contract strategy pair (f, b_2^*) and following a renegotiation offer (g, b_2) , beliefs put positive probability only on types for which (g, b_2) is not equilibrium-dominated, i.e., only on those types θ' for which $u_2(a_1, b_2(a_1, \theta'), \theta') - g(a_1, b_2(a_1, \theta')) \geq u_2(a_1, b_2^*(a_1, \theta'), \theta') - f(a_1, b_2^*(a_1, \theta'))$. This leads to the following definition.

Definition 8 (Strong Renegotiation-Proofness). We say that $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ is *strongly renegotiation-proof* if for all $a_1 \in A_1$ and $\theta \in \Theta$ for which there exists an incentive compatible (g, b_2) such that

$$u_2(a_1, b_2(a_1, \theta), \theta) - g(a_1, b_2(a_1, \theta)) > u_2(a_1, b_2^*(a_1, \theta), \theta) - f(b_2^*(a_1, \theta)) \quad (13)$$

there exists a $\theta' \in \Theta$ such that

$$f(a_1, b_2^*(a_1, \theta')) \geq g(a_1, b_2(a_1, \theta')) \quad (14)$$

and

$$u_2(a_1, b_2(a_1, \theta'), \theta') - g(a_1, b_2(a_1, \theta')) \geq u_2(a_1, b_2^*(a_1, \theta'), \theta') - f(a_1, b_2^*(a_1, \theta')) \quad (15)$$

This is exactly the same as renegotiation-proofness except that it adds condition (15), which allows us to construct beliefs that satisfy intuitive criterion after any renegotiation offer. It can be shown that when we work with this definition, Theorem 2 needs to be modified as follows.

Theorem 4. (f, b_2^*) is strongly renegotiation-proof if and only if for any $a_1 \in A_1$, $i \in \{1, 2, \dots, n\}$, and

$b_2 \in \mathfrak{B}(a_1, i, b_2^*)$ there exists a $k \in \{1, 2, \dots, i-1\}$ such that

$$u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) + \sum_{j=k}^{i-1} U(a_1, b_2)_{2j-1} \\ - \min\{0, u_2(a_1, b_2(a_1, \theta^k), \theta^k) - u_2(a_1, b_2^*(a_1, \theta^k), \theta^k)\} \leq f(a_1)_k - f(a_1)_i \quad (16)$$

or there exists an $l \in \{i+1, i+2, \dots, n\}$ such that

$$u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) + \sum_{j=i+1}^l U(a_1, b_2)_{2(j-1)} \\ - \min\{0, u_2(a_1, b_2(a_1, \theta^l), \theta^l) - u_2(a_1, b_2^*(a_1, \theta^l), \theta^l)\} \leq f(a_1)_l - f(a_1)_i \quad (17)$$

It is also easy to show that Proposition 3 goes through with strongly renegotiation-proof contracts whereas Propositions 4 and 5 go through with a minor modification similar to the one made in Theorem 4. Perhaps of more interest, Theorem 3 goes through without any modification with strongly renegotiation-proof contracts.¹⁶

6.2 Durability

Holmstrom and Myerson (1983) introduced “durability” as a notion of renegotiation proofness. A decision rule is durable if and only if the parties involved would never unanimously approve a change from this decision rule to any other decision rule. They also show that this is equivalent to interim incentive efficiency when there is only one player with private information. In our context, only player 2 has private information and hence a contract-strategy pair (f, b_2^*) is interim incentive efficient (and therefore durable) if and only if there is no $a_1 \in A_1$ and an incentive compatible (g, b_2) such that after a_1 every type of player 2 and the third-party do better under (g, b_2) , with at least one doing strictly better. More formally:

Definition 9 (Durability). We say that $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ is *durable* if there is no $a_1 \in A_1$ and incentive compatible $(g, b_2) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ such that

$$u_2(a_1, b_2(a_1, \theta), \theta) - g(a_1, b_2(a_1, \theta)) \geq u_2(a_1, b_2^*(a_1, \theta), \theta) - f(a_1, b_2^*(a_1, \theta)), \quad \text{for all } \theta \in \Theta \quad (18)$$

and

$$G(a_1, b_2) \geq F(a_1, b_2^*) \quad (19)$$

with at least one inequality holding strictly.

As we have mentioned before, this is also the renegotiation-proofness concept used in Dewatripont (1988). Therefore, it would be interesting to understand the implications of durability in a general model of third-party contracts as well as its relationship with our notion of renegotiation-proofness. Characterizing durable strategies for arbitrary games turns out to be quite difficult. However, we have a characterization for the two-type case, i.e., when $\Theta = \{\theta^1, \theta^2\}$.

¹⁶Proofs of Theorem 4 as well as these claims are available upon request.

Let us define a strategy $b_2 \in A_2^{A_1 \times \Theta}$ as durable if there exists a contract f such that (f, b_2) is incentive compatible and durable. Also, remember that for any $a_1 \in A_1$ and $b_2 \in A_2^{A_1 \times \Theta}$

$$\begin{aligned} U(a_1, b_2)_1 &= u_2(a_1, b_2(a_1, \theta^1), \theta^1) - u_2(a_1, b_2(a_1, \theta^2), \theta^1) \\ U(a_1, b_2)_2 &= u_2(a_1, b_2(a_1, \theta^2), \theta^2) - u_2(a_1, b_2(a_1, \theta^1), \theta^2) \end{aligned}$$

We have the following result:

Proposition 6. *A strategy $b_2^* \in A_2^{A_1 \times \Theta}$ is durable if and only if it is increasing and for any $a_1 \in A_1$ and increasing $b_2 \in A_2^{A_1 \times \Theta}$ such that*

$$\sum_{i=1}^2 p(\theta^i) \left[u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) \right] > 0 \quad (20)$$

one of the following is true:

$$u_2(a_1, b_2(a_1, \theta^1), \theta^1) - u_2(a_1, b_2^*(a_1, \theta^1), \theta^1) < p(\theta^2) [U(a_1, b_2^*)_2 - U(a_1, b_2)_2] \quad (21)$$

$$u_2(a_1, b_2(a_1, \theta^2), \theta^2) - u_2(a_1, b_2^*(a_1, \theta^2), \theta^2) < p(\theta^1) [U(a_1, b_2^*)_1 - U(a_1, b_2)_1] \quad (22)$$

Note that condition (20) states that the alternative strategy b_2 yields a higher expected surplus than does the candidate strategy b_2^* . Durability requires that, whenever this is the case, the extra surplus that comes from at least one type must be smaller than the bounds given in conditions (21) and (22).

The relationship between our concept of renegotiation-proofness and durability is quite subtle even in the two-type case. Remember that with two types Proposition 4 is a complete characterization of renegotiation-proofness, which we rewrite below in a form that facilitates comparison:

Proposition 7. *A strategy $b_2^* \in A_2^{A_1 \times \Theta}$ is renegotiation-proof if and only if it is increasing and for any $a_1 \in A_1$ and increasing $b_2 \in A_2^{A_1 \times \Theta}$ the following is true:*

$$\begin{aligned} u_2(a_1, b_2(a_1, \theta^1), \theta^1) - u_2(a_1, b_2^*(a_1, \theta^1), \theta^1) &> 0 \Rightarrow \\ u_2(a_1, b_2(a_1, \theta^1), \theta^1) - u_2(a_1, b_2^*(a_1, \theta^1), \theta^1) &< U(a_1, b_2^*)_2 - U(a_1, b_2)_2 \end{aligned} \quad (23)$$

and

$$\begin{aligned} u_2(a_1, b_2(a_1, \theta^2), \theta^2) - u_2(a_1, b_2^*(a_1, \theta^2), \theta^2) &> 0 \Rightarrow \\ u_2(a_1, b_2(a_1, \theta^2), \theta^2) - u_2(a_1, b_2^*(a_1, \theta^2), \theta^2) &< U(a_1, b_2^*)_1 - U(a_1, b_2)_1 \end{aligned} \quad (24)$$

Comparing Proposition 6 and 7, one can see that neither renegotiation-proofness nor durability immediately implies the other one. In fact, it is not difficult to find games and construct strategies that are durable but not renegotiation-proof and vice versa.

However, in games with externalities it can be shown that durability implies renegotiation-proofness (See Lemma 11 in Section 8). The entry-deterrence game is a game with externalities, and therefore, if entry can be deterred with durable contracts, it can also be deterred with renegotiation-proof

contracts. In fact, in the entry-deterrence game player 2's payoff function is single-peaked and for such environments we have a complete characterization of durable outcomes that is particularly easy to apply (See Proposition 10 in Section 8). Using this characterization, we can show that the relationship between durability and renegotiation-proofness is strict.

Proposition 8. *In the entry-deterrence game with two types, if $p_1(\theta^2 + \alpha) > (\theta^2 - \theta^1)$, then entry can be deterred with renegotiation-proof contracts but not with durable contracts.*

Remember that the harshest renegotiation-proof punishment strategy of the incumbent is to flood the market if entry occurs, except for the highest type (type θ^2), who has to best respond. Durability still requires that the highest type best responds. The difference is that flooding the market for type θ^1 is not a durable strategy: There is a restriction on how much the incumbent can produce in response to entry, which is given by condition (72) in Proposition 10 of Section 8. When applied to the entry game studied by Dewatripont (1988), this becomes condition (d) of his Proposition 1.

7 Conclusion

This paper characterizes equilibrium outcomes of two-stage games in which the second mover (player 2) has private information and can sign renegotiable contracts with a third-party. Our aim is to understand whether renegotiation-proof third-party contracts have any effect on the equilibrium outcomes of a game. In particular, in games where there is a first mover advantage, such as the Stackelberg and buyer-seller games, can the second mover undo this advantage using renegotiation-proof contracts?

Our analysis starts with non-renegotiable contracts and shows that any outcome in which the second mover best responds to the first mover's action and the first mover obtains his individually rational payoff can be supported. In other words, some kind of a "folk theorem" is true with non-renegotiable contracts. Renegotiation-proofness imposes some restrictions on the outcomes that can be supported. This is most transparent in games with externalities, i.e., games in which player 1's payoff increases (or decreases) in player 2's action. In such games, it is still true that any outcome in which the second mover best responds to the first mover's action and the first mover obtains his individually rational payoff can be supported. However, the individually rational payoff for player 1 would in general be higher in this case. This is because, in the minimization of player 1's payoff, renegotiation-proofness restricts player 2 to strategies where the highest (or lowest, for the case of negative externality) type best responds to every action of player 1. Still, even with renegotiation-proof contracts, one can support outcomes that are not perfect Bayesian equilibrium outcomes of the original game, and this may benefit the second mover in many games, such as the entry game.

8 Proofs

In the game with non-renegotiable contracts $\Gamma(G)$, player 2 has an information set at the beginning of the game, which we identify with the null history \emptyset , and an information set for each $(f, \theta, a_1) \in \mathcal{C} \times \Theta \times A_1$, where $\mathcal{C} = \mathbb{R}^{A_1 \times A_2}$. Player 1 has only one information set, given by \mathcal{C} , and player 3 has an information set for each $f \in \mathcal{C}$. In the game with renegotiable contracts $\Gamma_R(G)$, player 2 has additional information sets corresponding to each history (f, θ, a_1, g, y) and (f, θ, a_1, g, n) and player 3 has an additional information set of each (f, a_1, g) , which we denote by $I_3(f, a_1, g)$.

Proof of Proposition 1. [If] Let (b_1^*, b_2^*) be a Bayesian Nash equilibrium of G and f' satisfy the conditions of the proposition. For any $b_2 \in A_2^{A_1 \times \Theta}$ and $a_1 \in A_1$, let $b_2(a_1, \Theta)$ be the image of Θ under $b_2(a_1, \cdot)$. Define

$$f^*(a_1, a_2) = \begin{cases} f'(a_1, a_2), & \text{if } a_2 \in b_2^*(a_1, \Theta) \\ \max_{\theta} \{u_2(a_1, a_2, \theta) - u_2(a_1, b_2^*(a_1, \theta), \theta) + f'(a_1, b_2^*(a_1, \theta))\}, & \text{otherwise} \end{cases}$$

for any $(a_1, a_2) \in A_1 \times A_2$, and

$$b_{2,f}^*(a_1, \theta) = \begin{cases} b_2^*(a_1, \theta), & f = f^* \\ \in \operatorname{argmax}_{a_2} u_2(a_1, a_2, \theta) - f(a_1, a_2), & f \neq f^* \end{cases}$$

for any $f \in \mathcal{C}$, $a_1 \in A_1$, and $\theta \in \Theta$. Consider the assessment (β^*, μ^*) of $\Gamma(G)$, where $\beta_2^*[\emptyset] = f^*$, $\beta_3[f] = y$ iff $F(b_1^*, b_{2,f}^*) \geq \delta$, $\beta_1^*[\mathcal{C}] = b_1^*$, $\beta_2^*[f, \theta, a_1] = b_{2,f}^*(a_1, \theta)$ for all $f \in \mathcal{C}$, $a_1 \in A_1$, and $\theta \in \Theta$, and $\mu^*[\mathcal{C}](f^*) = 1$. It is easy to check that this assessment induces (b_1^*, b_2^*) and is a perfect Bayesian equilibrium of $\Gamma(G)$.

[Only if] Now, suppose that (b_1^*, b_2^*) can be supported. Then, there exists a perfect Bayesian equilibrium (β^*, μ^*) that induces (b_1^*, b_2^*) , i.e., $\beta_2^*[\emptyset] = f^*$, $\beta_3[f^*] = y$, $\beta_1^*[\mathcal{C}] = b_1^*$, $\beta_2^*[f^*, \theta, a_1] = b_2^*(a_1, \theta)$ for all $a_1 \in A_1$ and $\theta \in \Theta$. The fact that (b_1^*, b_2^*) is a Bayesian Nash equilibrium of G is a direct consequence of sequential rationality of players 1 and 2. We now show that f^* satisfies conditions 2 of Proposition 1. If $F^*(b_1^*, b_2^*) < \delta$, then player 3 rejects the equilibrium contract, which is a contradiction. Suppose, for contradiction, that $F^*(b_1^*, b_2^*) = \alpha > \delta$, and consider the contract $f'(a_1, a_2) = \delta + (\alpha - \delta)/2$ for all (a_1, a_2) . This contract is accepted by player 3 and yields player 2 a strictly higher expected payoff than f^* , a contradiction. Finally, sequential rationality of player 2 immediately implies condition 3. \square

Before we turn to the proof of Proposition 2 we introduce some notation and prove a supplementary lemma. Let the number of elements in Θ be equal to n and order its elements so that $\theta^n \succ_{\theta} \theta^{n-1} \succ_{\theta} \dots \theta^2 \succ_{\theta} \theta^1$. Let e_i be the i^{th} standard basis row vector for \mathbb{R}^n and define the row vector $d_i = e_i - e_{i+1}$, $i = 1, 2, \dots, n-1$. Let D be the $2(n-1) \times n$ matrix whose row $2i-1$ is d_i and row $2i$ is $-d_i$, $i = 1, \dots, n-1$. For any $a_1 \in A_1$ and $b_2 \in A_2^{A_1 \times \Theta}$ define $U(a_1, b_2)$ as a column vector with $2(n-1)$ components, where component $2i-1$ is given by $u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2(a_1, \theta^{i+1}), \theta^i)$ and component $2i$ is given by $u_2(a_1, b_2(a_1, \theta^{i+1}), \theta^{i+1}) - u_2(a_1, b_2(a_1, \theta^i), \theta^{i+1})$, $i = 1, 2, \dots, n-1$.

Notation 1. Given two vectors $x, y \in \mathbb{R}^n$

1. $x \geq y$ if and only if $x_i \geq y_i$, for all $i = 1, 2, \dots, n$;
2. $x > y$ if and only if $x_i \geq y_i$, for all $i = 1, 2, \dots, n$ and $x \neq y$;
3. $x \gg y$ if and only if $x_i > y_i$, for all $i = 1, 2, \dots, n$.

Similarly for \leq , $<$, and \ll .

For any $a_1 \in A_1$, $b_2 \in A_2^{A_1 \times \Theta}$ and $f \in \mathcal{C}$, let $f(a_1, b_2)$ be the column vector with n components, where i^{th} component is given by $f(a_1, b_2(a_1, \theta^i))$, $i = 1, 2, \dots, n$.

It is well-known that if b_2 is increasing, then, under increasing differences, incentive compatibility reduces to local incentive compatibility.¹⁷ We state it as a lemma for future reference.

¹⁷See, for example, Bolton and Dewatripont (2005), p. 78.

Lemma 1. If u_2 has increasing differences in $(\succsim_\theta, \succsim_2)$ and $b_2 \in A_2^{A_1 \times \Theta}$ is increasing in $(\succsim_\theta, \succsim_2)$, then for any $f \in \mathcal{C}$

$$u_2(a_1, b_2(a_1, \theta^i), \theta^i) - f(a_1, b_2(a_1, \theta^i)) \geq u_2(a_1, b_2(a_1, \theta^j), \theta^i) - f(a_1, b_2(a_1, \theta^j)), \text{ for all } i, j = 1, 2, \dots, n$$

holds if and only if

$$u_2(a_1, b_2(a_1, \theta^i), \theta^i) - f(a_1, b_2(a_1, \theta^i)) \geq u_2(a_1, b_2(a_1, \theta^{i-1}), \theta^i) - f(a_1, b_2(a_1, \theta^{i-1})), \text{ for all } i = 2, \dots, n,$$

and

$$u_2(a_1, b_2(a_1, \theta^i), \theta^i) - f(a_1, b_2(a_1, \theta^i)) \geq u_2(a_1, b_2(a_1, \theta^{i+1}), \theta^i) - f(a_1, b_2(a_1, \theta^{i+1})), \text{ for all } i = 1, 2, \dots, n-1.$$

Proof of Proposition 2. **[Only if]** Suppose that (b_1^*, b_2^*) can be supported with non-renegotiable contracts. Then, there exists a perfect Bayesian equilibrium (β^*, μ^*) that induces (b_1^*, b_2^*) , i.e., $\beta_2^*[\emptyset] = f^*$, $\beta_3[f^*] = y$, $\beta_1^*[\mathcal{C}] = b_1^*$, $\beta_2^*[f^*, \theta, a_1] = b_2^*(a_1, \theta)$ for all $a_1 \in A_1$ and $\theta \in \Theta$. Given Proposition 1 we only need to prove that b_2^* is increasing. Fix orders $(\succsim_\theta, \succsim_2)$ in which u_2 has strictly increasing differences. Take any $a_1 \in A_1$ and $\theta, \theta' \in \Theta$ and assume without loss of generality, that $\theta \succsim_\theta \theta'$. Suppose, for contradiction, that $b_2^*(a_1, \theta') \succ_2 b_2^*(a_1, \theta)$. Sequential rationality of player 2 implies

$$\begin{aligned} u_2(a_1, b_2^*(a_1, \theta), \theta) - f^*(a_1, b_2^*(a_1, \theta)) &\geq u_2(a_1, b_2^*(a_1, \theta'), \theta) - f^*(a_1, b_2^*(a_1, \theta')) \\ u_2(a_1, b_2^*(a_1, \theta'), \theta') - f^*(a_1, b_2^*(a_1, \theta')) &\geq u_2(a_1, b_2^*(a_1, \theta), \theta') - f^*(a_1, b_2^*(a_1, \theta)) \end{aligned}$$

and hence

$$u_2(a_1, b_2^*(a_1, \theta'), \theta) - u_2(a_1, b_2^*(a_1, \theta), \theta) \leq u_2(a_1, b_2^*(a_1, \theta'), \theta') - u_2(a_1, b_2^*(a_1, \theta), \theta'),$$

contradicting that u_2 has strictly increasing differences in $(\succsim_\theta, \succsim_2)$. Therefore, b_2^* must be increasing in $(\succsim_\theta, \succsim_2)$.

[If] Let (b_1^*, b_2^*) be a Bayesian Nash equilibrium of G such that b_2^* is increasing. Given Proposition 1, all we need to prove is the existence of a contract $f \in \mathcal{C}$ such that $F(b_1^*, b_2^*) = \delta$ and for all $a_1 \in A_1$

$$u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) - f(a_1, b_2^*(a_1, \theta^i)) \geq u_2(a_1, b_2^*(a_1, \theta^j), \theta^i) - f(a_1, b_2^*(a_1, \theta^j)), \text{ for all } i, j = 1, 2, \dots, n. \quad (25)$$

By Lemma 1, (25) holds if and only if $Df(a_1, b_2^*) \leq U(a_1, b_2^*)$. Therefore, we need to show that for any $a_1 \in A_1$ there exists $f(a_1, b_2^*) \in \mathbb{R}^n$ such that $Df(a_1, b_2^*) \leq U(a_1, b_2^*)$. By Gale's theorem for linear inequalities (Mangasarian (1994), p. 33), there exists such an $f(a_1, b_2^*) \in \mathbb{R}^n$ if and only if for any $y \in \mathbb{R}_+^{2(n-1)}$, $D'y = 0$ implies $y'U(a_1, b_2^*) \geq 0$. It is easy to show that $D'y = 0$ if and only if $y_1 = y_2, y_3 = y_4, \dots, y_{2(n-1)-1} = y_{2(n-1)}$. Let $U(a_1, b_2^*)_i$ denote the i^{th} row of $U(a_1, b_2^*)$ and note that since b_2^* is increasing and u_2 has strictly increasing differences, $U(a_1, b_2^*)_{2i-1} + U(a_1, b_2^*)_{2i} \geq 0$, for any $i = 1, 2, \dots, n-1$. Therefore,

$$y'U(a_1, b_2^*) = \sum_{i=1}^{n-1} (U(a_1, b_2^*)_{2i-1} + U(a_1, b_2^*)_{2i}) y_{2i-1} \geq 0$$

This proves the existence of a $f \in \mathcal{C}$ such that (25) is satisfied for all $a_1 \in A_1$. Now define

$$\tilde{f}(a_1, a_2) = \begin{cases} f(a_1, a_2), & \text{if } a_1 \neq b_1^* \\ f(a_1, a_2) + \delta - F(b_1^*, b_2^*), & \text{if } a_1 = b_1^* \end{cases}$$

It is easy to verify that \tilde{f} satisfies (25) for all $a_1 \in A_1$ and $\tilde{F}(b_1^*, b_2^*) = \delta$, and this completes the proof. \square

Proof of Proposition 3. [If] Let (b_1^*, b_2^*) be a Bayesian Nash equilibrium of G such that b_2^* is increasing and renegotiation-proof. This implies that there exists $f' \in \mathcal{C}$ such that (f', b_2^*) is incentive compatible and renegotiation-proof. Let $f^*(a_1, a_2) = f'(a_1, a_2) - F'(b_1^*, b_2^*) + \delta$ for all (a_1, a_2) and note that $F^*(b_1^*, b_2^*) = \delta$. Furthermore, using Theorem 2, it can be easily checked that (f^*, b_2^*) is incentive compatible and renegotiation-proof. For any $f \neq f^*$, a_1 , and θ , let $b_{2,f}(a_1, \theta) \in \arg\max_{a_2} u_2(a_1, a_2, \theta) - f(a_1, a_2)$ and $g_{(f, \theta, a_1)} \in \arg\max_g u_2(a_1, b_{2,g}(a_1, \theta), \theta) - g(a_1, b_{2,g}(a_1, \theta))$ subject to $g(a_1, b_{2,g}(a_1, \theta')) \geq f(a_1, b_{2,f}(a_1, \theta'))$ for all θ' .

Consider the following assessment (β^*, μ^*) of $\Gamma_R(G)$: $\beta_2^*[\emptyset] = f^*$; $\beta_3[f] = y$ iff $F(b_1^*, b_{2,f}^*) \geq \delta$, $\beta_1^*[\mathcal{C}] = b_1^*$, $\beta_2^*[f^*, \theta, a_1] = b_2^*(a_1, \theta)$ for all (a_1, θ) ;

$$\beta_2^*[f, \theta, a_1] = \begin{cases} g_{(f, \theta, a_1)}, & \text{if } u_2(a_1, b_{2,g_{(f, \theta, a_1)}}(a_1, \theta), \theta) - g_{(f, \theta, a_1)}(a_1, b_{2,g_{(f, \theta, a_1)}}(a_1, \theta)) \\ & > u_2(a_1, b_{2,f}(a_1, \theta), \theta) - f(a_1, b_{2,f}(a_1, \theta)) \\ b_{2,f}(a_1, \theta), & \text{otherwise} \end{cases}$$

for any $f \neq f^*$ and (θ, a_1) ; $\beta_2^*[f, \theta, a_1, g, y] = b_{2,g}(a_1, \theta)$ and $\beta_2[f, \theta, a_1, g, n] = b_{2,f}(a_1, \theta)$ for all $f \neq f^*$ and (a_1, θ, g) ; $\beta_2[f^*, \theta, a_1, g, n] = b_2^*(a_1, \theta)$ for all (a_1, θ, g) ;

$$\beta_3^*[I_3(f^*, a_1, g)] = \begin{cases} y, & g(a_1, b_{2,g}(a_1, \theta)) > f^*(a_1, b_2^*(a_1, \theta)) \quad \forall \theta \\ n, & \text{otherwise} \end{cases}$$

and

$$\beta_3^*[I_3(f, a_1, g)] = \begin{cases} y, & \text{if } g(a_1, b_{2,g}(a_1, \theta)) \geq f(a_1, b_{2,f}(a_1, \theta)) \quad \forall \theta \\ n, & \text{otherwise} \end{cases}$$

for any a_1, g and $f \neq f^*$; $\mu^*[\mathcal{C}](f^*) = 1$; $\mu^*[I_3(f^*, a_1, g)](\theta) = p(\theta)$ if $g(a_1, b_{2,g}(a_1, \theta)) > f^*(a_1, b_2^*(a_1, \theta))$ for all θ and $\mu^*[I_3(f^*, a_1, g)](\theta') = 1$ if there exists θ' such that $f^*(a_1, b_2^*(a_1, \theta')) \geq g(a_1, b_{2,g}(a_1, \theta'))$; For any $f \neq f^*$ and (a_1, g) , $\mu^*[I_3(f, a_1, g)](\theta) = p(\theta)$ if $g(a_1, b_{2,g}(a_1, \theta)) \geq f(a_1, b_{2,f}(a_1, \theta))$ for all θ and $\mu^*[I_3(f, a_1, g)](\theta') = 1$ if there exists θ' such that $f(a_1, b_{2,f}(a_1, \theta')) > g(a_1, b_{2,g}(a_1, \theta'))$. This assessment induces (b_1^*, b_2^*) and is a renegotiation-proof perfect Bayesian equilibrium.

[Only if] Suppose that $\Gamma_R(G)$ has a renegotiation-proof perfect Bayesian equilibrium (β^*, μ^*) that induces (b_1^*, b_2^*) . Letting $\beta_2^*[\emptyset] = f^*$, we have $\beta_1^*[\mathcal{C}] = b_1^*$, $\beta_2[f^*, \theta, a_1] = b_2^*(a_1, \theta)$ for all (a_1, θ) , and $\mu^*[\mathcal{C}](f^*) = 1$. Sequential rationality of player 1 implies that

$$b_1^* \in \arg\max_{a_1} U_1(a_1, b_2^*) \quad (26)$$

whereas that of player 2 implies $u_2(a_1, b_2^*(a_1, \theta), \theta) - f^*(a_1, b_2^*(a_1, \theta)) \geq u_2(a_1, b_2^*(a_1, \theta'), \theta) - f^*(a_1, b_2^*(a_1, \theta'))$ for all a_1 and θ, θ' , which, together with increasing differences, implies that b_2^* is increasing.

We also claim that

$$b_2^*(b_1^*, \theta) \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2, \theta) \quad \forall \theta. \quad (27)$$

Suppose, for contradiction, that this is not the case for θ' and let $\hat{a}_2 \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2, \theta')$ and define $\varepsilon = u_2(b_1^*, \hat{a}_2, \theta') - u_2(b_1^*, b_2^*(b_1^*, \theta'), \theta') > 0$. Define $f'(a_1, a_2) = F^*(b_1^*, b_2^*) + \varepsilon/2$ and note that the third-party accepts f' . Assume first that f' is not renegotiated after b_1^* and note that sequential rationality of player 2 implies that $\beta_2^*[f', \theta, b_1^*] \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2, \theta)$. Let $b_{2,f'}(a_1, \theta) = \beta_2^*[f', \theta, a_1]$. Player 2's expected payoff under f' is

$$U_2(b_1^*, b_{2,f'}) - F^*(b_1^*, b_2^*) - \varepsilon/2 > U_2(b_1^*, b_2^*) - F^*(b_1^*, b_2^*)$$

contradicting that (β^*, μ^*) is a PBE. A similar argument goes through if f' is renegotiated after b_1^* .

Therefore, by (26) and (27), (b_1^*, b_2^*) is a Bayesian Nash equilibrium of G and b_2^* is increasing. Finally, suppose that b_2^* is not renegotiation-proof. This implies that for any contract f such that (f, b_2^*) is incentive compatible, there exist a_1', θ' , and an incentive compatible (g, b_2) such that $u_2(a_1', b_2(a_1', \theta'), \theta') - g(a_1', b_2(a_1', \theta')) > u_2(a_1', b_2^*(a_1', \theta'), \theta') - f(a_1', b_2^*(a_1', \theta'))$ and $g(a_1', b_2(a_1', \theta')) > f(a_1', b_2^*(a_1', \theta'))$ for all θ . This implies that, in any perfect Bayesian equilibrium, after history (f, θ', a_1') player 2 strictly prefers to renegotiate and offer g and the third-party accepts it. In other words, there exists no renegotiation-proof perfect Bayesian equilibrium which induces (b_1^*, b_2^*) , completing the proof. \square

Proof of Theorem 2. By definition $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ is not renegotiation-proof if and only if there exist $a_1 \in A_1$, $i = 1, 2, \dots, n$ and an incentive compatible $(g, b_2) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ such that $u_2(a_1, b_2(a_1, \theta^i), \theta^i) - g(a_1, b_2(a_1, \theta^i)) > u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) - f(a_1, b_2^*(a_1, \theta^i))$ and $g(a_1, b_2(a_1, \theta^j)) > f(a_1, b_2^*(a_1, \theta^j))$ for all $j = 1, 2, \dots, n$. For any $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$, let $f(a_1, b_2^*) \in \mathbb{R}^n$ be a vector whose j -th component, $j = 1, 2, \dots, n$, is given by $f(a_1, b_2^*(a_1, \theta^j))$. Note that incentive compatibility of $(g, b_2) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ is equivalent to $Dg(a_1, b_2) \leq U(a_1, b_2)$ for all $a_1 \in A_1$. Therefore, $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ is not renegotiation-proof if and only if there exist $a_1 \in A_1$, $i = 1, 2, \dots, n$ and $(g(a_1, b_2), b_2) \in \mathbb{R}^n \times A_2^{A_1 \times \Theta}$ such that $Dg(a_1, b_2) \leq U(a_1, b_2)$, $u_2(a_1, b_2(a_1, \theta^i), \theta^i) - g(a_1, b_2(a_1, \theta^i)) > u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) - f(a_1, b_2^*(a_1, \theta^i))$, and $g(a_1, b_2) \gg f(a_1, b_2^*)$. Also note that $g(a_1, b_2) \gg f(a_1, b_2^*)$ if and only if there exists an $\varepsilon \gg 0$ such that $g(a_1, b_2) = f(a_1, b_2^*) + \varepsilon$. Therefore, we have the following

Lemma 2. $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ is not renegotiation-proof if and only if there exist $a_1 \in A_1$, $i = 1, 2, \dots, n$, $b_2 \in A_2^{A_1 \times \Theta}$, and $\varepsilon \in \mathbb{R}^n$ such that $D(f(a_1, b_2^*) + \varepsilon) \leq U(a_1, b_2)$, $\varepsilon_i < u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)$, and $\varepsilon \gg 0$.

We first state a theorem of the alternative, which we will use in the sequel.

Lemma 3 (Motzkin's Theorem). *Let A and C be given matrices, with A being non-vacuous. Then either*

1. $Ax \gg 0$ and $Cx \geq 0$ has a solution x

or

2. $A'y_1 + C'y_2 = 0$, $y_1 > 0$, $y_2 \geq 0$ has a solution y_1, y_2

but not both.

Proof of Lemma 3. See Mangasarian (1994), p. 28. \square

For any $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$, $a_1 \in A_1$, $b_2 \in A_2^{A_1 \times \Theta}$, and $i = 1, 2, \dots, n$, define $V = U(a_1, b_2) - Df(a_1, b_2^*)$, $C = \begin{pmatrix} V & -D \end{pmatrix}$, and

$$A = \begin{pmatrix} I_{n+1} \\ l_i \end{pmatrix}$$

where $l_i = (u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i))e_1 - e_{i+1}$. Note that C and A depend on and are uniquely defined by (f, b_2^*) , a_1 and (i, b_2) but we suppress this dependency for notational convenience. The following lemma uses Motzkin's Theorem to express renegotiation-proofness as an alternative.

Lemma 4. $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ is renegotiation-proof if and only if for any $a_1 \in A_1$, $i = 1, 2, \dots, n$ and $b_2 \in A_2^{A_1 \times \Theta}$ there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $A'y + C'z = 0$, $y > 0$, $z \geq 0$.

Proof of Lemma 4. By Lemma 2, (f, b_2^*) is not renegotiation-proof if and only if there exist $a_1 \in A_1$, $i = 1, 2, \dots, n$, $b_2 \in A_2^{A_1 \times \Theta}$, and $\varepsilon \in \mathbb{R}^n$ such that $D(f(a_1, b_2^*) + \varepsilon) \leq U(a_1, b_2)$, $\varepsilon_i < u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)$, and $\varepsilon \gg 0$. This is true if and only if for some a_1 , i and b_2 there exists an $x \in \mathbb{R}^{n+1}$ such that $Ax \gg 0$ and $Cx \geq 0$. To see this let $\xi > 0$ and define

$$x = \begin{pmatrix} \xi \\ \xi \varepsilon \end{pmatrix}$$

Then $D(f(a_1, b_2^*) + \varepsilon) \leq U(a_1, b_2)$ if and only if $Cx \geq 0$. Also, $\varepsilon \gg 0$ and $\varepsilon_i < u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)$ if and only if $Ax \gg 0$. The lemma then follows from Motzkin's Theorem. \square

For any $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$, $b_2 \in A_2^{A_1 \times \Theta}$, $a_1 \in A_1$, and $i = 1, 2, \dots, n$, let $U(a_1, b_2)_j$ denote the j -th component of vector $U(a_1, b_2)$ and define $\alpha_1 = 1$, $\alpha_{i+1} = u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)$, and

$$\alpha_{k+1} = \sum_{j=k}^{i-1} U(a_1, b_2)_{2j-1} + \alpha_{i+1} - f(a_1, b_2^*(a_1, \theta^k)) + f(a_1, b_2^*(a_1, \theta^i)), \quad \text{for } k = 1, 2, \dots, i-1,$$

$$\alpha_{l+1} = \sum_{j=i+1}^l U(a_1, b_2)_{2(j-1)} + \alpha_{i+1} - f(a_1, b_2^*(a_1, \theta^l)) + f(a_1, b_2^*(a_1, \theta^i)), \quad \text{for } l = i+1, i+2, \dots, n,$$

$$\beta_j = U(a_1, b_2)_{2j} + U(a_1, b_2)_{2j-1}, \quad \text{for } j = 1, 2, \dots, n-1.$$

Again, note that α_j and β_j depend on and are uniquely defined by (f, b_2^*) , a_1 and (i, b_2) but we suppress this dependency in the notation. We have the following lemma.

Lemma 5. For any $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$, $b_2 \in A_2^{A_1 \times \Theta}$, $a_1 \in A_1$ and $i = 1, 2, \dots, n$, there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $A'y + C'z = 0$, $y > 0$, and $z \geq 0$ if and only if there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and

$$\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j = 0 \quad (28)$$

Proof of Lemma 5. Fix $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$, $b_2 \in A_2^{A_1 \times \Theta}$, $a_1 \in A_1$ and $i = 1, 2, \dots, n$. First note that for

any y and z , $A'y + C'z = 0$ if and only if

$$y_1 + (u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i))y_{n+2} + V'z = 0 \quad (29)$$

$$D'z = [A'y]_{-1} \quad (30)$$

where $[A'y]_{-1}$ is the n -dimensional vector obtained from $A'y$ by eliminating the first row. Recursively adding row 1 to row 2, row 2 to row 3, and so on, we can reduce $(D' \quad [A'y]_{-1})$ to a row echelon form and show that (30) holds if and only if

$$z_{2j-1} = z_{2j} + \sum_{k=1}^j y_{k+1}, \quad j = 1, 2, \dots, i-1 \quad (31)$$

$$z_{2j} = z_{2j-1} + \sum_{k=j+1}^n y_{k+1}, \quad j = i, i+1, \dots, n-1 \quad (32)$$

$$y_{n+2} = \sum_{k=1}^n y_{k+1} \quad (33)$$

Substituting (31)-(33) into (29) we get

$$\begin{aligned} y_1 + \alpha_{i+1} \sum_{k=1}^n y_{k+1} + \sum_{j=1}^{i-1} U(a_1, b_2)_{2j-1} \sum_{k=1}^j y_{k+1} + \sum_{j=i}^{n-1} U(a_1, b_2)_{2j} \sum_{k=j+1}^n y_{k+1} + \sum_{j=1}^{i-1} (U(a_1, b_2)_{2j-1} + U(a_1, b_2)_{2j}) z_{2j} \\ + \sum_{j=i}^{n-1} (U(a_1, b_2)_{2j-1} + U(a_1, b_2)_{2j}) z_{2j-1} - \sum_{k=1}^n (f(a_1, b_2^*(a_1, \theta^k)) - f(a_1, b_2^*(a_1, \theta^i))) y_{k+1} = 0 \end{aligned} \quad (34)$$

Therefore, $A'y + C'z = 0$ if and only if equations (31) through (34) hold. Now suppose that there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $y > 0$, $z \geq 0$, and (31) through (34) hold. Define $\hat{y}_j = y_j$, for $j = 1, \dots, n+1$ and

$$\hat{z}_j = \begin{cases} z_{2j}, & j = 1, \dots, i-1 \\ z_{2j-1}, & j = i, \dots, n-1 \end{cases}$$

It is easy to verify that $\hat{y} > 0$, $\hat{z} \geq 0$, and $\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j = 0$.

Conversely, suppose that there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and (28) holds. Define $y_j = \hat{y}_j$ for $j = 1, \dots, n+1$ and $y_{n+2} = \sum_{j=1}^{n+1} \hat{y}_j$. For any $j = 1, \dots, i-1$, let $z_{2j-1} = \hat{z}_j + \sum_{k=1}^j \hat{y}_{k+1}$ and $z_{2j} = \hat{z}_j$, and for any $j = i, \dots, n-1$, let $z_{2j-1} = \hat{z}_j$ and $z_{2j} = \hat{z}_j + \sum_{k=j+1}^n \hat{y}_{k+1}$. It is straightforward to show that $y > 0$, $z \geq 0$, and (31) through (34) hold. This completes the proof of Lemma 5. \square

Lemma 4 and 5 imply that $(f, b_2^*) \in \mathcal{C} \times A_2^{A_1 \times \Theta}$ is renegotiation-proof if and only if for any $a_1 \in A_1$, $i \in \{1, 2, \dots, n\}$ and $b_2 \in A_2^{A_1 \times \Theta}$, there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and equation (28) holds. We can now complete the proof of Theorem 2.

[Only if] Suppose, for contradiction, that there exist $a_1 \in A_1$, $i = 1, 2, \dots, n$ and an increasing $b_2 \in A_2^{A_1 \times \Theta}$ such that $u_2(a_1, b_2(a_1, \theta^i), \theta^i) > u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)$, but there is no $k = 1, 2, \dots, i-1$ such that (5) holds and no $l = i+1, \dots, n$ such that (6) holds. This implies that $\alpha_j > 0$ for all $j = 1, \dots, n+1$. Since u_2 has increasing differences, $\beta_j \geq 0$ for all $j = 1, \dots, n-1$. Therefore, $\hat{y} > 0$ and $\hat{z} \geq 0$ imply that $\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j > 0$, which, by Lemma 5, contradicts that (f, b_2^*) is renegotiation-proof.

[If] Fix arbitrary $a_1 \in A_1$, $i = 1, 2, \dots, n$ and increasing $b_2 \in A_2^{A_1 \times \Theta}$ such that $u_2(a_1, b_2(a_1, \theta^i), \theta^i) > u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)$. Suppose first that there exists a $k \in \{1, \dots, i-1\}$ such that (5) holds. This im-

plies that $\alpha_{i+1} > 0$ and $\alpha_{k+1} \leq 0$. Let $\hat{y}_{k+1} = 1$, $\hat{y}_{i+1} = \frac{-\alpha_{k+1}}{\alpha_{i+1}} \geq 0$, and all the other $\hat{y}_j = 0$ and $\hat{z}_j = 0$. This implies that equation (28) holds and, by Lemma 4 and 5, that (f, b_2^*) is renegotiation-proof. Suppose now that there exists an $l \in \{i+1, \dots, n\}$ such that (6) holds. Then, $\alpha_{i+1} > 0$ and $\alpha_{l+1} \leq 0$. Let $\hat{y}_{l+1} = 1$, $\hat{y}_{i+1} = \frac{-\alpha_{l+1}}{\alpha_{i+1}} \geq 0$ and all the other $\hat{y}_j = 0$ and $\hat{z}_j = 0$. This, again, implies that (28) holds and that (f, b_2^*) is renegotiation-proof. \square

Proof of Proposition 4. Suppose that b_2^* is renegotiation-proof and fix a_1 , $i = 1, \dots, n$ and a $b_2(a_1, \theta^i) \in \mathfrak{B}(a_1, i, b_2^*)$. For any $j = 1, \dots, n$, let $c_j = e_i - e_j$, where e_j is the j^{th} standard basis row vector for \mathbb{R}^n , and define

$$E_j = \begin{pmatrix} D \\ c_j \end{pmatrix}$$

Also let

$$\begin{aligned} w_k &= u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) + \sum_{j=k}^{i-1} U(a_1, b_2)_{2j-1} \\ w_l &= u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) + \sum_{j=i+1}^l U(a_1, b_2)_{2(j-1)} \end{aligned}$$

for any $k \in \{1, \dots, i-1\}$ and $l \in \{i+1, \dots, n\}$ and define

$$V_j = \begin{pmatrix} U(a_1, b_2^*) \\ -w_j \end{pmatrix}$$

Incentive compatibility of (f, b_2^*) implies that $Df(a_1, b_2^*) \leq U(a_1, b_2^*)$. Renegotiation proofness, by Theorem 2, implies that $c_k f(a_1, b_2^*) \leq -w_k$ for some $k \in \{1, \dots, i-1\}$ or $c_l f(a_1, b_2^*) \leq -w_l$ for some $l \in \{i+1, \dots, n\}$. Suppose first that there exists a $k \in \{1, \dots, i-1\}$ such that $c_k f(a_1, b_2^*) \leq -w_k$. Then we must have $E_k f(a_1, b_2^*) \leq V_k$. By Gale's theorem of linear inequalities, this implies that $x \geq 0$ and $E'_k x = 0$ implies $x' V_k \geq 0$. Denote the first $2(n-1)$ elements of x by y and the last element by z . It is easy to show that $E'_k x = 0$ implies that $y_{2j-1} = y_{2j} + z$ for $j \in \{k, k+1, \dots, i-1\}$ and $y_{2j-1} = y_{2j}$ for $j \notin \{k, k+1, \dots, i-1\}$. Therefore,

$$\begin{aligned} x' V_k &= \sum_{j=1}^{n-1} U(a_1, b_2^*)_{2j} y_{2j} + \sum_{j=1}^{n-1} U(a_1, b_2^*)_{2j-1} y_{2j-1} - z w_k \\ &= \sum_{j=1}^{n-1} (U(a_1, b_2^*)_{2j} + U(a_1, b_2^*)_{2j-1}) y_{2j} + z(-w_k + \sum_{j=k}^{i-1} U(a_1, b_2^*)_{2j-1}) \\ &\geq 0 \end{aligned}$$

This implies that $-w_k + \sum_{j=k}^{i-1} U(a_1, b_2^*)_{2j-1} \geq 0$ and hence k is a blocking type.

Similarly, we can show that, if there exists an $l \in \{i+1, \dots, n\}$ such that $c_l f(a_1, b_2^*) \leq -w_l$, then l is a blocking type, and this completes the proof. \square

Proof of Proposition 5. Let $b_2^* \in A_2^{A_1 \times \Theta}$ be an increasing strategy satisfying the conditions of the proposition. We will show that there exist an $f \in \mathcal{C}$ such that (f, b_2^*) is incentive-compatible and renegotiation-proof. Fix an $a_1 \in A_1$ and for each $i = 1, \dots, n$ and $b_2^i \in \mathcal{B}(a_1, i, b_2^*)$ pick a blocking type $m(b_2^i) = 1, \dots, n$ that satisfies the conditions given in the proposition. For each $i = 1$ and $b_2^i \in \mathcal{B}(a_1, i, b_2^*)$ define

the n -dimensional row vector $c_{b_2^i} = e_i - e_{m(b_2^i)}$, where e_j is the j^{th} standard basis row vector for \mathbb{R}^n , and the scalar $w_{b_2^i}$ as

$$w_{b_2^i} = u_2(a_1, b_2^i(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) + \mathbf{1}_{\{m(b_2^i) \leq i-1\}} \sum_{j=m(b_2^i)}^{i-1} U(a_1, b_2^i)_{2j-1} + \mathbf{1}_{\{i \leq m(b_2^i)-1\}} \sum_{j=i+1}^{m(b_2^i)} U(a_1, b_2^i)_{2(j-1)}.$$

Note that for a given $a_1 \in A_1$ and $i = 1, \dots, n$, $\mathcal{B}(a_1, i, b_2^*)$ is finite and let $\sum_{i=1}^n |\mathcal{B}(a_1, i, b_2^*)| = p$. Denote with $C(a_1)$, the $p \times n$ matrix composed of all the rows $c_{b_2^i}$ and with $W(a_1)$ the p dimensional vector with component $w_{b_2^i}$ corresponding to each b_2^i . Let $E(a_1)$ be the matrix

$$E(a_1) = \begin{pmatrix} D \\ C(a_1) \end{pmatrix}$$

and $V(a_1)$ the column vector

$$V(a_1) = \begin{pmatrix} U(a_1, b_2^*) \\ -W(a_1) \end{pmatrix}$$

Now, if for each $a_1 \in A_1$, we can find an $f(a_1, b_2^*)$ such that $E(a_1)f(a_1, b_2^*) \leq V(a_1)$ the proof would be completed. In fact, if $E(a_1)f(a_1, b_2^*) \leq V(a_1)$, then $Df(a_1, b_2^*) \leq U(a_1, b_2^*)$, which implies that (f, b_2^*) incentive compatible. Furthermore, $E(a_1)f(a_1, b_2^*) \leq V(a_1)$ implies $W(a_1) \leq -C(a_1)f(a_1, b_2^*)$ and, by Theorem 2, that (f, b_2^*) is renegotiation-proof. Gale's theorem of linear inequalities implies that there exist $f(a_1, b_2^*) \in \mathbb{R}^n$ such that $E(a_1)f(a_1, b_2^*) \leq V(a_1)$ if and only if $x \in \mathbb{R}^{p+2(n-1)}$, $x \geq 0$ and $E(a_1)'x = 0$ implies $x'V(a_1) \geq 0$. Decompose x into two vectors so that the first $2(n-1)$ elements constitute y and the remaining p components constitute z . Notice that for any $i = 1, \dots, n$ and $b_2^i \in \mathcal{B}(a_1, i, b_2^*)$ there is a corresponding element of z , which we will denote $z_{b_2^i}$.

Recursively adding row 1 to row 2, row 2 to row 3, and so on, we can reduce $E(a_1)'$ to a row echelon form and show that $E(a_1)'x = 0$ if and only if

$$y_{2j-1} = y_{2j} + \sum_{b_2^i} z_{b_2^i} [\mathbf{1}_{\{m(b_2^i) \leq j \leq i-1\}} - \mathbf{1}_{\{i \leq j \leq m(b_2^i)-1\}}] \quad (35)$$

for $j = 1, \dots, n-1$.

Let $J_- = \{j \in \{1, \dots, n-1\} : \exists b_2^i \text{ such that } i \leq j \leq m(b_2^i)-1\}$ and $J_+ = \{j \in \{1, \dots, n-1\} : \exists b_2^i \text{ such that } m(b_2^i) \leq j \leq i-1\}$ and note that $J_- \cap J_+ = \emptyset$. To see this, suppose, for contradiction, that there exists a $j \in J_- \cap J_+$. Therefore, there exists a b_2^i such that $i \leq j \leq m(b_2^i)-1$ and $b_2^{i'}$ such that $m(b_2^{i'}) \leq j \leq i'-1$. This implies that $i < i'$, $m(b_2^i) > i$, $m(b_2^{i'}) < i'$, but $m(b_2^i) > m(b_2^{i'})$, contradicting the conditions of the proposition. We can therefore write (35) as

$$y_{2j} = y_{2j-1} + \sum_{b_2^i} z_{b_2^i} \mathbf{1}_{\{i \leq j \leq m(b_2^i)-1\}} \quad (36)$$

for $j \in J_-$ and

$$y_{2j-1} = y_{2j} + \sum_{b_2^i} z_{b_2^i} \mathbf{1}_{\{m(b_2^i) \leq j \leq i-1\}} \quad (37)$$

for $j \in J_+$.

Finally note that

$$x'V(a_1) = \sum_{j=1}^{n-1} U(a_1, b_2^*)_{2j} y_{2j} + \sum_{j=1}^{n-1} U(a_1, b_2^*)_{2j-1} y_{2j-1} - \sum_{b_2^i} z_{b_2^i} w_{b_2^i}$$

Substituting from (36) and (37) we obtain

$$\begin{aligned} x'V(a_1) = & \sum_{j \in J_-} [U(a_1, b_2^*)_{2j} + U(a_1, b_2^*)_{2j-1}] y_{2j-1} + \sum_{j \in J_+} [U(a_1, b_2^*)_{2j} + U(a_1, b_2^*)_{2j-1}] y_{2j} \\ & + \sum_{b_2^i} z_{b_2^i} \left[-w_{b_2^i} + \mathbf{1}_{\{m(b_2^i) \leq i-1\}} \sum_{j=m(b_2^i)}^{i-1} U(a_1, b_2^*)_{2j-1} + \mathbf{1}_{\{i \leq m(b_2^i)-1\}} \sum_{j=i}^{m(b_2^i)-1} U(a_1, b_2^*)_{2j} \right] \end{aligned}$$

Increasing differences, the definition of $m(b_2^i)$, and $y, z \geq 0$ imply that $x'V \geq 0$, and the proof is completed. \square

Proof of Theorem 3. We need the following definition

Definition 10. For any $b_2 \in A_2^{A_1 \times \Theta}$ we say that (a_1, i) , $i \in \{1, 2, \dots, n\}$ has *right deviation (left deviation)* at b_2 if there exists an $a_2 \in A_2$ such that $a_2 \succsim_2 b_2(a_1, \theta^i)$ ($b_2(a_1, \theta^i) \succsim_2 a_2$) and $u_2(a_1, a_2, \theta^i) > u_2(a_1, b_2(a_1, \theta^i), \theta^i)$. Otherwise, we say that i has *no right deviation (no left deviation)* at b_2 .

For any $b_2 \in A_2^{A_1 \times \Theta}$ and (a_1, i) , $i \in \{1, \dots, n\}$, that has right deviation at b_2 , define

$$R(a_1, i) = \{k > i : b_2(a_1, \theta^k) \in BR_2(a_1, \theta^k) \text{ and } i < j < k \text{ implies that } (a_1, j) \text{ has no left deviation at } b_2\}$$

Similarly, for any (a_1, i) with $i \in \{1, \dots, n\}$, that has a left deviation at b_2 , define

$$L(a_1, i) = \{k < i : b_2(a_1, \theta^k) \in BR_2(a_1, \theta^k) \text{ and } k < j < i \text{ implies that } (a_1, j) \text{ has no right deviation at } b_2\},$$

We need the following lemma:

Lemma 6. b_2^* is renegotiation-proof if for any (a_1, i_1) that has right deviation and any (a_1, i_2) that has left deviation at b_2^* , $R(a_1, i_1) \neq \emptyset$, $L(a_1, i_2) \neq \emptyset$, and $i_1 < i_2$ implies $R(a_1, i_1) \cap L(a_1, i_2) \neq \emptyset$.

Proof of Lemma 6. Fix $a_1 \in A_1$, $i \in \{1, \dots, n\}$, and $b_2^i \in \mathfrak{B}(a_1, i, b_2^*)$. Since A_2 is linearly ordered, we have $b_2^i(a_1, \theta_i) \succsim_2 b_2^*(a_1, \theta^i)$ or $b_2^*(a_1, \theta_i) \succsim_2 b_2^i(a_1, \theta^i)$. First, assume that $b_2^i(a_1, \theta_i) \succsim_2 b_2^*(a_1, \theta^i)$, i.e., (a_1, i) has right deviation at b_2^* , and note that $R(a_1, i) \neq \emptyset$ by assumption. Let $J = \{j \in \mathbb{N} : i+1 \leq j \leq \min R(a_1, i) - 1 \text{ and } b_2^*(a_1, \theta^j) \succ_2 b_2^i(a_1, \theta^j)\}$. If $J = \emptyset$, let $m(b_2^i) = \min R(a_1, i)$ and if $J \neq \emptyset$, let $m(b_2^i) = \min J$. It is simple to show that

$$\begin{aligned} & \sum_{j=i+1}^{m(b_2^i)} \left(u_2(a_1, b_2^i(a_1, \theta^{j-1}), \theta^j) - u_2(a_1, b_2^*(a_1, \theta^{j-1}), \theta^j) - [u_2(a_1, b_2^i(a_1, \theta^{j-1}), \theta^{j-1}) - u_2(a_1, b_2^*(a_1, \theta^{j-1}), \theta^{j-1})] \right) \\ & + u_2(a_1, b_2^*(a_1, \theta^{m(b_2^i)}), \theta^{m(b_2^i)}) - u_2(a_1, b_2^i(a_1, \theta^{m(b_2^i)}), \theta^{m(b_2^i)}) \geq 0 \quad (38) \end{aligned}$$

Inequality (38) implies that $m(b_2^i)$ is a blocking type.

Now assume that $b_2^*(a_1, \theta_i) \succsim_2 b_2^i(a_1, \theta^i)$, i.e., (a_1, i) has left deviation at b_2^* , and note that $L(a_1, i) \neq \emptyset$. Let $J = \{j \in \mathbb{N} : \max L(i) + 1 \leq j \leq i-1 \text{ and } b_2^i(a_1, \theta^j) \succ_2 b_2^*(a_1, \theta^j)\}$. If $J = \emptyset$, let $m(b_2^i) = \max L(i)$

and if $J \neq \emptyset$, let $m(b_2^i) = \max J$ and note that

$$\sum_{j=m(b_2^i)}^{i-1} \left(u_2(a_1, b_2^*(a_1, \theta^{j+1}), \theta^{j+1}) - u_2(a_1, b_2^i(a_1, \theta^{j+1}), \theta^{j+1}) - [u_2(a_1, b_2^*(a_1, \theta^j), \theta^j) - u_2(a_1, b_2^i(a_1, \theta^j), \theta^j)] \right) + u_2(a_1, b_2^*(a_1, \theta^{m(b_2^i)}), \theta^{m(b_2^i)}) - u_2(a_1, b_2^i(a_1, \theta^{m(b_2^i)}), \theta^{m(b_2^i)}) \geq 0 \quad (39)$$

Inequality (39) implies that $m(b_2^i)$ is a blocking type.

Finally assume that there exist (a_1, i_1) and (a_1, i_2) with $i_1 < i_2$ such that $m(b_2^{i_1}) > i_1$ and $m(b_2^{i_2}) < i_2$. This implies that (a_1, i_1) has right deviation and (a_1, i_2) has left deviation at b_2^* , which imply that $R(a_1, i_1) \neq \emptyset$, $L(a_1, i_2) \neq \emptyset$ and $R(a_1, i_1) \cap L(a_1, i_2) \neq \emptyset$. But this implies that $m(b_2^{i_1}) \leq m(b_2^{i_2})$ and the proof is completed by applying Proposition 5. \square

We can now proceed to the proof of Theorem 3.

[If] Assume that (a_1^*, a_2^*) satisfies conditions (1) and (2) of the theorem. Let $\hat{b}_2(a_1) \in \operatorname{argmin}_{a_2 \in BR_2(a_1, \theta^n)} u_1(a_1, a_2, \theta^n)$ and define $b_2^* \in A_2^{A_1 \times \Theta}$ as follows: $b_2^*(a_1^*, \theta^i) = a_2^*(\theta^i)$ for all $i = 1, \dots, n$ and

$$b_2^*(a_1, \theta) = \begin{cases} \underline{a}_2, & \theta < \theta^n \\ \hat{b}_2(a_1), & \theta = \theta^n \end{cases} \quad (40)$$

for $a_1 \neq a_1^*$.

First, note that (a_1^*, a_2^*) is the outcome induced by the strategy profile (a_1^*, b_2^*) because by construction $b_2^*(a_1^*, \theta^i) = a_2^*(\theta^i)$ for all $i = 1, \dots, n$. We now prove that the strategy profile (a_1^*, b_2^*) can be supported with renegotiation-proof contracts. By Proposition 3, we need to prove that (a_1^*, b_2^*) is a Bayesian Nash equilibrium of G and b_2^* is increasing and renegotiation proof.

Condition (1) of the theorem implies that $b_2^*(a_1^*, \theta^i) \in BR_2(a_1^*, \theta)$ for all θ , whereas condition (2) implies

$$U_1(a_1^*, b_2^*) = U_1(a_1^*, a_2^*) \geq \underline{U}_1^+ \geq U_1(a_1, b_2^*), \forall a_1 \neq a_1^*$$

Therefore, $a_1^* \in BR_1(b_2^*)$ and (a_1^*, b_2^*) is a Bayesian Nash equilibrium of G .

For any $a_1 \neq a_1^*$, $b_2^*(a_1, \theta)$ is increasing in θ by construction, and $b_2^*(a_1^*, \theta)$ is increasing because $a_2^*(\theta) \in BR_2(a_1^*, \theta)$ for all θ and u_2 has strictly increasing differences in $(\succsim_\theta, \succsim_2)$.

Finally, we prove that b_2^* is renegotiation proof. For any i , $\mathfrak{B}(a_1^*, i, b_2^*)$ is empty. For any $a_1 \neq a_1^*$, there is no (a_1, i) with left deviation by construction of b_2^* , and for any (a_1, i) with right deviation, we have $n \in R(i)$. Lemma 6, therefore, implies that b_2^* is renegotiation-proof.

[Only if] We will use the following lemma

Lemma 7. *If $b_2 \in A_2^{A_1 \times \Theta}$ is renegotiation-proof, then (a_1, θ^n) has no right deviation at b_2 for any $a_1 \in A_1$*

Proof of Lemma 7. Suppose, for contradiction, that there exists an $a_1' \in A_1$ such that (a_1', θ^n) has right deviation at b_2 , i.e., there exists an $a_2' \in A_2$ such that $a_2' \succsim_2 b_2(a_1', \theta^n)$ and $u_2(a_1', a_2', \theta^n) > u_2(a_1', b_2(a_1', \theta^n), \theta^n)$. Define

$$b_2'(a_1', \theta) = \begin{cases} a_2', & \theta = \theta^n \\ b_2(a_1', \theta^n), & \theta < \theta^n \end{cases}$$

Note that b'_2 is increasing and therefore $b'_2 \in \mathfrak{B}(a'_1, n, b_2)$. It is easy to show that for (a'_1, n, b'_2) there is no blocking type and therefore, by Proposition 4, b_2 is not renegotiation proof. \square

Proposition 3 implies that (a_1^*, a_2^*) can be supported with renegotiation-proof contracts only if there exists an increasing and renegotiation-proof $b_2^* \in A_2^{A_1 \times \Theta}$ such that (a_1^*, b_2^*) is a Bayesian Nash equilibrium of G and $b_2^*(a_1^*, \theta) = a_2^*(\theta)$ for all θ . This immediately implies that $a_2^*(\theta) \in BR_2(a_1^*, \theta)$ for all θ , i.e., condition (1) of the theorem holds. Suppose, for contradiction, that condition (2) does not hold, i.e., there exists an a'_1 such that $U_1(a_1^*, a_2^*) < U_1(a'_1, b_2^+)$. Note that Lemma 7 and the fact that u_1 is increasing in a_2 imply that $U_1(a_1, b_2) \geq U_1(a_1, b_2^+)$ for any a_1 and any increasing and renegotiation-proof $b_2 \in A_2^{A_1 \times \Theta}$, which, in turn, implies that $U_1(a'_1, b_2^*) \geq U_1(a'_1, b_2^+)$ for any a_1 . But then,

$$U_1(a_1^*, b_1^*) = U_1(a_1^*, a_2^*) < U_1(a'_1, b_2^+) \leq U_1(a'_1, b_2^*),$$

which contradicts that (a_1^*, b_2^*) is a Bayesian Nash equilibrium of G . \square

Before we continue towards the proof of Proposition 8, remember that for any $a_1 \in A_1$ and $b_2 \in A_2^{A_1 \times \Theta}$

$$\begin{aligned} U(a_1, b_2)_1 &= u_2(a_1, b_2(a_1, \theta^1), \theta^1) - u_2(a_1, b_2(a_1, \theta^2), \theta^1) \\ U(a_1, b_2)_2 &= u_2(a_1, b_2(a_1, \theta^2), \theta^2) - u_2(a_1, b_2(a_1, \theta^1), \theta^2) \end{aligned}$$

We first characterize durable strategy-contract pairs:

Proposition 9. $(f, b_2^*) \in \mathbb{R}^{A_1 \times A_2} \times A_2^{A_1 \times \Theta}$ is durable if and only if for any $a_1 \in A_1$ and increasing $b_2 \in A_2^{A_1 \times \Theta}$ such that

$$\sum_{i=1}^2 p(\theta^i) \left[u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) \right] > 0$$

one of the following is true:

$$u_2(a_1, b_2(a_1, \theta^1), \theta^1) - u_2(a_1, b_2^*(a_1, \theta^1), \theta^1) + p(\theta^2)U(a_1, b_2)_2 < p(\theta^2) \left[f(a_1, b_2^*(a_1, \theta^2)) - f(a_1, b_2^*(a_1, \theta^1)) \right] \quad (41)$$

or

$$u_2(a_1, b_2(a_1, \theta^2), \theta^2) - u_2(a_1, b_2^*(a_1, \theta^2), \theta^2) + p(\theta^1)U(a_1, b_2)_1 < p(\theta^1) \left[f(a_1, b_2^*(a_1, \theta^1)) - f(a_1, b_2^*(a_1, \theta^2)) \right] \quad (42)$$

Proof of Proposition 9. Fix $(f, b_2^*) \in \mathbb{R}^{A_1 \times A_2} \times A_2^{A_1 \times \Theta}$ and take any $a_1 \in A_1$ and $b_2 \in A_2^{A_1 \times \Theta}$. We say that (f, b_2^*) is not durable against (a_1, b_2) if there exists $g \in \mathbb{R}^{A_1 \times A_2}$ such that (g, b_2) is incentive compatible and (18) and (19) hold (with at least one inequality being strict). Otherwise, we say that (f, b_2^*) is durable against (a_1, b_2) . Obviously, (f, b_2^*) is not durable if and only if it is not durable against some (a_1, b_2) .

Fix a_1, b_2 and define the matrices $U(b_2)$, D , f as before (this time omitting the reference to a_1).

Also define an n -vector k , whose i th row is given by

$$k_i = u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i), \quad i = 1, \dots, n$$

and

$$V = U - Df, \quad C = \begin{bmatrix} V & -D \end{bmatrix}, \quad A = \begin{pmatrix} 1 & 0_{1 \times n} \end{pmatrix}, \quad B = \begin{pmatrix} k & -I_{n \times n} \\ 0 & p_{1 \times n} \end{pmatrix}$$

where $p_i = p(\theta^i)$.

Using these definitions we can show that (f, b_2^*) is not durable against (a_1, b_2) if and only if there exists an $x \in R^{n+1}$ such that $Ax \gg 0$, $Bx > 0$, and $Cx \geq 0$. It then follows from Slater's theorem of the alternative (see Mangasarian (1994), p. 27) that (f, b_2^*) is durable against (a_1, b_2) if and only if there exist y, t, z such that $A'y + B't + C'z = 0$ and $y > 0$, $t \geq 0$, and $z \geq 0$ or $y \geq 0$, $t \gg 0$, and $z \geq 0$. We first prove the following claim:

Lemma 8. *(f, b_2^*) is durable against (a_1, b_2) such that b_2 is increasing and $p_1 k_1 + p_2 k_2 > 0$ if and only if $p_1 v_1 + k_2 < 0$ or $p_2 v_2 + k_1 < 0$.*

Proof. As shown above, (f, b_2^*) is durable against (a_1, b_2) if and only if there exist y, t, z such that $A'y + B't + C'z = 0$ and $y > 0$, $t \geq 0$, and $z \geq 0$ or $y \geq 0$, $t \gg 0$, and $z \geq 0$. Simple algebra shows that $A'y + B't + C'z = 0$ if and only if

$$y + t_1 + k_1 t_2 + k_2 t_3 + v_1 z_1 + v_2 z_2 = 0 \quad (43)$$

$$-t_2 + p_1 t_4 - z_1 + z_2 = 0 \quad (44)$$

$$-t_3 + p_2 t_4 + z_1 - z_2 = 0 \quad (45)$$

Solve for t_2 and t_3 from (44) and (45) and substitute into (43) so that (43) becomes

$$H \equiv y + t_1 + (p_1 k_1 + p_2 k_2) t_4 + (v_1 + k_2 - k_1) z_1 + (v_2 + k_1 - k_2) z_2 = 0. \quad (46)$$

Therefore, (f, b_2^*) is durable against (a_1, b_2) if and only if there exist y, t, z such that (44), (45), (46) hold and $y > 0$, $t \geq 0$, and $z \geq 0$ or $y \geq 0$, $t \gg 0$, and $z \geq 0$.

To prove the lemma, suppose that (f, b_2^*) is durable against (a_1, b_2) such that b_2 is increasing and $p_1 k_1 + p_2 k_2 > 0$. Then, as shown above, (44), (45), (46) hold and $y > 0$, $t \geq 0$, and $z \geq 0$ or $y \geq 0$, $t \gg 0$, and $z \geq 0$. Assume first that $y > 0$, $t \geq 0$, and $z \geq 0$. Equation (46), $p_1 k_1 + p_2 k_2 > 0$, $y > 0$, $t \geq 0$, and $z \geq 0$ imply that $v_1 + k_2 - k_1 < 0$ or $v_2 + k_1 - k_2 < 0$.

Assume first that $v_1 + k_2 - k_1 < 0$. Then, $t_2 \geq 0$ and (44) imply that $z_1 \leq p_1 t_4 + z_2$ and hence $(v_1 + k_2 - k_1) z_1 \geq (v_1 + k_2 - k_1)(p_1 t_4 + z_2)$. Therefore,

$$\begin{aligned} H &\geq y + t_1 + (p_1 k_1 + p_2 k_2) t_4 + (v_1 + k_2 - k_1)(p_1 t_4 + z_2) + (v_2 + k_1 - k_2) z_2 \\ &= y + t_1 + (p_1 v_1 + k_2) t_4 + (v_1 + v_2) z_2 \\ &\geq y + (p_1 v_1 + k_2) t_4 \end{aligned}$$

since $t_1 \geq 0$, $z_2 \geq 0$, and $v_1 + v_2 \geq 0$ because b_2 is increasing and u_2 has increasing differences in (a_2, θ) . $H = 0$, $y > 0$, and $t_4 \geq 0$ imply that $p_1 v_1 + k_2 < 0$.

Assume now that $v_2 + k_1 - k_2 < 0$. Then, $t_3 \geq 0$ and (45) imply that $z_2 \leq p_2 t_4 + z_1$ and hence $(v_2 + k_1 - k_2)z_2 \geq (v_2 + k_1 - k_2)(p_2 t_4 + z_1)$. Therefore,

$$\begin{aligned} H &\geq y + t_1 + (p_1 k_1 + p_2 k_2) t_4 + (v_1 + k_2 - k_1) z_1 + (v_2 + k_1 - k_2)(p_2 t_4 + z_1) \\ &= y + t_1 + (p_2 v_2 + k_1) t_4 + (v_1 + v_2) z_1 \\ &\geq y + (p_2 v_2 + k_1) t_4 \end{aligned}$$

since $t_1 \geq 0$, $z_1 \geq 0$, and $v_1 + v_2 \geq 0$ because b_2 is increasing and u_2 has increasing differences in (a_2, θ) . $H = 0$, $y > 0$, and $t_4 \geq 0$ imply that $p_2 v_2 + k_1 < 0$.

Assume now that $y \geq 0$, $t \gg 0$, and $z \geq 0$. Equation (46), $p_1 k_1 + p_2 k_2 > 0$, $y \geq 0$, $t \gg 0$, and $z \geq 0$ imply that $v_1 + k_2 - k_1 < 0$ or $v_2 + k_1 - k_2 < 0$.

Assume first that $v_1 + k_2 - k_1 < 0$. Then, $t_2 > 0$ and (44) imply that $z_1 < p_1 t_4 + z_2$ and hence $(v_1 + k_2 - k_1)z_1 > (v_1 + k_2 - k_1)(p_1 t_4 + z_2)$. Therefore,

$$\begin{aligned} H &> y + t_1 + (p_1 k_1 + p_2 k_2) t_4 + (v_1 + k_2 - k_1)(p_1 t_4 + z_2) + (v_2 + k_1 - k_2) z_2 \\ &= y + t_1 + (p_1 v_1 + k_2) t_4 + (v_1 + v_2) z_2 \\ &> (p_1 v_1 + k_2) t_4 \end{aligned}$$

since $y \geq 0$, $t_1 > 0$, $z_2 \geq 0$, and $v_1 + v_2 \geq 0$ because b_2 is increasing and u_2 has increasing differences in (a_2, θ) . $H = 0$ and $t_4 > 0$ imply that $p_1 v_1 + k_2 < 0$.

Similarly, one can show that $v_2 + k_1 - k_2 < 0$ implies that $p_2 v_2 + k_1 < 0$. Therefore, we have shown that durability implies $p_1 v_1 + k_2 < 0$ or $p_2 v_2 + k_1 < 0$.

Conversely, suppose that $p_1 v_1 + k_2 < 0$. Let $y = -(p_1 v_1 + k_2)$, $t_1 = t_2 = 0$, $t_3 = t_4 = 1$, $z_1 = p_1$, $z_2 = 0$. It is easy to check that (44), (45), (46) hold and $y > 0$, $t \geq 0$, and $z \geq 0$. Similarly, suppose that $p_2 v_2 + k_1 < 0$, and let $y = -(p_2 v_2 + k_1)$, $t_1 = t_3 = 0$, $t_2 = t_4 = 1$, $z_1 = 0$, $z_2 = p_2$. One can easily show that (44), (45), (46) are satisfied and $y > 0$, $t \geq 0$, and $z \geq 0$. We have shown that if $p_1 v_1 + k_2 < 0$ or $p_2 v_2 + k_1 < 0$, then (f, b_2^*) is durable against (a_1, b_2) . \square

Lemma 9. (f, b_2^*) is durable against (a_1, b_2) if $p_1 k_1 + p_2 k_2 \leq 0$.

Proof. Fix any (a_1, b_2) and assume $p_1 k_1 + p_2 k_2 \leq 0$. Suppose, for contradiction, that (f, b_2^*) is not durable against (a_1, b_2) . This implies that there exists a contract $g \in \mathbb{R}^{A_1 \times A_2}$ such that (g, b_2) is incentive compatible and (18) and (19) hold (with at least one inequality being strict). Define $U(\theta) = u_2(a_1, b_2(a_1, \theta), \theta)$ and $U^*(\theta) = u_2(a_1, b_2^*(a_1, \theta), \theta)$, $g(\theta) = g(a_1, b_2(a_1, \theta))$, $f(\theta) = f(b_2^*(a_1, \theta))$. Then, we have

$$U(\theta_1) - U^*(\theta_1) \geq g(\theta_1) - f(\theta_1) \tag{47}$$

$$U(\theta_2) - U^*(\theta_2) \geq g(\theta_2) - f(\theta_2) \tag{48}$$

$$p_1(g(\theta_1) - f(\theta_1)) + p_2(g(\theta_2) - f(\theta_2)) \geq 0 \tag{49}$$

with at least one strict inequality. Now, $p_1(U(\theta_1) - U^*(\theta_1)) + p_2(U(\theta_2) - U^*(\theta_2)) = p_1 k_1 + p_2 k_2 \leq 0$ implies that

$$0 \leq p_1(g(\theta_1) - f(\theta_1)) + p_2(g(\theta_2) - f(\theta_2)) \leq p_1(U(\theta_1) - U^*(\theta_1)) + p_2(U(\theta_2) - U^*(\theta_2)) \leq 0$$

which, in turn, implies that

$$p_1(g(\theta_1) - f(\theta_1)) + p_2(g(\theta_2) - f(\theta_2)) = 0$$

Therefore, (47) or (48) must hold with strict inequality. Suppose (47) holds strictly. Then,

$$0 = p_1(g(\theta_1) - f(\theta_1)) + p_2(g(\theta_2) - f(\theta_2)) < p_1(U(\theta_1) - U^*(\theta_1)) + p_2(U(\theta_2) - U^*(\theta_2)) \leq 0$$

a contradiction. Similarly, assuming that (48) holds strictly leads to a contradiction. This proves the lemma. \square

We are now ready to finish the proof of Proposition 9.

[Only If] Suppose, for contradiction, that there exist an $a_1 \in A_1$ and an increasing $b_2 \in A_2^{A_1 \times \Theta}$ such that $p_1 k_1 + p_2 k_2 > 0$ and $p_1 v_1 + k_2 \geq 0$ and $p_2 v_2 + k_1 \geq 0$. Lemma 8 implies that (f, b_2^*) is not durable against (a_1, b_2) , a contradiction.

[If] Suppose now that the conditions of the proposition hold. Fix any $a_1 \in A_1$ and incentive compatible $(g, b_2) \in \mathbb{R}^{A_1 \times A_2} \times A_2^{A_1 \times \Theta}$. If $p_1 k_1 + p_2 k_2 \leq 0$, then Lemma 9 implies that (f, b_2^*) is durable. So, suppose that $p_1 k_1 + p_2 k_2 > 0$ and note that incentive compatibility of (g, b_2) implies that b_2 is increasing. Therefore, Lemma 8 implies that (f, b_2^*) is durable. \square

Proof of Proposition 6. [Only if] Assume that b_2^* is durable. Fix any $a_1 \in A_1$ and an increasing b_2 such that $p_1 k_1 + p_2 k_2 > 0$. Definition of durability and Proposition 9 imply that there exist a contract f such that (f, b_2^*) is incentive compatible and $k_2 + p_1 v_1 < 0$ or $k_1 + p_2 v_2 < 0$. Assume first that $k_2 + p_1 v_1 < 0$, that is $[u_2(a_1, b_2(a_1, \theta_2), \theta_2) - u_2(a_1, b_2^*(a_1, \theta_2), \theta_2)] + p_1[u_2(a_1, b_2(a_1, \theta_1), \theta_1) - u_2(a_1, b_2(a_1, \theta_2), \theta_1)] < p_1[f(a_1, b_2^*(a_1, \theta_1)) - f(a_1, b_2^*(a_1, \theta_2))]$. Since (f, b_2^*) is incentive compatible, then $f(a_1, b_2^*(a_1, \theta_1)) - f(a_1, b_2^*(a_1, \theta_2)) \leq u_2(a_1, b_2^*(a_1, \theta_1), \theta_1) - u_2(a_1, b_2^*(a_1, \theta_2), \theta_1)$, which implies (22). Similarly, assuming $k_1 + p_2 v_2 < 0$ implies (21).

[If] Suppose that all the conditions of the proposition are satisfied and fix any $a_1 \in A_1$ and incentive compatible (g, b_2) . If $p_1 k_1 + p_2 k_2 \leq 0$, then Lemma 9 implies that (f, b_2^*) is durable. So, suppose that $p_1 k_1 + p_2 k_2 > 0$ and note that incentive compatibility of (g, b_2) implies that b_2 is increasing. First assume that (22) is satisfied; define $f(a_1, a_2)$ such that $u_2(a_1, b_2^*(a_1, \theta_1), \theta_1) - f(a_1, b_2^*(a_1, \theta_1)) = u_2(a_1, b_2^*(a_1, \theta_2), \theta_1) - f(a_1, b_2^*(a_1, \theta_2))$ and $f(a_1, a_2) = \infty$ if $a_2 \neq b_2^*(a_1, \theta_1)$ or $a_2 \neq b_2^*(a_1, \theta_2)$. Such an f is incentive compatible because, first, $u_2(a_1, b_2^*(a_1, \theta_1), \theta_1) - f(a_1, b_2^*(a_1, \theta_1)) \geq u_2(a_1, b_2^*(a_1, \theta_2), \theta_1) - f(a_1, b_2^*(a_1, \theta_2))$ by construction. Secondly, since u_2 has increasing differences in $(\succsim_2, \succsim_\theta)$ and b_2^* is increasing in θ the following is true:

$$\begin{aligned} u_2(a_1, b_2^*(a_1, \theta_2), \theta_2) - u_2(a_1, b_2^*(a_1, \theta_1), \theta_2) &\geq u_2(a_1, b_2^*(a_1, \theta_2), \theta_1) - u_2(a_1, b_2^*(a_1, \theta_1), \theta_1) \\ &= f(a_1, b_2^*(a_1, \theta_2)) - f(a_1, b_2^*(a_1, \theta_1)) \end{aligned}$$

This implies $u_2(a_1, b_2^*(a_1, \theta_2), \theta_2) - f(a_1, b_2^*(a_1, \theta_2)) \geq u_2(a_1, b_2^*(a_1, \theta_1), \theta_2) - f(a_1, b_2^*(a_1, \theta_1))$. Finally (f, b_2^*) is durable because

$$\begin{aligned} u_2(a_1, b_2(a_1, \theta_2), \theta_2) - u_2(a_1, b_2^*(a_1, \theta_2), \theta_2) + p_1[u_2(a_1, b_2(a_1, \theta_1), \theta_1) - u_2(a_1, b_2(a_1, \theta_2), \theta_1)] \\ < p_1[u_2(a_1, b_2^*(a_1, \theta_1), \theta_1) - u_2(a_1, b_2^*(a_1, \theta_2), \theta_1)] = p_1[f(a_1, b_2^*(a_1, \theta_1)) - f(a_1, b_2^*(a_1, \theta_2))] \end{aligned}$$

Finally assume that (21) is satisfied. By setting f such that $u_2(a_1, b_2^*(a_1, \theta_2), \theta_2) - f(a_1, b_2^*(a_1, \theta_2)) = u_2(a_1, b_2^*(a_1, \theta_1), \theta_2) - f(a_1, b_2^*(a_1, \theta_1))$, we similarly obtain that (f, b_2^*) is incentive compatible and durable. \square

We will now prove results that will help prove Proposition 8 but might also be of interest on their own.

Definition 11. For any $b_2 \in A_2^{A_1 \times \Theta}$ we say that type $i = 1, 2$ has *right deviation* at b_2 if there exist $a_1 \in A_1$ and $a_2 \in A_2$ such that $a_2 \succsim_2 b_2(a_1, \theta^i)$ and $u_2(a_1, a_2, \theta^i) > u_2(a_1, b_2(a_1, \theta^i), \theta^i)$. Otherwise, we say that i has *no right deviation* at b_2 . Definition of *left deviation* is similar.

Lemma 10. A strategy $b_2^* \in A_2^{A_1 \times \Theta}$ is durable only if type 1 has no left deviation and type 2 has no right deviation at b_2^* .

Proof of Lemma 10. Suppose, for contradiction, that type 1 has left deviation at b_2^* . This implies that there exist a_1 and $\hat{a}_2 \succsim_2 b_2^*(a_1, \theta^1)$ such that $u_2(a_1, \hat{a}_2, \theta^1) > u_2(a_1, b_2^*(a_1, \theta^1), \theta^1)$. Define

$$b_2(a_1, \theta) = \begin{cases} \hat{a}_2, & \theta = \theta^1 \\ b_2^*(a_1, \theta^2), & \theta = \theta^2 \end{cases}$$

and note that

$$\sum_{i=1}^2 p(\theta^i) [u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)] = p(\theta^1) [u_2(a_1, b_2(a_1, \theta^1), \theta^1) - u_2(a_1, b_2^*(a_1, \theta^1), \theta^1)] > 0$$

Also note that

$$U(a_1, b_2^*)_1 - U(a_1, b_2)_1 = U(a_1, b_2^*(a_1, \theta^1), \theta^1) - U(a_1, b_2(a_1, \theta^1), \theta^1) < 0 = u_2(a_1, b_2(a_1, \theta^2), \theta^2) - u_2(a_1, b_2^*(a_1, \theta^2), \theta^2)$$

contradicting, (22). We will first show that

$$u_2(a_1, b_2(a_1, \theta^1), \theta^1) - u_2(a_1, b_2^*(a_1, \theta^1), \theta^1) + p(\theta^2) [u_2(a_1, b_2^*(a_1, \theta^1), \theta^2) - u_2(a_1, b_2(a_1, \theta^1), \theta^2)] \geq 0 \quad (50)$$

If, $u_2(a_1, b_2^*(a_1, \theta^1), \theta^2) - u_2(a_1, b_2(a_1, \theta^1), \theta^2) \geq 0$, then (50) holds trivially. Suppose therefore that $u_2(a_1, b_2^*(a_1, \theta^1), \theta^2) - u_2(a_1, b_2(a_1, \theta^1), \theta^2) < 0$. In this case, (50) holds if

$$u_2(a_1, b_2(a_1, \theta^1), \theta^1) - u_2(a_1, b_2^*(a_1, \theta^1), \theta^1) + u_2(a_1, b_2^*(a_1, \theta^1), \theta^2) - u_2(a_1, b_2(a_1, \theta^1), \theta^2) \geq 0,$$

which follows from $b_2(a_1, \theta^1) = \hat{a}_2 \succsim_2 b_2^*(a_1, \theta^1)$ and increasing differences. But

$$u_2(a_1, b_2^*(a_1, \theta^1), \theta^2) - u_2(a_1, b_2(a_1, \theta^1), \theta^2) = U(a_1, b_2)_2 - U(a_1, b_2^*)_2$$

which, together with (50), contradicts (21). The case of right deviation is proved similarly. \square

Lemma 11. Suppose that u_1 is increasing in a_2 . If an outcome $(a_1^*, a_2^*(\theta))$ can be supported with durable contracts, it can also be supported with renegotiation-proof contracts.

Proof. Suppose $(a_1^*, a_2^*(\theta))$ can be supported with a durable strategy b_2^* . Lemma 10 implies that $b_2^*(a_1, \theta^1) \precsim_2 br_2(a_1, \theta^1)$ and $b_2^*(a_1, \theta^2) \precsim_2 br_2(a_1, \theta^2)$ for any best response selection br_2 . Also, we must have $b_2^*(a_1^*, \theta) \in BR_2(a_1^*, \theta)$, $\forall \theta$ and $U_1(a_1^*, b_2^*) \geq U_1(a_1, b_2^*)$, $\forall a_1$. Consider

$$\hat{b}_2(a_1, \theta) = \begin{cases} b_2^*(a_1, \theta^1), & \theta = \theta^1 \\ br_2(a_1, \theta^2), & \theta = \theta^2 \end{cases}$$

and note that this strategy is increasing and renegotiation-proof (RP). Now,

$$U_1(a_1^*, b_2^*) \geq U_1(a_1, b_2^*) \geq U_1(a_1, \hat{b}_2), \forall a_1,$$

where the last inequality follows from the fact that u_2 is increasing in a_2 . We also have $b_2(a_1^*, \theta) \in BR(a_1^*, \theta) \forall \theta$, which implies that (a_1^*, \hat{b}_2) supports $(a_1^*, a_2^*(\theta))$. \square

Definition 12. u_2 is *single-peaked* in \precsim_2 if for all $br_2 \in BR_2$ and $(a_1, \theta) \in A_1 \times \Theta$, $br_2(a_1, \theta) \precsim_1 a'_2 \precsim_2 a_2$ implies $u_2(a_1, a'_2, \theta) \geq u_2(a_1, a_2, \theta)$ and $a_2 \precsim_2 a'_2 \precsim_2 br_2(a_1, \theta)$ implies $u_2(a_1, a'_2, \theta) \geq u_1(a_1, a_2, \theta)$.

Lemma 12. Assume that u_2 is single-peaked and let $b_2^* \in A_2^{A_1 \times \Theta}$ be such that $b_2^*(a_1, \theta^1) \precsim_2 br_2(a_1, \theta^1)$ and $b_2^*(a_1, \theta^2) \precsim_2 br_2(a_1, \theta^2)$ for any best response selection br_2 . Fix $br_2 \in BR_2$ and define

$$\hat{b}_2(a_1, \theta) = \begin{cases} b_2^*(a_1, \theta^1), & \theta = \theta^1 \\ br_2(a_1, \theta^2), & \theta = \theta^2 \end{cases}$$

If b_2^* is durable, then \hat{b}_2 is durable.

Proof of Lemma 12. Suppose, for contradiction, that b_2^* is durable but \hat{b}_2 is not. Proposition 6 implies that there exist a_1 and increasing b_2 such that

$$p_1[u(1, 1) - \hat{u}(1, 1)] + p_2[u(2, 2) - \hat{u}(2, 2)] > 0 \quad (51)$$

$$u(2, 2) - \hat{u}(2, 2) + p_1[u(1, 1) - u(2, 1)] \geq p_1[\hat{u}(1, 1) - \hat{u}(2, 1)] \quad (52)$$

$$u(1, 1) - \hat{u}(1, 1) + p_2[u(2, 2) - u(1, 2)] \geq p_2[\hat{u}(2, 2) - \hat{u}(1, 2)] \quad (53)$$

where $u(i, j) = u_2(a_1, b_2(a_1, \theta^i), \theta^j)$ and $\hat{u}(i, j) = u_2(a_1, \hat{b}_2(a_1, \theta^i), \theta^j)$. Assume that $b_2(a_1, \theta^1) \precsim_2 b_2^*(a_1, \theta^2)$.

We first show that this assumption is without loss of generality since if it does not hold, then there is an increasing b'_2 such that $b'_2(a_1, \theta^1) \precsim_2 b_2^*(a_1, \theta^2)$ for which (51) - (53) hold when $u(i, j)$ is replaced with $u'(i, j) = u_2(a_1, b'_2(a_1, \theta^i), \theta^j)$. To this end define

$$b'_2(a_1, \theta) = \begin{cases} b_2^*(a_1, \theta^2), & \theta = \theta^1 \\ b_2(a_1, \theta^2), & \theta = \theta^2 \end{cases}$$

We first note that b'_2 is increasing since b_2 is increasing and $b'_2(a_1, \theta^1) = b_2^*(a_1, \theta^2) \prec_2 b_2(a_1, \theta^1) \precsim_2 b_2(a_1, \theta^2) = b'_2(a_1, \theta^2)$.

Note that $u'(2, 2) = u(2, 2)$ and $u'(2, 1) = u(2, 1)$. Furthermore, $u'(1, 1) \geq u(1, 1)$ and $u'(1, 2) \geq u(1, 2)$ because u_2 is single-peaked and $br_2(a_1, \theta^1) \precsim_2 br_2(a_1, \theta^2) \precsim_2 b_2^*(a_1, \theta^2) = b'_2(a_1, \theta^1) \prec_2 b_2(a_1, \theta^1)$. It immediately follows that (51) and (52) hold when u is replaced with u' . Finally, since u_2 exhibits

increasing differences and $b'_2(a_1, \theta^1) <_2 b_2(a_1, \theta^1)$, we have $u(1, 2) - u'(1, 2) \geq u(1, 1) - u'(1, 1)$. This implies that $u'(1, 1) - u(1, 1) \geq u'(1, 2) - u(1, 2) \geq p_2[u'(1, 2) - u(1, 2)]$, where the last inequality follows from $u'(1, 2) \geq u(1, 2)$. Rearranging we have $u'(1, 1) - p_2 u'(1, 2) \geq u(1, 1) - p_2 u(1, 2)$. Since $u(2, 2) = u'(2, 2)$, this is equivalent to $u'(1, 1) + p_2[u'(2, 2) - u'(1, 2)] \geq u(1, 1) + p_2[u(2, 2) - u(1, 2)]$, which implies that (53) holds when u is replaced with u' .

So, we can assume $b_2(a_1, \theta^1) \prec_2 b_2^*(a_1, \theta^2)$, which implies that \tilde{b}_2 defined below is increasing.

$$\tilde{b}_2(a_1, \theta^1) = \begin{cases} b_2(a_1, \theta^1), & \theta = \theta^1 \\ b_2^*(a_1, \theta^2), & \theta = \theta^2 \end{cases}$$

We will now show that b_2^* is not durable. First note that $\hat{u}(2, 2) \geq u(2, 2)$ and (51) imply that $\tilde{u}(1, 1) = u(1, 1) > \hat{u}(1, 1) = u^*(1, 1)$. Together with $\tilde{u}(2, 2) = u^*(2, 2)$, this implies that

$$p_1[\tilde{u}(1, 1) - u^*(1, 1)] + p_2[\tilde{u}(2, 2) - u^*(2, 2)] > 0. \quad (54)$$

Secondly, $\tilde{u}(2, 2) = u^*(2, 2)$, $\tilde{u}(2, 1) = u^*(2, 1)$, and $\tilde{u}(1, 1) > u^*(1, 1)$ imply that

$$\tilde{u}(2, 2) - u^*(2, 2) + p_1[\tilde{u}(1, 1) - \tilde{u}(2, 1)] \geq p_1[u^*(1, 1) - u^*(2, 1)]. \quad (55)$$

Suppose, for contradiction, that

$$\tilde{u}(1, 1) - u^*(1, 1) + p_2[\tilde{u}(2, 2) - \tilde{u}(1, 2)] < p_2[u^*(2, 2) - u^*(1, 2)] \quad (56)$$

Now, $\tilde{u}(1, 1) = u(1, 1)$, $\hat{u}(1, 1) = u^*(1, 1)$, and (53) imply that

$$u^*(2, 2) - \tilde{u}(2, 2) - u^*(1, 2) + \tilde{u}(1, 2) > \hat{u}(2, 2) - u(2, 2) - \hat{u}(1, 2) + u(1, 2). \quad (57)$$

Since $u^*(1, 2) = \hat{u}(1, 2)$, $\tilde{u}(1, 2) = u(1, 2)$, and $\tilde{u}(2, 2) = u^*(2, 2)$, (57) implies that $\hat{u}(2, 2) < u(2, 2)$, which is a contradiction because $\hat{b}_2(a_1, \theta^2) \in BR_2(a_1, \theta^2)$.

The following result follows from Lemma 10 and 12.

Lemma 13. Assume that u_2 is single-peaked and fix $b_2^* \in A_2^{A_1 \times \Theta}$ and $br_2 \in BR_2$. Define

$$\hat{b}_2(a_1, \theta) = \begin{cases} b_2^*(a_1, \theta^1), & \theta = \theta^1 \\ br_2(a_1, \theta^2), & \theta = \theta^2 \end{cases}$$

If b_2^* is durable, then \hat{b}_2 is durable.

□

We next prove

Lemma 14. Assume that u_1 is increasing in a_2 and u_2 is single-peaked. An outcome $(a_1^*, a_2^*(\theta))$ can be supported with durable contracts if and only if it can be supported with a durable strategy \hat{b}_2 such that $\hat{b}_2(a_1, \theta^2) \in BR(a_1, \theta^2)$ for all $a_1 \in A_1$.

Proof. [If] Obvious.

[Only If] Suppose that $(a_1^*, a_2^*(\theta))$ can be supported with durable contracts. This implies that there exists a durable strategy b_2^* such that

$$b_2^*(a_1^*, \theta) = a_2^*(\theta), \forall \theta \quad (58)$$

$$a_2^*(\theta) \in BR_2(a_1^*, \theta), \forall \theta \quad (59)$$

$$U_1(a_1^*, b_2^*) \geq U_1(a_1, b_2^*) \quad (60)$$

Also, Lemma 10 implies that

$$b_2^*(a_1, \theta^2) \succsim_2 br_2(a_1, \theta^2), \forall br_2 \in BR_2, a_1 \in A_1. \quad (61)$$

Fix $br_2 \in BR_2$ and define $\hat{b}_2(a_1^*, \theta) = b_2^*(a_1^*, \theta)$ for all θ and for any $a_1 \neq a_1^*$

$$\hat{b}_2(a_1, \theta) = \begin{cases} b_2^*(a_1, \theta^1), & \theta = \theta^1 \\ br_2(a_1, \theta^2), & \theta = \theta^2 \end{cases}$$

Note that \hat{b}_2 is durable by Lemma 13. Furthermore, $\hat{b}_2(a_1^*, \theta) = a_2^*(\theta) \in BR_2(a_1^*, \theta)$ for all θ and

$$U_1(a_1^*, \hat{b}_2) = U_1(a_1^*, b_2^*) \geq U_1(a_1, b_2^*) \geq U_1(a_1, \hat{b}_2)$$

where the last inequality follows from $\hat{b}_2(a_1, \theta^1) = b_2^*(a_1, \theta^1)$, $\hat{b}_2(a_1, \theta^2) = br_2(a_1, \theta^2) \precsim_2 b_2^*(a_1, \theta^2)$, and the fact that u_1 is increasing in a_2 . Finally, note that Lemma 10 implies that $\hat{b}_2(a_1, \theta^1) = b_2^*(a_1, \theta^1) \precsim_2 br_2(a_1, \theta^1) \precsim_2 br_2(a_1, \theta^2) = \hat{b}_2(a_1, \theta^2)$ for any a_1 , which implies that \hat{b}_2 is increasing. We conclude that \hat{b}_2 is durable and increasing and supports $(a_1^*, a_2^*(\theta))$. \square

This result implies that, when u_1 is increasing and u_2 is single-peaked, in order to characterize outcomes that can be supported by durable contracts we may limit ourselves to durable and increasing strategies that plays a best response for any a_1 after θ^2 . A similar result is true when u_1 is decreasing: We can limit ourselves to strategies that play a best response after θ^1 .

Lemma 15. Suppose that u_2 is single-peaked, $b_2^* \in A_2^{A_1 \times \Theta}$ is increasing and $b_2^*(a_1, \theta^2) \in BR_2(a_1, \theta^2)$ for all a_1 . Then, b_2^* is durable if and only if it is durable against all increasing strategies b_2 such that

$$\sum_{i=1}^2 p(\theta^i) \left[u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) \right] > 0$$

and $b_2(a_1, \theta^2) = b_2^*(a_1, \theta^2)$ for all a_1 .

Proof of Lemma 15. **[Only If]** Obvious.

[If] Suppose the conditions on b_2^* stated in the lemma hold but, for contradiction, it is not durable. This implies that there exists an a_1 and increasing b_2 such that the following hold:

$$p_1[u(1, 1) - u^*(1, 1)] + p_2[u(2, 2) - u^*(2, 2)] > 0 \quad (62)$$

$$u(2, 2) - u^*(2, 2) + p_1[u(1, 1) - u(2, 1)] \geq p_1[u^*(1, 1) - u^*(2, 1)] \quad (63)$$

$$u(1, 1) - u^*(1, 1) + p_2[u(2, 2) - u(1, 2)] \geq p_2[u^*(2, 2) - u^*(1, 2)] \quad (64)$$

Define

$$\hat{b}_2(a_1, \theta^1) = \begin{cases} b_2(a_1, \theta^1), & \theta = \theta^1 \\ b_2^*(a_1, \theta^2), & \theta = \theta^2 \end{cases}$$

As it has been shown in the proof of Lemma 12, we may assume without loss of generality that $b_2(a_1, \theta^1) \preceq_2 b_2^*(a_1, \theta^1)$, which implies that \hat{b}_2 is increasing. Furthermore,

$$p_1 \hat{u}(1, 1) + p_2 \hat{u}(2, 2) = p_1 u(1, 1) + p_2 u^*(2, 2) \geq p_1 u(1, 1) + p_2 u(2, 2) > p_1 u^*(1, 1) + p_2 u^*(2, 2)$$

where the first inequality follows from $b_2^*(a_1, \theta^2) \in BR_2(a_1, \theta^2)$ and the second from (62). Therefore,

$$p_1 [\hat{u}(1, 1) - u^*(1, 1)] + p_2 [\hat{u}(2, 2) - u^*(2, 2)] > 0. \quad (65)$$

Also, $u^*(2, 2) \geq u(2, 2)$ and (62) imply that $u(1, 1) > u^*(1, 1)$, which in turn implies that

$$\hat{u}(2, 2) - u^*(2, 2) + p_1 [\hat{u}(1, 1) - \hat{u}(2, 1)] - p_1 [u^*(1, 1) - u^*(2, 1)] = p_1 [u(1, 1) - u^*(1, 1)] > 0 \quad (66)$$

Lastly, definition of \hat{b}_2 , $\hat{u}(2, 2) \geq u(2, 2)$, and (64) imply

$$\begin{aligned} \hat{u}(1, 1) - u^*(1, 1) + p_2 [\hat{u}(2, 2) - \hat{u}(1, 2)] - p_2 [u^*(2, 2) - u^*(1, 2)] \\ \geq u(1, 1) - u^*(1, 1) + p_2 [u(2, 2) - u(1, 2)] \geq p_2 [u^*(2, 2) - u^*(1, 2)] \geq 0 \end{aligned} \quad (67)$$

Therefore, by (65), (66), and (67) imply that b_2^* is not durable against \hat{b}_2 , which is increasing

$$\sum_{i=1}^2 p(\theta^i) \left[u_2(a_1, \hat{b}_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) \right] > 0$$

and $\hat{b}_2(a_1, \theta^2) = b_2^*(a_1, \theta^2)$ for all a_1 . This completes the proof. \square

Lemma 14 and 15 imply that if u_1 is increasing in a_2 and u_2 is single-peaked, then an outcome can be supported with durable contracts if and only if it can be supported by an increasing strategy b_2^* which has $b_2^*(a_1, \theta^2) \in BR_2(a_1, \theta^2)$ for all a_1 and which is durable against any increasing b_2 with $b_2(a_1, \theta^2) = b_2^*(a_1, \theta^2)$ for all a_1 and $u_2(a_1, b_2(a_1, \theta^1), \theta^1) > u_2(a_1, b_2^*(a_1, \theta^1), \theta^1)$ for some a_1 . In other words, durability condition becomes the following: For each a_1

$$a_2 \preceq_2 b_2^*(a_1, \theta^2) \text{ and } u_2(a_1, a_2, \theta^1) > u_2(a_1, b_2^*(a_1, \theta^1), \theta^1) \Rightarrow \quad (68)$$

$$u_2(a_1, a_2, \theta^1) - u_2(a_1, b_2^*(a_1, \theta^1), \theta^1) < p_2 [u_2(a_1, a_2, \theta^2) - u_2(a_1, b_2^*(a_1, \theta^1), \theta^2)] \quad (69)$$

We have $a_2 \preceq_2 b_2^*(a_1, \theta^2)$ because the strategy we are testing against is the same as b_2^* at θ^2 and it has to be increasing. We have $u_2(a_1, a_2, \theta^1) > u_2(a_1, b_2^*(a_1, \theta^1), \theta^1)$ since the strategy we are testing against is the same as b_2^* at θ^2 . These also imply that (22) cannot hold and hence (21) must hold, which is equivalent to (69). Finally, since a durable strategy cannot have a left deviation at θ^1 we have the following result.

Proposition 10. *Assume that u_1 is increasing in a_2 and u_2 is single-peaked. Then, an outcome $(a_1^*, a_2^*(\theta))$ can be supported with durable contracts if and only if for any $a_1 \in A_1$, there exist $(a_2'(a_1), a_2''(a_1))$ such*

that

$$a_2'(a_1) \preceq_2 br_2(a_1, \theta^1) \text{ for any } br_2 \in BR_2 \quad (70)$$

$$a_2''(a_1) \in BR_2(a_1, \theta^2) \quad (71)$$

$$a_2 \preceq_2 a_2''(a_1) \text{ and } u_2(a_1, a_2, \theta^1) - u_2(a_1, a_2'(a_1), \theta^1) > 0 \Rightarrow$$

$$u_2(a_1, a_2, \theta^1) - u_2(a_1, a_2'(a_1), \theta^1) < p_2[u_2(a_1, a_2, \theta^2) - u_2(a_1, a_2'(a_1), \theta^2)] \quad (72)$$

$$a_2^*(\theta) \in BR_2(a_1^*, \theta) \text{ for all } \theta \quad (73)$$

$$p_1 u_1(a_1^*, a_2^*(\theta^1), \theta^1) + p_2 u_1(a_1^*, a_2^*(\theta^2), \theta^2) \geq p_1 u_1(a_1, a_2'(a_1), \theta^1) + p_2 u_1(a_1, a_2''(a_1), \theta^2) \quad (74)$$

Proof of Proposition 8. Follows from Proposition 10. \square

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