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## Research among Copycats: R&D, Spillovers, and Feedback Strategies

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# Research among Copycats: R&D, Spillovers, and Feedback Strategies

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# Research among Copycats: R&D, Spillovers, and Feedback Strategies\*

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## Abstract

We study a stochastic dynamic game of process innovation in which firms can initiate and terminate R&D efforts and production at different times. We discern the impact of knowledge spillovers on the investments in existing markets, as well as on the likely structure of newly forming markets, for all possible asymmetries between firms. We show that the relation between spillovers, R&D efforts, and surpluses is non-monotonic and dependent on both the relative and absolute efficiency of firms. Larger spillovers increase the likelihood that a new technology is brought to production, but they do not necessarily make the industry more competitive.

*Keywords:* Differential game, Feedback Nash equilibrium, Numerical partial differential equations, R&D, Spillovers,

*JEL:* C61, C63, C73, D43, D92, L13, O31

## 1 Introduction

Contemporary markets are flooded with imitations – it is hard to find a business model, good, or service that is not a variation or an adaptation of some earlier version. Dell and HP are only two out of many firms that cloned IBM’s Personal Computer. Atari’s video game attracted as many as 75 imitators, led by Nintendo. More recently, Samsung’s lawyers could not tell the difference between Samsung’s Galaxy Tab and Apple’s iPad in court.<sup>1</sup> While more latent than product imitations, imitations of business processes abound as well and often even transcend the sector in which they were first introduced. Walmart’s automated supply chain management strategies were

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<sup>1</sup><http://gizmodo.com/5849803/even-samsung-cant-tell-the-difference-between-its-tablet-and-ipad> (accessed August 2, 2012)

imitated by its competitors (e.g., Kmart, Tesco), but also by companies in other sectors, such as Ryanair. South West Airlines' innovative business model which led to the low cost revolution in air travel was successfully imitated by both Ryanair and easyJet. Henry Ford's introduction of the moving assembly line did not only reduce the cost of his Model T car, but also revolutionized manufacturing processes across industries worldwide. Today a similar revolution is taking place as e-commerce adopted by various sorts of organizations is reducing the need for paperwork.

This paper studies how information leakages, or spillovers, affect industry dynamics and structure through their impact on innovation incentives of firms. Its focus is process innovation, interpreted as any improvement in "the way things are done" that enables a firm to satisfy a given consumer need at lower cost. This focus is motivated by the fact that product innovations often cannot be successfully introduced in the market without accompanying process innovations<sup>2</sup> and that over time relative productivity becomes decisive for surviving in the market. In fact, a higher emphasis on process innovation by Japanese companies is believed to be one of the main reasons for their increasing competitiveness over their American counterparts (Bhoovaraghavan et al., 1996).

Hardly any business idea is immune to imitation,<sup>3</sup> the pace of which is increasing with the increased codification of knowledge and the advance of globalization: "In the early 1990s Cardizem lost 80 percent of the market to generic substitutes within five years; a decade later, Cardura lost a similar share in nine months; and Prozac [...] lost the same market share in only two months" (Shenkar, 2010, p. 6).<sup>4</sup> Imitators often even outperform innovators in business results. For instance, both Visa and MasterCard enjoy larger market shares than the first credit card issuer Diners Club, Japanese producers of roll-film cameras switched to digital photography and eventually overtook Sony, the digital photography pioneer, and Toyota and Honda dominate the market for minivans over both the pioneering Chrysler, as well as its earlier followers Ford and GM. Examples abound (see Shenkar, 2010). Similar observations led Levitt (1966, p. 63) to

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<sup>2</sup> A good example of a technology which had been subject to considerable investment before it finally became suitable for the market is the modern plasma display, the concept of which was first conceived at the University of Illinois in 1964. At that time, it was too expensive to mass produce using the existing technology. It took several years for IBM to launch a 48-cm monochrome plasma display destined for commercial use in 1983. Several additional years of research and improvements on the licensed technology of first innovators were needed for Fujitsu to present the first 53-cm fully-colored hybrid display in 1992, and the 107-cm display later in 1997. Philips' plasma display of the same resolution claimed to be the first to be presented to the retail sector on a large scale in 1997 at a price of no less than \$14,999. Later, Pioneer, Sony, LG, Samsung, and a few others also entered this market. Thanks to subsequent improvements in the technology and concomitant reductions in the production costs, different variants of plasma TVs are nowadays available for less than \$1,000.

Yet another illustrative example is polyethylene. Previously, the material was extremely expensive due to its high production costs and was used only to insulate underwater electrical transmission cables. As Utterback (1994) reports, thanks to subsequent improvements in the production process, polyethylene is today "so cheap and plentiful that it is used for throwaway bags and milk bottles."

<sup>3</sup> As Arrow (1962, p. 615) explains, "no amount of legal protection can make a thoroughly appropriable commodity of something so intangible as information. The very use of the information in any productive way is bound to reveal it, at least in part. Mobility of personnel among firms provides a way of spreading information. Legally imposed property rights can provide only a partial barrier, since there are obviously enormous difficulties in defining in any sharp way an item of information and differentiating it from similar sounding items."

<sup>4</sup> Mansfield (1985) finds that rivals obtain information about new products or processes in 12 months or less. Similarly, Caballero and Jaffe (1993) in their analysis of patent citations conclude that diffusion of information about innovations is so rapid that it can be regarded as being instantaneous. In their study of 48 specific innovations, Mansfield et al. (1981) report that 60% of all patented innovations were imitated within 4 years and that imitation occurred on average at about two thirds of cost and time expended by the innovator.

claim, “Imitation is not only more abundant than innovation, but actually a much more prevalent road to business growth and profits.” [Shenkar \(2010\)](#) even writes about the “innovation challenge” – companies that want to succeed, need to fuse innovation and imitation as in the future it will not be possible anymore “to rely on innovation or imitation alone to drive competitive advantage.”

Spillovers are prevalent in business practice, yet our understanding of their role in innovation activities of firms is still incomplete. On the one hand, spillovers could improve market performance as imitators make the market more competitive. On the other hand, innovators facing the danger of having their returns on research investments reduced by imitators could be less willing to innovate in the first place. Furthermore, the effect of spillovers on the innovation of asymmetric firms appears ambiguous as well. Empirical studies show that the laggard often strives to catch up with the leader (see, e.g., [Lerner, 1997](#)). If imitating the leader becomes easier, the laggard is able to progress faster and the industry is less likely to become monopolized. However, the final result depends also on the way the leader responds to spillovers. The leader might increase his innovation efforts in order to widen the lead and hopefully drive the laggard out of the market, or he might decrease his innovation efforts as the laggard is free-riding on them.

Due to their prominence in practice, this paper puts asymmetries between firms and their dynamic evolution to the center of its investigation. It has been a long-standing indication in the literature, confirmed in a dynamic framework by [Petit and Tolwinski \(1999\)](#), that larger spillovers can prevent the monopolization of the industry by easing imitation. Using a richer analytical framework and considering all possible absolute and relative differences in firms’ production costs, this paper shows that while intuitive, the above result is not universally true. It is true only when initial production costs of the leader are high and, by extension, his incentives to exert R&D efforts to reduce these production costs. At relatively lower unit costs of the leader, additional R&D efforts benefit the leader progressively less and the laggard progressively more, and the incentives of the leader to exert R&D efforts can be rather low. This makes it harder for the laggard to catch up with the leader. Notably, the ability to copy is not worth much when there is little to copy. Consequently, *lower* cost asymmetries can suffice to induce the monopolization of the industry at *larger* spillovers. We also show that the relation between spillovers and welfare is more involved than previously thought – it is highly non-monotonic and affected by both the relative and absolute initial efficiency of firms.

Strategic R&D decisions of firms in the presence of spillovers were first analyzed in a static framework by [d’Aspremont and Jacquemin \(1988\)](#), whose work fostered many generalizations and extensions. Surveying this literature, [De Bondt \(1997\)](#) reports that while the amount of R&D done by each firm tends to decrease as spillovers increase (due to a free-riding effect), industry output and welfare increase as long as the level of spillovers is below some critical intermediate value (0.5 in the homogenous goods case of [d’Aspremont and Jacquemin \(1988\)](#)). [Petit and Tolwinski \(1999\)](#) consider asymmetric firms in a dynamic, though deterministic, context. They show that when firms are symmetric, high spillovers are not desirable from the consumers’ perspective as they lead to low R&D efforts and high prices. However, when firms are asymmetric, high spillovers might be preferred as they can reduce market concentration, and thereby prices, by enabling less efficient firms to survive. [Hinloopen et al. \(2013\)](#) point out that the received models give an incomplete picture of R&D: while patent race models abstract from the problem

of production, strategic models of R&D impose the coexistence of production and R&D at all times, which contradicts the observation that for many new technologies research starts long before production. The present article extends the dynamic framework of [Hinloopen et al. \(2013\)](#) in two important directions. First, rather than a monopoly, we consider a proper differential game with two competing firms. Second, given that innovation is inherently uncertain, we introduce stochasticity to the R&D process.

At the outset of the game, each firm has an initial unit cost of production  $c_i(0)$ , corresponding to the initial level of a particular technology. This value may differ between firms, for instance if the imitator lacks the production experience of the innovator. Firms can increase their production efficiency by exerting R&D efforts, which in turn are subject to imitation. Higher production efficiency in turn makes them stronger competitors on the Cournot product market. In the present set-up, the initial unit production cost of a firm is allowed to be above the choke price (the lowest price at which the quantity sold is zero); in such a situation, it is optimal for a firm to postpone production. Firms' product market participation constraints are taken into account explicitly. Consequently, the R&D process and production do not need to coexist at all times and firms can enter or exit the product market and initiate or cease their R&D processes at different times.

The analysis of spillovers in the existing literature is typically limited to their effects on the innovation activities already in place. In fact, to this day, all related static and dynamic models of strategic R&D assume initial unit costs to be below the choke price and are silent on situations where initial production costs of a new technology exceed the highest willingness to pay in the market (see [Hinloopen et al. \(2013\)](#) for a critical discussion). Hence, the existence of a market and R&D process is already assumed and the question left is how spillovers affect the (size of) R&D efforts. A distinguishing feature of our approach is that we do not limit ourselves to the question of how much to invest in a given market but recognize that this question is preceded by the question of whether to invest in a given technology at all.<sup>5</sup> We pay special attention to the determination of those critical cost levels at which a firm's goal switches from entering the market to exiting the market. Consequently, we are able to analyze not only how spillovers affect the investments on existing markets, but also how they influence the likelihood that a new market will be formed, and if so, how its likely structure (monopoly or duopoly) relates to the level of spillovers, and this so for every possible combination of firms' initial unit costs. Our framework therefore puts conclusions of the previous literature, typically holding only for a subset of possible cost levels, into a broader perspective.

We analyze both the deterministic and the stochastic version of the dynamic game. This enables us to analyze the impact of uncertainty on R&D decisions of firms. It turns out that it is much harder to obtain the solution to the deterministic game. The reason for this is that in the deterministic version, a firm's policy functions may exhibit discontinuities at initial technology levels where entering or not entering the market is equally profitable. These discontinuities pose significant problems to numerical schemes. We progress by first exploiting the fact that for a stochastic R&D process policy functions and value functions are smooth, making numerical schemes easier to implement. We then analyze the policy functions under uncertainty, which

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<sup>5</sup>Elmer Bolton, a scientist-manager at DuPont, one of the most innovative corporations in American business history, was famous for saying to the company's chemists who in his opinion lacked the awareness that the success of the company depends on its products being commercially exploitable: "This is very interesting chemistry, but somehow I don't hear the tinkle of the cash register" ([Hounshell, 1988](#)).

are interesting in their own right, as approximating solutions to the deterministic game when the uncertainty level tends to zero. That is, we obtain a solution to the deterministic game as a vanishing viscosity solution (see [Başar and Olsder, 1999](#), Ch. 5.7).

We solve for a feedback Nash equilibrium of the differential game, which is characterized by a system of highly nonlinear implicit partial differential equations, by a variant of the numerical method of lines ([Schiesser, 1991](#)): by discretizing technology levels, but not time, the system of partial differential equations is approximated by a system of ordinary differential equations. The solution to this system is then obtained by standard methods. This solution method requires the values of the value functions of the players at the boundaries of the state space over which we seek a solution. However, the true boundary values are not known *ex ante*. To circumvent this, we exploit the fact that the characteristics of the first-order Hamilton-Jacobi-Bellman partial differential equations, which correspond to the deterministic situation, leave the state space at the boundaries. This implies that the solution in the interior of the state space is unaffected by the precise specification of the boundaries; for modest noise levels, this still holds true, excepting a small layer along the boundaries. This enables us to obtain accurate approximating solutions for value and policy functions.

We find that a duopoly arises in the product market generally only, if initial asymmetries between the firms are not too large.<sup>6</sup> The duopoly in the product market is characterized by the regression toward the mean phenomenon: asymmetries between the firms tend to vanish over time.

Through increasing complementarities in R&D, larger spillovers always increase the chance that an expensive technology calling for investments in advance of production will be brought to production. The level to which such a technology is developed can however be lower due to lower R&D investments of firms that try to free-ride on each other along the way. In this latter sense, spillovers increase production efficiency only up to a point. Here our model emphasizes the importance of considering both dimensions in which spillovers affect markets: first, through affecting the formation of markets, and second, through affecting investments *in* the formed markets. These effects can be of opposite sign.

We find that larger random shocks to firms' production costs are increasing the likelihood that a technology will be developed further as firms are stimulated by the possibility of a favorable shock to their production costs in the future more than they are discouraged by the possibility of an unfavorable shock. They rather invest more today than be sorry later for giving up too early. Uncertainty also increases the likelihood that the product market will be competitive as the chance of a larger favorable shock in the future increases the perseverance of the laggard.

At low spillovers, we find large investments of firms at high initial unit costs for both firms, if the difference between the unit cost levels is small. That is, a small cost advantage of one firm leads to a behavior that can be considered predatory: the leader exerts high R&D efforts which are profitable in that they induce the laggard to give up. However, when firms start from a perfectly symmetric situation, their behavior resembles a preemption race: each firm invests a lot trying to win the race in which a small lead suffices for gaining a monopoly position, resulting in very fast technology development.

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<sup>6</sup>This conclusion is similar to [Doraszelski \(2003\)](#) who finds action-reaction behavior in a patent race model with history-dependent R&D stocks: the laggard catches up with the leader provided his initial stock of knowledge is of sufficient size and gives up otherwise.



The ranking of spillovers based on consumer surplus and total surplus depends on both the relative and absolute value of firms' initial unit costs of production, as well as on the variance of random shocks. Roughly speaking, at high initial unit costs of firms the intermediate level of spillovers is associated with the highest consumer surplus, except if larger spillovers prevent the leader from monopolizing the market, or if symmetric initial costs at small spillovers induce firms to engage in the preemptive race characterized by high R&D investments. The latter two possibilities diminish in importance as the variance of the shocks increases or as the initial unit costs of firms decrease. For low initial unit costs of firms, large spillovers bring about the lowest consumer surplus. This lower desirability of large spillovers at low initial costs characterizes total surplus as well. While the latter tends to increase with spillovers at high initial unit costs, increases in spillovers beyond the intermediate level diminish total surplus at low initial unit costs of firms, especially when asymmetries between firms increase. When initial costs are already quite low, savings on R&D costs induced by large spillovers are comparably less important and the main effect of large spillovers is the perverse free-riding effect that diminishes R&D efforts.

The remainder of the paper is organized as follows. Section 2 describes the model specification. Section 3 explains our computational approach to obtaining a feedback Nash equilibrium solution. Section 4 discusses equilibrium strategies and corresponding industry dynamics. Section 5 concludes.

## 2 Model

The dynamic game is defined in continuous time and over an infinite horizon:  $t \in [0, \infty)$ . There are two firms which potentially compete in a market for a homogenous good with demand

$$p(t) = \max \{A - q_i(t) - q_j(t), 0\}, \quad (1)$$

where  $p(t)$  is the market price,  $q_i(t)$  is the quantity produced by firm  $i = \{1, 2\}$ ,  $q_j(t)$  is the quantity produced by its rival ( $i \neq j$ ), and  $A$  is the choke price (the lowest price at which the quantity sold is zero). At the outset of the game, each firm obtains an exogenous technology, represented by its unit cost level  $c_i(0)$ . For simplicity, we assume that firms may differ in their production cost, being identical in every other aspect. While both firms produce with constant returns to scale, each firm can reduce its unit cost  $c_i(t) > 0$  by investing in R&D. This process is subject to spillovers. Firm  $i$  exerts R&D effort  $k_i(t) \geq 0$  and as a consequence of these investments, its unit cost (state variable) evolves over time according to

$$\frac{dc_i}{c_i} = (-k_i - \beta k_j + \delta) dt + \sigma_i dB_i, \quad (2)$$

where  $k_j = k_j(t)$ ,  $j \neq i$ , is the R&D effort exerted by the rival,  $\beta \in [0, 1]$  is a degree of spillovers, and  $\delta > 0$  reflects the constant rate of efficiency reduction due to the aging of technology and organizational forgetting (Besanko et al., 2010; Hinlopen et al., 2013). Moreover, the  $B_i(t)$  are standard Wiener processes with  $\sigma_i > 0$  denoting their strength. The processes  $B_i$  are assumed to be independent and of equal strength, that is  $\sigma_1 = \sigma_2 = \sigma$ ; this restriction is for definiteness, as other correlation structures could also be analyzed using our methods. Equation (2) is of

the form  $dc = \mu(c, k) dt + \sigma(c) dB$ , where  $\mu$  and  $\sigma$  are drift and diffusion, respectively, of a controllable Itô process  $c$  (see Kloeden and Platen, 1992). Hence, firms face some randomness in the evolution of their unit costs. The random variations in  $c$  are high when  $c$  is high, i.e. when the technology is not fully developed yet.

Note that Equation (2) is not linear in state and controls, and that consequently the game is not linear-quadratic. Low values of  $\beta$  correspond to strong intellectual property protection and the ability of firms to prevent involuntary leaks of information. The reverse is true for high values of  $\beta$ .<sup>7</sup> We treat the value of  $\beta$  as given for firms.<sup>8</sup> Observe in (2) that the smaller the  $c_i$ , the smaller the effect of particular  $k_i$  on  $\dot{c}_i$ . Further innovations require increasingly more R&D efforts. Exerting R&D effort is costly. This cost equals  $bk_i(t)^2$  per unit of time, where  $b > 0$  is inversely related to the cost-efficiency of the R&D process. In assuming decreasing returns to R&D, we follow the bulk of the literature (see Hinlopen et al. (2013) for a discussion of the model's assumptions). Both firms discount the future with the same constant rate  $\rho > 0$ . The instantaneous profit of firm  $i$  is:

$$\pi_i(t) = \begin{cases} (A - q_i(t) - q_j(t) - c_i(t))q_i(t) - bk_i(t)^2 & \text{if } p(t) > 0, \\ -c_i(t)q_i(t) - bk_i(t)^2 & \text{if } p(t) = 0, \end{cases} \quad (3)$$

yielding its expected total discounted profits over time:

$$\Pi_i = \mathbb{E}_0 \int_0^\infty \pi_i(t) e^{-\rho t} dt; \quad (4)$$

here and below  $\mathbb{E}_t$  is the expectation operator conditional on information available up to time  $t$ .

## 2.1 Rescaling

Our model depends on five parameters:  $A, b, \delta, \beta$ , and  $\rho$ . Some of these can be set to 1 by choosing the measurement scale of units appropriately. The five-dimensional parameter space then reduces to a three-dimensional one.

**Lemma 1.** *After choosing units of variables and parameters appropriately, we can assume  $A = 1$ ,  $b = 1$ , and  $\delta = 1$ . In the new units, the state equations read as*

$$dc_i = \left(1 - (k_i + \beta k_j)\phi\right) c_i dt + c_i \sqrt{2\varepsilon} dB_i \quad (5)$$

with  $\phi = A/\delta\sqrt{b}$  and  $\varepsilon = \sigma^2/2\delta$ .

*Proof.* See Appendix A. □

<sup>7</sup>Some caution should be taken when considering high spillovers. Even with no legal or other constraints on information flows, it is questionable that there is perfect additivity between R&D efforts of competing firms. Some R&D results of a competitor will necessary involve duplication or be only an imperfect match to a firm's own knowledge or organizational culture (see discussion in Hinlopen (2003)).

<sup>8</sup>In general,  $\beta$  may be one of a firm's strategic variables. See Katsoulacos and Ulph (1998) and Amir et al. (2003) for an attempt to endogenize the degree of spillovers. Von Hippel (1988) provides empirical evidence for firms being consensually involved in information sharing. See also Shenkar (2010). Amir et al. (2000) allow for spillovers to differ between firms.

The new parameter  $\phi$  introduced by the rescaling captures the profit potential of a technology (cf. [Hinloopen et al., 2013](#)). Higher potential revenues come with a higher  $A$ , and each unit of R&D effort costs more if  $b$  increases, while it reduces unit cost by less the higher is  $\delta$ . In sum, a lower (higher)  $\phi$  corresponds to a lower (higher) profit potential.

## 2.2 Product market and equilibrium output levels

Firms compete in a product market by strategically setting their output levels. The analysis of this market is simplified by the fact that production, unlike R&D efforts, does not influence the evolution of the unit cost levels. Hence, the firms play a Cournot duopoly game with respect to production at each instant of time, and the output levels, while depending on unit costs,  $q_i = \psi_i(c_1, c_2)$ , are static Cournot-Nash equilibria of the instantaneous game.

**Proposition 1.** *The strategy profile  $\psi^*(c_1, c_2) = (\psi_1^*(c_1, c_2), \psi_2^*(c_1, c_2))$ , where*

$$i) \quad \psi_i^*(c_1, c_2) = \frac{1 - 2c_i + c_j}{3} \quad \text{if } 2c_1 - c_2 < 1, \quad 2c_2 - c_1 < 1; \quad (6)$$

$$ii) \quad \psi_i^*(c_1, c_2) = 0, \quad \psi_j^*(c_1, c_2) = \frac{1 - c_j}{2} \quad \text{if } 2c_i - c_j \geq 1, \quad c_j < 1; \quad (7)$$

$$iii) \quad \psi_1^*(c_1, c_2) = 0, \quad \psi_2^*(c_1, c_2) = 0 \quad \text{if } c_1 \geq 1, \quad c_2 \geq 1, \quad (8)$$

is a Cournot-Nash equilibrium of a quantity setting duopoly in the product market ( $i, j \in \{1, 2\}, i \neq j$ ).

These equilibria are illustrated in [Figure 1](#). The curves  $E_1$  and  $E_2$  are the ‘entry/exit’ curves for firms 1 and 2 respectively: to the right of  $E_1$ , firm 1 is not active in the product market; above  $E_2$ , firm 2 is not active. The two curves divide state space into four regions: ‘Duopoly’, ‘Monopoly of Firm  $i$ ’, for  $i = 1$  or 2, and ‘No production’.

Both firms produce positive amounts only for combinations of unit costs in the ‘Duopoly’ region. There, the market price is higher than the unit cost of each firm (the first case in the above proposition), and firm  $i$  earns a profit of

$$g_i(c_1, c_2) = \frac{(1 - 2c_i + c_j)^2}{9}. \quad (9)$$

In the region ‘Monopoly of Firm 1’, the market price is lower than firm 2’s unit cost; firm 2 therefore optimally sells nothing. Firm 1 earns there a monopoly profit

$$g_i(c_1, c_2) = \frac{(1 - c_i)^2}{4}. \quad (10)$$

The roles are reversed in the region ‘Monopoly of Firm 2’.

Finally, in the ‘No production’ region, the unit costs of both firms are higher than the choke price ( $A = 1$ ), and as firms could sell a positive amount only at negative mark-ups, neither firm produces.

In sum, the sales profit of firm  $i$  is given by

$$g_i(c_1, c_2) = \begin{cases} (1 - 2c_i + c_j)^2/9 & \text{if } 2c_1 - c_2 < 1, \quad 2c_2 - c_1 < 1, \\ (1 - c_i)^2/4 & \text{if } 2c_j - c_i \geq 1, \quad c_i < 1, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

The function  $g_i$  is continuous. The total instantaneous profit is the sales profit  $g_i$  diminished by the R&D expenditure  $k_i^2$ :

$$\pi_i = g_i(c_1, c_2) - k_i^2. \quad (12)$$

The substitution of equilibrium output levels in firms' profit functions has resulted in the profit function of firm  $i$  being dependent only on unit costs and R&D effort. Consequently, the problem of the firms is reduced to finding optimal R&D efforts.

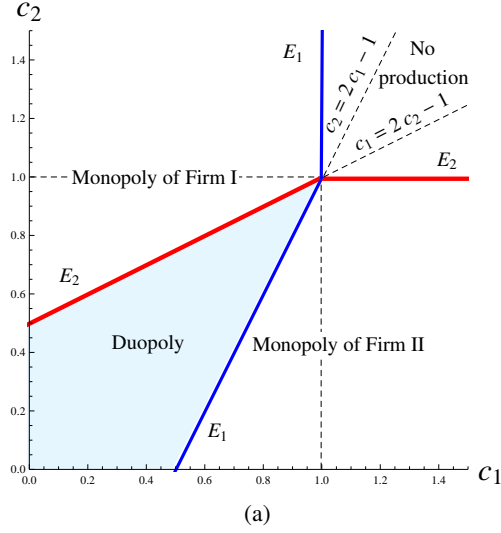


Figure 1: *Product-market activity.*

### 2.3 Problem statement

To sum up formally, in the two-firm differential game, each firm tries to maximize its expected pay-off

$$\Pi_i = \mathbb{E}_0 \int_0^\infty [g_i(c_1, c_2) - k_i^2] e^{-\rho t} dt \quad (13)$$

through its choice of R&D effort  $k_i \geq 0$ , subject to the state equations

$$dc_i = (1 - (k_i + \beta k_j)\phi) c_i dt + c_i \sqrt{2\varepsilon} dB_i, \quad i = 1, 2, \quad (14)$$

where  $c_i > 0$  for  $i = 1, 2$ .

## 2.4 Equilibrium

In the game, each firm tries to maximize its total discounted profits by selecting a strategy which specifies its action, that is its R&D effort, at each instant of time. In the stochastic context, it is most natural to consider feedback strategies, where the actions are given as functions of the current state of the system (see [Başar and Olsder, 1999](#)). The corresponding feedback Nash equilibria are characterized by a dynamic programming equation. The resulting equilibrium strategies are strongly time consistent or ‘subgame perfect’. Here, the state of the game is determined by the pair  $(c_1, c_2)$  of unit production costs of the firms, and a strategy for firm  $i$  is a function  $k_i = \Gamma_i(c_1, c_2)$ , specifying its R&D effort  $k_i$  as functions of the state.

In this context, a ‘feedback Nash equilibrium’ is a pair  $(\Gamma_1^*, \Gamma_2^*)$  of feedback strategies, such that the choice

$$k_i(t) = \Gamma_i^*(t, c_1(t), c_2(t)) \quad (15)$$

for all  $t \geq t_0$  maximizes the present discounted value of firm  $i$ ’s profits, given that the other firm chooses its R&D level by (15) with  $i$  replaced by  $j$ . This implies that choosing  $k_i$  according to (15) maximizes

$$\mathbb{E}_{t_0} \int_{t_0}^{\infty} \pi_i(k_i, \Gamma_j^*(t, c_1, c_2)) e^{-\rho t} dt, \quad (16)$$

subject to the state equations. Introduce the value  $W^i = W^i(t_0, c_{1,0}, c_{2,0})$  at  $c_{i,0} = c_i(t_0)$ ,  $i = 1, 2$ , as the value of the integral. Then  $W^i$  is the maximum discounted value of profits at time  $t = t_0$  that can be earned by firm  $i$ , given that the other firm pursues its equilibrium strategy. If  $\Gamma^*$  is a feedback Nash equilibrium solution to our dynamic game, there exist functions  $W^i$ , satisfying the so-called Hamilton-Jacobi-Bellman equations (see [Başar and Olsder, 1999](#), Ch. 6.5) either everywhere, or at least in the sense of viscosity solutions ([Crandall and Lions, 1983](#)):

$$\begin{aligned} -W_t^i &= \varepsilon c_1^2 W_{c_1 c_1}^i + \varepsilon c_2^2 W_{c_2 c_2}^i \\ &+ \max_{k_i \geq 0} \left[ (g_i(c_1, c_2) - k_i^2) e^{-\rho t} + W_{c_i}^i (1 - (k_i + \beta \Gamma_j^*(t, c_1, c_2)) \phi) \right. \\ &\left. + W_{c_j}^i (1 - (\Gamma_j^*(t, c_1, c_2) + \beta k_i) \phi) \right], \end{aligned} \quad (17)$$

where  $i \in \{1, 2\}, i \neq j$ . We adopt the convention that a subscript to the value function indicates a partial derivative of that function with respect to each subscripted variable. This equation is complemented by the asymptotic terminal condition

$$\lim_{t \rightarrow \infty} W^i(t, c_1, c_2) = 0, \quad (18)$$

which is a ‘No-Ponzi’ condition.

Introduce the current-time value function  $V^i(t, c_1, c_2)$  by setting  $W^i = V^i e^{-\rho t}$ . That is,  $V^i$  equals the profits earned when firm  $i$  starts in the state  $(c_1, c_2)$  at time  $t$  and invests optimally, while firm  $j$  pursues its equilibrium strategy. Equation (17) can then be written as

$$\begin{aligned} \rho V^i - V_t^i = & \varepsilon c_1^2 V_{c_1 c_1}^i + \varepsilon c_2^2 V_{c_2 c_2}^i \\ & + \max_{k_i \geq 0} \left[ g_i(c_1, c_2) - k_i^2 + V_{c_i}^i (1 - (k_i + \beta \Gamma_j^*(t, c_1, c_2)) \phi) \right. \\ & \left. + V_{c_j}^i (1 - (\Gamma_j^*(t, c_1, c_2) + \beta k_i) \phi) \right]. \end{aligned} \quad (19)$$

These equations are complemented by the following analogue of Equation (18)

$$\lim_{t \rightarrow \infty} V^i(t, c_1, c_2) e^{-\rho t} = 0. \quad (20)$$

If the value functions  $V^i$  are continuously differentiable, equations (19) hold everywhere. If this assumption is not valid, which may typically happen for  $\varepsilon = 0$ , the  $V^i$  do not satisfy the equations (19) in a classical sense. In such a case, these equations are understood in the sense of viscosity solution (see [Crandall and Lions, 1983](#)). If  $V^i$  is differentiable at some  $(t, c_1, c_2)$ , then

$$\Gamma_i^*(t, c_1, c_2) = \max \left\{ -\frac{1}{2} \phi \left( V_{c_i}^i + \beta V_{c_j}^i \right), 0 \right\}. \quad (21)$$

This relation can be used to eliminate the  $\Gamma_i^*$  from equations (19). The result is a coupled system of parabolic partial differential equations for the value functions  $V^i$ ,  $i = 1, 2$ . For a single firm, under deterministic cost dynamics ( $\varepsilon = 0$ ) there are situations where the value function is not differentiable over the entire space ([Hinloopen et al., 2013](#)), as ‘indifference’ or ‘Skiba’ points appear for certain parameter values. These are points where a firm is indifferent between developing and not developing a technology further. At such a point, the optimal investment function is multiple valued, and the corresponding value function is kinked. We expect the same phenomenon to occur in the case of two competing firms in the deterministic limit  $\varepsilon \rightarrow 0$ . [Hinloopen et al. \(2013\)](#) were able to show the existence of the value function using geometrical arguments. For the present dynamic game, we are not able to proceed along those lines. We therefore propose a method to obtain numerical approximations of the value function.<sup>9</sup>

### 3 Computation

This section discusses the numerical solution strategy. First, a number of preliminary space and time transformations bring the Hamilton-Jacobi-Bellman equations into the form of a system of quasi-linear parabolic partial differential equations with constant and isotropic diffusion tensors with given initial values. In order to obtain numerical solutions, artificial boundary conditions have to be imposed on the problem; this is done in such a way that the solutions are not essentially changed. A standard numerical scheme, the method of lines, can then be used to obtain numerical approximations of the solutions to the resulting system.

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<sup>9</sup>[Kossioris et al. \(2008\)](#) numerically compute a non-linear feedback Nash equilibrium for a differential game with a *single* state variable, limiting themselves to a class of continuous feedback rules. [Dockner and Wagener \(2014\)](#) study necessary conditions for feedback equilibria in games with a single state variable. Through an auxiliary system of differential equations they are also able to find non-continuous feedback strategy equilibria.

### 3.1 Preliminary transformations

The structure of equation (19) for  $V^i$  is that of a parabolic partial differential equation with state-dependent diffusion tensor

$$D = \varepsilon \begin{pmatrix} c_1^2 & 0 \\ 0 & c_2^2 \end{pmatrix}.$$

The variable transformation

$$c_i = e^{-x_i}, \quad (22)$$

converts  $D$  to a constant multiple of the identity. For applying Itô's theorem shows that in  $x_i$  coordinates the state equation (5) takes the form<sup>10</sup>

$$dx_i = ((k_i + \beta k_j)\phi - 1 + \varepsilon) dt - \sqrt{2\varepsilon} dB_i, \quad (23)$$

for  $i, j \in \{1, 2\}$ ,  $i \neq j$ .

Set  $\tilde{V}^i(x_1, x_2) = V^i(e^{-x_1}, e^{-x_2})$ , etc. In the new variables, the Hamilton-Jacobi-Bellman equations read as

$$\begin{aligned} \rho \tilde{V}^i - \tilde{V}_t^i &= \varepsilon \tilde{V}_{x_1 x_1}^i + \varepsilon \tilde{V}_{x_2 x_2}^i \\ &+ \max_{k_i \geq 0} \left[ \tilde{g}_i(x_1, x_2) - k_i^2 + \tilde{V}_{x_i}^i \left( (k_i + \beta \tilde{\Gamma}_j^*(t, x_1, x_2))\phi - 1 + \varepsilon \right) \right. \\ &\left. + \tilde{V}_{x_j}^i \left( (\tilde{\Gamma}_j^*(t, x_1, x_2) + \beta k_i)\phi - 1 + \varepsilon \right) \right], \end{aligned} \quad (24)$$

with terminal condition

$$\lim_{t \rightarrow \infty} \tilde{V}^i(t, x_1, x_2) e^{-\rho t} = 0.$$

In order not to overburden notation, we shall drop the tildes, and refer to 'functions in  $c_i$ -coordinates' and 'functions in  $x_i$ -coordinates' instead.

The terminal condition, which specifies the value at infinite times, is actually numerically inconvenient. To circumvent it, the model is modified in two steps. First, the corresponding finite horizon model, where  $t$  ranges from 0 to  $T > 0$ , is considered instead, with value functions  $V_T^i$  and terminal condition

$$V_T^i(T, x_1, x_2) = 0, \quad i = 1, 2. \quad (25)$$

Then, by replacing the variable  $t$ , denoting elapsed time, with the time-to-completion  $s = T - t$ , the terminal condition is transformed into an initial condition. Introduce the 'time-reversed value functions' by the relation

$$V_T^i(t, x_1, x_2) = U_T^i(T - t, x_1, x_2), \quad i = 1, 2. \quad (26)$$

The terminal condition (25) is then replaced by the initial condition

$$U_T^i(0, x_1, x_2) = 0, \quad i = 1, 2.$$

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<sup>10</sup>The transformation  $c_i = e^{x_i}$  would also give rise to a constant diffusion tensor. By the choice made in the article, the main region of interest ( $0 < c_i < 1$ ) corresponds to positive values of the  $x_i$ .

As all time-reversed value functions  $U_T^i$  satisfy the same Hamilton-Jacobi-Bellman equation for each  $T$ , and as they all have the same initial value, it follows by standard uniqueness results that they are equal. Hence we may drop the subscript  $T$ . The time-reversed Hamilton-Jacobi-Bellman equation for  $U^i$  then reads as

$$\begin{aligned} \rho U^i + U_s^i &= \varepsilon U_{x_1 x_1}^i + \varepsilon U_{x_2 x_2}^i \\ &+ \max_{k_i \geq 0} \left[ g_i(x_1, x_2) - k_i^2 + U_{x_i}^i \left( (k_i + \beta \Gamma_j^*(s, x_1, x_2)) \phi - 1 + \varepsilon \right) \right. \\ &\quad \left. + U_{x_j}^i \left( (\Gamma_j^*(s, x_1, x_2) + \beta k_i) \phi - 1 + \varepsilon \right) \right]. \end{aligned} \quad (27)$$

A solution to this equation will yield all value functions  $V_T^i$  by relation (26).

In the infinite horizon game, profit functions as well as state equations are autonomous, not depending explicitly on time. As a consequence, when the game is stopped at any point in time, the continuation game is identical to the original game — this could be termed the autonomous dynamic programming principle. Consequently, the present-time value functions are time-invariant. Moreover, if

$$V_T^i(t, x_1, x_2) \rightarrow v^i(x_1, x_2) \quad \text{and} \quad \frac{\partial V_T^i}{\partial t}(t, x_1, x_2) \rightarrow 0$$

as  $T \rightarrow \infty$ , then  $v^i$  solves a stationary equation. Equivalently, the time-reversed value function must become asymptotically time-invariant:  $U^i(s, x_1, x_2) \rightarrow u^i(x_1, x_2)$ , with  $u^i$  solving the stationary Hamilton-Jacobi-Bellman equations

$$\begin{aligned} \rho u^i &= \varepsilon u_{x_1 x_1}^i + \varepsilon u_{x_2 x_2}^i \\ &+ \max_{k_i \geq 0} \left[ g_i(x_1, x_2) - k_i^2 + u_{x_i}^i \left( (k_i + \beta \Gamma_j^*(x_1, x_2)) \phi - 1 + \varepsilon \right) \right. \\ &\quad \left. + u_{x_j}^i \left( (\Gamma_j^*(x_1, x_2) + \beta k_i) \phi - 1 + \varepsilon \right) \right]. \end{aligned} \quad (28)$$

Once the value functions  $u^i$  have been determined from these equations, time-independent equilibrium feedback strategies are obtained as the maximizers of the right-hand side of Equation (28):

$$\Gamma_i^*(x_1, x_2) = \max \left\{ \frac{1}{2} \phi \left( u_{x_i}^i + \beta u_{x_j}^i \right), 0 \right\}. \quad (29)$$

To obtain an approximating numerical solution to the infinite horizon game, the system (27) is considered as an ordinary differential equation in the space of pairs  $(U^1, U^2)$  of value functions depending on  $x_1$  and  $x_2$  with a given initial value. A solution to the stationary equation (28) is a steady state of this differential equation.

By integrating equation (27), or rather a discretized approximation, over time-to-completion  $s$ , and stopping once the time derivative  $(U_s^1, U_s^2)$  is sufficiently close to zero, an approximation of an attracting steady state of the ordinary differential equation in function space is obtained.<sup>11</sup>

<sup>11</sup>This approach resembles what is in the literature known as a *method of false transients* (see Schiesser, 1991), where a time derivative which is not part of the original problem is added to a partial differential equation in order to transform it into a well-posed initial value (Cauchy) problem. It is then expected that this additional term will have an insignificant effect on the final solution. In our case, it is rather a method of *true transients* as  $U_s^i$  (or  $V_t^i$ ) is a true part of the Hamilton-Jacobi-Bellman equation (corresponding to a finite-horizon game) and approaches true zero only in



## 3.2 Numerical Method of Lines

The *numerical method of lines* (Schuesser, 1991) is used to obtain an approximation of the solutions to (27). Solutions to the differential equations are considered on a symmetric square region

$$\Omega = (\underline{M}, \overline{M}) \times (\underline{M}, \overline{M}).$$

The region is discretized using a uniform tensor grid with grid spacing  $h = (\overline{M} - \underline{M})/(n + 1)$ . Grid points are of the form  $(x_{1,k}, x_{2,m})$ , with

$$x_{1,k} = \underline{M} + kh, \quad x_{2,m} = \underline{M} + mh, \quad 1 \leq k, m \leq n.$$

Function values at grid points are denoted as

$$U_{k,m} = U(x_{1,k}, x_{2,m}), \quad 1 \leq k, m \leq n.$$

At each grid point  $(x_{1,k}, x_{2,m})$ , derivatives are approximated by second-order central finite differences, e.g.,

$$\begin{aligned} \frac{\partial U^i}{\partial x_1}(s, x_{1,k}, x_{2,m}) &= \frac{U_{k+1,m}^i - U_{k-1,m}^i}{2h} + O(h^2), \\ \frac{\partial^2 U^i}{\partial x_1^2}(s, x_{1,k}, x_{2,m}) &= \frac{U_{k-1,m}^i - 2U_{k,m}^i + U_{k+1,m}^i}{h^2} + O(h^2), \quad \text{etc.} \end{aligned}$$

Discretizing in this way results in a system of  $2n^2$  ordinary differential equations. This system is however not closed, as the derivative discretizations at near-boundary points, e.g. for  $k = 1$ , refer to undefined values, like  $U_{0,m}^i$ . These values are supplied by the boundary conditions of the problem, which are discussed in the next subsection.

The time variable is however still continuous. After discretization, a system of ordinary differential equations is obtained, approximating the partial differential equation (27). It is solved using a third-order Runge-Kutta method (Judd, 1998).<sup>12</sup>

### 3.2.1 Boundary conditions

We already motivated our choice of the initial condition  $U^i(0, x_1, x_2) = 0$ . To solve the system of differential equations (27), we also need to specify boundary conditions corresponding to the four sides of the grid square. The problem is that the value of a solution at all boundaries is *ex ante* not known to us. We address this in Appendix B where we argue that the misspecification of the boundary conditions only results in a significant error in a small region along the boundaries.

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limit (when the horizon of the game approaches infinity and the game itself becomes stationary).

<sup>12</sup>We wrote the code for computations in Fortran 95, using double precision arithmetic. The criterion for convergence is that the value of the  $L^2$ -norm of  $\mathbf{U}_s$  is below  $1 \times 10^{-12}$ . Auxiliary calculations and plots were executed in MATLAB and Mathematica. In presented plots,  $\underline{M} = -2.5$ ,  $n = 200$  and  $h = 0.035$ . To prevent the solution from becoming unstable, the time step  $\Delta t$  in the Runge-Kutta method has to be taken sufficiently small, in order to satisfy both the Courant-Friedrichs-Lewy condition  $\Delta t < \frac{h}{v}$ , where  $v$  is a maximum drift velocity, and the diffusion condition  $\Delta t < \frac{h^2}{2\varepsilon}$ . For small  $\varepsilon$ , as used in the article, the latter condition is usually not binding.

We used standard Neumann boundary conditions:

$$\frac{\partial}{\partial \vec{n}} U^i(s, x) = 0, \quad i = 1, 2, \quad (30)$$

where for a boundary point  $x \in \partial\Omega$ ,  $\vec{n}$  is the outward pointing normal vector, and  $\frac{\partial U}{\partial \vec{n}} = \vec{n} \cdot \nabla U$  is the normal derivative of  $U$  at  $x$ .

### 3.2.2 Time paths

Having obtained numerical approximations of the value functions in (27) and consequently the equilibrium feedback strategies, the investment paths of firms can be simulated. Stochastic time paths are calculated using the Euler-Maruyama scheme (Kloeden and Platen, 1992). For low values of  $\varepsilon$ , the drift part of equation (23) generates a good approximation of the evolution of the state variables over time. That is, we solve

$$\dot{x}_i = (\hat{\Gamma}_i^* + \beta \hat{\Gamma}_j^*)\phi - 1 + \varepsilon, \quad x_i(0) = x_i^0, \quad i = 1, 2, \quad (31)$$

where  $\hat{\Gamma}_i^*(x_1, x_2)$  and  $\hat{\Gamma}_j^*(x_1, x_2)$  are obtained from (29) after replacing derivatives of the value functions with their numerical approximations, and where  $x_i^0$  is firm  $i$ 's initial value of unit cost. Any value of variables between grid points is obtained using cubic spline interpolation (Judd, 1998). Steady states of the drift vector field are analyzed in the usual way.

## 4 Equilibrium strategies and industry dynamics

This section presents the results of the numerical analysis. It discusses strategic interactions between firms as implied by their value and policy functions. Furthermore, state vector fields and time paths of certain variables of interest are analyzed, to obtain insight into possible evolutions of the game.

As baseline, we take the evolution under moderate spillover effects ( $\beta = 0.5$ ). These are then compared with low and high spillover dynamics. The presented plots are all drawn for  $\phi = 8$  and  $\tilde{\rho} = 1$ .<sup>13</sup> This parameterization is representative of situations in which firms have an incentive to develop further a technology which requires R&D efforts prior to production (cf. also Section 4.8).

### 4.1 Value function

Figure 2 shows the value functions and R&D efforts for  $\varepsilon_1 = 0.125$  and  $\varepsilon_2 = 0.0156$ . As the value functions are symmetric in the sense that  $U^2(x_2, x_1) = U^1(x_1, x_2)$ , it is sufficient to consider just  $U^1$ .<sup>14</sup> Large values of  $x_i$  correspond to small values of  $c_i$ .<sup>15</sup> As Figure 2 shows, a

<sup>13</sup> Recall that  $\tilde{\rho} = \rho/\delta$  is a rescaled discount factor (see Appendix A).

<sup>14</sup> While we cannot analytically rule out a possible multiplicity of equilibria, our computations always lead to the same value function irrespective of the particularities of the numerical scheme. This leads us to conjecture that the equilibrium is unique.

<sup>15</sup> Recall that  $c_i = e^{-x_i}$ . Hence, negative (positive) values of  $x_i$  correspond to unit costs above (below) the choke price ( $A = 1$ ).

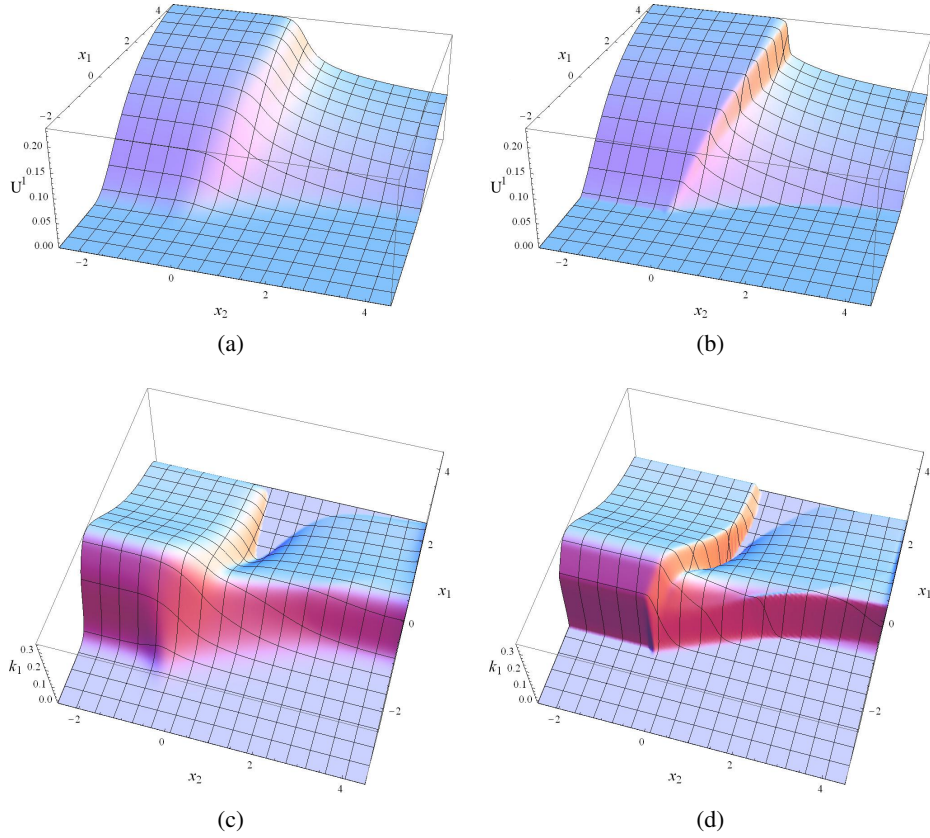


Figure 2: Value functions (top) and R&D efforts (bottom) for  $\varepsilon = 0.125$  (left) and  $\varepsilon = 0.0156$  (right). In both cases  $\beta = 0.5$ .

firm's value function is negatively related to a firm's own unit cost and positively related to the unit cost of its competitor. The smaller a firm's unit costs for a given cost of its competitor, the better a firm's competitive position and so the larger the profits a firm is able to reap. The highest, left part of the value function corresponds to unit costs for which firm 1 is a monopolist. Firm 1's relative cost advantage keeps its competitor out of the market. For lower values of firm 2's unit costs, both firms are (eventually) active in the market (recall Figure 1). This change of the regimes is marked by a steep decline in the value function of the incumbent firm. For relatively high values of own unit costs (the region of the southern valley), the value of the game for firm 1 is zero as the firm finds it optimal to stay inactive.

## 4.2 Policy function

The profits a firm is able to reap from the product market are determined by a firm's cost efficiency. The latter is costly in the sense that due to a positive rate of technology depreciation, a firm needs to invest in R&D not only to increase its efficiency (relative to its competitor), but also to maintain it. The equilibrium R&D efforts are shown in Figure 3. As equation (29) indicates, we can decompose two effects underlying R&D efforts. We call the first effect, which corresponds to the relation between the firm's value of the game and its own unit cost,  $U_{x_i}^i$ , a *pure cost effect*, and the second one, which corresponds to the relation between the firm's value of the game and

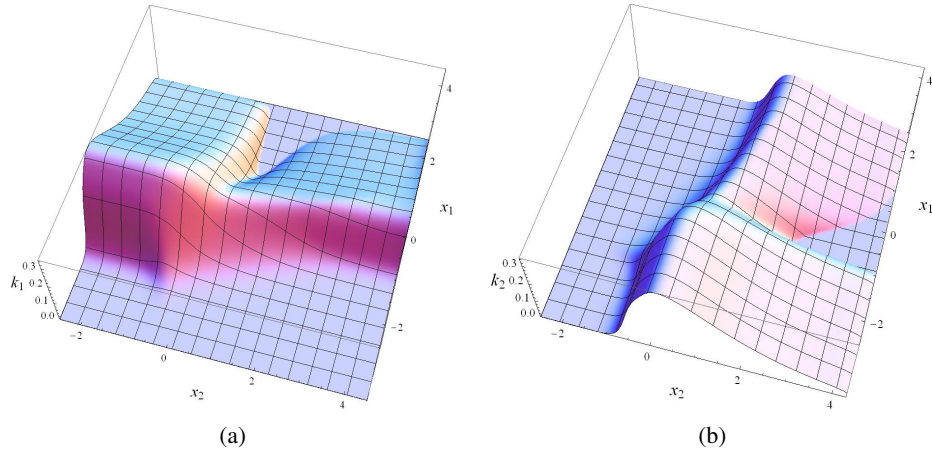


Figure 3: *Symmetry between the R&D efforts of firm 1 (left) and firm 2 (right).*  $\beta = 0.5$ ,  $\varepsilon = 0.125$ .

its competitor's unit cost,  $\beta U_{x_j}^i$ , a *feedback cost effect*.

Observe that, for a given unit cost of its competitor, the R&D effort of a firm increases with decreasing own unit cost over the region of zero production and decreases shortly thereafter. This is driven by the pure cost effect. This effect is always positive and is present whether the competing firm is active or not. It is independent of strategic considerations in the sense that it concerns the relation between a firm's own production costs and its profits. When initial unit costs are high (but still low enough for a firm to pursue further development), there are huge benefits for a firm to exert R&D efforts as this reduces the amount of time needed to reach the production phase. Consequently, R&D efforts are high. The lower the unit costs, the more efforts it takes to reduce them further. This, together with lower tendency of technology to depreciate for lower unit costs, leads the firm to optimally reduce R&D efforts as the unit costs decrease.

On the contrary, the feedback cost effect corresponds to the fact that due to spillovers, any R&D effort a firm exerts contributes also to the reduction of a competitor's production costs, which retroactively affects the firm's profits through the product market competition. The feedback cost effect is always negative and depends positively on the level of spillovers. In the extreme case of zero spillovers, this effect is null. The feedback cost effect underlies industry dynamics through strategic considerations discussed in what follows.

### 4.3 Vector field and dynamics

Recall from Figure 1 that there are three possibilities in the product market depending on the value of unit costs: no production at all, duopoly, or monopoly by one of the two firms. They are delimited by the the product market "entry/exit" curves of firm 1 and firm 2 ( $E_1$  and  $E_2$ , respectively), which are in Figure 4 redrawn in the new coordinates. For any starting value of unit costs, we are interested in how the spillovers affect the way in which firms steer their unit costs as the game evolves. There are many possibilities; some of them are indicated in Figure 4. High initial unit costs (point A) can lead to no market (if both firms refrain from development), to monopoly (if only one firm pursues development), or to duopoly (if both firms enter the market

either simultaneously or sequentially). Monopoly by one firm can either sustain or transform into duopoly if the laggard firm catches up (point B). Likewise, duopoly can persist or change into monopoly if one firm is squeezed out of the market by the other, more efficient firm (point C).

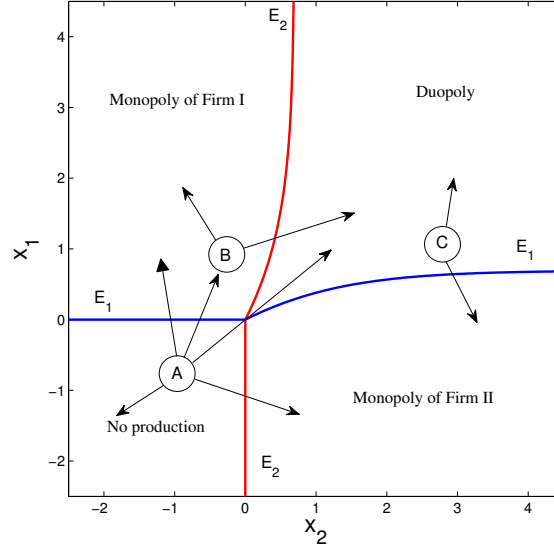


Figure 4: *Evolving market structure.*

The R&D efforts that both firms exert influence the way in which unit costs evolve over time through the drift term (see equation (23)). This evolution of costs as governed by the drift term is summarized by the drift vector field in Figure 5.

Solutions  $x_i^\varepsilon(t)$  to the stochastic evolution equation (23) tend, for finite times, to the corresponding solution  $x_i^0(t)$  to the deterministic equation as  $\varepsilon \rightarrow 0$  (Freidlin and Wentzell, 1998). In the following, drift paths are used as an approximation of the evolution of the system over time.

In Figure 5, the  $\dot{x}_1 = 0$  loci (labeled  $I_1$ ) and  $\dot{x}_2$  loci (labeled  $I_2$ ) intersect in four steady states of the drift vector field:  $S_1$  and  $S_2$  are saddles,  $S_3$  is a nodal source, whereas  $S_4$  is a nodal sink. Invariant manifolds of the two saddles are labeled by letter  $W$ . Stable and unstable manifolds of  $S_1$  are labeled by  $W_1^S$  and  $W_1^U$ , respectively. Similarly,  $W_2^S$  and  $W_2^U$  are, respectively, a stable and an unstable manifold of  $S_2$ .<sup>16</sup>  $E_1$  and  $E_2$  are the product market “entry/exit” curves of firm 1 and firm 2, respectively. In the region above the indicated 45-degree diagonal, firm 1 has a cost advantage over firm 2, whereas the reverse is true in the region below the diagonal. For combinations of unit costs lying exactly on the diagonal, the firms are equally efficient. Observe that the vector field below the indicated 45-degree diagonal is a mirror image of the field above the diagonal. This follows from the symmetry of the feedback equilibrium which is best visible in Figure 3.

Notice that  $W_1^S$  and  $W_2^S$  are separatrices which together with isoclines  $I_1^*$  and  $I_2^*$  around  $S_3$  divide the state space into four domains. The first domain is a basin of attraction of the asymptotically stable steady state  $S_4$ . Every motion starting in this domain converges to  $S_4$  as  $t \rightarrow \infty$ . In this domain, eventually both firms are active in the product market. In any other

<sup>16</sup>The stable and the unstable manifold are the only two trajectories that pass through the saddle point. On any trajectory other than the stable path, the direction of motion is always away from the saddle point.

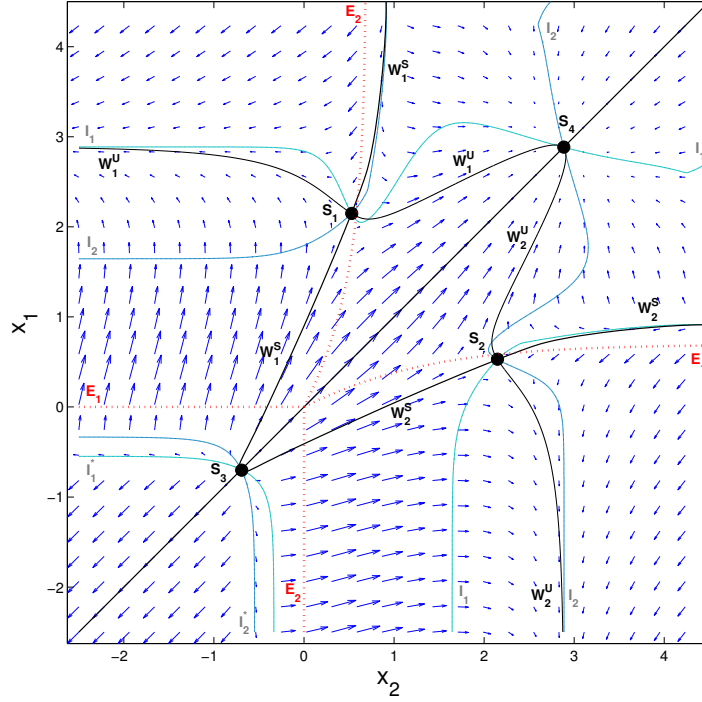


Figure 5: *Drift vector field.*  $\beta = 0.5$ ,  $\varepsilon = 0.125$ .

domain, the unit cost of at least one firm diverges to infinity; we are left either with a monopoly of firm 1 (domain 2), a monopoly of firm 2 (domain 3), or no market at all (domain 4).

We now analyze possible evolutions of the game by jointly looking at Figure 3 and Figure 5. In the region south-west from  $S_3$ , the unit costs of both firms are “very” high and above the choke price, such that both firms decide to refrain from developing further the initial technology. Future expected profits are not high enough to compensate for investments needed to bring the technology to the production phase. Technically, unit costs flow towards infinity due to the positive depreciation rate. Intuitively, though, we always interpret any situation in which a firm stays inactive as one in which it has left the market.

Left to the stable manifold of  $S_1$ , labeled in Figure 5 by  $W_1^S$ , the cost advantage of firm 1 over firm 2 is so large that the latter gives up on R&D (see figure 3). When cost asymmetries are large, the profits the less efficient firm earns in the product market are low. This reduces the ability of firm 2 to compensate for R&D investments needed to bring its technology to the product market and catch up with firm 1. It turns out that left to  $W_1^S$  the cost asymmetries are just so large that firm 2 cannot even afford to battle depreciation of its own technology, thereby succumbing to its more efficient competitor. Firm 2 does produce only when its initial unit costs are already sufficiently low (the region between  $E_2$  and  $W_1^S$  curve), such that it can profitably sell a positive quantity in a competitive product market. However, its product market activity is only temporary as the large cost advantage enables the more efficient firm 1 to squeeze firm 2 out of the market. Thus, firm 2 does eventually neither produce nor invest in R&D. Its unit costs in this region always tend to flow towards infinity, which is, again, due to the positive depreciation rate.

The investment of firm 1 depends on its initial unit cost. The firm decides to enter the market for all initial costs that in Figure 5 correspond to  $x_1$  above the  $I_1^*$  curve which flows through  $S_3$ . For initial unit costs above the choke price ( $x_1$  below the horizontal part of  $E_1$  curve), the firm does at first produce nothing but invests increasingly in the reduction of its unit costs. Once its unit costs have been reduced below the choke price, the firm starts producing as it can now sell at positive mark-ups. The level of R&D efforts and unit costs then gradually decrease to their long-run optimal levels ( $x_1$  approaches the unstable manifold  $W_1^U$  which asymptotically converges with  $I_1$  isocline). For unit costs above the choke price, instantaneous profits of firm 1 are negative as there is no production yet. Firm 1 initiates R&D as it expects future profits will more than compensate for initial investments. There exists a finite upper bound on unit costs beyond which expected future profits are not enough to compensate for short run losses (unit costs corresponding to  $x_1$  below the  $I_1^*$  curve). In this case, the initial technology is not developed further. Observe how the direction of vectors in Figure 5 changes its sign when passing through the  $I_1^*$  curve. While firm 2 benefits from R&D efforts of firm 1 through spillovers, this effect is not strong enough to bring firm 2 into the market. It, however, slows down the rise in the discrepancy between the two firms' unit costs.

In the south-eastern part of the state space, below the  $W_2^S$  curve, the situation is reversed. It is now firm 2 whose cost advantage leads to its monopoly. For all initial unit costs on the right side of the  $I_2^*$  curve passing through  $S_3$ , firm 2 brings a technology into the market, while firm 1 is sooner or later forced out of business.

In the north-east region of the state space, between the  $W_1^S$  and  $W_2^S$  manifolds, the cost asymmetries are moderate. Eventually, a product market duopoly emerges as for all initial costs in this region, each firm sooner or later brings a technology into the product market. It is interesting to observe that the asymptotically stable steady state  $S_4$  lies on the 45-degree diagonal. This implies a kind of a *regression toward the mean* phenomenon, where any initial difference in the unit costs between firms tends to vanish over time.<sup>17</sup> We have noted that above the 45-degree diagonal, firm 1 has a cost advantage, which is to the left of  $W_1^S$  large enough to squeeze firm 2 out of the market. However, to the right of  $W_1^S$ , this is not the case any more. Notice that  $W_1^S$  curve travels along the edge of the precipice in the policy function of firm 2 (see the right plot in Figure 3). While left to  $W_1^S$  firm 2 gives up on R&D, right to  $W_1^S$ , it invests heavily to catch up with firm 1. Firm 1 exerts less R&D efforts than firm 2, however, it prolongs its cost supremacy through positive spillover effects arising from relatively high R&D efforts of firm 2. When its initial costs are very low, firm 1 for some time even sits back on R&D (observe the basin in the northern region of the firm 1's policy function in Figure 3) and retards its technology decay optimally by relying mostly on spillovers from the R&D efforts of its zealous counterpart.<sup>18</sup> Namely, when unit costs of firm 1 decrease relative to firm 2, an additional unit of firm 1's R&D effort benefits firm 2 progressively more than firm 1 itself, which diminishes firm 1's incentives for own R&D (this follows directly from the formulation of unit costs in (2)). The story is analogous when we are on the other side of the diagonal, where firm 2 has a relative cost advantage. In both cases, a dominant firm gradually loses its lead.

<sup>17</sup>We say "tends to" as it *de facto* vanishes only in light of a deterministic game interpretation. In a stochastic interpretation, only the gap between the mean values of the two unit costs narrows and eventually closes.

<sup>18</sup>A typical example of a large firm relying on inventions by smaller firms is Microsoft, whose competitors "have long complained that the rest of the industry has served as Microsoft's R&D lab" (Pollack, 1991).

#### 4.4 Leader versus laggard

We have noted above that whenever cost asymmetries are large, the less efficient firm is squeezed out of the market. Only when initial asymmetries are not too large (i.e., when we are within the basin of attraction of  $S_4$ ) both firms steer the evolution of their costs so that they remain active in the product market. In this latter case, any initial asymmetry between the firms tends to vanish over time.<sup>19</sup> Figure 6 illustrates further how in the latter case, costs and R&D efforts evolve over time along the drift path. In the first two plots, firms already start with relatively low costs,  $(x_1, x_2) = (1.5, 1)$  or  $(c_1, c_2) = (0.22, 0.37)$ . We see that the laggard exerts more R&D efforts than the leader, which gradually reduces the gap between the two firms' unit costs. In plots 6c and 6d, both firms start with unit costs above the choke price,  $(x_1, x_2) = (-0.2, -0.3)$  or  $(c_1, c_2) = (1.22, 1.35)$ . With own unit costs very large, the more efficient firm 1 initially invests a lot in R&D, in particular, more than firm 2. Firm 1 exploits its own cost advantage to enter the product market first and earn temporary monopoly profits. The situation changes in the course of time and eventually the laggard invests more than the leader. This causes the gap between the unit costs to shrink (the evolution of costs is similar to that in 6a and is omitted for brevity). The quantity each firm produces increases over time together with decreasing unit costs. As firm 1 is more efficient than firm 2, firm 1 at all times produces more. However, with the gap between unit costs gradually narrowing, the gap between quantities is narrowing as well.

#### 4.5 Stochasticity and R&D

It is an interesting question how the R&D efforts of firms relate to uncertainty. A look at Figure 2 reveals that both the value function and the policy function are smoother for higher levels of noise in unit costs. To investigate this further, we plot the value function and policy function of firm 1 for different fixed values of firm 2's unit cost. In Figure 7, we fix  $c_2$  at such a high value that firm 1 is a monopolist ( $c_2 = 11.76$ ; for reference,  $A = 1$ ). Then, we can directly compare our solution with the deterministic monopoly solution, obtained in Hinloopen et al. (2013). We observe that while the deterministic value function (the solid curve) has a kink, the stochastic value function is smooth. The fact that the value function corresponding to a higher noise level lies above the one corresponding to a lower noise level suggests that stochasticity increases expected profits. We see that the stochastic value function converges to the deterministic monopoly value function as  $\varepsilon \downarrow 0$ . For  $\varepsilon = 0.0156$ , the stochastic solution is already almost indistinguishable from the deterministic one, the absolute difference between the two solutions at the kink being 0.0012. The deterministic policy function is discontinuous at the point of indifference, where the firm is indifferent between developing a technology further or staying out. This discontinuity is smoothed out by stochasticity. The policy function of the stochastic model is smooth and everywhere differentiable. It is interesting to observe that stochasticity makes a firm invest in R&D over the values of unit costs for which a firm in the deterministic setting already gives up. The firm in the stochastic setting still invests a bit at larger costs in hope of a favorable shock, for which it sacrifices some investments at lower unit costs – the R&D efforts are smoothed out. While R&D efforts exerted at large costs might as such not be sufficient to bring a technology to

<sup>19</sup>Once the drift path has reached the asymptotically stable steady state  $S_4$ , the actual unit costs fluctuate around  $S_4$ . Of course, over a very long time, eventually a large shock may occur, driving one of the firms out of the market (its unit cost diverges to infinity).



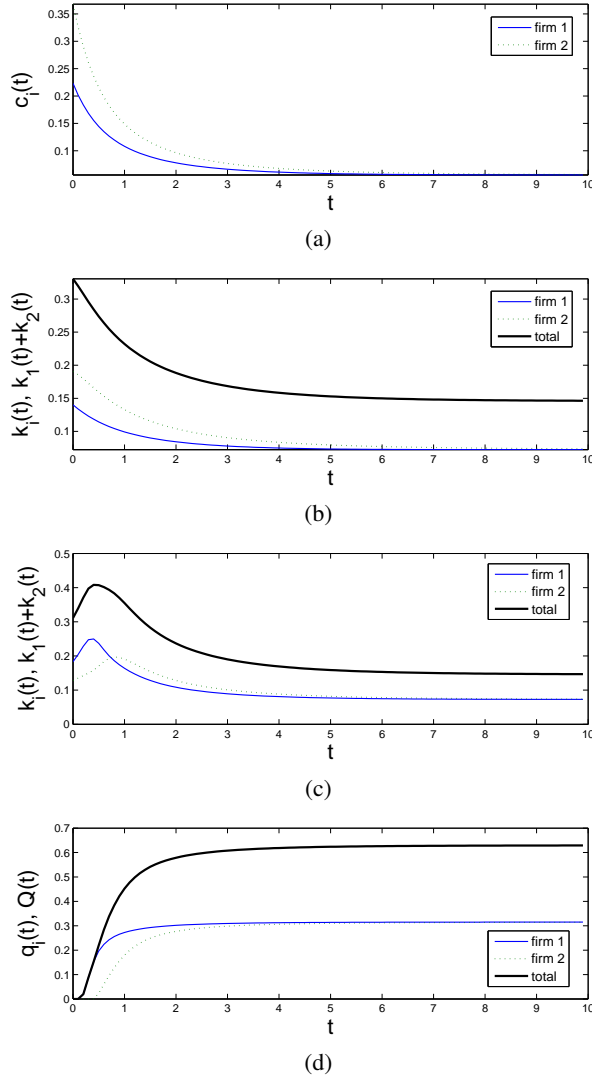


Figure 6: *Time paths: laggard versus leader.*

the production phase, they at least retard the decay of a technology for some time during which hopefully a favorable shock arises. Higher uncertainty, therefore, leads to more opportunistic behavior of firms, which increases the chance that the development of expensive technologies will be pursued further. Computations shows that this opportunistic behavior also increases the relative size of the region of the state space for which eventually duopoly tends to appear in the product market (in the drift vector field, the basin of attraction of  $S_4$  spreads out with increasing noise levels).

#### 4.6 Deterministic game and indifference curves

In this section, we take a closer look at the deterministic game. Its solution will be close to the solution to the stochastic game with a small noise level.

As shown in Figure 7, the policy function in the deterministic monopoly solution is discontinuous at the indifference point. Only for initial costs below the indifference point a monopolist

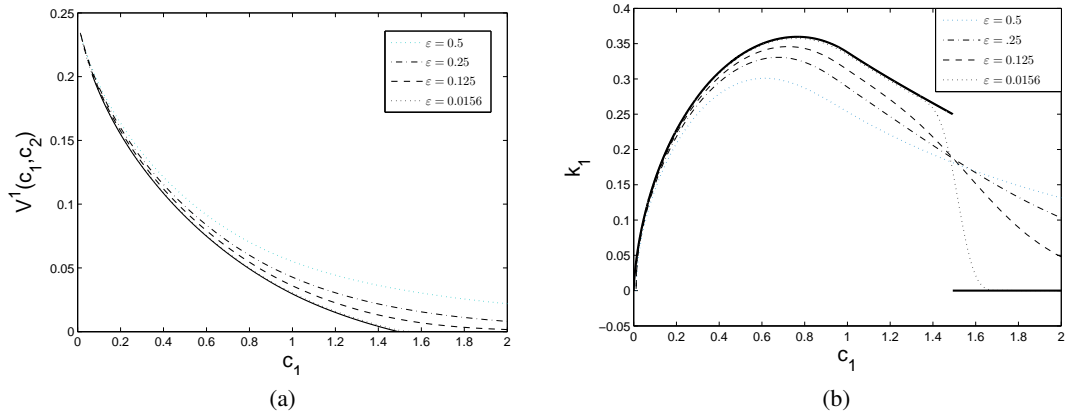


Figure 7: Value functions (a) and policy functions (b) of firm 1 for varying levels of noise  $\varepsilon$  when the unit cost of firm 2 is fixed at  $c_2 = 11.76$ . The solid line corresponds to the deterministic monopoly solution.

continues to invest in R&D and stays active in the product market. In what follows, we analyze the existence of indifference points in the deterministic competitive game. We define the deterministic indifference point of a firm as a value of its unit cost at which a firm is indifferent between developing a technology further and exiting the market. In general, indifference points do not coincide with points at which the R&D effort of a firm is zero. This is the case only when an indifference point is in the region of zero production.<sup>20</sup> When an indifference point is at the value of a unit cost at which a firm produces, an exiting firm might still invest a bit in order to slow down the speed at which it leaves the product market (cf. the notion of the ‘exit trajectory’ in Hinlopen et al. (2013)).

In our stochastic setting, firms steer the evolution of their unit costs through the drift term. The direction of this steering is summarized in the drift vector field (see Figure 5). The lower the noise level and so the lower the random shocks to the unit costs, the smaller the deviations of actual costs from their drift path. For zero noise, the drift path is the actual path. We saw in our discussion of the drift vector field above that the isoclines  $I_1^*$  and  $I_2^*$  around  $S_3$  and the separatrices  $W_1^S$  and  $W_2^S$  play an important role in the motion of the drift – they are boundaries of different basins of attraction. With noise approaching zero, these curves therefore converge to the boundaries of basins of attraction of actual unit costs. Consequently, our conjecture, on which we further elaborate below, is that these curves converge to the union of indifference points of the deterministic game.

Consider first the state space left to  $S_3$  in Figure 5, where the cost of firm 2 is relatively high. We have already observed that firm 1 steers its unit cost downwards for initial unit costs corresponding to  $x_1$  above the  $I_1^*$  curve, and upwards otherwise. In Figure 8, we consider the left most part of the state space as, again, there we can compare our solution with the deterministic monopoly solution. We know that the deterministic policy function is discontinuous at the

<sup>20</sup>Clearly, when there is no uncertainty involved, a firm has no reason to initiate investment if it never plans to enter the product market. However, when  $\varepsilon > 0$ , the R&D efforts are always positive on both sides of an approximating indifference point due to smoothing-out.

point of indifference. As Figure 8b shows, the deterministic drift is also discontinuous at the indifference point, it is negative left to the indifference point (the unit cost continues to decrease) and positive to the right of it (the unit cost flows towards infinity as a technology decays due to zero research activity). On the contrary, in the stochastic setting, the policy function is smooth, as is the drift. The point of zero drift in Figure 8b corresponds to the point on the  $I_1^*$  isocline. The difference between the latter and the deterministic indifference point is for  $\varepsilon = 0.0156$  already within the second decimal point. This difference decreases further with lowering the noise level as the point on  $I_1^*$  converges to the deterministic indifference point. We expect the same relation between the points on the isocline and indifference points to hold also for other parts of the  $I_1^*$  isocline. Analogously, for firm 2, indifference points correspond to points on the  $I_2^*$  isocline below  $S_3$ .

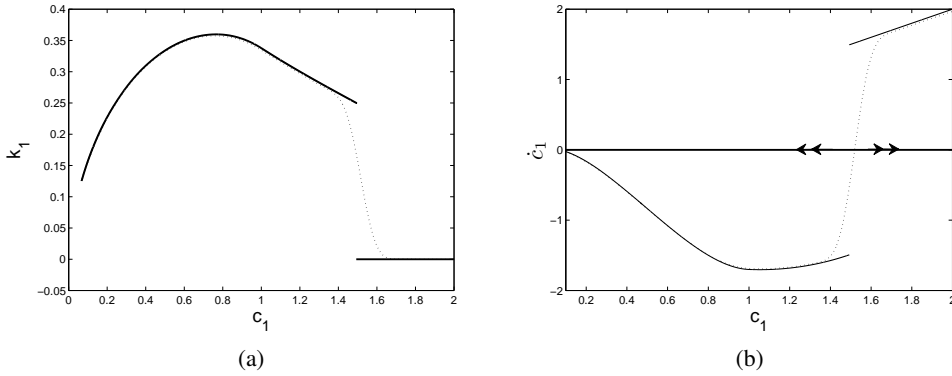


Figure 8: Policy function of firm 1 (a) and drift of its unit cost (b). The solid line corresponds to the deterministic monopoly solution, the dotted line to the stochastic game with  $\varepsilon = 0.0156$ . The unit cost of firm 2 is fixed at  $c_2 = 11.76$ .

We have already noted that the stable manifold  $W_2^S$  act as a separatrix. For unit costs above it, firm 1 invests relatively a lot and steers its unit cost towards  $S_4$ . For unit costs below it, the firm still invests a bit in hope of a shock (random discovery) that would reduce its unit cost, but progressively less so as its unit cost increases due to technology depreciation, thereby steering its unit cost towards infinity. In the absence of a sufficiently large favorable shock, the firm gradually gives up on R&D. Figure 9 shows plots of firm 1's R&D efforts for a given value of firm 2's unit cost. The vertical line corresponds to the point on the  $W_2^S$  curve. As plots show, with decreasing  $\varepsilon$ , the policy function straightens up at the point corresponding to  $W_2^S$  as the latter converges to the deterministic indifference point at which the policy function breaks off and the discontinuity arises. Right to the indifference point, the less efficient firm gives up, whereas left to it, the firm invests heavily in R&D in order to catch up with the more efficient competitor.

Analogously, points on  $W_1^S$  converge to the indifference points of firm 2 as  $\varepsilon \rightarrow 0$ . We also observe that steady states  $S_1$ ,  $S_2$ , and  $S_3$  can lie in the regions of zero production (see Figure 5). Clearly, in a deterministic game, there cannot be steady-state points in such regions as that would imply a situation in which a firm invests at all times but never produces. These steady states are the implication of the continuity of the stochastic drift. They correspond to points at which the drift of both unit costs is zero. Like other points on the two manifolds, when a stochastic game

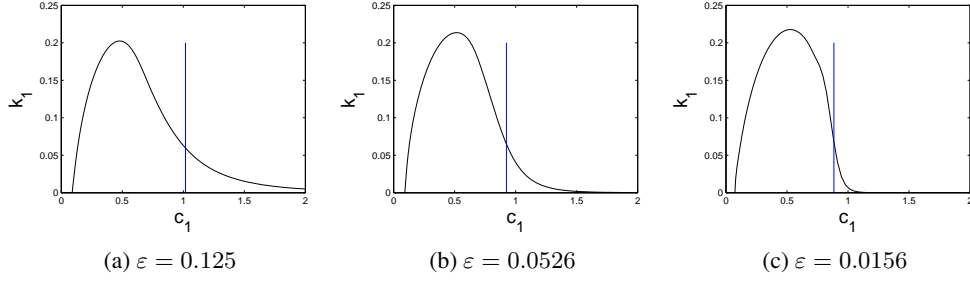


Figure 9: Policy functions of firm 1 for a given value of firm 2's unit cost,  $c_2 = 0.4232$ . The vertical line corresponds to the point on the stable manifold  $W_2^S$ .

transforms into a deterministic one, these steady states convert into the points of discontinuity. While at a stochastic steady state the drift is zero, at the indifference point the drift is multi-valued and so there can be no steady state corresponding to such a point. This is visible in plot (b) of Figure 8 above and fits nicely into our picture of convergence.

We summarize our observations in the following conjecture.

**Conjecture 1.** *In the deterministic game, the union of the stable manifold  $W_2^S$  and the  $I_1^*$  isocline to the left of  $S_3$  approximates an indifference curve of firm 1 as  $\varepsilon \rightarrow 0$ . Likewise, the union of the stable manifold  $W_1^S$  and the  $I_2^*$  isocline below  $S_3$  approximates an indifference curve of firm 2 as  $\varepsilon \rightarrow 0$ .*

This conjecture is illustrated in Figure 10. Observe that indifference curves divide the state space into four regions *i*) the region of eventual duopoly, *ii*) the region of eventual monopoly of firm 1, *iii*) the region of eventual monopoly of firm 2, and *iv*) the no market region, in which initial unit costs are too high for either firm to consider further development of a technology. Observe also that the indifference curves do not coincide with product market “entry/exit” curves. The indifference curve of each firm roughly outlines the “foothills” of the respective firm’s policy function (see the bottom right plot in Figure 2). Finally, recall that these delimiting curves play an important role also in a stochastic game – they are boundaries of different basins of attraction of the drift vector field.<sup>21</sup>

## 4.7 Spillover effects

In this section, we compare the case with moderate spillovers ( $\beta = 0.5$ ) with that of low and high spillovers. The larger the level of spillovers (the larger the  $\beta$ ), the more the R&D efforts

<sup>21</sup>Notice a few interesting properties of these separatrices. Consider the one which is above defined as the indifference curve of firm 1 (properties of the curve corresponding to firm 2 are analogous). First, the subspace of the state space above the curve is the union of all those unit costs for which firm 1 steers its unit costs toward the level characterized by positive production (goal I: ‘in the market’), whereas the subspace of the state space below the curve is the union of all those unit costs for which firm 1 steers its unit costs toward the level characterized by no production (goal II: ‘out of the market’). Second, for any  $\mathbf{x}$  in a given subspace it holds that for every unit cost in the  $\varepsilon$ -ball around  $\mathbf{x}$ , the firm steers the cost in the same direction as for  $\mathbf{x}$ . That is, only a large shock will change the direction in which the firm steers its unit costs given its initial value of  $\mathbf{x}$ . Third, a ball around any point on the separating curve contains points not on the curve. In other words, for every initial  $\mathbf{x}$  on the curve, even so small a shock can change the direction of movement. As (large) shocks are purely random, it is in this sense that in a stochastic setting we can say that on different sides of separatrices firms *pursue*, though not necessarily also achieve, a different goal.

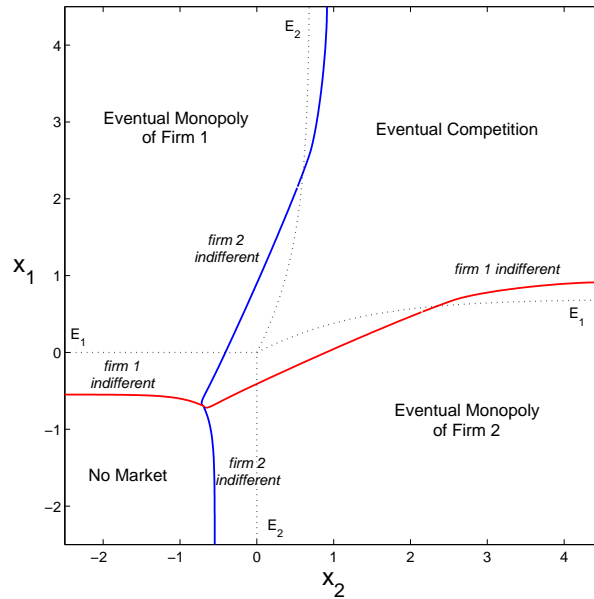


Figure 10: *Conjectured indifference curves of the deterministic game.*

that a firm exerts benefit its competitor, and thus the larger the role of feedback cost effects in shaping a firm's policy function.

#### 4.7.1 Low spillovers

For low levels of spillovers, the policy function exhibits a sharp and narrow bulge, as visible in Figure 11.<sup>22</sup> The lower the level of spillovers and/or noise, the more pronounced and sharp the bulge. Figure 12 shows the drift vector field corresponding to the case with  $\beta = 0$ . Observe that  $W_1^S$  and  $W_2^S$  separatrices are in the vicinity of  $S_3$  practically indistinguishable from the diagonal – a small difference in unit costs is enough to drive the less efficient firm out of the market. This region of proximity corresponds to the location of the bulge. With increasing levels of spillovers and/or noise, the bulge becomes thicker and lower; the separatrices shift away from the diagonal, implying that a larger cost advantage is needed to drive the opponent out of the market (compare with Figure 5).

As Figure 13 shows, on the diagonal within the region of proximity, each firm invests a lot trying to reduce its production costs as fast as possible and so increase its chances of survival (the vertical line in the figure corresponds to symmetric costs). For a symmetric initial position, firms are engaged in a preemption race where the one that falls sufficiently behind the other is driven out of the market.

Observe that the bulge attains its top above the diagonal (left to the vertical line in Figure 13) and sweeps sharply down on the other side of the diagonal (the policy function has a steep slope on the right side of the vertical line in Figure 13). The firm with a cost advantage therefore

<sup>22</sup>The case with low spillovers is the hardest one to integrate numerically. For small values of  $\varepsilon$ , the bulge increases and sharpens dramatically, posing problems for the stability of the numerical scheme. Solutions for lower  $\varepsilon$  require increasing refinements of the grid, which in consequence rather considerably affects the speed of calculations. For this reason, the lowest noise level we currently present for this case is  $\varepsilon = 0.125$ .

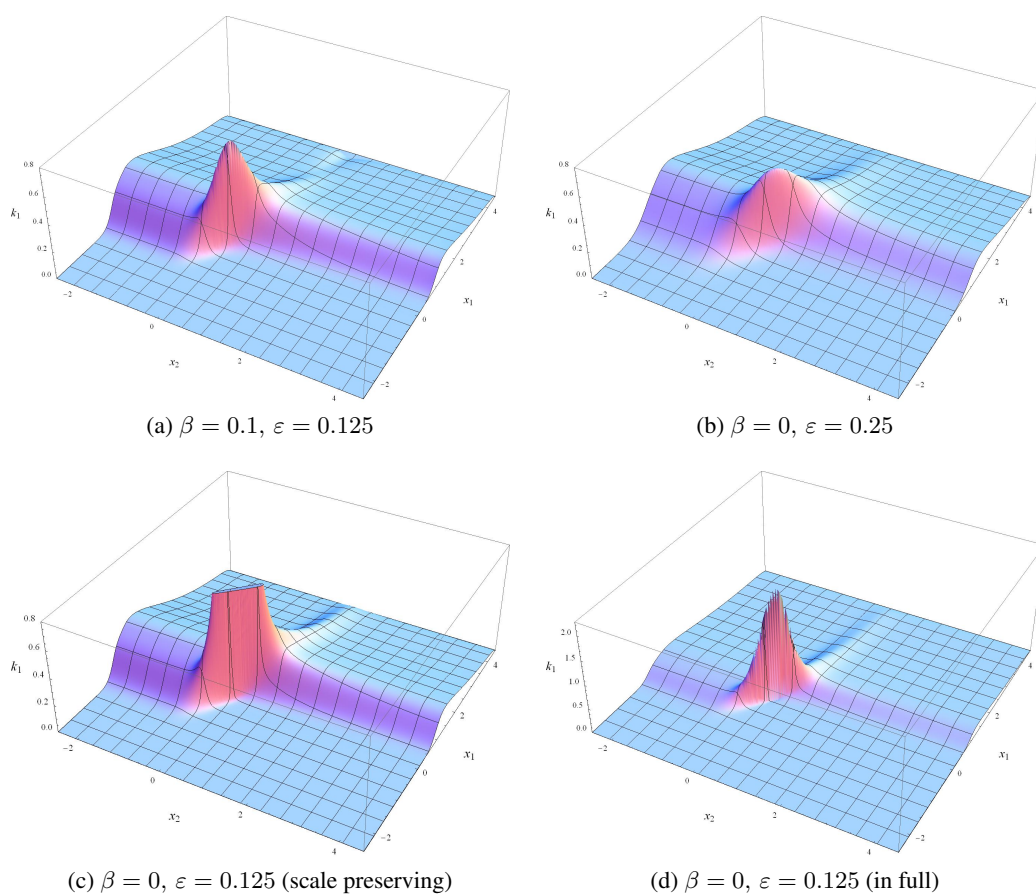


Figure 11: *Preemption & predation at low spillovers*. The plots show the policy function of firm 1 for different levels of spillovers and uncertainty.

invests heavily (but briefly), whereas the laggard is induced to give up. This additional R&D effort of the leader can be considered predatory in a sense that it is profitable only for its effect on the exit decision of the laggard, but unprofitable otherwise.<sup>23</sup> The predatory nature of these investments is confirmed by the fact that such large investment asymmetries never occur when the likelihood that a rival remains viable is negligible (e.g., at very high levels of a rival’s unit cost) or the ability of a firm to influence this likelihood is negligible (e.g., in the case of large spillovers where large investments would to a great extent benefit the competitor).

The extent of predatory efforts is positively related to the easiness with which the leader can induce the laggard to give up. Recall that the bulge grows with the spillovers and noise level approaching zero (see Figure 11). At low spillovers, it is easier for the leader to induce the laggard to exit as the latter cannot count on catching up with the leader by copying the results of the leader’s R&D efforts. Thus, the lower the spillovers, the easier it is for the leader to achieve his dominance by exerting R&D efforts and so the higher are his incentives for extensive predation. Next, when the probability of large unexpected changes in costs is high, the laggard

<sup>23</sup>In declaring an action predatory, we follow Cabral and Riordan (1997) who define an action as predatory if “i) a different action would increase the likelihood that rivals remain viable, and ii) the different action would be more profitable under the counterfactual hypothesis that the rival’s viability were unaffected” (p. 160). Our interpretation is similar to that of Borkovsky et al. (2012) who consider predatory investment in a dynamic quality ladder model.

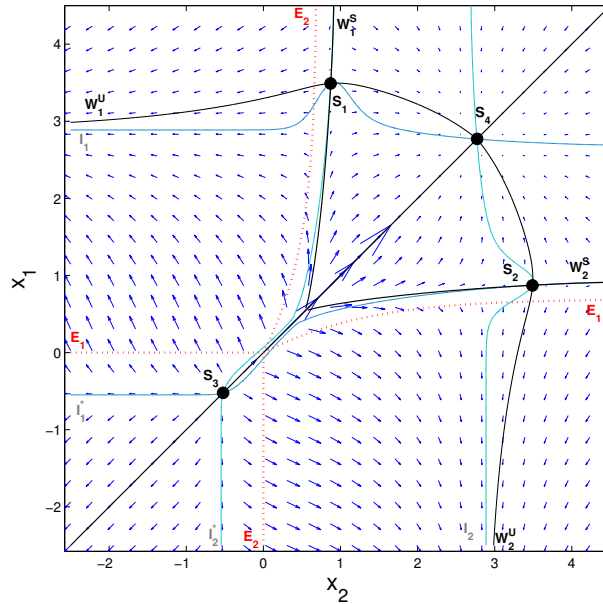


Figure 12: *Drift vector field for  $\beta = 0$ ,  $\varepsilon = 0.125$ .*

does not give up that fast when falling behind as it is still possible for him to catch up the leader if he has a run of luck. In this case, the leader needs to achieve a relatively large cost advantage to induce the laggard to give up. However, due to large randomness in costs, the effect of the leader's R&D efforts on the likelihood of achieving such an advantage is small. Consequently, his incentives for predatory investments are low. On the contrary, when there is low uncertainty in cost movements, a small cost advantage is sufficient to drive the other firm out of the market and the effect of the leader's R&D efforts on the likelihood of achieving a needed cost advantage is large. As a consequence, the leader's incentives to engage in extensive predation are high.<sup>24</sup>

#### 4.7.2 High spillovers

Figure 14 and Figure 15 show the policy function and the drift vector field, respectively, for  $\beta = 0.9$ . Figure 16 jointly plots the indifference curves (separatrices) for  $\beta = 0$  (L),  $\beta = 0.5$  (M), and  $\beta = 0.9$  (H).

When the level of spillovers is high, the R&D efforts of one firm benefit the other firm to a large extent. As each firm tries to free-ride on the other firm's R&D efforts, the incentives to exert much R&D efforts can be rather low. This standard conclusion in the literature is in part

<sup>24</sup>We have seen that the situation in which one firm invests heavily while the other negligibly small can be a feature of the equilibrium of the stochastic game as large investments are optimal in that they influence the likelihood the rival is induced to give up on R&D and exit the market. Clearly, it is hard to imagine that such a situation could be a feature of the equilibrium of a deterministic game as there is no probability of the rival's market viability involved. We observe that the bulge narrows and sharpens as  $\varepsilon \downarrow 0$  (refer also to Figure 13). We therefore conjecture that in a deterministic case, the bulge corresponds to the point of discontinuity in the policy function, whereas his left and right foothills determine the R&D efforts of the leader and the laggard, respectively. While still asymmetric, the difference in R&D efforts is comparably much smaller. In the region of the bulge, indifference curves are tangent to the diagonal (recall the tangent behavior of  $W_1^S$  and  $W_2^S$  separatrices in Figure 12). For symmetric initial costs, both firms eventually produce, whereas a minuscule asymmetry already leads to a monopoly.

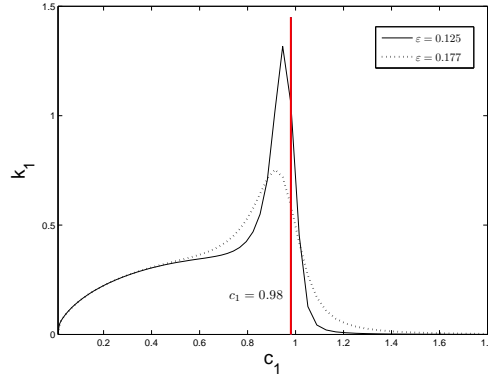


Figure 13: Policy function of firm 1 for a value of firm 2's unit cost corresponding to the region of the bulge.  $\beta = 0$ ,  $c_2 = 0.98$ .

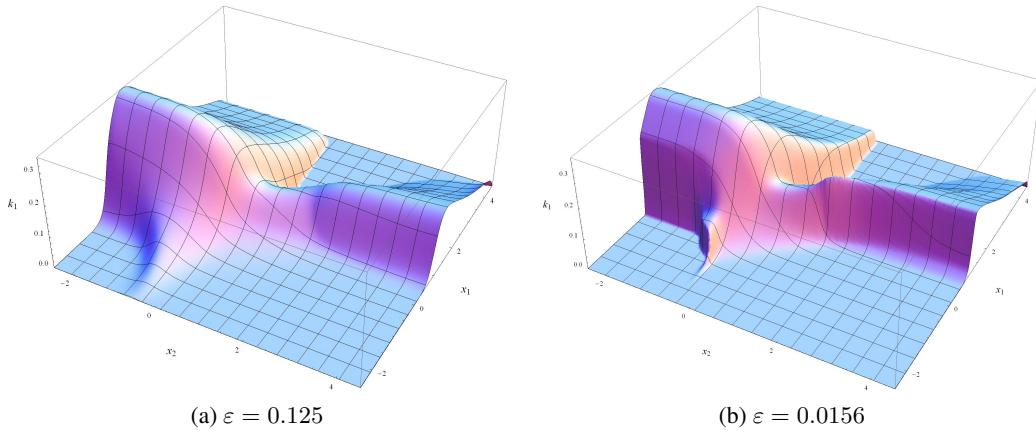


Figure 14: Policy function of firm 1 for  $\beta = 0.9$ .

confirmed by our calculations – R&D efforts decrease over the bulk of the state space as the level of spillovers approach one. However, there is an important exception, depicted in Figure 14. The policy function for large spillovers exhibits a pronounced bulge spreading into the “southern” part of the state space (i.e.,  $x_1$  low,  $x_2 \approx -0.1$ ). The size of this bulge increases with spillovers (first traces of it appear in the policy function for  $\beta = 0.5$  in Figure 3). The intuition is the following. Exerting R&D efforts is costly (recall the quadratic R&D cost function). When spillovers are large and so R&D efforts of the firms complement each other well, the firms facing a convex cost function are able to circumvent diseconomies of scale in R&D to a large extent. This reduction in the costs of R&D enables the two firms to competitively bring into the market a technology which a single firm cannot profitably develop itself. This explains why, provided the cost asymmetries are not too large, a firm in a duopoly market sometimes does invest in further development of a technology whereas a monopolist does not (the bulge). The implication of this is best visible in the drift vector field in Figure 15, where we observe that the steady state  $S_3$  (the indifference point of two firms) corresponds to a higher value of the firm 1's unit cost than the leftmost part of the  $I_1^*$  isocline (the indifference point of a monopolist). Furthermore, Figure 16 shows that  $S_3^H$  corresponds to a higher value of unit costs than either  $S_3^L$  or  $S_3^M$ . The larger the spillovers,



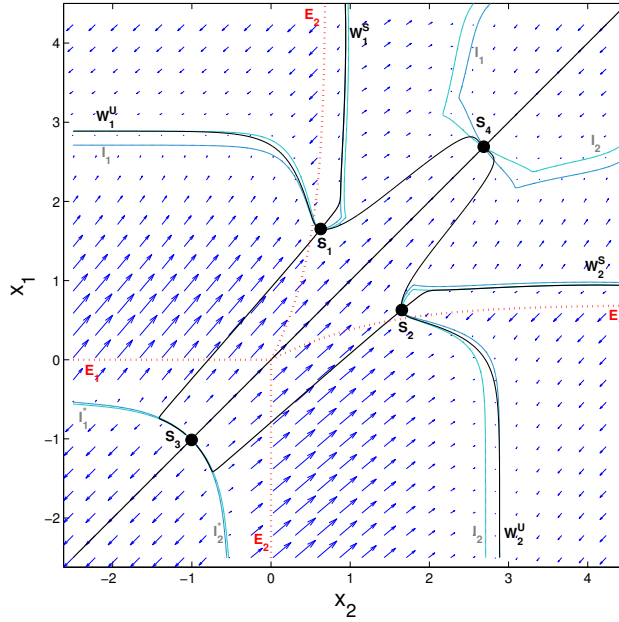


Figure 15: *Drift vector field for  $\beta = 0.9$ ,  $\varepsilon = 0.125$ .*

the larger the joint savings in R&D costs and so the more infant the initial technology that the firms can afford to develop further. In contrast to the cases with lower spillovers, in the case with high spillovers,  $W_1^S$  and  $W_2^S$  separatrices also form a wider arc around  $S_3$ . Therefore, higher spillovers not only increase the range of initial technologies that are developed further, but also the likelihood that the ensuing product market will be competitive.

Comparing the drift vector field of  $\beta = 0$  with that of  $\beta = 0.5$ , we observe that the basin of attraction of the steady state  $S_4$  is wider in the latter case (see Figure 16). The  $W_1^S$  and  $W_2^S$  separatrices spread out. This suggests that it takes a larger cost asymmetry for the less efficient firm to leave the market when spillovers are higher. In particular, the exit of any firm is much less likely when both firms already produce (for  $\beta = 0.5$ , larger parts of separatrices lie outside the production region bounded by  $E_1$  and  $E_2$  curves). The larger the spillovers, the more the laggard can benefit from the R&D investments of the leader and so the more disadvantaged it must be to give up. This point was already raised by [Petit and Tolwinski \(1999, p. 204\)](#) claiming that “[...] for a duopoly consisting of unequal competitors free diffusion of knowledge may be a way to avoid market concentration.”

In contrast to the aforementioned authors, our analytical framework allows us to draw much more precise conclusions as it makes it possible for us to obtain indifference sets over the entire state space, which in turn enables us to compare entry-exit investment decisions of firms for all possible initial positions. In particular, we show that the pro-competitive benefit of larger spillovers does not hold for all levels of spillovers and costs. Observe how in Figure 16 the separatrices corresponding to  $\beta = 0.9$  intersect those corresponding to  $\beta = 0.5$ . While for high initial unit costs of firms larger spillovers still make duopoly in the ensuing product market more likely, this does not hold for lower values of initial unit costs as there the less efficient firm is sooner squeezed out of the market when spillovers are *larger*! Behind this result are

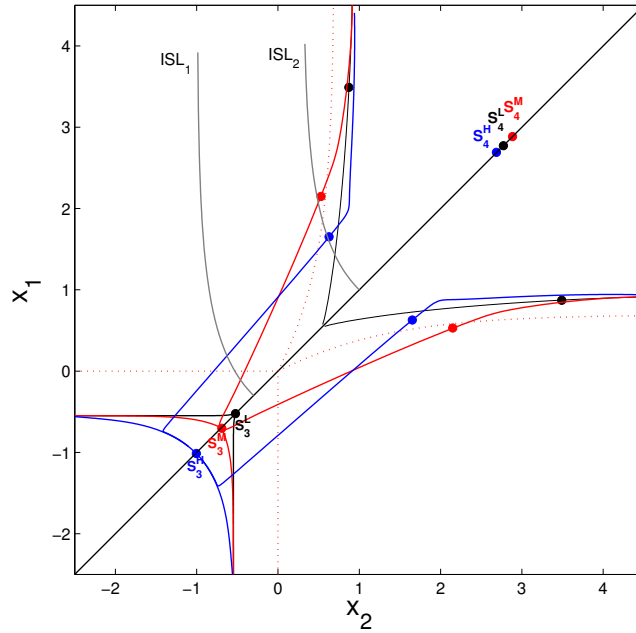


Figure 16: Comparison of indifference curves (separatrices) between  $\beta = 0$  (black),  $\beta = 0.5$  (red), and  $\beta = 0.9$  (blue).  $\varepsilon = 0.125$ .

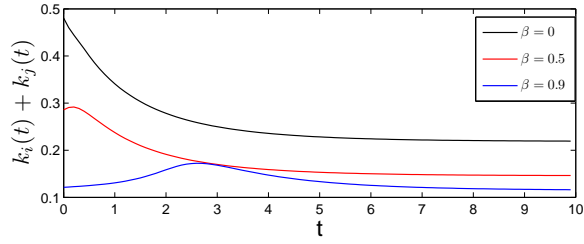
two countervailing effects of spillovers. The first effect is a pure spillover effect – the larger the spillovers, the more the laggard is able to free-ride on the leader’s R&D efforts and so the easier it is for him to overcome any initial asymmetries. This effect is positively related to the level of spillovers and contributes to widening the region of eventual product market duopoly. The second effect is the feedback cost effect, which is also positively related to the level of spillovers, however, it contributes to narrowing the region of eventual product market duopoly. When the unit cost of a firm is large, an additional unit of R&D effort benefits this firm a lot (the pure cost effect dominates). However, when the unit cost of the firm is lower, so is the impact of an additional unit of R&D on its costs (the factor  $c_i k_i$  in (2) decreases with  $c_i$  for a given  $k_i$ ). If the unit cost of the laggard is sufficiently larger, it can well happen that the additional R&D effort of the leader benefits the laggard more than the leader himself ( $c_i k_i < c_j \beta k_i$ ). As lower costs of the laggard through fiercer product market competition negatively affects the leader’s profits, this reduces the leader’s incentives to invest in R&D. This feedback effect, which negatively affects the leader’s R&D efforts, is stronger, the larger the spillovers. Consequently, the larger the spillovers, the less asymmetry in costs it takes for the leader to optimally stop his R&D efforts. Observe how the region of zero R&D efforts above the diagonal spreads out in the policy function as the spillovers increase (compare Figure 3, Figure 11, and Figure 14). This explain why larger spillovers might in fact increase the likelihood that the market will be monopolistic. After a certain level, further increases in spillovers decrease the leader’s incentives to invest rather significantly, which makes it harder for the laggard to catch up with the leader. The laggard’s possibilities to copy the leader’s R&D results do increase further with increasing spillovers, however, the problem is that there is now very little or nothing to copy. In the two north-eastern regions between the intersecting separatrices in Figure 16, the leader in case of

$\beta = 0.9$  invests relatively less than in case of  $\beta = 0.5$  and this effect of lower investments by the leader dominates the pure spillover effect. Consequently, while for larger spillovers, the laggard is driven out of the market, for smaller spillovers, he continues to catch up with the leader. In sum, *increasing spillovers favors a socially desirable outcome only up to a point.*

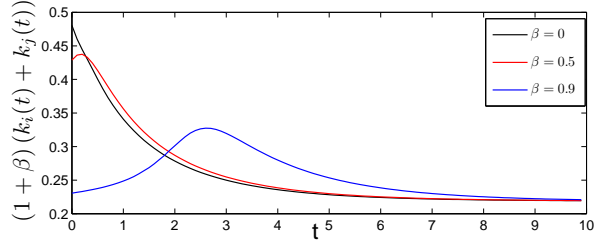
Whenever one firm gives up, the unit cost of the remaining firm converges to the same long-run optimal monopolist's level irrespective of the level of spillovers. The separatrices  $W_1^U$  for the three different levels of  $\beta$  asymptotically converge as  $t \rightarrow \infty$  (not shown). The same holds for separatrices  $W_2^S$ . However, the long-run steady-state level of unit costs in case of duopoly ( $S_4$ ) does depend on the level of spillovers. As Figure 16 shows, in our rescaled coordinates  $S_4^H < S_4^L < S_4^M$ . That is, the long-run unit costs are the lowest in case of  $\beta = 0.5$ , the second lowest in case of  $\beta = 0$ , and the highest in case of  $\beta = 0.9$ . This suggests that the spillovers decrease steady-state costs only up to a certain level, beyond which further increases in spillovers start to increase the long-run costs. We illustrate this further by time plots in Figure 17. We select an asymmetric initial position which lies in the basin of attraction of  $S_4$  for all the three levels of spillovers:  $(x_1, x_2) = (1.68, 0.81)$ , corresponding to  $(c_1, c_2) = (0.186, 0.440)$ .

Figure 17a shows total R&D efforts of the two competing firms over time for different levels of spillovers. We see a typical effect of increasing spillovers – the total industry R&D efforts decrease as firms increasingly free-ride on each other. However, due to larger complementarities between R&D efforts at larger spillovers, the effective efforts of the firm  $i$  and the firm  $j$  ( $k_i + \beta k_j$  and  $k_j + \beta k_i$ , respectively) might be larger at larger spillovers despite the firms' lower *de facto* R&D efforts ( $k_i$  and  $k_j$ , respectively). Figure 17b shows that this is indeed the case when spillovers increase from  $\beta = 0$  to  $\beta = 0.5$ . While for  $\beta = 0.5$ , the industry R&D efforts are relatively lower at all times, the effective industry R&D efforts are larger for most of the time. This explains why in the latter case, the unit costs converge to a lower long-run level than in the case with  $\beta = 0$ . We see that among the three regimes, the industry R&D efforts are comparably the lowest for  $\beta = 0.9$ . At the beginning, the leader in the latter case invests very little as he free-rides on the efforts of the laggard. These smaller investments are not offset by larger spillovers, such that the effective efforts are much lower than in the other two cases. This changes over time as the leader himself starts to invest more when the laggard gradually reduces his efforts over time. However, as Figure 17c reveals, lower effective investments at the beginning very much slow down the speed at which unit costs decrease. In the case of  $\beta = 0.9$ , the unit costs of both the leader (solid line) and the laggard (dotted line) decrease much slower than in the other two cases. Moreover, the gap between the laggard and the leader also closes more slowly. These slower and lower reductions of costs as a consequence of smaller investments are the reason that the total quantity offered in the market is for  $\beta = 0.9$  at all times the lowest among the cases considered (see Figure 17d). The largest total quantity is offered for  $\beta = 0.5$ , whereas the total quantity for  $\beta = 0$  is close to that for  $\beta = 0.5$  but slightly lower.

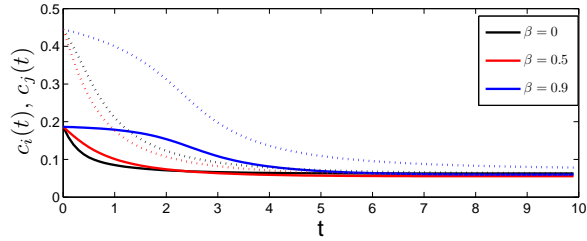
Calculations show that in the above example total profits increase with spillovers. This is the effect of higher complementarities in R&D outputs that allow for significant savings on R&D costs. However, the consumers are not necessarily any better for it. As our comparisons indicate, there exists a threshold level of spillovers after which further increases in spillovers do not benefit consumers. At large spillovers, the free riding effect induces firms to invest less and the consequent lower production efficiency, to the detriment of consumers, also induces them to



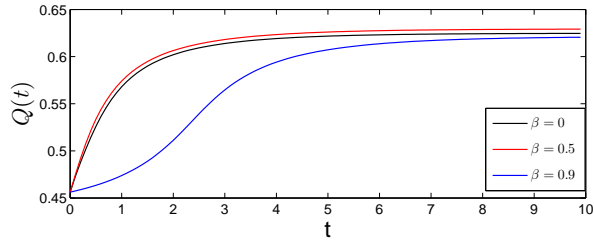
(a)



(b)



(c)



(d)

Figure 17: Time paths for varying levels of spillovers.  $\varepsilon = 0.125$ . In plot (c), the solid lines correspond to the leader, whereas the dotted lines correspond to the laggard.

produce less.

### 4.7.3 Spillovers and Welfare

In this section, we analyze welfare effects of spillovers in greater detail. As spillovers play a crucial role in the laggard-leader relationship, we simulate the model for different relative initial unit costs of firms. We are interested in how welfare measures relate to spillovers when we vary the initial difference in units costs,  $d = c_1 - c_2$ . For this, we pick up an initial point on the diagonal of the state space ( $d = 0$ ) and vary the costs such that their sum remains unchanged

(i.e.,  $e^{-x_1} + e^{-x_2} = \text{const.}$ ). Two such “iso-sum” loci (ISL) are shown in Figure 16.<sup>25</sup>  $ISL_1$  intersects the diagonal at  $x_1 = x_2 = -0.3 < 0$ , whereas  $ISL_2$ , which corresponds to a lower sum of unit costs, intersects it at  $x_1 = x_2 = 1 > 0$ . For every initial point on the loci, we then calculate the total discounted value of profits, consumer surplus, and total surplus.<sup>26</sup>

Notice that Figure 16 already reveals that a sufficiently high level of spillovers might be necessary for a technology to be developed by any firm at all ( $S_3^H < S_3^M < S_3^L$ ). Insofar a given level of spillovers prevent the development of a new technology (and so brings about zero profits and consumer surplus), it is dominated by all higher levels of spillovers for which the market is formed (and which thereby bring about positive profits and consumer surplus). In this section, we are interested in how spillovers influence surpluses through their impact on the structure of the ensuing product market (monopoly vs duopoly). Therefore, we select initial values of unit costs such that  $x_i(0) > S_3^L$ .

We calculate surpluses in two ways. In the first, we calculate total discounted values of surpluses as they evolve along a drift path. In the second, we calculate discounted values of surpluses for each of 1,000 stochastic paths that start in the same initial point, and take their mean values.<sup>27</sup> The first method approximates surpluses in a game with a low noise (and in a deterministic game), the second one approximates the expected value of surpluses in a proper stochastic game.

Figure 18 shows standard Marshallian consumer surplus calculated in both ways for  $\beta = 0.5$ . We see that for low noise, consumer surplus has a local maximum at a symmetric initial point and decreases as the initial asymmetry is increased. A turnaround occurs when we pass through the separatrices  $W_1^S$  and  $W_2^S$  (notice kinks in the drift surplus) and enter the domain of a monopoly. Clearly, the lower the initial cost of a monopolist, the higher the total consumer surplus. Notice how stochasticity smooths up the surplus (compare thin curves to bold ones). The drift surplus and stochastic surplus converge as noise is reduced, but the drift surplus for  $\varepsilon = 0.125$  is still a relatively good approximation of the surplus in a stochastic game with low noise. In what follows, we compare surpluses between different levels of spillovers along the two ISL loci. We do this first for drift surpluses as noise can mask some important differences. Later we also compare stochastic surpluses to see what the effect of noise is. Due to computational constraints at low spillovers (see footnote 22), we compare different scenarios for  $\varepsilon = 0.125$ .

#### 4.7.4 High sum of initial unit costs ( $ISL_1$ locus)

Figure 19 compares surpluses along the drift path for three different levels of spillovers ( $\beta = 0.1$ ,  $\beta = 0.5$ , and  $\beta = 0.9$ ). As Figure 19a reveals, the effect of spillovers on consumer surplus is

<sup>25</sup>The figure shows only the part of loci above the diagonal. As loci have a diagonal line of symmetry, the second part is computationally redundant.

<sup>26</sup>It is a standard feature of Cournot equilibrium that as long as both firms produce, industry output, price, and consumer surplus remain unchanged for exogenous changes in marginal costs that preserve their sum (see Salant and Shaffer, 1999). However, in our model, changes in initial unit costs (which equal marginal costs) induce strategic responses of firms, such that all these variables do change in course of time.

<sup>27</sup>As the game is infinite, we need to truncate the time interval of simulations. We impose that a simulation stops once a surplus increment over a time step has fallen below  $1 \times 10^{-6}$ . If the boundary of the state space is reached within the simulated time interval, we impose that from the moment the boundary is reached, the instantaneous surplus remains constant. Alternatively, we assume that from the boundary onwards the monopoly solution (cf. Hinlopen et al., 2013) applies. The difference in discounted values turns out to be negligible.

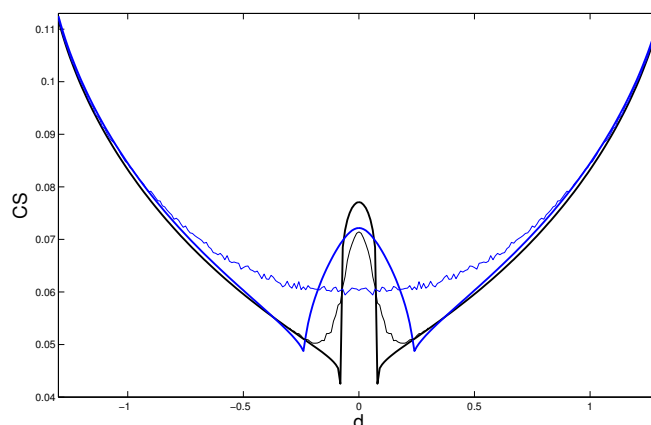


Figure 18: Total discounted value of consumer surplus for different levels of initial asymmetry and noise;  $\varepsilon = 0.125$  (blue),  $\varepsilon = 0.015$  (black); drift path (bold), average of 1,000 stochastic paths (thin);  $\beta = 0.5$ .

highly non-monotonic. In all three cases, consumer surplus has a peak when firms start symmetric. This is due to the preemptive nature of investment, which is, as we saw, highly intensified at low spillovers. As there both firms invest a lot, costs are reduced and quantity increased fast, which is of benefit to consumers. Thereby, for initially symmetric firms, the lower the spillovers, the better off are consumers. Things change when we introduce asymmetries. For lower asymmetries, consumer surplus is now the lowest for  $\beta = 0.1$ . The reason is that low spillovers lead to a market monopoly as they prevent the laggard from catching up with the leader. The highest consumer surplus is achieved for  $\beta = 0.5$ , followed by  $\beta = 0.9$ . There are two effects of spillovers at work behind this ranking. At  $\beta = 0.5$ , firms invest relatively more in R&D than at  $\beta = 0.9$  due to lower free-riding incentives at lower spillovers, which is of benefit to consumers. On the other hand, lower spillovers make it harder for the laggard to catch up with the leader, which is through softer competition in the product market detrimental to consumers. Initially the first effect is dominant, but it succumbs to the second one as asymmetries increase. Consequently,  $\beta = 0.9$  eventually starts to dominate  $\beta = 0.5$  as we move away from the diagonal. The difference becomes especially pronounced once we pass through the separatrix for  $\beta = 0.5$  (the kink in the red curve). There  $\beta = 0.5$  leads to a monopoly, whereas for  $\beta = 0.9$ , there is still a duopoly in the product market. However, as asymmetries increase further, the benefits of larger spillovers gradually decline and  $\beta = 0.9$  starts to bring about the lowest consumer surplus. The reason is that at high spillovers the dominant firm holds back its R&D efforts to a rather considerable extent in order not to empower the laggard through spillovers and thereby risk his entry into the market.

Figure 19b shows total discounted profit (implied from the value function) as it depends on initial asymmetries and spillovers. We see that its value increases with spillovers, which is due to the cost-saving effect that complementaries in R&D bring about. For  $\beta = 0.1$ , we also plot individual profits (dotted for firm 1, dashed for firm 2). While the profit of the leader increases and that of the follower decreases as we move away from the diagonal, the total profit increases.

Figure 19c shows total surplus, which is the sum of profits and consumer surplus. We see that high consumer surplus induced by preemptive investments at low spillovers prevail over

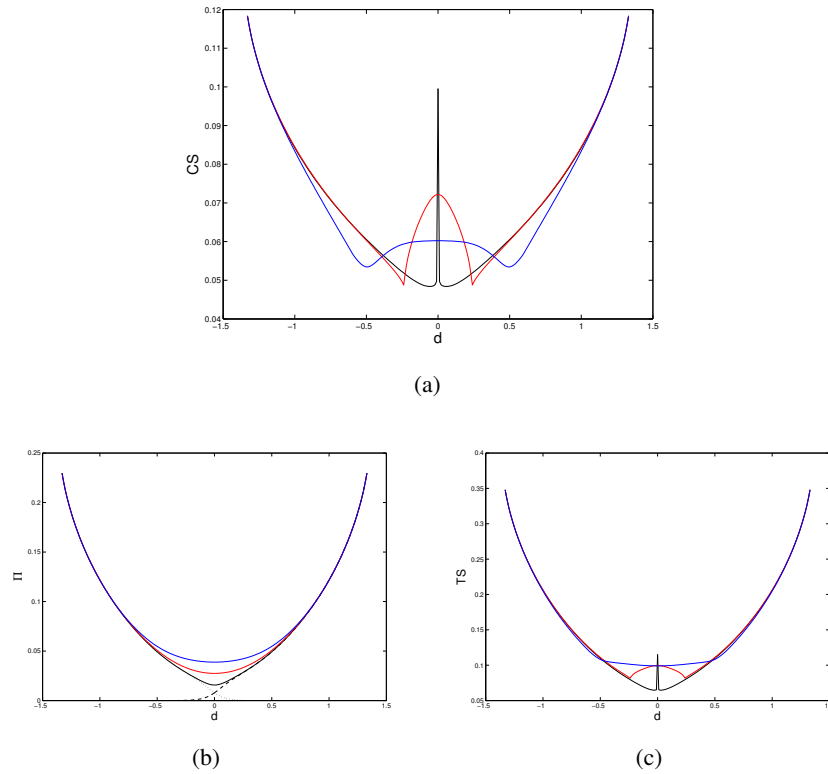


Figure 19: Total discounted value of consumer surplus (a), profits (b), and total surplus (c) along the drift path for different levels of initial asymmetry on  $ISL_1$  locus;  $\beta = 0.1$  (black),  $\beta = 0.5$  (red), and  $\beta = 0.9$  (blue).

rather low profits caused by such investments. Hence, with symmetric firms total surplus is maximized at low spillovers. As we increase initial asymmetries, the situation becomes rather different as total surplus then increases with increasing spillovers. When initial costs of firms are large, the cost-saving effect of larger spillovers seems predominant. Eventually, the situation is reversed again and high spillovers, whose effect is mostly to hold back the leader's investments, bring about the lowest total surplus. However, the difference is comparably small and diminishes as asymmetries increase further and with them the risk that the follower will catch up through free-riding vanishes.

Figure 20 shows expected values of total discounted consumer surplus and total surplus, calculated as averages over 1,000 stochastic trajectories for each initial point on the iso-sum locus. We see that noise smooths up surpluses. In particular, the bulge at  $\beta = 0.1$  is now considerably less pronounced. The reason for this that as noise increases, firms move faster away from the diagonal, characterized by high investments.  $\beta = 0.1$  now seems to be strictly dominated by larger spillovers. The intermediate value of spillovers brings about the highest value of consumer surplus for a wide range of initial unit costs past the diagonal. Excepting the bulge, the ranking for total surplus is roughly similar as before: total surplus is positively related to spillovers.

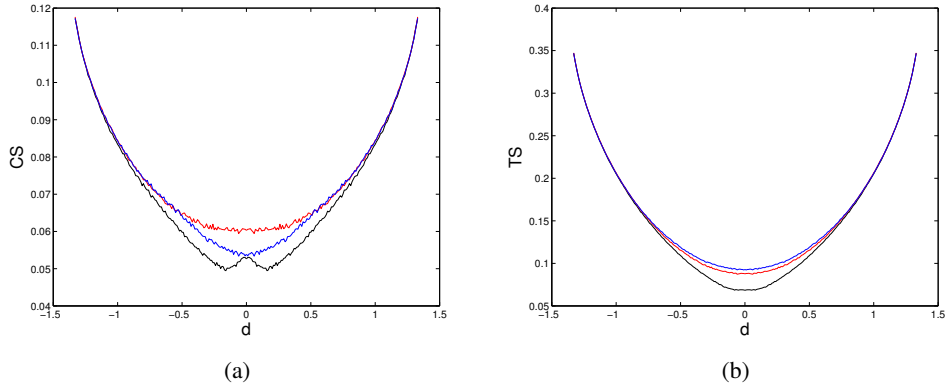


Figure 20: Total discounted value of consumer surplus (a) and total surplus (b); the average over 1,000 stochastic paths for different levels of initial asymmetry on  $ISL_1$  locus;  $\beta = 0.1$  (black),  $\beta = 0.5$  (red), and  $\beta = 0.9$  (blue).

#### 4.7.5 Low sum of initial unit costs ( $ISL_2$ locus)

To see whether spillover effects depend on the initial level of a technology, we now consider comparably lower initial values of unit costs. For unit costs on  $ISL_2$  locus, production can start immediately. Figure 21 reveals that there indeed are significant differences in surpluses. With lower unit costs, preemptive and predatory stimuli at low spillovers have disappeared. For a wide range of unit costs, an intermediate level of spillovers brings about the highest value of consumer surplus. High spillovers ( $\beta = 0.9$ ) are strictly dominated by lower ones, and this so even for unit costs that at low spillovers induce a monopoly market structure. This is due to the free-riding effect that at high spillovers considerably slows down technological progress, in line with our discussion of time paths in Figure 17. Therefore, for technologies with relatively low initial unit costs, intermediate spillovers lead to the highest consumer surplus. They also lead to the lowest long-run equilibrium level of unit costs, as we saw in Figure 16. Again, for very asymmetric initial costs, the higher the spillovers, the lower the consumer surplus, as there the main effect of spillovers is to stifle monopolist's investments.

For small initial asymmetries, total surplus is the lowest for  $\beta = 0.1$  and the highest for  $\beta = 0.9$ . This is induced by larger savings on R&D costs and thereby comparably much higher profits at higher levels of spillovers. As asymmetries increase, high spillovers lead to lower and lower total surplus as negative effects of a slow technological progress induced by free-riding gradually prevail over any advantage of larger complementarities in R&D. Recall that the more efficient firm has very small incentives to exert R&D efforts when its unit costs are small and spillovers high. This much worse performance of high spillovers at larger asymmetries is a notable difference in comparison with Figure 19c.

Conclusions based on stochastic simulations (see Figure 22) are similar. Consumers are worst off for  $\beta = 0.9$  and best off for  $\beta = 0.5$ . The difference between  $\beta = 0.1$  and  $\beta = 0.9$  is virtually indistinguishable at low levels of asymmetry. Figure 22b shows that total surplus increases with spillovers, but only up to a point. Notably,  $\beta = 0.9$  brings about considerably higher total surplus than  $\beta = 0.1$ , but still lower surplus than  $\beta = 0.5$ .

A notable observation when comparing surpluses between  $ISL_1$  and  $ISL_2$  is that with lower



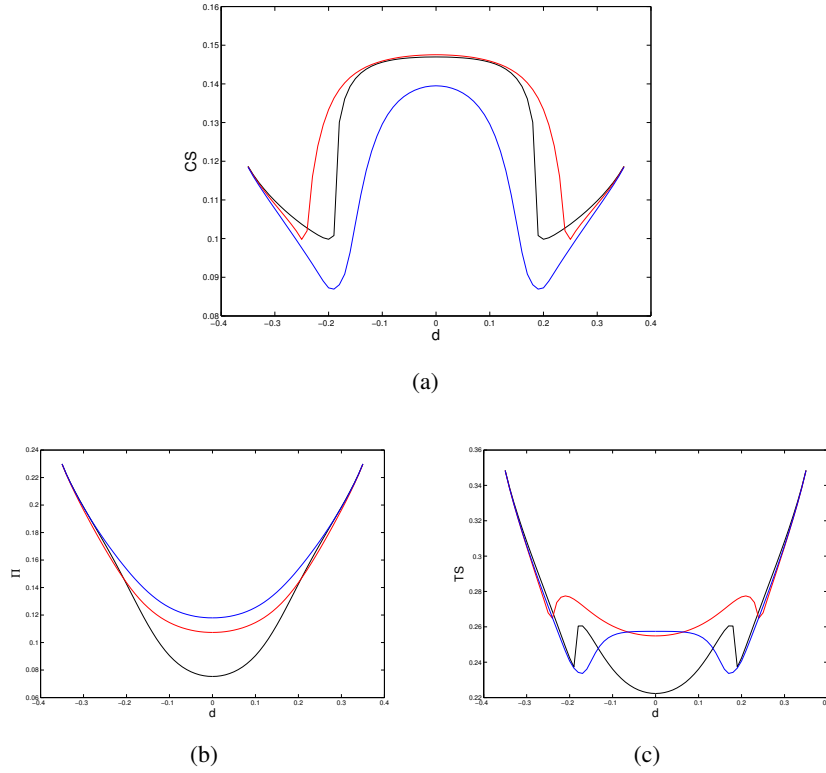


Figure 21: Total discounted value of consumer surplus (a), profits (b), and total surplus (c) along the drift path for different levels of initial asymmetry on  $ISL_2$  locus;  $\beta = 0.1$  (black),  $\beta = 0.5$  (red), and  $\beta = 0.9$  (blue).

values of initial unit costs high levels of spillovers become progressively less desirable from a social standpoint. The reason for this is that with more developed initial technologies, savings on R&D costs realized through spillovers become less important simply because there is less R&D left to be done to reach the long-run optimal level of production costs. Moreover, when initial production costs already permit profitable production, the free-riding effect of spillovers is worse as then once high R&D incentives of firms in anticipation of entering the product market have subsided.

#### 4.8 Market size and industry dynamics

The presentation has so far been focused on the dynamics that occurs at parameters  $(\phi, \tilde{\rho}) = (8, 1)$ . Our calculations over a wide range of parameterizations show that the dynamics at this particular parameterization is representative of the subset of the parameter space for which both coordinates of the unstable steady state  $S_3$  are negative, that is,  $S_3$  corresponds to unit costs above the choke price. This means that it can be profitable for firms to develop further a technology which requires R&D efforts before the production can profitably start. We focused on this subset of the parameter space, which characterizes a ‘promising technology’, as we find it most relevant – for great many new technologies, research starts long before a prototype sees the light of day.

We now briefly consider two other possibilities that occur when for a given discount rate  $\tilde{\rho}$  we

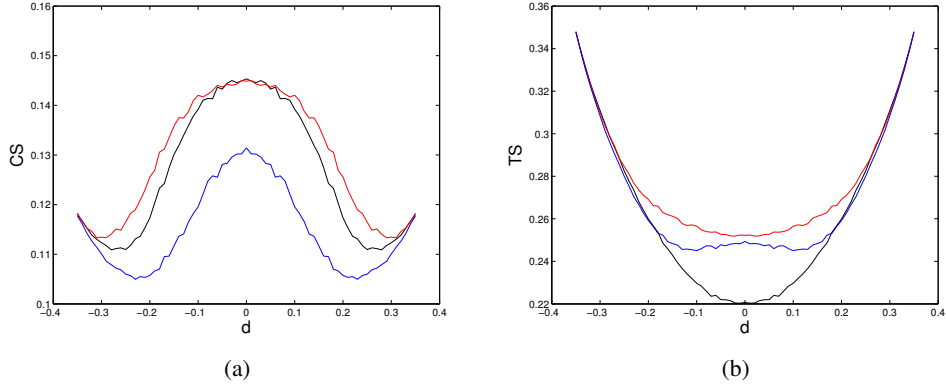


Figure 22: Total discounted value of consumer surplus (a) and total surplus (b); the average over 1,000 stochastic paths for different levels of initial asymmetry on  $ISL_2$  locus;  $\beta = 0.1$  (black),  $\beta = 0.5$  (red), and  $\beta = 0.9$  (blue).

reduce the profitability potential of a technology that is captured by  $\phi$ .<sup>28</sup> In the first one, which we call ‘strained market’, a technology corresponding to initial unit costs above the choke price is never developed further, whereas that already in the production phase is developed further only if it is already sufficiently developed, such that it does not require “too much” additional R&D efforts.

Reducing  $\phi$  further, we obtain the second case, which we call ‘obsolete technology’. In this case, it is always in the interest of any firm to exit the market at some optimal speed as a low demand makes it unprofitable to maintain a decaying technology.

Consider first the case of a strained market, represented by  $(\phi, \tilde{\rho}) = (5, 1)$ . Notice that a lower  $\phi$  for a given discount rate corresponds to a lower demand and/or higher costs of R&D (recall the rescaling in Lemma 1). In general, lowering  $\phi$  moves  $S_3$  and  $S_4$  closer together, contracting the region of the state space for which there is a duopoly in the product market (the region between the  $W_1^S$  and  $W_2^S$  separatrices). When demand decreases or R&D costs rise, a much smaller lead is needed to induce the laggard to give up. Figure 23b indicates this for the case of  $\beta = 0.5$ . Observe that neither a monopolist nor any of the two competing firms develop further a technology which would require investments prior to production – the basin of attraction of  $S_4$  is compressed and fully contained within the production area delimited by the  $E_1$  and  $E_2$  curves (compare with Figure 5).

Figure 23a shows the drift vector field for  $(\phi, \tilde{\rho}) = (5, 1)$  with  $\beta = 0.1$ . There are now only two steady states – a nodal source  $S_3$  and a saddle-point steady state  $S_1^*$ . The two saddles ( $S_1$  and  $S_2$ ) and the nodal sink ( $S_4$ ) have colluded and formed a new steady state  $S_1^*$ . This new saddle has two manifolds – the unstable manifold  $W_1^U$  and the stable manifold  $W_1^S$  which lies on the diagonal of the state space. The implication of this is that the region of duopoly is now compressed into a line segment which originates in  $S_3$ , passes through  $S_1^*$  and continues to infinity. Only for symmetric initial position lying on this line both firms keep producing and steer their unit costs towards the long-run equilibrium level of  $S_1^*$ .<sup>29</sup> However, any initial asymmetry

<sup>28</sup>These possibilities are suggested by the bifurcation diagram of the monopoly regime in Hinloopen et al. (2013).

<sup>29</sup>If an initial position happens to lie on the diagonal above  $S_1^*$ , firms find it optimal to decrease their efficiency

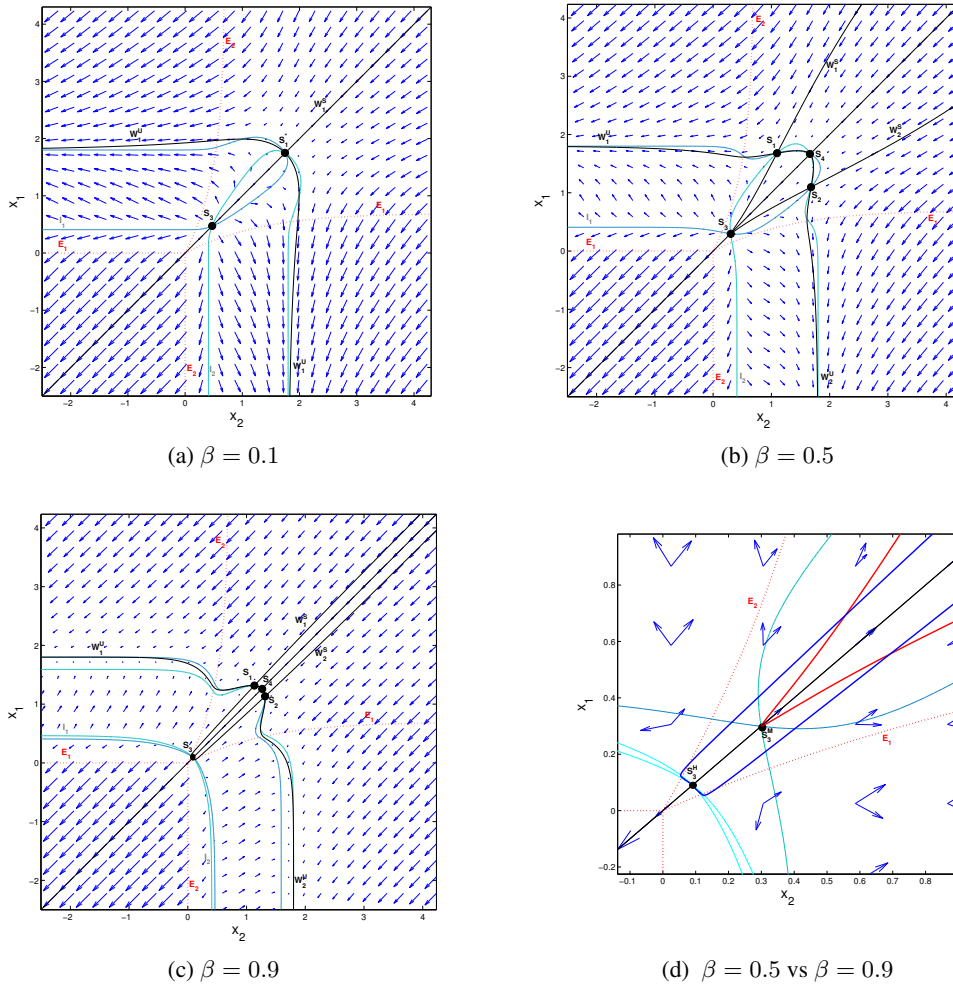


Figure 23: Drift vector field for three different values of  $\beta$ ,  $(\phi, \tilde{\rho}, \varepsilon) = (5, 1, 0.125)$ .

makes the less efficient firm to gradually exit the market. At low spillovers, a small lead is enough to induce the laggard to give up. Observe how the diagonal acts like a repeller – on each side of it, the motion is away from it. Clearly, in a stochastic game interpretation, any symmetry is only temporary, such that for most of the time one firm always diverges out of the market.

In sum, it is noteworthy that for less favorable market and R&D conditions, the asymmetry emerges for low spillovers – initial asymmetries lead to an asymmetric outcome (a firm with an initial cost advantage becomes a monopolist, whereas the other firm exits the market).

As spillovers increase,  $S_1^*$  transforms into two saddles and a nodal sink. The region of duopoly becomes a proper region, as in the case of  $\beta = 0.5$ . With increasing spillovers, the two saddles ( $S_1$  and  $S_2$ ) move aside and the region of duopoly enlarges. After some point, however, the saddles start approaching each other and so the region of duopoly starts to contract. This effect is visible when comparing the drift vector fields for  $\beta = 0.1$  and  $\beta = 0.5$  with the drift vector field for  $\beta = 0.9$  (see Figure 23c).

Figure 23d, which shows a zoomed comparison between  $\beta = 0.5$  and  $\beta = 0.9$ , indicates that towards a higher long-run level which is less costly to maintain.

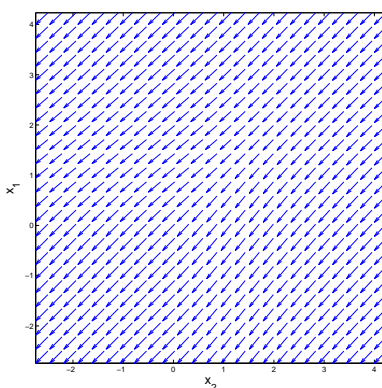


Figure 24: *Drift vector field when exiting the market is always optimal.*  $(\beta, \phi, \tilde{\rho}, \varepsilon) = (0.5, 2, 1, 0.125)$ .

this contraction of the duopoly region at higher spillovers is again not universal as the region of duopoly for  $\beta = 0.9$  remains wider at larger levels of unit costs. All in all, our conclusion is similar as before – after a certain level, larger spillovers start to reduce the duopoly region (the region of regression toward the mean) as smaller investments of the unmotivated leader makes it harder for the laggard to catch up.

The drift vector field in Figure 24 shows the drift motion for the case of obsolete technology represented by  $(\phi, \tilde{\rho}) = (2, 1)$ . We see that both unit costs diverge towards infinity. At this parameters, the demand is so low and the R&D process so costly that both firms find it optimal to eventually leave the market. They might still invests in R&D at some smaller rate that retards the decay of the technology optimally, but eventually both the R&D and production will terminate and the firms will exit the market.

## 5 Concluding remarks

In this paper, we study feedback Nash equilibria of a dynamic game in which firms enhance their production efficiency through R&D endeavors. Firms' product market participation constraints are explicitly taken into account. As a result, R&D efforts and production do not necessarily coexist at all times. In particular, R&D efforts can precede production, as it holds for the development phase of great many new technologies in practice. Our model allows us to study not only how spillovers affect investments in existing markets, but also how they affect the formation of markets and the evolution of their structure over time. We show that the relation between spillovers, R&D efforts, and surpluses is highly non-linear, non-monotonic, and depends both on the relative as well as the absolute level of competing technologies.

In the sense that patents obstruct the use of R&D results of a competitor, we can (with some reservation, as in footnote 7) associate high patent protection with low spillovers, and vice versa. Our analysis then contributes to the lively debate on the relation between patents and innovation. It shows that the likelihood that some initial technology which requires investments in advance of production will be developed further always increases with spillovers. With a convex R&D cost function, larger spillovers enable larger savings on R&D costs and thereby allow firms to bring

into the market more expensive technologies. This result stands in a stark contrast to claims that weak patent protection reduces incentives of firms to pursue the development of a new technology. It also supports the hands-off policy towards R&D cooperatives involving infant technologies, under the assumption that such cooperatives increase the flow of R&D information. However, the level to which a promising technology is developed increases with spillovers only up to a point due to the free-riding effect that dominates at larger spillovers.

While weak patents tend to prevent monopolization of the market by the more efficient firm, our analysis reveals that this indication in the received literature is not universally true. When the leader progresses on the development ladder, his incentives to exert further R&D efforts can be at high spillovers rather low as innovations can start to benefit the laggard relatively more than the leader himself. This obstructs the laggard's efforts to catch up with the leader through free-riding. Consequently, a smaller lag can induce the laggard to exit the market at larger spillovers. Put differently, weak patents make R&D results easier to copy, but if they also lead to less patents being taken out by innovators, they can well have a contra-competitive effect. Besides, in our simulations of surpluses, we obtain a relatively worse performance of large spillovers for low initial unit costs, especially regarding consumer surplus. This indicates that patent protection might be relatively more important for technologies that are already associated with fairly efficient production (and/or have comparably low R&D potential) as in case of weak patents firms may have very little incentives for further innovations. For them, weak patents might in fact favor market concentration.

There are several interesting venues for future research opened up by our analysis. Our model is focused on a market for a single product and so does not consider a possibility that firms can simultaneously work on several products of different levels of substitution in the product market. Indeed, in practice, introducing new versions of existing products seems to be a viable response to imitators by many high-tech firms. The kind of technologies pursued can affect imitation capabilities and firms can intentionally make their research tracks more or less complementary to those of their competitors. This endogenous determination of R&D complementarities and absorption capacities is an interesting alternative to the current exogenous formulation of spillovers. In our analysis we introduce stochasticity through random shocks to unit costs, keeping market demand fixed. This simplification is not crucial to our results as shocks to the common demand would not change firms' relative positions, but is expected to play a more important role in models combining process and product innovations. The analysis in the paper is readily extended to different forms of cooperation between firms and now allows a comprehensive study of their stability and desirability over all possible combinations of firms' costs. Furthermore, our discussion has been limited to some representative parameter combinations. We believe it is an interesting task to unveil a complete bifurcation diagram for a dynamic game which would precisely indicate those critical values of parameters at which important qualitative features of the solution change (e.g., a new steady state appears or its stability changes). Recently, [Hinloopen et al. \(2013\)](#) have obtained such a bifurcation diagram for a model with a single firm investing in R&D. More complete analysis in this direction awaits the development of bifurcation methods for non-convex optimization problems with multidimensional state space, which are still in their infancy.

# Appendices

## A Proof of Lemma 1

A scaled variable or parameter is distinguished by a tilde. Define the scaled variables by the following conversion equations:  $q_i = A\tilde{q}_i$ ,  $k_i = \frac{A}{\sqrt{b}}\tilde{k}_i$ ,  $c_i = A\tilde{c}_i$ ,  $\pi_i = A^2\tilde{\pi}_i$ ,  $B_i(t) = \delta^{-1/2}\tilde{B}_i(\tilde{t})$  for  $i = 1, 2$ , as well as  $t = \tilde{t}/\delta$  and  $\tilde{\rho} = \rho/\delta$ . Here the  $\tilde{B}_i$  for  $i = 1, 2$  are independent standard Wiener processes. Introduce new parameters

$$\phi = \frac{A}{\delta\sqrt{b}} \quad \text{and} \quad \varepsilon = \frac{\sigma^2}{2\delta}.$$

The state equation in the scaled variables then reads as

$$d\tilde{c}_i = \tilde{c}_i \left( 1 - \left( \tilde{k}_i + \beta\tilde{k}_j \right) \phi \right) d\tilde{t} + \sqrt{2\varepsilon} d\tilde{B}_i. \quad (32)$$

The scaled instantaneous profit function takes the form

$$\tilde{\pi}_i = \left( 1 - \tilde{q}_i - \tilde{q}_j - \tilde{c}_i \right) \tilde{q}_i - \tilde{k}_i^2. \quad (33)$$

## B Boundary conditions

To fully specify the system of coupled Hamilton-Jacobi-Bellman differential equations on a compact state space  $\Omega$ , we have specified the Neumann-type boundary conditions

$$\frac{\partial U^i}{\partial \vec{n}}(s, x) = 0$$

for all points  $x \in \partial\Omega$ , and  $\vec{n}$  being the outward pointing normal vector at  $x$ .

Consider the unrestricted problem in  $(x_1, x_2)$  coordinates, and choose a fixed initial state  $x(0) = (x_1(0), x_2(0))$  in the open set  $\Omega$ . The current time value function of a player equals

$$V^i(x_1(0), x_2(0)) = \mathbb{E}_0 \left( \int_0^\tau \left( g_i(x_1, x_2) - (\Gamma_i^*)^2 \right) e^{-\rho t} dt + e^{-\rho\tau} V^i(x_1(\tau), x_2(\tau)) \right), \quad (34)$$

where  $\tau = \tau^\varepsilon$  is the (first) exit time from  $\Omega$ , and where the time dynamics are given as

$$dx_i = \left( (\Gamma_i^* + \beta\Gamma_j^*)\phi - 1 + \varepsilon \right) dt - \sqrt{2\varepsilon} dB_i.$$

Consider first the dynamics without the Brownian motion term. As the drift vector field is bounded, it follows that the exit time  $\tau^0$  is bounded from below by some constant  $c$ , which is, for large values of  $\underline{M}$  and  $\overline{M}$ , proportional to the distance  $d$  of the initial state  $x(0)$  to the boundary  $\partial\Omega$  of  $\Omega$ .

For the full dynamics, including the Brownian motion term, the probability that  $\tau^\varepsilon < \tau^0/2$  is exponentially small in  $d$ . But the contribution of the ‘boundary term’  $e^{-\rho\tau} V^i$  to the right hand

side of (34) is then

$$\begin{aligned}\mathbb{E}_0 \left( e^{-\rho\tau} V^i(x(\tau)) \right) &= \mathbb{E}_0 \left( e^{-\rho\tau} V^i(x(\tau)) \mid \tau \leq \frac{\tau^0}{2} \right) + \mathbb{E}_0 \left( e^{-\rho\tau} V^i(x(\tau)) \mid \tau > \frac{\tau^0}{2} \right) \\ &\leq \max_{x \in \partial\Omega} |V^i(x)| \mathbb{P} \left( \tau \leq \frac{\tau^0}{2} \right) + e^{-\rho\tau^0/2} \mathbb{E}_0 \left( V^i(x(\tau)) \mid \tau > \frac{\tau^0}{2} \right) \\ &\leq c_1 e^{-c_2 d}.\end{aligned}$$

This contribution constitutes the maximal error from misspecifying the boundary conditions. We see that it exponentially declines towards 0 as  $\underline{M}, \overline{M} \rightarrow \infty$ .

This is illustrated in Figure 25, which shows the policy function of firm 1 at the upper boundary of  $x_1$  for three different values of  $\overline{M}$ :  $\overline{M} = 4.5$ ,  $\overline{M} = 5$ , and  $\overline{M} = 5.5$ , respectively. We see that consecutive solutions diverge only in the very close proximity of the boundary, where the solution corresponding to a smaller grid range sweeps sharply down to zero. Hence, by specifying a large enough range of the grid, we can always obtain a good approximation over the interior region of interest. This is further confirmed by the first plot in Figure 7, which shows that for a given large value of firm  $j$ 's unit cost, the solution for firm  $i$  converges to the deterministic monopoly solution as  $\varepsilon \downarrow 0$ .

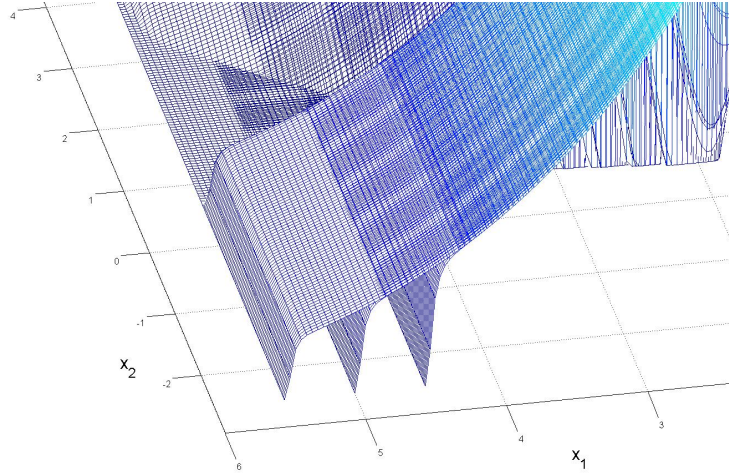


Figure 25: *Solution at the boundary.*

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