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## Public Debt and Total Factor Productivity

Leo Kaas

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# Public Debt and Total Factor Productivity

## Abstract

This paper explores the role of public debt and fiscal deficits on factor productivity in an economy with credit market frictions and heterogeneous firms. When credit market conditions are sufficiently weak, low interest rates permit the government to run Ponzi schemes so that permanent primary deficits can be sustained. For small enough deficit ratios, the model has two steady states of which one is an unstable bubble and the other one is stable. The stable equilibrium features higher levels of credit and capital, but also a lower interest rate, lower total factor productivity and output. The model is calibrated to the US economy to derive the maximum sustainable deficit ratio and to examine the dynamic responses to changes in debt policy. A reduction of the primary deficit triggers an expansion of credit and capital, but it also leads to a deterioration of total factor productivity since more low-productivity firms prefer to remain active at the lower equilibrium interest rate.

JEL-Code: D920, E620, H620.

Keywords: credit constraints, unbacked public debt, dynamic inefficiency, sustainable deficits.

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# 1 Introduction

Large fiscal deficits and rising stocks of public debt in many advanced economies, especially in the aftermath of the Great Recession, raise concerns about debt sustainability and about adverse consequences for the private sector, in particular for credit supply and investment. Empirical studies point at a non-monotonic relationship between public debt and economic growth (see e.g. Kumar and Woo (2010), Reinhart and Rogoff (2010) and Baum et al. (2013)). Examining the channels through which debt impacts growth, Pattillo et al. (2011) find for a sample of 93 developing countries that the majority of the effect of public debt on output growth occurs via total factor productivity (TFP) rather than via capital accumulation. On the other hand, theoretical and quantitative studies on debt policy (see the literature surveyed below) traditionally focus on the effects of public debt on capital investment and interest rates, while treating the dynamics of TFP as an exogenous process.

To fill this gap, this paper examines the effect of public debt and fiscal deficits on TFP, using a tractable dynamic general equilibrium model with credit market imperfections and heterogeneous firms facing idiosyncratic productivity shocks. Due to binding collateral constraints, the capital allocation among firms, and therefore TFP, are endogenous variables. If the credit constraints are sufficiently tight, low interest rates permit the government to run Ponzi schemes, i.e. to roll over unbacked government debt indefinitely, so that permanent primary deficits can be sustained. Such economies typically have two steady states. One steady state has a larger stock of public debt, a higher interest rate and a more efficient factor allocation, but at the same time less private credit and capital. This steady state is generally unstable and can only be sustained if the price of government bonds is a bubble. In the absence of such a bubble, the economy converges to the other, stable steady state at which TFP and the interest rate are lower, while private credit and capital are larger.

After describing the model and deriving the key dynamic relationships in Section 2, I prove in Section 3 several properties about the different stationary equilibria. In particular, I derive conditions for existence and uniqueness and show that TFP is always increasing if credit market conditions improve, regardless of the existence of a bubble. Further, it is shown that primary deficits up to a maximal level can be sustained whenever the economy is dynamically inefficient, i.e. when its interest rate is smaller than the growth rate in the absence of a bubble.

To determine the maximum sustainable deficit ratio and to address further quantitative questions, I calibrate the model in Section 4 to match long-run features of the US econ-

omy. It turns out that the maximum sustainable primary deficit is actually quite small, amounting to less than one percent of output. This number is much lower than the 5.2% threshold that Chalk (2000) obtains for the maximal deficit ratio based on a quantitative overlapping generations model.

Another finding for the calibrated model is that changes of the deficit ratio have opposite effects on credit and capital, on the one hand, and on TFP, on the other hand. A reduction of the deficit, for example, reduces the interest rate which triggers a surge in private credit and capital investment. At the same time, however, the lower interest rate makes it more attractive for low-productivity firms to stay in business which has a detrimental effect on aggregate productivity. I show that the second effect dominates the overall impact on aggregate output which necessarily falls in response to the deficit reduction in the very long run. Yet, this long run effect only materializes after several decades and is offset by substantial output gains in the medium run. A further quantitative finding concerns the effect of changes in financial conditions. It turns out that even significant expansions of private credit induced by a relaxation of collateral constraints have quite modest effects on aggregate output and TFP. For example, a 10 percent expansion of firm credit merely raises TFP by 0.3 percent and output by 0.5 percent. This contrasts with some of the recent business-cycle literature that finds large output responses to financial shocks (e.g. Jermann and Quadrini (2012)).

This paper relates to the literature on overlapping generations models in which dynamic inefficiency can give rise to the existence of rational bubbles and to the possibility of debt Ponzi schemes so that governments are able to finance primary deficits indefinitely (Diamond (1965), Tirole (1985), Blanchard (1985) and Weil (1987)).<sup>1</sup> Bullard and Russell (1999) and Chalk (2000) consider quantitative overlapping generations economies with low interest rates and unbacked debt, and the latter explicitly calculates maximum sustainable deficit levels. Rankin and Roffia (2003) are interested in maximum sustainable debt (rather than deficits) in an overlapping generations model. In all these papers, however, TFP is exogenous. The same is true in the literature on incomplete markets with infinitely-lived households facing idiosyncratic income risk where a dynamically inefficient overaccumulation of capital is possible, so that unbacked public debt can be sustained (e.g. Aiyagari and McGrattan (1998)).

The paper also relates to the literature on rational bubbles in economies with infinitely-lived agents and credit market frictions (e.g. Kocherlakota (2008) and Hellwig and Lorenzoni

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<sup>1</sup>The tight connection between dynamic inefficiency and the possibility of Ponzi schemes is only valid in the absence of aggregate uncertainty (see e.g. Blanchard and Weil (2001)).

(2009)). As in this paper, Kocherlakota (2009) and Miao and Wang (2012, 2013) argue that rational bubbles can improve production efficiency when credit constraints are tight. The model of this paper builds upon Azariadis et al. (2014) who show that the dynamics of unsecured credit, which responds to changes in self-fulfilling beliefs, affects aggregate factor productivity. Different from these contributions, this paper explores the impact of unbacked government debt and fiscal deficits on TFP.

## 2 The Model

### 2.1 The Environment

Consider a general equilibrium model in discrete time with a representative worker household and a continuum  $i \in [0, 1]$  of firms, each owned by a single shareholder. The government runs primary surpluses or deficits and issues public debt. Firms face idiosyncratic productivity shocks. Their owners can invest either in their own business, which is optimal if productivity is sufficiently high, or in financial assets. There is an asset market for business credit which operates in each period, after firm productivities are revealed. Besides lending to other firms, firm owners can also hold government bonds. While government liabilities can principally be interpreted as either bonds or fiat money, this paper favors the first interpretation. This is because fiscal governments in modern economies finance their deficits primarily by issuing bonds rather than by seignorage. Moreover, in this model money has no essential role that is separate from the other assets.

At any time  $t$ , firm owner  $i$  maximizes expected discounted utility

$$\mathbb{E}_t \sum_{\tau \geq t} \beta^{\tau-t} \ln(C_\tau^i) \quad (1)$$

over future consumption streams. Firm  $i$  has access to a production technology which produces the single consumption/investment good  $Y_t^i = F(z_t^i K_t^i, A_t L_t^i)$  from inputs of capital  $K_t^i$  and labor  $L_t^i$ .  $F$  is a constant-returns and concave production function.  $z_t^i$  is an idiosyncratic capital productivity shock, which is realized in period  $t - 1$  when firms decide about capital investment.<sup>2</sup> These idiosyncratic productivities are distributed with continuous cumulative distribution function  $G(z)$  and they are uncorrelated across time. The labor efficiency parameter  $A_t$  grows with constant rate  $g \geq 0$ . Capital depreciates during the period at rate  $\delta$ .

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<sup>2</sup>The timing notation follows the convention in the macroeconomic literature in that capital employed at date  $t$  carries index  $t$ , although it is decided at time  $t - 1$ . Idiosyncratic productivity therefore has the same time index.

The market for business credit operates in period  $t$  after idiosyncratic productivities are revealed. Owners of more productive firms borrow from the owners of less productive firms at gross interest rate  $R_{t+1}$ .<sup>3</sup> As borrowers cannot promise to repay their debts in the next period  $t+1$ , they need to provide collateral to their lenders. I assume that a fixed fraction  $\lambda \in (0, 1]$  of the firm's capital stock provides secure collateral. Thus, when  $B_{t+1}^i$  denotes the firm owner's debt issued in period  $t$ , the collateral constraint<sup>4</sup> in period  $t$  reads as

$$R_{t+1}B_{t+1}^i \leq \lambda(1 - \delta)K_{t+1}^i . \quad (2)$$

Owners of low-productivity firms optimally decide to not invest in their own businesses but rather to hold financial assets. In period  $t$ , these firm owners supply  $-B_{t+1}^i > 0$  units of credit to other firms or buy  $D_{t+1}^i$  units of government bonds at price  $q_t$ . The budget constraint of firm owner  $i$  in period  $t$  reads as follows:

$$C_t^i + K_{t+1}^i - B_{t+1}^i + q_t D_{t+1}^i = Y_t^i - w_t L_t^i + (1 - \delta)K_t^i - R_t B_t^i + q_t D_t^i . \quad (3)$$

The right-hand side of this budget constraint represents the owner's wealth in period  $t$ . Labor is hired at real wage  $w_t$  to produce output  $Y_t^i$ , the firm owner repays credit (or is repaid) and holds the undepreciated stock of capital and possibly some government bonds. The firm owner can spend his wealth on consumption, on investment in his own business, with equity  $K_{t+1}^i - B_{t+1}^i$ , or on public bonds.

The aggregate supply of government bonds  $D_t$  changes over time when the government runs deficits or surpluses. The government budget constraint in period  $t$  is

$$q_t(D_{t+1} - D_t) = X_t , \quad (4)$$

where  $X_t$  is the primary deficit (or primary surplus, when negative). To simplify the role of the government, it is assumed that deficits (surpluses) are transferred to (collected from) the worker household in a lump-sum fashion.

The worker household supplies labor and is not active in asset markets, thus consumes labor income net of taxes/transfers in every period. The absence of the worker household from asset markets is not a severe restriction. Indeed, if the worker household has the same preferences as firm owners, he would decide not to save in any steady state equilibrium of

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<sup>3</sup>One can think of this credit as including corporate bonds (directly held by the lender) and loans that are granted through financial intermediaries (which are not explicitly modeled).

<sup>4</sup>Loss of collateral is the only punishment of a defaulting owner. Azariadis et al. (2014) additionally allow for unsecured credit that rests on the borrower's reputation which is harmed in a default event. Neither setting has default in equilibrium because productivities are revealed before credit is exchanged. Cui and Kaas (2014) consider a related model with default in equilibrium.

this model because the interest rate is always smaller than  $(1 + g)/\beta$  (see Propositions 1 and 2 below), and the absence of collateral prevents any borrowing. To keep the model simple, I assume further that labor supply is constant, normalized to unity. Due to these assumptions, workers' consumption is simply  $c_t^w = w_t + X_t$ .

**Definition:** Given deficit policy  $(X_t)_{t \geq 0}$ , and given an initial distribution of capital, credit and bond holdings  $(K_0^i, B_0^i, D_0^i)_{i \in [0,1]}$ , a competitive equilibrium is a list of consumption, investment, and production plans for all firms,  $(C_t^i, K_{t+1}^i, B_{t+1}^i, D_{t+1}^i, L_t^i)_{i \in [0,1], t \geq 0}$ , consumption of workers,  $c_t^w = w_t + X_t$ ,  $t \geq 0$ , public bonds  $(D_{t+1})_{t \geq 0}$ , and prices for labor, credit and public bonds,  $(w_t, R_{t+1}, q_t)_{t \geq 0}$ , such that

(i)  $(C_t^i, K_{t+1}^i, B_{t+1}^i, D_{t+1}^i, L_t^i)_{t \geq 0}$  maximizes firm owner  $i$ 's expected discounted utility (1) subject to credit constraints (2) and budget constraints (3).

(ii) Markets for labor, credit and public bonds clear. That is, in every period  $t \geq 0$ ,

$$\int_0^1 L_t^i di = 1, \quad \int_0^1 B_{t+1}^i di = 0, \quad \int_0^1 D_{t+1}^i di = D_{t+1}.$$

(iii) The government's flow budget constraints (4) are satisfied for all  $t \geq 0$ .

## 2.2 Equilibrium Characterization

Consider firm  $i$  which operates the capital stock  $K_t^i$  in period  $t$  at productivity  $z_t^i$ . The firm's labor demand can be written as  $L_t^i = \phi(w_t/A_t) z_t^i K_t^i/A_t$ , where the downward-sloping function  $\phi$  is the inverse of the marginal product of labor, i.e.  $F_2(1, \phi(\omega)) = \omega$ . Writing

$$K_t \equiv \int_0^1 K_t^i di \quad \text{and} \quad Z_t \equiv \int_0^1 z_t^i \frac{K_t^i}{K_t} di \tag{5}$$

to measure aggregate capital and aggregate productivity, the labor market is in equilibrium if

$$1 = \phi\left(\frac{w_t}{A_t}\right) \frac{Z_t K_t}{A_t}. \tag{6}$$

This also implies that firm  $i$ 's profit before interest payments can be written as

$$Y_t^i - w_t L_t^i = f'\left(\frac{Z_t K_t}{A_t}\right) z_t^i K_t^i,$$

with  $f(k) \equiv F(k, 1)$ . Aggregating over the right-hand sides of budget constraints (3) yields aggregate wealth

$$W_t \equiv f'\left(\frac{Z_t K_t}{A_t}\right) Z_t K_t + (1 - \delta) K_t + q_t D_t. \tag{7}$$



Now consider the firm owners' investment possibilities in period  $t - 1$ . Each unit of capital invested in firm  $i$  yields gross capital return  $f'(Z_t K_t / A_t) z_t^i + 1 - \delta$  in period  $t$ . Hence, the firm owner will invest in his business if this return exceeds  $R_t$ ; otherwise all funds will be invested in financial assets. For the owner of an unproductive firm to be willing to supply credit, the inequality  $R_t \geq q_t / q_{t-1}$  must hold.<sup>5</sup> Furthermore, for government bonds to be held by investors, the arbitrage condition

$$R_t = \frac{q_t}{q_{t-1}} \quad (8)$$

must hold.

If  $f'(Z_t K_t / A_t) z_t^i + 1 - \delta > R_t$ , firm owner  $i$  is borrowing constrained in period  $t - 1$  and borrows up to the leverage limit  $B_t^i / K_t^i = \theta_t$  with

$$\theta_t \equiv \frac{\lambda(1 - \delta)}{R_t} . \quad (9)$$

In this case, the owner's return on equity investment  $K_t^i - B_t^i = (1 - \theta_t) K_t^i$  is

$$\tilde{R}_t^i \equiv \frac{f'(Z_t K_t) z_t^i + 1 - \delta - \theta_t R_t}{1 - \theta_t} , \quad (10)$$

which exceeds the interest rate and the return on government bonds; hence,  $D_t^i = 0$ . Conversely, if  $f'(Z_t K_t / A_t) z_t^i + 1 - \delta \leq R_t$ , the financial return  $R_t$  exceeds the capital return, so that the owner will not invest in its own business (i.e.  $K_t^i = 0$ ) if this inequality is strict. The owner then supplies credit and possibly holds government bonds as well. These considerations imply that there is a productivity cutoff  $z_t$  defined by

$$R_t = f' \left( \frac{Z_t K_t}{A_t} \right) z_t + 1 - \delta , \quad (11)$$

such that any owner's return on savings is equal to  $\tilde{R}_t^i$  if  $z_t^i > z_t$  and equal to  $R_t$  if  $z_t^i \leq z_t$ . When  $S_t^i$  denotes savings, budget constraints (3) can be expressed as  $C_t^i + S_{t+1}^i = \hat{R}_t^i S_t^i$  with  $\hat{R}_t^i \in \{R_t, \tilde{R}_t^i\}$ . Because of logarithmic utility, any firm owner saves fraction  $\beta$  of wealth and consumes the rest. Therefore, the aggregate equity that firm owners invest in their own businesses in period  $t$  is  $\beta W_t [1 - G(z_{t+1})]$  so that the aggregate capital stock is

$$K_{t+1} = \beta W_t \frac{1 - G(z_{t+1})}{1 - \theta_{t+1}} . \quad (12)$$

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<sup>5</sup>If that inequality would fail, all wealth of the owners of unproductive firms would be invested in government bonds, so that credit supply would be zero. This cannot be an equilibrium because credit demand is positive since  $\lambda > 0$ .

Fraction  $\theta_{t+1}$  of the aggregate capital stock is financed externally, hence  $\theta_{t+1}K_{t+1}$  is credit demand in period  $t$ . Aggregate credit supply is equal to aggregate savings of the owners of less productive firms,  $\beta W_t G(z_{t+1})$ , less than the demand for government bonds.<sup>6</sup> Therefore, the credit market is in equilibrium if

$$\beta W_t G(z_{t+1}) - q_t D_{t+1} = \theta_{t+1} K_{t+1} . \quad (13)$$

Since idiosyncratic productivity draws are uncorrelated across time,  $z_t^i$  is not correlated with the equity, and thus with the capital stock  $K_t^i$  across firms. Hence, aggregate capital productivity, as defined in (5), can be written

$$Z_t = \frac{1}{1 - G(z_t)} \int_{z_t} z dG(z) = E(z|z > z_t) \equiv Z(z_t) . \quad (14)$$

With these considerations, a competitive equilibrium can be conveniently characterized as a solution

$$\left( w_t, R_t, z_t, Z_t, K_t, W_t, \theta_t, q_t, D_t \right) \quad (15)$$

to the nine equations (4), (6)–(14).

Given an equilibrium, aggregate output is

$$Y_t = f\left(\frac{Z_t K_t}{A_t}\right) A_t .$$

With a Cobb-Douglas production function  $f(k) = k^\alpha$ , for example, TFP is simply expressed as  $Z_t^\alpha A_t^{1-\alpha}$ . Although the previous exposition does not consider stochastic shocks, the term  $Z_t$  (and therefore TFP) can possibly fluctuate in response to exogenous changes in the idiosyncratic productivity distribution  $G$  (e.g. uncertainty shocks), to variations in financial conditions (modeled as shocks to parameter  $\lambda$ , for example), or to the government's debt policy. In all these cases, the cutoff productivity level  $z_t$  shifts endogenously which directly impacts TFP.

Since this is a growing economy, it helps to describe the dynamics in terms of a number of stationary variables. For this purpose, define  $k_t \equiv K_t/A_t$ ,  $y_t = Y_t/A_t$  and  $d_t = (q_t D_t)/A_t$ . Further suppose that the government's deficit policy is exogenously specified in terms of the deficit-output ratio, denoted by  $\xi_t \equiv X_t/Y_t = X_t/(A_t y_t)$ .

In the Appendix (Lemma 1), I show that the model dynamics can be reduced to three dynamic equations in the variables  $k_t$ ,  $z_t$  and  $d_t$  which, together with a given deficit policy

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<sup>6</sup>Due to the assumption of a continuous productivity distribution, there is an atomistic mass of firms whose productivity equals exactly  $z_{t+1}$ . Those firm owners are indifferent between investing in their own business or in financial assets, but their behavior does not affect the equilibrium.

$(\xi_t)_{t \geq 0}$ , describe a competitive equilibrium. To save notation, write  $Z_t = Z(z_t)$  for average capital productivity and  $R_t = R(k_t, z_t) = z_t f'(Z(z_t)k_t) + 1 - \delta$  for the interest rate. The first dynamic equation describes the equality between aggregate savings (right-hand side) and aggregate investment in capital and in government bonds (left-hand side):

$$k_{t+1} + \frac{d_{t+1}}{R_{t+1}} = \frac{\beta}{1+g} \left[ f'(Z_t k_t) Z_t k_t + (1 - \delta)k_t + d_t \right]. \quad (16)$$

The second equation follows from aggregate capital investment (12):

$$k_{t+1}[R_{t+1} - \lambda(1 - \delta)] = \frac{\beta R_{t+1}}{1+g} [1 - G(z_{t+1})] \left[ f'(Z_t k_t) Z_t k_t + (1 - \delta)k_t + d_t \right]. \quad (17)$$

The third equation is a reformulation of the government budget constraint (4):

$$\xi_t f(Z_t k_t) = \frac{1+g}{R_{t+1}} d_{t+1} - d_t. \quad (18)$$

### 3 Stationary Equilibria

This section describes all stationary equilibria (balanced growth paths) of this model for stationary deficit policy  $\xi_t = \xi$ . It is convenient to specify steady states in terms of the interest rate  $R = z f'(Zk) + 1 - \delta$  and the productivity threshold  $z$ . Solving (16) and (17) for the stationary debt-capital ratio gives

$$\frac{d}{k} = \frac{RG(z) - \lambda(1 - \delta)}{1 - G(z)}. \quad (19)$$

Substitution back into (17) gives rise to

$$(1+g)[R - \lambda(1 - \delta)] = \beta R \left\{ (1 - \delta)(1 - \lambda) + (R - 1 + \delta) \left[ G(z) + (1 - G(z)) \frac{Z(z)}{z} \right] \right\}. \quad (20)$$

The capital stock in efficiency units  $k(z, R)$  follows from  $R = f'(Z(z)k)z + 1 - \delta$ . Then, write the elasticity of the production function as  $\alpha(z, R) = \frac{f'(Z(z)k(z, R))Z(z)k(z, R)}{f(Z(z)k(z, R))}$  and solve (18) for

$$\frac{d}{k} = \frac{R}{1+g-R} \frac{Z(z)}{z} (R - 1 + \delta) \frac{\xi}{\alpha(z, R)}. \quad (21)$$

Equating to (19) yields

$$\frac{\xi}{\alpha(z, R)} [1 - G(z)] \frac{Z(z)}{z} (R + \delta - 1) = [RG(z) - \lambda(1 - \delta)] \left[ \frac{1+g}{R} - 1 \right]. \quad (22)$$

Any solution  $(z, R)$  to equations (20) and (22) specifies a steady state equilibrium, provided that government debt is non-negative, which entails the restriction  $RG(z) \geq \lambda(1 - \delta)$  (see equation (17)).<sup>7</sup>

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<sup>7</sup>A model solution with  $RG(z) < \lambda(1 - \delta)$  would describe an implausible situation in which the government is a net creditor to the private sector, earning the same return  $R$  as private investors.

I start with the case of zero deficits ( $\xi = 0$ ) so that the (nominal) stock of debt  $D$  is constant. Due to the presence of credit constraints, this economy can permit an equilibrium in which the government runs a Ponzi scheme, rolling over the existing stock of debt from period to period. In such situations, the price of government debt is a bubble which has positive value to investors because the rate of return equals the economy's growth rate. As shown below, the bubble equilibrium exists if and only if the economy with unvalued outside assets is "dynamically inefficient"; that is, if the interest rate at the (unique) no-bubble equilibrium is smaller than the growth rate. This feature is similar to well-known results for overlapping-generations economies (Diamond (1965)) or for incomplete-markets models (Aiyagari and McGrattan (1998)). Different from this literature, I show that the bubble equilibrium has higher TFP than any equilibrium without a bubble.

A *no-bubble stationary equilibrium* has  $d = 0$  which implies  $RG(z) = \lambda(1 - \delta)$ . Then (20) implies that the productivity threshold in a no-bubble equilibrium  $z^N$  solves the equation

$$\lambda = G(z) \left[ 1 + \frac{z}{Z(z)} \left( \frac{1+g}{\beta(1-\delta)} - 1 \right) \right]. \quad (23)$$

For existence and uniqueness, the following two assumptions on the productivity distribution will be required.

**Assumption 1:** *The distribution function  $G$  has connected support  $\text{supp}G$  and is such that  $z/Z(z)$  is non-decreasing for all  $z \in \text{supp}G$ .*

**Assumption 2:** *The distribution function  $G$  is such that  $z/Z(z)$  tends to one when  $z$  converges to the supremum of  $\text{supp}G$ .*

Assumption 1 is a relatively mild condition which is satisfied for uniform, exponential, Pareto or log-normal distributions. Assumption 2 obviously holds for any bounded distribution, but also for exponential or log-normal distributions.

**Proposition 1:** *A no-bubble stationary equilibrium  $(z^N, R^N)$  exists if either Assumption 2 holds or if  $\lambda$  is sufficiently small. The interest rate satisfies  $R^N = \frac{\lambda(1-\delta)}{G(z^N)} < \beta/(1+g)$ . Under Assumption 1, a no-bubble stationary equilibrium is unique.*

The possibility of a *bubble stationary equilibrium* arises in those cases where the no-bubble equilibrium is dynamically inefficient. In a bubble stationary equilibrium, the price of the fixed stock of debt  $D$  grows at rate  $g$ , so that  $R = 1 + g$ . It follows from (20) that the

productivity threshold  $z^B$  in a bubble equilibrium solves

$$\frac{1-\beta}{\beta}[1+g-\lambda(1-\delta)] = (1-G(z))\left[\frac{Z(z)}{z}-1\right](g+\delta). \quad (24)$$

A solution to this equation is a bubble equilibrium only if  $G(z^B) > \lambda(1-\delta)/(1+g)$  which implies that public bonds have strictly positive value.

**Proposition 2:** *Suppose that Assumption 1 holds. Then a unique bubble stationary equilibrium  $(z^B, R^B)$  with  $R^B = 1+g$  exists if and only if  $R^N < 1+g$ . Moreover,  $z^B > z^N$  holds so that TFP is higher in the bubble stationary equilibrium.*

The existence of a bubble stationary equilibrium depends on the tightness of credit constraints. With tighter constraints, lower demand for credit reduces the interest rate so that dynamic inefficiency (overaccumulation of capital) can arise which is a prerequisite for a bubble equilibrium. While the bubble asset (here unbacked government debt) crowds out capital, it has a positive impact on factor productivity via the capital allocation among heterogeneous firms. This is because some low-productivity firms decide to stay out of business at the higher interest rate so that capital is employed more productively. While the ultimate effect of the bubble on output seems ambiguous, I show in the calibrated example of the next section that output is higher at the bubble equilibrium. That is, the increase in TFP more than offsets the decline in the aggregate capital stock.

Regardless of the existence of a bubble, one can show that a relaxation of credit constraints has an unambiguous positive effect on TFP. In the absence of a bubble, this is because low-productivity firms find it less valuable to borrow when the interest rate rises in response to a credit expansion. If the economy exhibits a bubble, a credit expansion does not affect the interest rate (which equals the growth rate), yet it also has a positive effect on factor productivity: higher demand for credit leads to more capital investment and to less investment in government bonds. This reduces the capital return for all firms which makes it again less valuable for low-productivity firms to borrow and to invest. Consequently, capital is more efficiently allocated and TFP rises. These assertions are summarized as follows:

**Proposition 3:** *A credit expansion induced by an increase of parameter  $\lambda$  raises  $z^N$  and  $z^B$ , and therefore total factor productivity at any stationary equilibrium.*

While Proposition 2 establishes the precise condition under which unbacked government debt can be rolled over indefinitely, the very same condition also permits the perpetual

financing of primary deficits, as long as these are not too large. To obtain an analytical result, it simplifies to consider a Cobb-Douglas production function so that the elasticity  $\alpha(z, R)$  is constant.

**Proposition 4:** *Suppose that the production function is  $f(Zk) = (Zk)^\alpha$ . Under Assumption 1 and  $R^N < 1 + g$ , there is a maximum sustainable deficit  $\bar{\xi} > 0$  such that any economy with deficit-output ratio  $\xi \in [0, \bar{\xi})$  has two stationary equilibria  $(z_i, R_i)_{i=1,2}$  with  $z^N \leq z_1 < z_2 \leq z^B$  and  $R^N \leq R_1 < R_2 \leq 1 + g$ .*

This proposition extends Proposition 2 to positive deficit levels. The low-productivity equilibrium  $(z_1, R_1)$  is a continuation of the no-bubble equilibrium which is identical to  $(z^N, R^N)$  when  $\xi = 0$ . Conversely, the high-productivity equilibrium  $(z_2, R_2)$  coincides with the bubble equilibrium when  $\xi = 0$ . Consequently, it is appropriate to label  $(z_1, R_1)$  a no-bubble steady state and  $(z_2, R_2)$  a bubble steady state. Both equilibria collapse when the primary deficit reaches the maximum sustainable level  $\bar{\xi}$ . Values of the deficit above  $\bar{\xi}$  are not sustainable, i.e. they lead to exploding debt stocks and hence do not permit a stationary equilibrium.

Without providing a formal stability analysis for the three-dimensional dynamic system (16)–(18) in  $(k_t, z_t, d_t)$ , it is worth noting that the low-productivity (no bubble) equilibrium at  $z_1$  is typically a sink while the bubble equilibrium is a saddle, which is similar to related findings for overlapping-generations models with unbacked government debt (cf. Chalk (2000)). If the bond price  $q_t$  (and hence  $d_t$ ) is allowed to jump in response to contemporaneous shocks, the bubble equilibrium should be regarded as a (locally) determinate equilibrium (i.e., there is no other equilibrium nearby the stationary equilibrium) while the no-bubble equilibrium is locally indeterminate: that is, bond prices may be subject to sunspot fluctuations, so that an indeterminate equilibrium dynamics arises around the stationary equilibrium at  $(k_1, z_1, d_1)$ . Without such instantaneous adjustments of bond prices (e.g. if bond yields were indexed so that government debt pays always the same return as private credit), the bubble equilibrium should be regarded as an unstable state, while the no-bubble equilibrium is locally stable (see e.g. Chalk (2000)). While both interpretations are principally acceptable, this paper favors the second view and treats  $d_t$  as a predetermined variable. This can be justified by the observation that empirical bond returns are rather stable and do not exhibit the large fluctuations (relative to capital returns) that would be required to keep the economy at the determinate saddle path in response to fundamental economic shocks.

## 4 Quantitative Implications

This section describes how the model is calibrated to match certain long-run features for the US economy. Subsequently the calibrated model is used to address several quantitative questions: First, what is the maximum sustainable level of the primary deficit? Similarly, what is the maximum sustainable stock of public debt, given the calibrated value of the primary deficit? Second, what are the consequences of temporary or permanent changes in fiscal deficits on TFP and output? Third, how does the economy respond to a shock to credit market conditions?

### 4.1 Calibration

I calibrate the model at annual frequency and set parameters to match selected data targets for the period 1970–2013. Regarding the technology, I choose a Cobb-Douglas production function  $y = (Zk)^\alpha$  and set  $\alpha = 0.3$  to match a labor income share of 70 percent. The growth rate of labor efficiency is set to  $g = 0.028$  to match the average growth rate of US real GDP (output). The depreciation rate  $\delta$  is set to the data average of 5.4 percent and the discount factor  $\beta$  is set to match a capital-output ratio of 2.68.<sup>8</sup>

For the idiosyncratic capital productivity distribution, it is convenient to use a Pareto distribution of the form  $G(z) = 1 - (z_0/z)^\varphi$  with scale parameter  $z_0$  and shape parameter  $\varphi > 1$ . As the scale parameter can be arbitrarily normalized, I set it to the value such that TFP (i.e.,  $Z^\alpha$ ) equals unity in steady state. Because of  $Z(z) = E(z'|z' \geq z) = \frac{\varphi z}{\varphi - 1}$ , the gross interest rate can be written

$$R = z\alpha(kZ)^{\alpha-1} + 1 - \delta = \frac{z}{Z}\alpha\frac{(Zk)^\alpha}{k} + 1 - \delta = \frac{\varphi - 1}{\varphi}\alpha\frac{Y}{K} + 1 - \delta. \quad (25)$$

To obtain a data target for the real interest rate (i.e., the real return on government bonds), I calculate the difference between the average annualized 3-months treasury bill rate and the average growth rate of the GDP deflator. This gives  $R = 1.016$  which implies  $\varphi = 2.56$ . Because this shape parameter is crucial for the following quantitative findings, it is appropriate to compare its implications for other data moments. First, note that the gap between  $Z$  and  $z$  not only pins down the safe interest rate but also the average equity premium in this model, which is the difference between the average equity return  $[f'(Zk)Z + 1 - \delta - \theta R]/(1 - \theta)$  (see equation (10)) and the safe interest rate  $R$ . For the calibrated parameter values, the equity premium attains the reasonable value of 5.4%.

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<sup>8</sup>The capital stock is calculated from real net domestic investment using the perpetual inventory method based on the assumption that the capital-output ratio is constant.

Second, parameter  $\varphi$  has direct implications for the cross-sectional productivity dispersion. Since  $\ln(z)$  is exponentially distributed with parameter  $\varphi$ , the standard deviation of  $\ln(\text{TFP}) = \alpha \ln(z)$  across active firms is  $\alpha/\varphi$  which is 0.12 for the calibrated parameter values. If firms are weighted by size (note that employment and output are proportional to the idiosyncratic draw of  $z^i$ ), the weighted standard deviation of  $\ln(\text{TFP})$  turns out to be  $\alpha/(\varphi - 1) \approx 0.19$ . Both measures are smaller than the empirical measures of standard deviations of  $\ln(\text{TFP})$  for US firms within narrowly defined industries which average around 0.4 (see Hsieh and Klenow (2009) and Bartelsman et al. (2013)). While model calibrations with more dispersion (i.e. lower  $\varphi$ ) are principally feasible, they would require either a negative real interest rate or implausibly low values of the capital-output ratio, see equation (25). For instance, with  $\varphi = 1.75$  (and the same calibration target for  $K/Y$ ), the employment-weighted standard deviation of  $\ln(\text{TFP})$  would increase to 0.4, but the real interest rate would fall to  $-0.6\%$  and the equity premium would increase to  $8.2\%$ . The collateral share parameter  $\lambda$  is calibrated so that the volume of business credit is 58 percent of annual output, while the primary deficit ratio  $\xi$  is calibrated to match a public-debt-output ratio of 50 percent.<sup>9</sup> The calibrated parameter value for the (primary) deficit-output ratio  $\xi$  is  $0.59\%$ , which slightly exceeds the data average<sup>10</sup> of  $0.31\%$ . All parameter choices are summarized in Table 1. It is worth mentioning that the calibration targets are hit exactly and pin down the model parameters uniquely (see the Appendix).

## 4.2 Results

To determine the maximum sustainable deficit for this calibration, Figure 1 shows how the steady states depend on the deficit-output ratio (parameter  $\xi$ ). The solid curves in these graphs show the no bubble steady states which feature lower TFP than the bubble steady states shown by the dashed curves (cf. Proposition 4). At the same time, no-bubble steady states exhibit lower interest rates, more private credit and hence a larger capital stock, which is similar to dynamically inefficient overlapping-generations production economies (Diamond (1965)). But while in those economies the no-bubble equilibrium has higher output, the opposite is true in this model: despite a higher capital stock, output is *smaller* at the no-bubble equilibrium compared with the bubble equilibrium. This is so because of

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<sup>9</sup>To obtain these data targets (averages over 1970–2013), I use credit market liabilities of the non-financial business sector and credit market liabilities of the general government (consolidated) from the Flow of Funds Accounts of the Federal Reserve (Tables L.101 and L.105.c).

<sup>10</sup>The deficit-output ratio is calculated as net borrowing minus interest payments of the general government (consolidated) from the Flow of Funds Accounts (Table F.105.c), divided by nominal GDP.



Table 1: Parameter choices.

Parameter	Value	Description	Target
$\delta$	0.054	Depreciation rate	Consumption of fixed capital
$\alpha$	0.3	Production fct. elasticity	Capital income share
$g$	0.028	Labor efficiency growth	GDP growth
$\beta$	0.978	Discount factor	Capital-output ratio
$\varphi$	2.67	Pareto shape	Real interest rate
$z_0$	0.535	Pareto scale	Normalization $Z = 1$
$\xi$	0.0059	Deficit-output ratio	Public-debt-output ratio
$\lambda$	0.236	Collateral share	Firm-credit-output ratio

lower TFP: at low interest rates more low-productivity firms are active which leads to a less efficient capital allocation.

The dot in Figure 1 shows the calibrated steady state which is a no-bubble equilibrium and therefore stable. Note that this feature is not imposed in the calibration but is a result of the relatively large difference between the average output growth ( $g = 2.8\%$ ) and the safe interest rate ( $R - 1 = 1.6\%$ ). A higher target for the interest rate would imply that the calibrated steady state is a bubble (and hence unstable).

As can also be seen from Figure 1, the maximum sustainable deficit ratio implied by this calibration is at  $\bar{\xi} = 0.834\%$ . This is much lower than the value for the maximum deficit obtained from quantitative overlapping generations models: Chalk (2000) finds that primary deficits up to 5.2% are sustainable, while Bullard and Russell (1999) calibrate a similar model with a primary deficit of 1.9%. Both papers have an even larger gap between the growth rate and the real interest rate. In the model of this paper, raising the deficit to the maximum sustainable level would increase the public-debt-output ratio to 114% while output increases slightly, which is due to higher TFP albeit at a reduced capital stock. A related policy issue is to determine the maximum sustainable debt stock under the presumption that the government implements the actual (calibrated) primary deficit ratio  $\xi = 0.59\%$ . Simulations reveal that debt stocks above 203% of output lead to explosive dynamics whereas any lower debt level is sustainable under the current deficit policy.

While these considerations compare steady state outcomes, it is also instructive to explore the transition dynamics under a change of policy. Figure 2 considers a scenario in which the government starts to implement a zero-deficit policy from year  $t = 1$  onwards, thus reducing the debt-output ratio to zero in the very long run. Indeed, it takes quite some

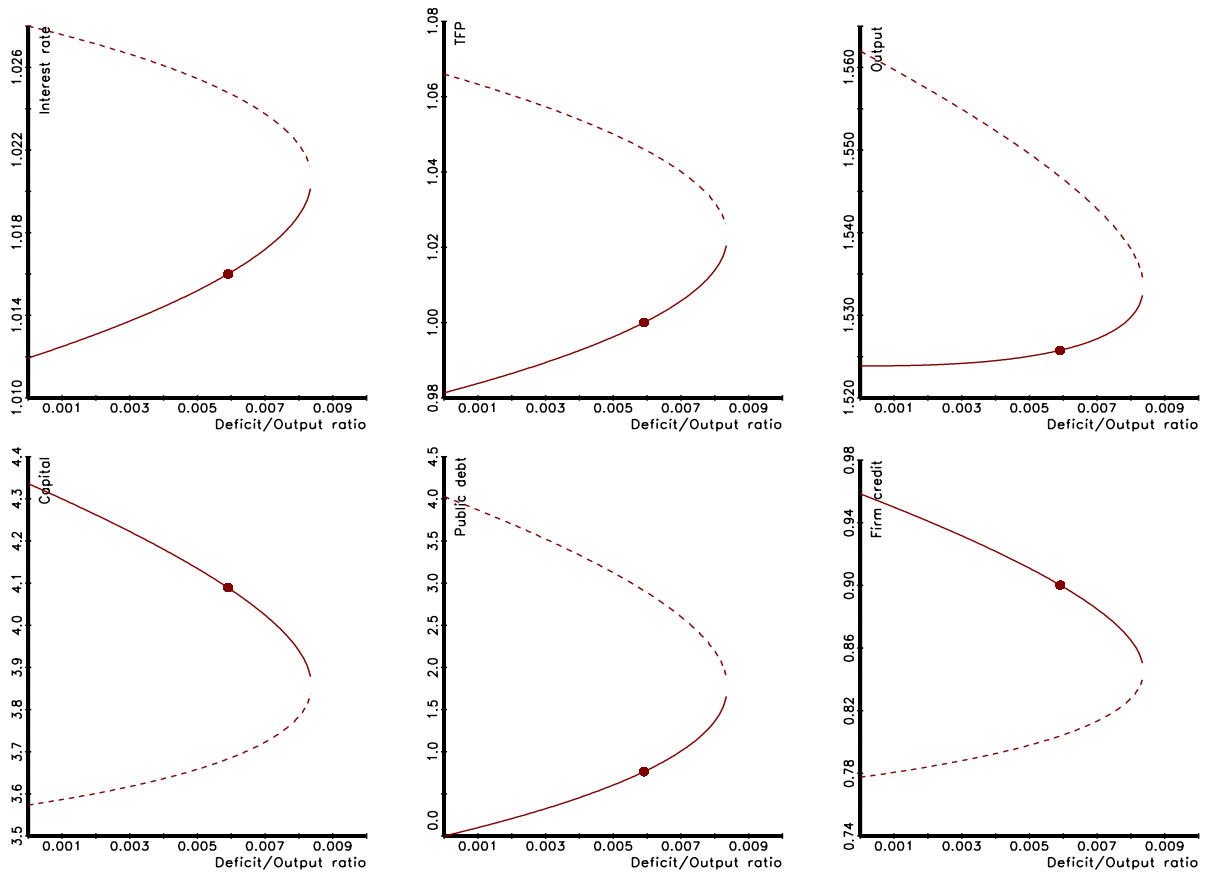


Figure 1: Steady state values for varying deficit-output ratio  $\xi$ . The dot shows the calibrated steady state. Solid curves show no-bubble (sink) steady states and dashed curves show bubble (saddle) steady states.

time to deplete the debt stock: after 50 (100; 200) years, the debt-output ratio falls to 25.5% (12.3%; 2.7%). As shown by the intersection between the solid curves and the vertical axes in Figure 1, the zero-deficit steady state has a higher capital stock, lower TFP and slightly lower output. However, the transition dynamics in Figure 2 reveals that it takes a very long time (almost 140 years) until (detrended) output falls below the initial level. This is because the private credit and investment boom induced by the public debt reduction dominates output dynamics initially, while the TFP reductions operate much later and are quantitatively less important in the first century after the policy change.

A more topical policy experiment concerns the strong increase of the public debt stock induced by expansive fiscal policy in the aftermath of the Great Recession. To study the effects of this policy, Figure 3 shows a scenario in which the government raises the primary

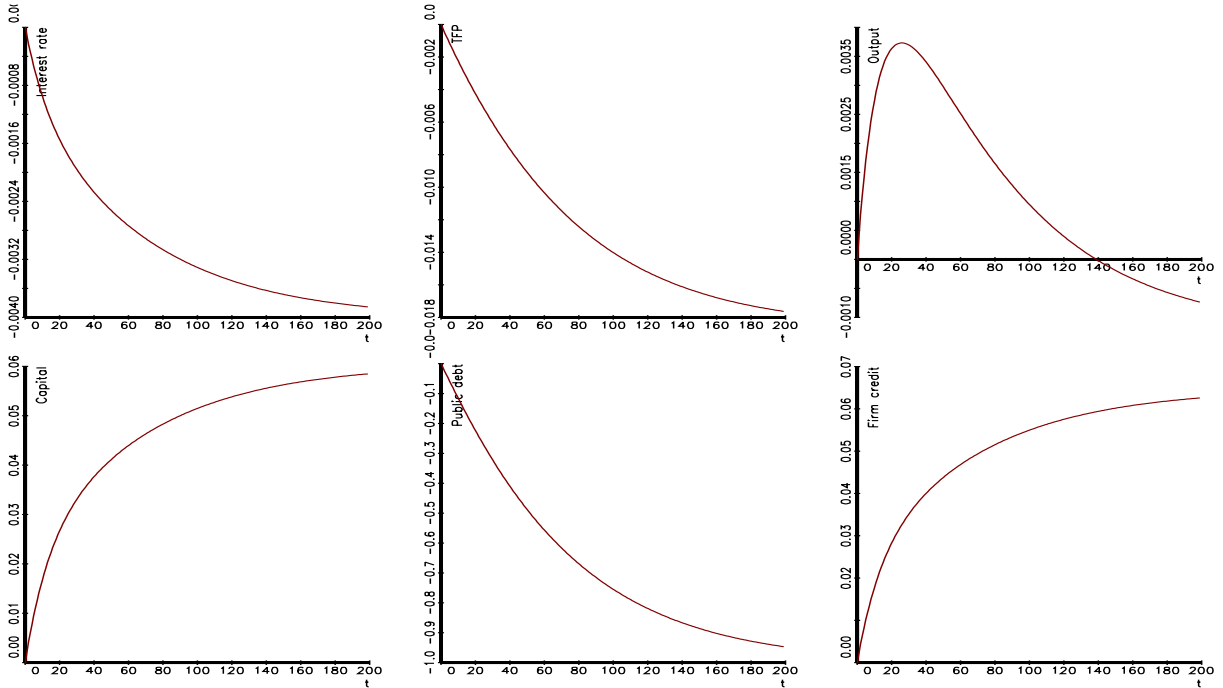


Figure 2: Adjustment dynamics to a zero-debt steady state (zero primary deficit from  $t = 1$  onwards). All variables are expressed as percentage deviations from their steady state values.

deficit ratio to  $\xi = 6\%$  over five years (the average value over the period 2009-2013) before it returns to the stationary (calibrated) deficit policy. As shown by the lower graphs in the figure, public debt increases by 54% which substantially crowds out private credit and capital. It also takes a long time to reduce the public debt stock: after 50 years, debt is still 45% above the initial level. Yet, the upside of the deficit policy is an increase of TFP by about 1% which dampens the initial decline in output, although the quantitative effect of the crowding out of capital dominates output dynamics, similar to the previous findings. After three decades, however, detrended output slightly exceeds the level before the policy change, until it converges back to the initial level.

Since this economy features productivity losses because of binding credit constraints, it is also interesting to consider the response to a change in credit market conditions (“financial shock”). Figure 4 shows the model adjustment to a permanent 10% increase of the collateral parameter  $\lambda$ , capturing the effect of a credit expansion which is similar in magnitude to the period 1995–2005. Relaxed borrowing constraints have a twofold (albeit quantitatively small) effect on output. First, TFP increases by about 0.3% which is induced by a

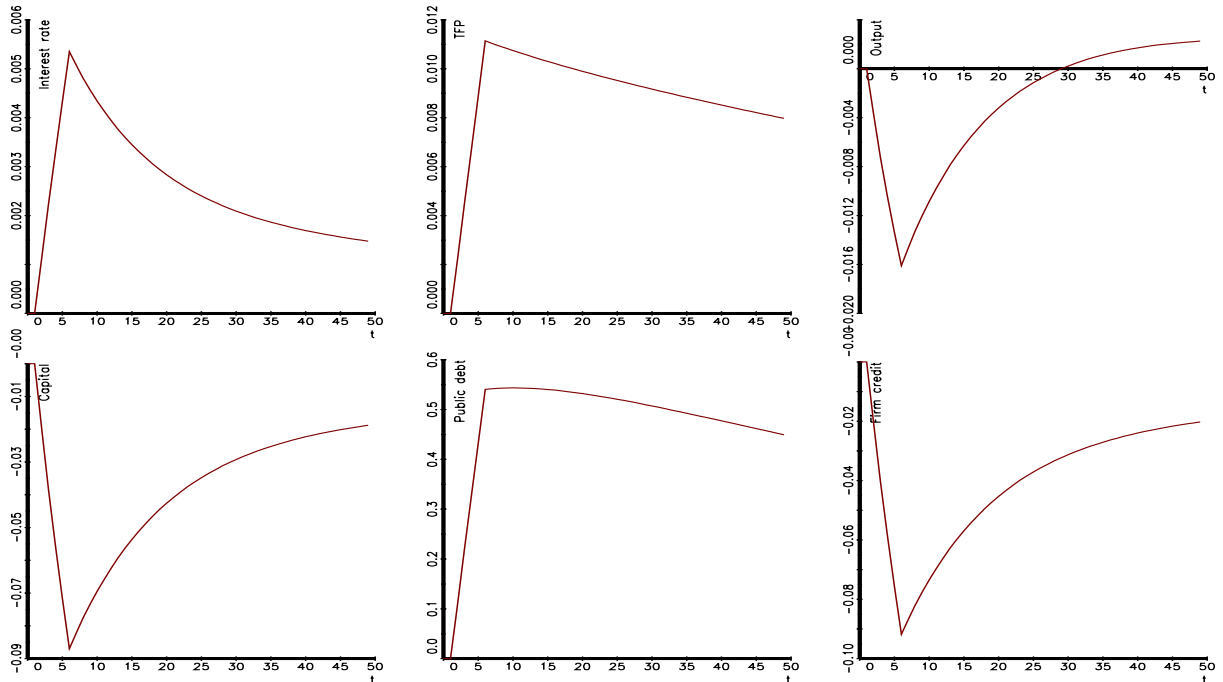


Figure 3: Adjustment dynamics with primary deficit  $\xi_t = 6\%$  for  $t = 1, \dots, 5$  and  $\xi_t = 0.0059$  for  $t \geq 6$ . All variables are expressed as percentage deviations from their steady state values.

small increase of the interest rate. Over time, the capital stock increases by  $0.45\%$  which reduces the capital return and the interest rate, leaving TFP unaffected. The combined long-term effect on output is just below half a percentage point. Hence in this economy the quantitative effect of credit market conditions on output is quite modest.

## 5 Conclusions

This paper presents a macroeconomic model that includes two key ingredients: (i) endogenous TFP coming from financially constrained firms operating at different productivity levels; (ii) the possibility of sustainable primary deficits that arises from low equilibrium interest rates. The second feature is closely linked to the first one: if financial constraints are sufficiently tight, the interest rate falls below the growth rate in which case the government is able to role over unbacked public debt indefinitely. Those dynamically inefficient economies typically feature two steady states: one unstable “bubble” steady state with a higher interest rate, higher public debt and a lower capital stock, and a stable “no bubble”

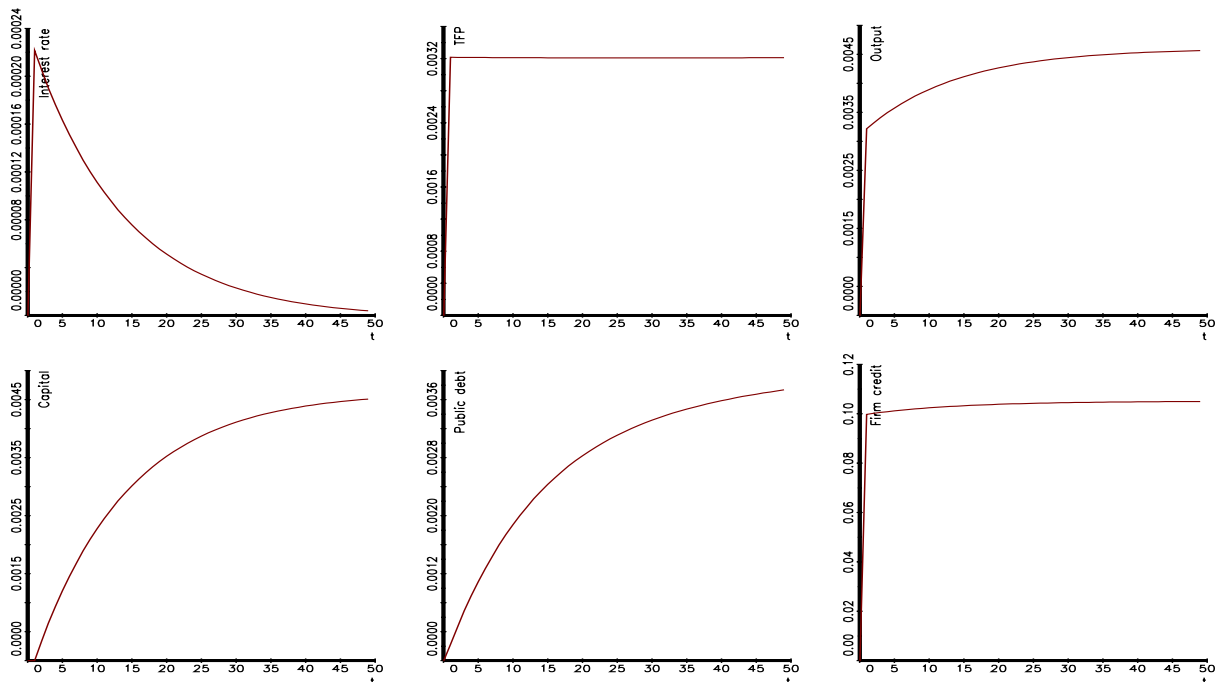


Figure 4: Adjustment dynamics to a permanent financial shock ( $\lambda_t = 0.26$  from  $t = 1$  onwards). All variables are expressed as percentage deviations from their steady state values.

steady state with a lower interest rate, less public debt and more private capital. While these features are also obtained in overlapping generations models, the endogenous TFP of this model reverses the implications for aggregate output: the bubble steady state has higher TFP and output (despite a lower capital stock) than the no-bubble steady state.

When calibrated to the US economy, the model is used to study a few quantitative questions. I find that the maximum sustainable primary deficit ratio is less than one percent, and hence much smaller than related numbers from quantitative studies based on overlapping-generations models. Moreover, the endogenous responses of TFP to changes in debt policy are quantitatively modest and are typically dominated (at least in the medium run) by the crowding out of private credit and capital.

An attractive feature of this model is that it is quite tractable, described by only few state variables and dynamic equations, which makes it rather flexible for extensions. It should be straightforward to augment the model by adding flexible labor supply and by fleshing out more detailed fiscal policy instruments. Such extensions would make the model more interesting for business-cycle analysis in order to quantify the importance of various shocks,

including uncertainty shocks or financial shocks. Model extensions could also be used to study the impact of fiscal rules on the dynamics of output, credit and TFP.

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## Appendix

**Lemma 1:** *A competitive equilibrium is described by solutions in  $(z_t, k_t, d_t)$  to the dynamic system defined by (16), (17) and (18).*

**Proof:** Rewrite equations (7), (12) and (13) in terms of  $k_t = K_t/A_t$ ,  $\omega_t = W_t/A_t$ ,  $d_t = (q_t D_t)/A_t$ :

$$\omega_t = f'(k_t Z_t) Z_t k_t + (1 - \delta)k_t + d_t, \quad (26)$$

$$k_{t+1} = \beta \omega_t \frac{R_{t+1}(1 - G(z_{t+1}))}{(1 + g)[R_{t+1} - \lambda(1 - \delta)]}, \quad (27)$$

$$\frac{\beta R_{t+1}}{1 + g} \omega_t G(z_{t+1}) = \lambda(1 - \delta)k_{t+1} + d_{t+1}. \quad (28)$$

Substitution of (26) into (27) directly yields (17). Combine (27) and (28) to obtain

$$\begin{aligned} k_{t+1}[R_{t+1} - \lambda(1 - \delta)] &= \beta \frac{\omega_t}{1 + g} [1 - G(z_{t+1})] R_{t+1} \\ &= \beta \frac{\omega_t}{1 + g} R_{t+1} - d_{t+1} - \lambda(1 - \delta)k_{t+1}. \end{aligned}$$

Substitution of (26) yields (16). With  $X_t/A_t = \xi_t y_t = \xi_t f(Z_t k_t)$ , the government budget constraint (4) is

$$\xi_t f(Z_t k_t) = \frac{(1 + g)d_{t+1}}{R_{t+1}} - d_t,$$

which is equation (18). □

**Proposition 1:** *A no-bubble stationary equilibrium  $(z^N, R^N)$  exists if either Assumption 2 holds or if  $\lambda$  is sufficiently small. The interest rate satisfies  $R^N = \frac{\lambda(1 - \delta)}{G(z^N)} < \beta/(1 + g)$ . Under Assumption 1, a no-bubble stationary equilibrium is unique.*

**Proof:** Let  $\underline{z} \geq 0$  and  $\bar{z} > 0$  denote the upper and lower bounds of the support of  $G$ , with  $\bar{z} = \infty$  if  $G$  has unbounded support. The right-hand side (RHS) of (23) is zero at  $z = \underline{z}$  and positive at higher values of  $z$ . Hence a solution exists if  $\lambda$  is sufficiently small. Moreover, under Assumption 2 the RHS tends to  $(1 + g)/(\beta(1 - \delta)) > 1$  for  $z \rightarrow \bar{z}$ , so that (23) has a solution for any  $\lambda \leq 1$ . Because of  $z/Z(z) < 1$ , it follows that  $\lambda < G(z^N) \frac{1 + g}{\beta(1 - \delta)}$  and

$$\text{hence } R^N = \frac{\lambda(1 - \delta)}{G(z^N)} < \frac{1 + g}{\beta}.$$

Under Assumption 1, the RHS is strictly monotonically increasing. Hence, there can be at most one solution to equation (23). □



**Proposition 2:** *Suppose that Assumption 1 holds. Then a bubble stationary equilibrium exists if and only if  $R^N < 1 + g$ . Moreover,  $z^B > z^N$  holds so that TFP is higher in the bubble stationary equilibrium.*

**Proof:** Let  $\hat{z}$  be the unique productivity value where  $G(\hat{z}) = \lambda(1 - \delta)/(1 + g) < 1$ . A stationary bubble equilibrium is a solution  $z^B$  of (24) such that  $z^B > \hat{z}$ . Write  $\phi(z)$  for the RHS of (24). Since  $\phi$  is strictly decreasing and tends to zero for  $z \rightarrow \bar{z}$ , a bubble stationary equilibrium exists if and only if the LHS of (24) is smaller than  $\phi(\hat{z})$  which is equivalent to

$$(1 + g) \frac{1 - \beta}{\beta} < \left[ \frac{Z(\hat{z})}{\hat{z}} - 1 \right] (g + \delta) . \quad (29)$$

Now suppose that  $R^N < 1 + g$ . Because of

$$R^N = \frac{\lambda(1 - \delta)}{G(z^N)} = 1 - \delta + \frac{z^N}{Z(z^N)} \left( \frac{1 + g}{\beta} - 1 + \delta \right) < 1 + g ,$$

it follows that  $z^N > \hat{z}$ , and therefore

$$\frac{1 + g}{\beta} - 1 + \delta < \frac{Z(z^N)}{z^N} (g + \delta) \leq \frac{Z(\hat{z})}{\hat{z}} (g + \delta) ,$$

where the last inequality follows since  $Z(z)/z$  is non-increasing. This proves the existence of a bubble stationary equilibrium.

Conversely, suppose that

$$R^N = \frac{\lambda(1 - \delta)}{G(z^N)} = 1 - \delta + \frac{z^N}{Z(z^N)} \left( \frac{1 + g}{\beta} - 1 + \delta \right) \geq 1 + g .$$

This implies that  $z^N \leq \hat{z}$  and therefore

$$\frac{1 + g}{\beta} - 1 + \delta \geq \frac{Z(z^N)}{z^N} (g + \delta) \geq \frac{Z(\hat{z})}{\hat{z}} (g + \delta) .$$

This proves that no bubble stationary equilibrium exists if  $R^N \geq 1 + g$ .

Finally, it remains to show  $z^B > z^N$  whenever a bubble stationary equilibrium exists. This holds iff

$$\frac{1 - \beta}{\beta} [1 + g - \lambda(1 - \delta)] = \phi(z^B) < \phi(z^N) . \quad (30)$$

Because of

$$R^N = \frac{\lambda(1 - \delta)}{G(z^N)} = 1 - \delta + \frac{z^N}{Z(z^N)} \left( \frac{1 + g}{\beta} - 1 + \delta \right)$$

one has

$$\phi(z^N) = [1 - G(z^N)] \left[ \frac{Z(z^N)}{z^N} - 1 \right] (g + \delta) = [1 - G(z^N)] \frac{g + \delta}{\beta(1 - \delta)} \frac{G(z^N)(1 + g) - \beta(1 - \delta)\lambda}{\lambda - G(z^N)}.$$

To show (30), it must be proven that

$$\frac{1 - \beta}{\beta} [1 + g - \lambda(1 - \delta)] < \underbrace{[1 - G] \frac{g + \delta}{\beta(1 - \delta)} \frac{G(1 + g) - \beta(1 - \delta)\lambda}{\lambda - G}}_{\equiv \psi(G)} \quad (31)$$

holds at  $G = G(z^N)$ . Because a bubble equilibrium exists,  $R^N < 1 + g$  implies that  $G(z^N) > \lambda(1 - \delta)/(1 + g)$ , and (23) implies that  $G(z^N) < \lambda$ . It follows  $\psi(\lambda(1 - \delta)/(1 + g)) = (1/\beta - 1)[1 + g - \lambda(1 - \delta)]$  and  $\psi(\lambda) = \infty$ . Therefore (31) holds if  $\psi'(G) > 0$  for  $G \in [\lambda(1 - \delta)/(1 + g), \lambda)$  which is easy to verify. This proves that  $z^B > z^N$ .  $\square$

**Proposition 3:** *Under Assumption 1, a credit expansion induced by an increase of parameter  $\lambda$  raises  $z^N$  and  $z^B$ , and therefore total factor productivity at any stationary equilibrium.*

**Proof:** Consider first the no-bubble stationary equilibrium at  $z^N$ . Because the RHS in (23) strictly increases in  $z$  under Assumption 1,  $z^N$  (and hence TFP which depends positively on  $Z(z)$ ) increases if  $\lambda$  increases. Second consider a bubble stationary equilibrium at  $z^B$ . Again Assumption 1 implies that the RHS of (24) decreases in  $z$ , and since the LHS decreases in  $\lambda$ ,  $z^B$  must strictly increase if  $\lambda$  increases.  $\square$

**Proposition 4:** *Suppose that the production function is  $f(Zk) = (Zk)^\alpha$ . Under Assumption 1 and  $R^N < 1 + g$ , there is a maximum sustainable deficit  $\bar{\xi} > 0$  such that any economy with deficit-output ratio  $\xi \in [0, \bar{\xi})$  has two stationary equilibria  $(z_i, R_i)_{i=1,2}$  with  $z_1 < z_2$  and  $R_1 < R_2 < 1 + g$ .*

**Proof:** A steady state equilibrium  $(R, z)$  is a solution to equations (20) and (22). For any given  $z$ , the RHS of (20) is a quadratic function of  $R$  which intersects the LHS uniquely at some  $R > 1 - \delta$ ; see Figure 5. This follows because  $\text{LHS}(R = 1 - \delta) > \text{RHS}(R = 1 - \delta) > 0$  and  $\text{RHS}(R = 0) = 0$ , so that the RHS is quadratic in  $R$  and strictly increasing in  $R > 1 - \delta$ . Since the RHS is strictly decreasing in  $z$ , equation (20) defines an upward-sloping solution  $R(z) \geq 1 - \delta$  such that  $R(z) \rightarrow (1 + g)/\beta$  when  $z \rightarrow \bar{z}$ .

It will now be shown that  $R(z)$  intersects twice with the curve that defines implicit solutions  $(R, z)$  to the other equilibrium condition (22) for small enough values of the deficit ratio  $\xi$ . Because two solutions at  $z = z^N$  and at  $z = z^B$  exist for  $\xi = 0$ , the implicit function

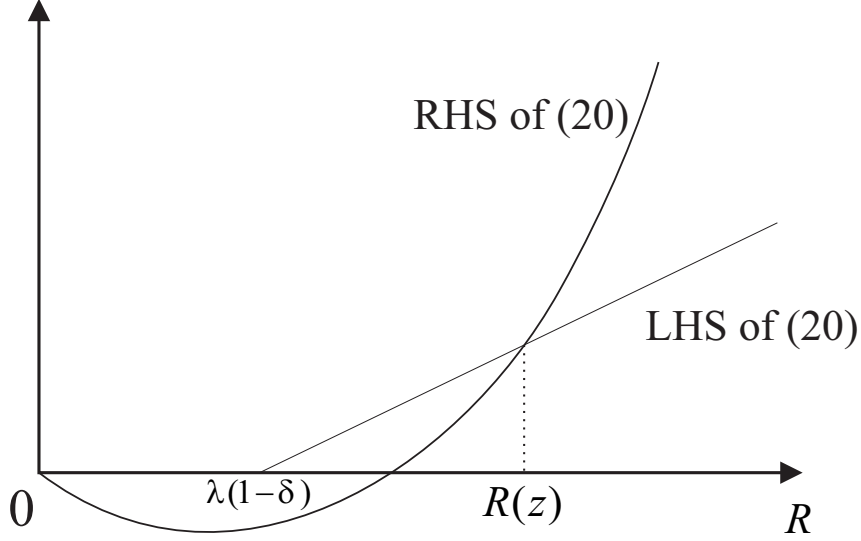


Figure 5: Equation (20) defines  $R(z)$ .

theorem also guarantees two (locally unique) solutions for small values of  $\xi$ , provided that the two curves (20) and (22) have different slopes at  $z^B$  and at  $z^N$  for  $\xi = 0$ . But this is easy to verify. At  $z = z^B$  and  $\xi = 0$ , (22) defines the flat curve  $R = 1 + g$ , whereas at  $z = z^N$ , equation (22) defines the downward-sloping  $R = \lambda(1 - \delta)/G(z)$ ; see Figure 6. As shown above, (20) defines an upward-sloping curve  $R(z)$ . Therefore, for any small enough  $\xi > 0$ , the two equations have two solutions  $(z_i, R_i)$ ,  $i = 1, 2$  such that  $z_1 \rightarrow z^N$  and  $z_2 \rightarrow z^B$  when  $\xi \rightarrow 0$ .

It still remains to verify that public debt is non-negative at either solution. This follows trivially by continuity for the solution  $z_2$  because of  $R^B G(z^B) > \lambda(1 - \delta)$ . For the other solution  $(z_1, R_1)$ , rewrite equation (22) as

$$\phi(z, R, \xi) \equiv [RG(z) - \lambda(1 - \delta)][1 + g - R] - \frac{\xi}{\alpha}(1 - G(z))R(R + \delta - 1)\frac{Z(z)}{z} .$$

The local derivatives at  $(z, R, \xi) = (z^N, R^N, 0)$  have signs

$$\phi_1 = G'(z^N)R^N(1 + g - R^N) + \frac{\xi}{\alpha}R^N(R^N + \delta - 1)\frac{(z^N)^2 G'(z^N) + \int_{z^N} z dG(z)}{(z^N)^2} > 0 ,$$

and  $\phi_3 < 0$  which implies that  $\frac{dz}{d\xi} = -\phi_3/\phi_1 > 0$  at any given  $R$ . Hence, the curve (22) lies above the curve  $R = \lambda(1 - \delta)/G(z)$  in  $(z, R)$  space for  $\xi > 0$ , which implies that  $z_1 > z^N$  and  $R_1 > R^N$ , and hence  $R_1 G(z_1) > \lambda(1 - \delta)$ , so that public debt is positive.

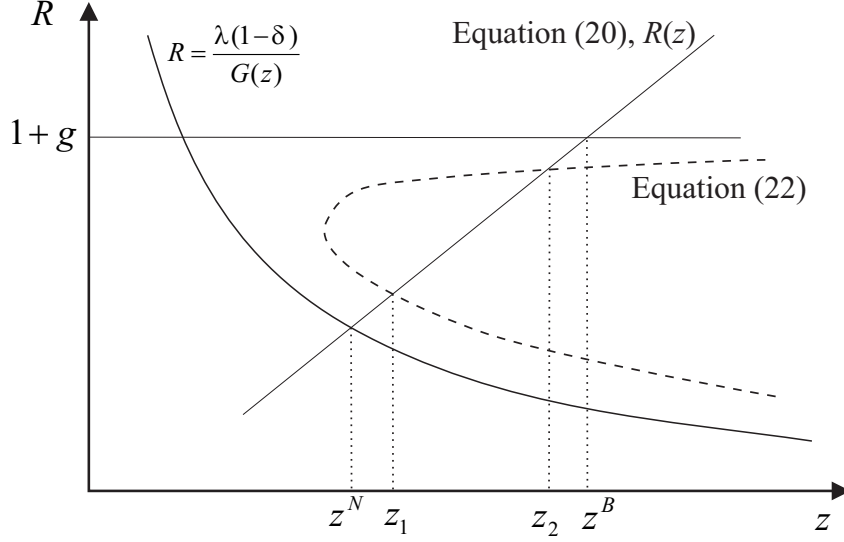


Figure 6: Intersections between equations (20) and (22) define  $z_1$  and  $z_2$ .

Lastly, because both solutions  $(z_i, R_i)$ ,  $i = 1, 2$ , satisfy the upward-sloping relation  $R = R(z)$ , the equilibrium with the higher interest rate has higher TFP.  $\square$

### Calibration

Parameters  $\alpha$ ,  $\delta$  and  $g$  are directly calibrated as mentioned in the text. Given the calibration targets for the interest rate  $R$  and for the debt-output ratio  $D/Y$ , the deficit-output ratio follows from  $\xi = (D/Y) \cdot (g - R)/R$ . The calibration target for  $K/Y$  together with the normalization  $Z = 1$  yields the steady-state value of  $k = K/A = (K/Y)^{1/(1-\alpha)}$ . This together with  $R = z f'(Zk) + 1 - \delta$  yields the steady-state value of  $z = (K/Y) \cdot (R + \delta - 1)/\alpha$ , so that the Pareto shape parameter follows from  $\varphi = 1/(1 - z)$ . The calibration target for the firm-credit-output ratio  $Credit/Y = \theta K/Y = \lambda(1 - \delta)/R \cdot (K/Y)$  identifies parameter  $\lambda = (Credit/Y) \cdot R/(1 - \delta)/(K/Y)$ . Then steady-state equation (22) can be solved for  $G(z)$ :

$$G(z) = \frac{\frac{\xi}{\alpha z}(R + \delta - 1) + \lambda(1 - \delta)\frac{g - R}{R}}{g - R + \frac{\xi}{\alpha z}(R + \delta - 1)}.$$

With  $G(z) = 1 - (z_0/z)^\varphi$ , this yields the Pareto scale parameter  $z_0$ . Finally, the discount factor follows uniquely from steady-state equation (20).