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Volatility and GMM: Monte Carlo studies and empirical estimations

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Volatility and GMM - Monte Carlo studies and empirical estimations

Hartmut Nagel
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Abstract [1]

In this paper we examine small sample properties of a generalized method of moments (GMM) estimation using Monte Carlo simulations. We assume that the generated time series describe the stochastic variance rate of a stock index. We use a mean reverting square-root process to simulate the dynamics of this instantaneous variance rate. The generated time series consist of 63, 250, and 1000 data points, respectively. They are used to estimate the parameters of the assumed variance rate process by applying GMM. The results obtained are described and compared to our estimates from empirical volatility data. We use the German volatility index VDAX, historical volatilities of the German stock index DAX over 10, 22 and 33 trading days as well as daily volume data of the German stock market.

[1] We wish to acknowlege helpful comments of Robert Jung and Roman Liesenfeld. Special thanks to Roman Liesenfeld for his support with the computer programs. We thank the Deutsche Börse AG for providing us with the data. Financial support from the Deutsche Forschungsgemeinschaft is also gratefully acknowledged.

1. Introduction

Since empirical studies indicate a mean reverting behavior of the volatility of the returns of stocks or stock indices, [2] it has become popular to model stochastic volatility as a continuous-time square-root process. This results in a system of two stochastic variables, where the stock price or stock price index is denoted by $S(t)$, and its instantaneous variance rate $v(t)$ serves as a second state variable. Following Heston (1993), the dynamics of such a system can be specified as

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dw_1(t) \tag{1.1}$$

$$dv(t) = \kappa\left[\theta - v(t)\right]dt + \sigma\sqrt{v(t)}dw_2(t). \tag{1.2}$$

In equation (1.1), the parameter μ is the expected instantaneous rate of return of $S(t)$, and $\sqrt{v(t)}$ is its instantaneous stochastic volatility. The corresponding instantaneous variance rate $v(t)$, described by equation (1.2), follows a square-root process with mean reversion parameter κ, a long-run target variance θ, and volatility parameter σ. (Hence, σ can be thought of as the "volatility of volatility.") The system is driven by a two-dimensional Brownian motion $[w_1(t), w_2(t)]$ with correlation parameter ρ.

One advantage of this specification is the fact that $v(t)$, once started from the positive half of the real axis, subsequently never becomes negative. [3] Actually, equation (1.2) implies that $v(t)$ can reach zero if $\sigma^2 > 2\kappa\theta$, in which case the boundary at $v(t) = 0$ is of the reflecting type. Otherwise $v(t) = 0$ is a natural boundary. [4]

In this paper we restrict ourselves to the estimation of the three parameters κ, θ and σ of equation (1.2). Using this simplified setting, our objective is twofold: First, we want to examine the small sample properties of the GMM estimation procedure using Monte Carlo simulation techniques. Second, we want to compare these results with those we get when using empirical volatility data for parameter estimation.

We assume that not the stock price movements but the dynamics of the stock index itself can be described according to equations (1.1) and (1.2). To measure volatility empirically, we choose the

[2] See Bühler/Grünbichler.

[3] An alternative specification is given in Stein/Stein (1991), where the instantaneous volatility, not the variance rate, can become negative.

[4] For details see Cox/Ingersoll/Ross (1985).

VDAX, [5] some historical volatility estimates of daily DAX returns, or data for the daily trading volume of German stocks.

2. The GMM estimation procedure

Our estimation procedure for the parameters of the volatility process (1.2) utilizes the generalized method of moments (GMM) developed by Hansen (1982). [6] We estimate the parameters of the continuous-time model (1.2) using the discrete-time Euler approximation [7]

$$v_t - v_{t-1} = \kappa\theta\Delta t - \kappa v_{t-1}\Delta t + \varepsilon_t, \tag{2.1}$$

with

$$\mathrm{E}\left[\varepsilon_t\right] = 0, \qquad \mathrm{E}\left[\varepsilon_t^2\right] = \sigma^2 \Delta t v_{t-1} = \sigma^{*2} v_{t-1}. \tag{2.2}$$

This leads to

$$\begin{aligned} \varepsilon_t &= v_t - v_{t-1} - \kappa\theta\Delta t + \kappa v_{t-1}\Delta t \\ &= v_t - v_{t-1} - \kappa^*\theta + \kappa^* v_{t-1}. \end{aligned} \tag{2.3}$$

We define φ to be the parameter vector with elements κ^*, θ and σ^*. [8] Let the vector $f_t(\varphi)$ be

$$f_t(\varphi) = \begin{bmatrix} \varepsilon_t \\ \varepsilon_t v_{t-1} \\ \varepsilon_t^2 - \sigma^{*2} v_{t-1} \\ \left(\varepsilon_t^2 - \sigma^{*2} v_{t-1}\right) v_{t-1} \end{bmatrix}. \tag{2.4}$$

[5] The VDAX is a volatility index for the German stock index DAX. It is computed implicitely using DAX-option prices. The procedure is similar to the one used by the CBOE to calculate the VIX. For a detailed description see Redelberger (1994).

[6] Chan/Karolyi/Longstaff/Sanders (1992) use a similar specification to compare a variety of continuous-time models of the short-term interest rate.

[7] If changes of the volatility process are measured over short periods of time, the approximation error is known to be of second-order importance. See Duffie (1992), p. 200.

[8] We estimate κ^*, θ and σ^* instead of κ, $\sqrt{\theta}$ and σ, since more accurate results are obtained if parameter values have about the same magnitude.

Under the null hypothesis that the model described by (2.1) and (2.2) is true, $\mathrm{E}[f_t(\varphi)]$ has to be zero. Replacing $\mathrm{E}[f_t(\varphi)]$ by its sample counterpart $g_T(\varphi)$, that is

$$g_T(\varphi) = \frac{1}{T}\sum_{t=1}^{T} f_t(\varphi), \qquad t = 1, ..., T, \tag{2.5}$$

we get parameter estimates of κ^*, θ and σ^*, by minimizing the quadratic form

$$J_T(\varphi) = g_T'(\varphi)\, W_T(\varphi)\, g_T(\varphi), \tag{2.6}$$

where $W_T(\varphi)$ is a positive-definite symmetric weighting matrix. In general, the GMM estimates of the parameter vector φ do depend on the choice of $W_T(\varphi)$. However, due to a result by Hansen (1982), the weighting matrix $W_T(\varphi) = S^{-1}(\varphi)$, where

$$S(\varphi) = E\left[f_t(\varphi)\, f_t'(\varphi)\right], \tag{2.7}$$

gives us the GMM estimator of φ with the *smallest asymptotic covariance matrix.* Using an approach developed by Newey and West (1987), we replace $S(\varphi)$ by its estimator $\widehat{S}(\varphi)$ and get for the asymptotic covariance matrix for the GMM estimate of φ the expression

$$\frac{1}{T}\left[\left(\frac{\partial g_T(\varphi)}{\partial \varphi}\right)' \left(\widehat{S}(\varphi)\right)^{-1} \left(\frac{\partial g_T(\varphi)}{\partial \varphi}\right)\right]^{-1}\Bigg|_{\varphi=\hat{\varphi}} . \tag{2.8}$$

(2.9)

To get a first estimation of φ, we choose $W_T(\varphi) = I$. This leads to a consistent but inefficient estimate $\widehat{\varphi_1}$. With this first estimation of φ we can proceed as previously described to get a second and efficient estimate $\widehat{\varphi_2}$. However, this second estimate still depends on the value of $\widehat{\varphi_1}$, because of its use in calculating $W_T(\widehat{\varphi_1})$. Therefore, we also carry out a third run of the GMM procedure. This time we are able to use the efficient estimate $\widehat{\varphi_2}$ in calculating $W_T(\widehat{\varphi_2})$. Especially for short time series, this leads to a distinct increase in the accuracy of our estimations.

The number of sample autocovariances used in the calculation of $W_T(\varphi) = \left(\widehat{S}(\varphi)\right)^{-1}$ are chosen to be either $l = 1$, $l = 9$, or the automatic lag-truncation parameter for the Bartlett kernel ($l = auto$) as described in Andrews (1991).

The minimized value of $J_T(\varphi)$ using (2.6) is distributed χ^2 under the null hypothesis that the model is true with degrees of freedom equal to the number of orthogonality conditions minus the number of parameters to be estimated. This χ^2 measure is used in the Hansen specification test. A high value of this test statistic means that the model is misspecified.

The calculations are done with GAUSS. We use the Davidon-Fletcher-Powell (DFP) optimization algorithm with a maximum of 2000 iterations.

3. Monte Carlo simulations

3.1. The data-generating process

Unfortunately, statistical properties of GMM estimators are known only asymptotically. Thus, in practice, exact finite-sample theory cannot be used to interpret these estimates or test statistics. Unless the sample size is sufficiently large, it is difficult to know whether asymptotic approximations are misleading or not. In this situation, Monte Carlo simulations allow us to check the validity of estimators and test statistics for finite samples. To investigate the finite-sample properties of the GMM estimators of our econometric model (2.1) and (2.2), we specify a *data-generating process* of the following form:

$$\begin{aligned} v_t &= v_{t-1} + \kappa\theta\Delta t - \kappa v_{t-1}\Delta t + \sigma\sqrt{v_{t-1}}\omega_t \\ &= \kappa\theta\Delta t + (1-\kappa\Delta t)v_{t-1} + \sigma\sqrt{v_{t-1}}\omega_t \\ &= \kappa^*\theta + (1-\kappa^*)v_{t-1} + \sigma\sqrt{v_{t-1}}\omega_t, \qquad \text{with } \omega_t \sim N(0,\Delta t), \quad t = 1,...,T \end{aligned} \tag{3.1}$$

Hence, the error term ω_t has a normal distribution with mean 0 and variance Δt. The starting value v_0 is chosen to be equal to θ. In other words, it is assumed that the volatility at the beginning of the simulation is exactly at its long-run level. The length of the simulated time series are $T = 63$, $T = 250$, or $T = 1000$ trading days, respectively. This corresponds to observation periods of three months, one year, or four years. Each Monte Carlo experiment has been replicated $n = 5000$

times for each T. Parameter estimates and test statistics are calculated for each single replication according to the GMM procedure described in the preceding section. To preserve comparability, all calculations within one group ($T = 63$, $T = 250$ and $T = 1000$) were done by using the same n simulated time series. Hence, all *GMM estimates for different lags* within one group are based on identical data sets. The "true" parameter values used for all Monte Carlo simulations are given in Table 3.1.

Parameter	Value
θ	0.0225
κ^*	0.0160
σ^*	0.0126
$\sqrt{\theta}$	0.1500
κ	4.0000
σ	0.2000
Δt	0.0040

Table 3.1.: True parameter values for Monte Carlo simulations.

It should be noted that the results of a Monte Carlo experiment are themselves estimates, and therefore subject to experimental error. However, using a sufficiently large number of replications, this error can be made arbitrarily small.

3.2. Parameter estimates

The resulting parameter estimates due to our GMM estimation procedure are given in Table 3.2 and Figures 3.1a, 3.1b and 3.1c. [9] As expected, the estimates for $T = 1000$ are more precise than for $T = 250$, which also outperform those for $T = 63$. Not only the standard deviations of the estimated values are lower, but the sample means of $\widehat{\theta}$, $\widehat{\kappa^*}$ and $\widehat{\sigma^*}$ are also closer to their true values, or at least at the same level. The same can be observed for the means of $\sqrt{\widehat{\theta}}$, $\widehat{\kappa}$ and $\widehat{\sigma}$. [10] When looking at the medians, all results for $T = 1000$ strictly dominate those for $T = 250$. On the other

[9] The figures only show the distributions for $l = 1$, since the results for $l = 9$ and $l = auto$ are almost identical.

[10] A better mean of the estimated values for the target variance $\widehat{\theta}$ does not necessarily imply a better mean for the target volatility $\sqrt{\widehat{\theta}}$, too. This is because $\sqrt{\widehat{\theta}}$ is not just the square root of $\widehat{\theta}$, but is calculated as $\frac{1}{T}\sum_{i=1}^{N}\sqrt{\widehat{\theta}_i}$, which is not equal to $\sqrt{\frac{1}{T}\sum_{i=1}^{N}\widehat{\theta}_i}$.

hand $T = 250$, also leads to better results compared to $T = 63$, except for the medians for $\widehat{\theta}$ and $\sqrt{\widehat{\theta}}$, respectively, which are closer to the true values for $T = 63$.

The precision of the estimated parameter values differs significantly. For $\widehat{\theta}$ the standard deviations of the estimates are considerably larger than for $\widehat{\sigma^*}$, not only in absolute terms, but also relative to their means. However, compared with $\widehat{\kappa^*}$, the precision of these two parameter estimates is

		$\widehat{\theta}$ $(.0225)^d$	$\widehat{\kappa^*}$ $(.0160)^d$	$\widehat{\sigma^*}$ $(.0126)^d$	$\sqrt{\widehat{\theta}}$ $(.1500)^d$	$\widehat{\kappa}$ $(4.0000)^d$	$\widehat{\sigma}$ $(.2000)^d$
				$\mathbf{l = 1}^a$			
$T = 63$	mean	.0232	.1033	.0120	.1485	25.8137	.1894
	std.dev.	.0120	.0690	.0012	.0331	17.2455	.0187
	median	.0216	.0876	.0120	.1470	21.8957	.1896
$T = 250$	mean	.0230	.0366	.0125	.1495	9.1513	.1976
	std.dev.	.0092	.0208	.0006	.0253	5.1947	.0093
	median	.0215	.0328	.0125	.1467	8.1944	.1977
$T = 1000$	mean	.0225	.0210	.0126	.1494	5.2537	.1994
	std.dev.	.0038	.0075	.0003	.0123	1.8850	.0046
	median	.0222	.0201	.0126	.1489	5.0237	.1994
				$\mathbf{l = 9}^a$			
$T = 63$	mean	.0231	.1034	.0121	.1480	25.8452	.1909
	std.dev.	.0148	.0677	.0012	.0346	16.9265	.0182
	median	.0215	.0887	.0121	.1466	22.1714	.1906
$T = 250$	mean	.0228	.0380	.0125	.1488	9.4936	.1977
	std.dev.	.0086	.0213	.0006	.0250	5.3242	.0092
	median	.0213	.0339	.0125	.1458	8.4775	.1977
$T = 1000$	mean	.0224	.0214	.0126	.1490	5.3604	.1994
	std.dev.	.0037	.0077	.0003	.0123	1.9181	.0046
	median	.0221	.0206	.0126	.1486	5.1416	.1994
				$\mathbf{l = auto}^a$			
$T = 63$	mean	.0232	.1036	.0120	.1485	25.9074	.1895
$l = 1.43$ ^{a,b}	std.dev.	.0122	.0692	.0012	.0333	17.2951	.0187
$(.78)^c$	median	.0216	.0879	.0120	.1469	21.9640	.1896
$T = 250$	mean	.0230	.0367	.0125	.1495	9.1813	.1976
$l = 1.48$ ^{a,b}	std.dev.	.0086	.0208	.0006	.0250	5.2070	.0093
$(.81)^c$	median	.0215	.0327	.0125	.1466	8.1853	.1977
$T = 1000$	mean	.0225	.0210	.0126	.1494	5.2619	.1994
$l = 1.53$ ^{a,b}	std.dev.	.0038	.0075	.0003	.0123	1.8861	.0046
$(.83)^c$	median	.0222	.0201	.0126	.1489	5.0373	.1994

Table 3.2: Parameters estimated from time series generated by Monte Carlo simulations.

[a] l: lag

[b] mean of the lag values when using automatic bandwith

[c] standard deviation of the distribution of the lag values when using automatic bandwith

[d] true values

acceptable. This confirms common knowledge, that it is in general very difficult to estimate the mean reversion parameter κ with sufficient precision. Fortunately, this finding is counterbalanced by the fact that the sensitivity of valuation models often is low with respect to variations in the mean reversion parameter κ.

A closer look at the GMM estimators from our simulated data in Table 3.2 shows that the number of autocovariances used to calculate $W_T(\varphi)$ does not affect the results in a significant way. If we compare the results for the cases $l = 1$ and $l = 9$, we see that they are almost identical. Assuming our model is a correct description of reality, we can conclude from the similarity of the results that the value for v_{t-1} seems to be the most relevant one, when dealing with daily autocorrelations. The one-day-autocorrelation is decisive for parameter estimation. To check for this, we also use a Bartlett kernel with automatic bandwidth, i.e. the number of autocorrelations considered was chosen dependent on the data of the time series. The lags for $l = auto$ are within a range from 0.0081 to 4.5844 for $T = 63$, 0.0082 to 4.8017 for $T = 250$, and from 0.0099 to 4.6405 for $T = 1000$. The associated standard deviations of the distributions of the lag values are 0.78, 0.81 and 0.83, respectively. That is, the distributions of the lag values are very similar. The same can be said about the quality of the estimates, when we compare them to those for $l = 1$. The results documented in panels $l = 1$ and $l = auto$ are almost identical. Therefore, using simulated data, we can conclude that the number of data points per time series is not an

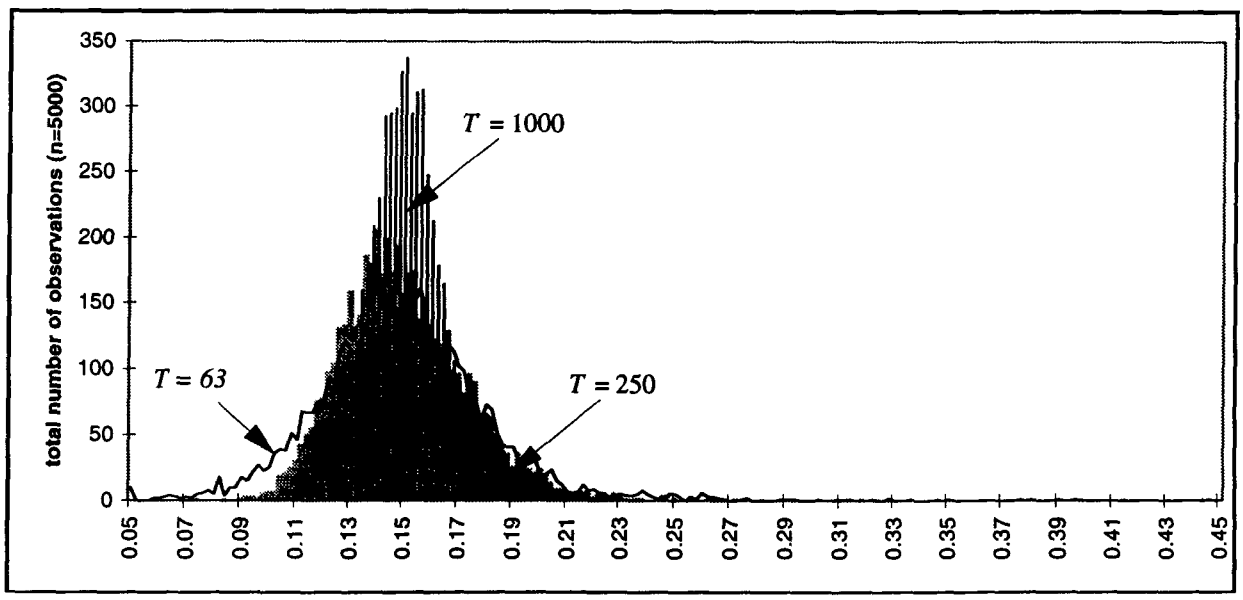

Figure 3.1a: Distributions of $\sqrt{\hat{\theta}}$ for $T = 63$, $T = 250$, and $T = 1000$ with $l = 1$. The numbers on the ordinate correspond to the total number of observations out of samples of $n = 5000$.

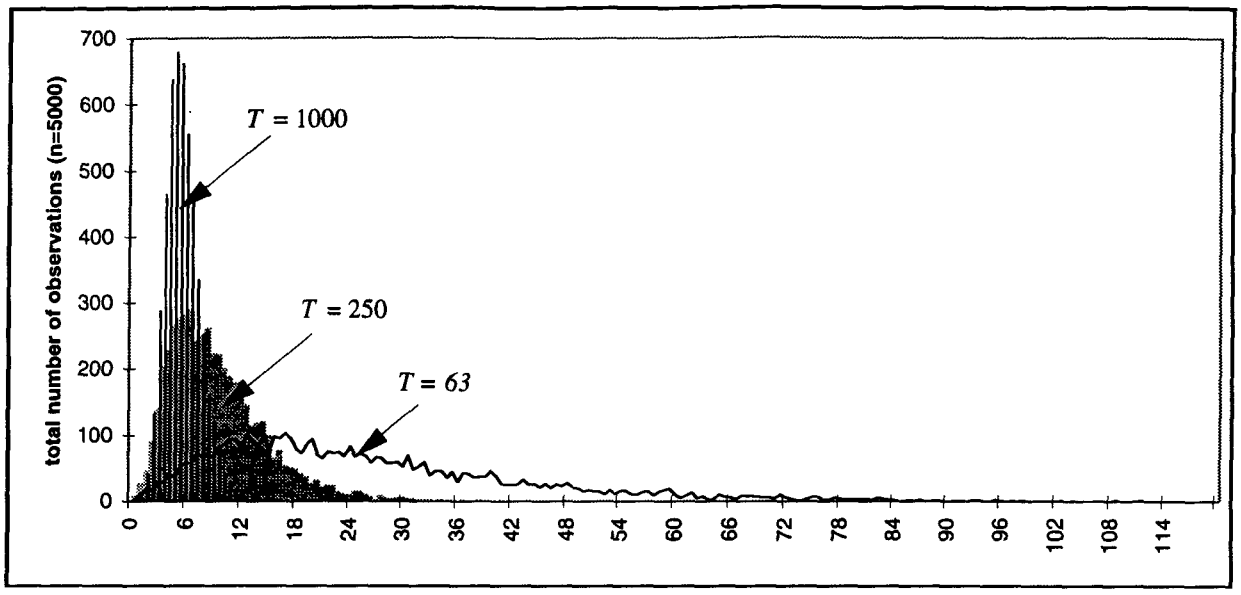

Figure 3.1b: Distributions of $\widehat{\kappa}$ for $T = 63$, $T = 250$, and $T = 1000$ with $l = 1$. The numbers on the ordinate correspond to the total number of observations out of samples of $n = 5000$.

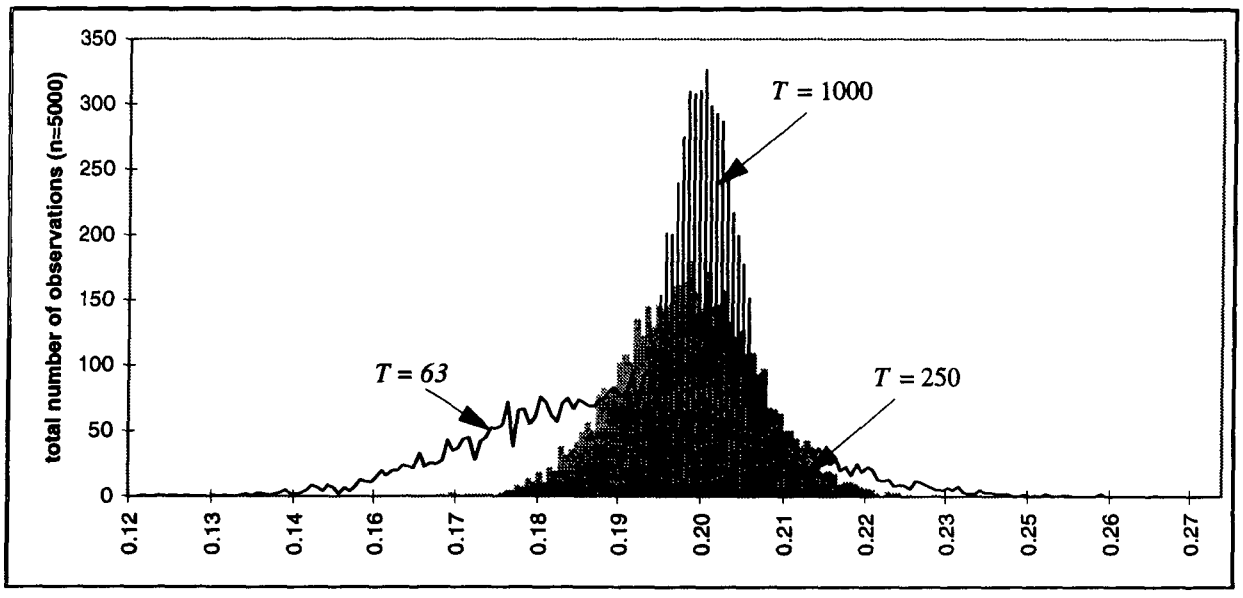

Figure 3.1c: Distributions of $\widehat{\sigma}$ for $T = 63$, $T = 250$, and $T = 1000$ with $l = 1$. The numbers on the ordinate correspond to the total number of observations out of samples of $n = 5000$.

important factor when choosing the number of autocorrelations. On the other hand, including more than one autocorrelation does not considerably improve the quality of the GMM estimates of our simulated data. However, using real world data, this pattern may be different.

3.3. t statistics

Every single estimation produces not only an estimate for each parameter, but also one for the corresponding standard deviation of the estimation. Hence, it is a common proceeding to calculate t statistics for GMM estimates. Usually one favors estimates which are significantly different from zero at a certain level, e.g. the 5%-level. Those who fulfill this criterion are regarded as good estimates, otherwise they rise doubts in regard to the specification of the underlying model. Therefore, we examine the t statistics of all GMM estimates to check for the accuracy of this procedure in the context of our Monte Carlo simulation. Table 3.3 shows the results.

Since the standard deviations of our estimates decrease with longer time series, one must expect decreasing proportions of insignificance across the different time series. This is exactly what we observe. Longer time series lead in general to fewer rejections of the null hypothesis.

The ranges of the t statistics of $\widehat{\theta}$ are smaller for longer time series, which is mainly due to a decrease in the maximum values of the statistics. As mentioned above, one is usually interested in estimates which have t statistics significantly different from zero. In our case the estimates are significant at a 5%-level, if they have a t statistic of at least 1.646 for $T = 1000$, 1.651 for $T = 250$, and 1.670 for $T = 63$ [11]. This is given for the large majority of our estimates. But looking at Figure 3.2a [12], we see that we have estimations of up to 0.52 for $\sqrt{\widehat{\theta}}$. These extreme estimations are clearly far from the true value, but all are not significant. That is, by applying the significance criterion we can exclude all estimations that are extremely far away from the true value. On the other hand, there seems to be a tendency for larger t statistics to correspond to better estimations. This phenomenon can be seen best for $T = 63$ where we also have the largest t statistics. Since the estimates of $\sqrt{\theta}$ are very good for these cases, the only explanation can be found in small standard deviations, which obviously are much smaller than for comparable estimation results for longer time series.

In the case of $\widehat{\kappa^*}$, the mean values of the t statistics given in Table 3.3 show, that depending on the length of the time series, up to 42.1% of the estimates are not significantly different from zero according to their individual t statistic. As Figure 3.2b shows, the estimations with smaller t statistics are often better than those with larger t statistics compared with the true value. This is

[11] We apply one-sided t tests, since our estimates cannot become negative.

[12] The t statistics are calculated using the actually estimated parameter values, i.e. in our case $\widehat{\theta}$, $\widehat{\kappa^*}$ and $\widehat{\sigma^*}$. But to keep all figures comparable, we show the relation between the t statistics and the estimates $\sqrt{\widehat{\theta}}$, $\widehat{\kappa}$ and $\widehat{\sigma}$.

		$\widehat{\theta}$	$\widehat{\kappa^*}$	$\widehat{\sigma^*}$
l = 1[a]				
$T = 63$	t (mean)	9.7	1.9	12.0
	t (median)	7.9	1.8	11.9
	t (min)	.0	.1	6.3
	t (max)	54.0	7.6	20.0
	not significant (%)[b]	7.8	42.1	.0
$T = 250$	t (mean)	6.7	2.0	22.6
	t (median)	5.9	2.0	22.6
	t (min)	.1	.0	15.3
	t (max)	36.5	5.3	30.5
	not significant (%)[b]	3.5	32.6	.0
$T = 1000$	t (mean)	7.8	2.9	44.1
	t (median)	7.5	2.9	44.1
	t (min)	.4	.3	35.2
	t (max)	20.3	5.4	51.2
	not significant (%)[b]	.1	3.3	.0
l = 9[a]				
$T = 63$	t (mean)	10.8	2.6	14.5
	t (median)	8.4	2.4	14.0
	t (min)	.0	.0	5.8
	t (max)	84.9	9.9	35.5
	not significant (%)[b]	6.8	25.7	.0
$T = 250$	t (mean)	6.9	2.3	23.7
	t (median)	6.1	2.2	23.5
	t (min)	.0	.0	14.4
	t (max)	40.1	6.4	38.7
	not significant (%)[b]	3.3	23.8	.0
$T = 1000$	t (mean)	7.9	3.1	44.7
	t (median)	7.6	3.1	44.6
	t (min)	.5	.3	32.6
	t (max)	22.5	6.7	55.8
	not significant (%)[b]	.1	3.0	.0
l = auto[a]				
$T = 63$	t (mean)	9.7	2.0	12.1
	t (median)	7.9	1.9	12.0
	t (min)	.0	.1	6.5
	t (max)	54.0	7.6	21.9
	not significant (%)[b]	7.5	41.2	.0
$T = 250$	t (mean)	6.7	2.0	22.6
	t (median)	5.9	2.0	22.6
	t (min)	.1	.1	15.3
	t (max)	36.8	5.4	30.7
	not significant (%)[b]	3.4	32.0	.0
$T = 1000$	t (mean)	7.8	2.9	44.1
	t (median)	7.5	2.9	44.2
	t (min)	.3	.2	36.0
	t (max)	20.0	5.9	51.7
	not significant (%)[b]	.1	3.2	.0

Table 3.3: t statistics of the estimations from time series generated by Monte Carlo simulations.
[a] l: lag
[b] at a 5%-level for a one-sided t test

especially obvious for small and medium sized time series, i.e. $T = 63$ and $T = 250$. But even for $T = 1000$ we have similar behavior in the t statistics, i.e. they increase with an increase in $\widehat{\kappa}$. In contrast to $\sqrt{\widehat{\theta}}$, the ranges of the t statistics do not change much across different time series. For GMM estimations from empirical data this implies that a small t statistic for $\widehat{\kappa}$ is not necessarily a problem of the model specification.

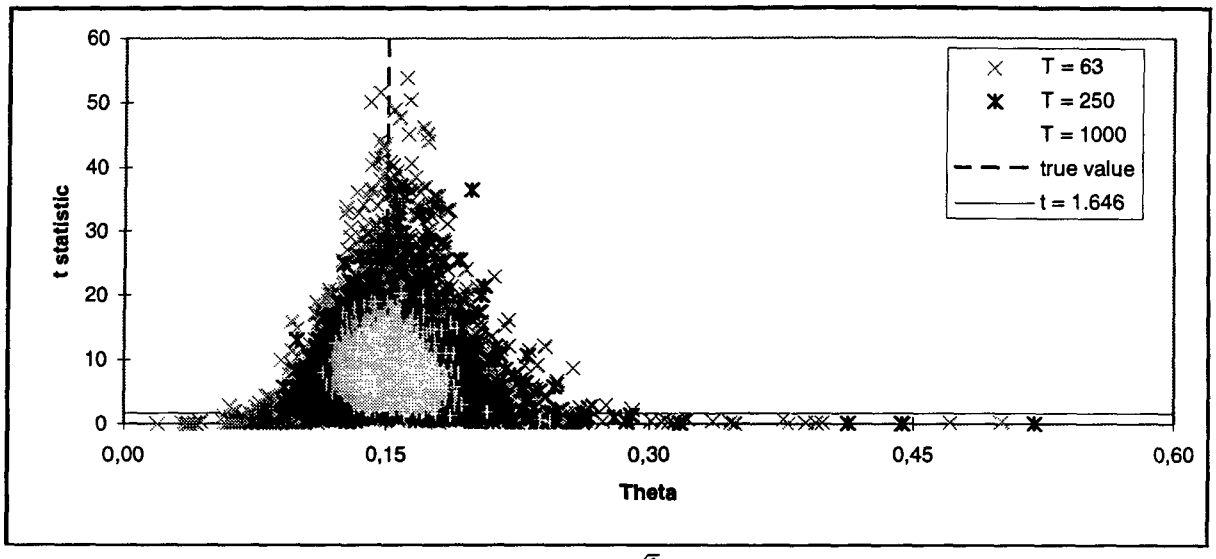

Figure 3.2a: Relationships of the estimated parameters $\sqrt{\widehat{\theta}}$ to the corresponding t statistics for all time series and $l = 1$.

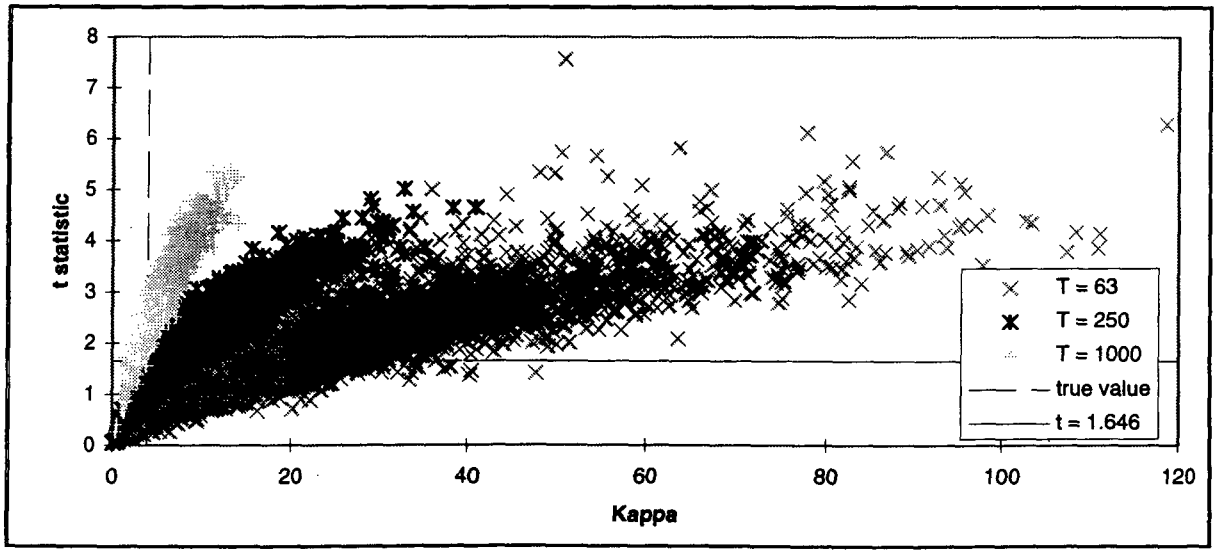

Figure 3.2b: Relationships of the estimated parameters $\widehat{\kappa}$ to the corresponding t statistics for all time series and $l = 1$.

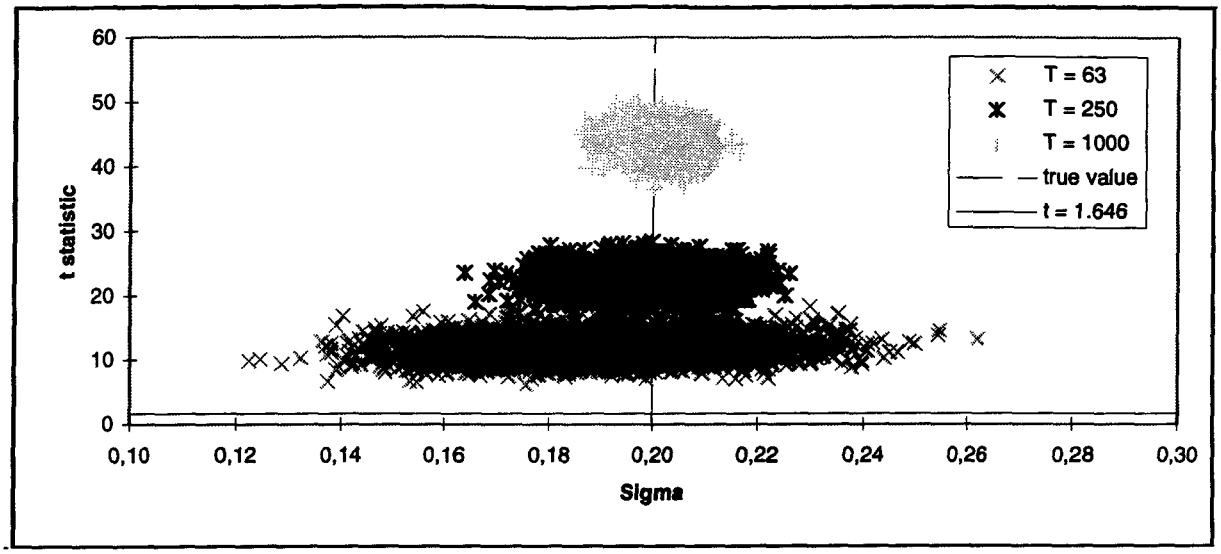

Figure 3.2c: Relationships of the estimated parameters $\hat{\sigma}$ to the corresponding t statistics for all time series and $l = 1$.

This finding is in contrast with Chan/Karolyi/Longstaff/Sanders (1992). They conclude from the insignificance of their parameter β, which corresponds to our κ, that allowing their state variable to be mean reverting can lead to unnecessarily complex models. As we show above, this is a conclusion that cannot be drawn based on the insignificance of the mean reversion parameter, especially since the time series observed in their paper has some 300 data points, i.e. it is of comparable size as our time series for $T = 250$. A further problem arises from the length of their observation period which covers more than 25 years. As we will show later, this leads to problems in regard to the stability of the parameters which is reflected in lower t statistics.

The t statistics for σ^* in Table 3.3 indicate that all GMM estimates are significantly different from zero. Furthermore, the differences between the extreme values of the t statistics are very similar, whereas the means increase for longer time series. These results are depicted in Figure 3.2c, where we can also see that the standard deviations of the estimates must be approximately proportional to the estimated values of σ^*. Otherwise we should have more variation within the values of the t statistics within each group. σ^* is obviously the only parameter where the quality of the GMM estimations is reflected in the size of the t statistics, i.e. better estimations on the average lead to larger t statistics. It seems that the size of the t statistics for σ^* can indeed be used as a measure of the accuracy of the estimations.

4. Volatility estimates using empirical time series

4.1. The data

The data for the German stock index (DAX) and its volatility index (VDAX) used in our study are supplied by the Deutsche Börse AG. The data for the DAX contain 9068 observations for the period October 1, 1959 until December 29, 1995. The VDAX data were available for the period January 2, 1992 until December 29, 1995, consisting of 1006 observations. The volume data used in the study are taken from the Handelsblatt, the leading German financial newspaper, for the same period as the VDAX. If not otherwise stated, time is measured in trading days rather than calendar days. We assume an average of 250 trading days per year. That is, our basic observation interval is $\Delta t = 0.004$ years or one trading day.

In principle, the instantaneous volatility is unobservable. Therefore, the raw data were used to generate the following time series as possible measures of instantaneous volatility:

1. Series *VDAX*:

   We use the original time series provided by the Deutsche Börse AG.

2. Series *std10, std22* and *std33*:

   These are historical standard deviations calculated over moving intervals for the last 10, 22 or 33 trading days for the day-to-day log-returns of the DAX. This enables us to examine the behavior of the DAX for periods of two weeks, one month and 45 calendar days. The latter is the hypothetical time to expiration used in the evaluation of the VDAX.

3. Series *NoT*:

   We use the number of daily transactions (NoT) in German stocks. This is the only volatility measure in our study which is not an average of subsequent daily volatilities. [13]

Estimations based on these time series are for observation periods that are of comparable size as those used for the Monte Carlo simulations. Hence, we choose time series that consist of $T = 68$, $T = 251$ and $T = 1006$ datapoints. All time periods end on December 29, 1995. An additional

[13] According to Jones/Kaul/Lipson (1994), it is the occurence of transactions per se rather than the size of the transactions that generates volatility.

observation period is used for *std10, std22* and *std33*, which covers the period from November 20, 1959 until December 29, 1995 (9034 observations). [14]

4.2. Empirical results

In this section we want to compare the GMM results from simulated data with those from empirical data. The latter are reported in Tables 4.1 through 4.4. The Monte Carlo simulations show that the precision of the estimated parameter values differs significantly. The parameter with the highest precision is σ^* followed by θ and κ^*. This is reflected in the values of the corresponding t statistics. The t statistics for $\widehat{\sigma^*}$ are much larger than those for $\widehat{\theta}$, and both are considerably larger, on the average, than the t statistics for $\widehat{\kappa^*}$. The empirical time series *std10*, *std22* and *std33* show the same behavior. For each time series the t statistics for $\widehat{\sigma^*}$, $\widehat{\theta}$ and $\widehat{\kappa^*}$ display the same kind of ordering. The results for *VDAX* are ambiguous. For $T = 68$ and $T = 251$ the t statistics for $\widehat{\theta}$ are larger than those for $\widehat{\sigma^*}$ whereas for $T = 1006$ the opposite applies. Concerning the series *NoT*, the t statistics for $\widehat{\sigma^*}$ are smaller than those for $\widehat{\theta}$. But obviously they are much larger than those for the other time series.

Again, we examine the impact of the length of the empirical time series on the estimated values. In the case of the Monte Carlo simulations, the estimated parameter values are closer to the true values, and the standard deviations are smaller for longer time series. These results do not always hold for the empirical data. This can be seen best for the estimates of $\widehat{\sigma^*}$. As we know from simulated data, $\widehat{\sigma^*}$ can be estimated very accurately, whereby the t statistics can be used as a relative measure of accuracy. Since longer simulated time series lead to better estimations of $\widehat{\sigma^*}$, we get larger t statistics for longer time series. The results for the empirical time series $T = 68$, $T = 251$ and $T = 1006$ seem to confirm this finding. Here we have a distinct increase in the t statistics of $\widehat{\sigma^*}$. Hence, the accuracy of the estimates obviously increases. Therefore one should expect a further increase in the t statistics for time series containing even more data points. But for $T = 9034$ we have t statistics that are significantly smaller than those for $T = 1006$. Moreover, the parameter estimates also differ significantly from those for $T = 1006$. One explanation for this observation could be the instability of the parameters over time. Since the 9034 observations cover a period of more than 36 years, it would be unlikely to have constant parameter values during

[14] The data from October 1, 1959 until November 19, 1959 are used to estimate the first value of the time series *std33*.

the entire time period. This assertion is supported by the smaller t statistics for $T = 9034$. This implies that it is much more difficult for our GMM procedure to find specific parameter values. This fact can be explained by unstable parameter values. Hence, when estimating the parameters for empirical time series and faced with instabilities in the data, shorter time series are more reliable. Having this in mind we see that NoT is a very interesting time series in this regard. Its t statistics are the highest, which could be a sign of good stability in the parameters. At first glance the t statistics seem to be too high compared to the other results. But we have to consider that NoT is not a moving average, as are the series *std10*, *std22* and *std33*. As can be seen in Tables 4.1 through 4.4, a decrease in the number of trading days used for generating a single data point leads to an increase in $\widehat{\kappa^*}$ and a decrease in $\widehat{\sigma^*}$. [15] To get parameter estimates that can be compared in magnitude with the other series, NoT was transformed, such that the sample mean of NoT would equal the standard deviation of the daily log-returns of the DAX for the same period. Therefore, it is not a big surprise that the estimates $\widehat{\theta}$ for the series *std10*, *std22*, *std33* and NoT are similar in magnitude.

What is the impact of the number of autocovariances used in the calculation of $W_T(\varphi)$? The results of the Monte Carlo simulations lead us to the conclusion that the size of l is rather unimportant for these estimations. When looking at the ranges of the estimates obtained for different values of l, we see that the results do not differ much. Compared to the differences in the parameter estimates when using different sizes for the time series, the length of the time series has a much bigger impact on the results than the choice of l. Therefore we can conclude from this that the size of l is not very important for the estimation of the parameters.

The results stated thus far in this section seem to favor "medium-sized" time series compared to long time series. This conclusion is supported by the p-values of the Hansen specification test, which tests for the correctness of the underlying model. That is, a large test statistic leads to a small value of p, which is a measure for the positive probability that the underlying model is correctly specified for a given time series. For the historical volatility data *std10*, *std22* and *std33*, the values of p are significantly larger for time series of up to 4 years than for $T = 9034$. Hence, the shorter time series are much better described by our model than the long ones.

[15] This does not hold for *std33* and *std22* in the case of $T = 68$. But here we also have the only case where the estimate of θ is not significant. Hence, this time series obviously is too short when it comes down to the estimation of the parameters.

	$\widehat{\theta}$	$\widehat{\kappa^*}$	$\widehat{\sigma^*}$	$\sqrt{\widehat{\theta}}$	$\widehat{\kappa}$	$\widehat{\sigma}$	p [c]
			l = 1 [a]				
VDAX ($T = 68$) [b]	.0215*** (.0022) [9.7]	.0838** (.0380) [2.2]	.0110*** (.0018) [6.1]	.1465	20.9584	.1737	.870
std33 ($T = 68$) [b]	.0207** (.0097) [2.1]	.0260 (.0365) [.7]	.0128*** (.0021) [6.1]	.1438	6.5017	.2022	.388
std22 ($T = 68$) [b]	.0175 (.0137) [1.3]	.0201 (.0269) [.7]	.0156*** (.0027) [5.8]	.1324	5.0326	.2471	.353
std10 ($T = 68$) [b]	.0196*** (.0050) [3.9]	.0994* (.0629) [1.6]	.0416*** (.0061) [6.8]	.1399	24.8455	.6585	.535
NoT ($T = 68$) [b]	.0265*** (.0011) [23.0]	.6191*** (.0821) [7.5]	.0333*** (.0032) [10.6]	.1627	154.7815	.5266	.016
			l = 9 [a]				
VDAX ($T = 68$) [b]	.0213*** (.0022) [9.6]	.0843*** (.0261) [3.2]	.0110*** (.0019) [5.7]	.1460	21.0670	.1747	.870
std33 ($T = 68$) [b]	.0196 (.0176) [1.1]	.0211 (.0379) [.6]	.0123*** (.0014) [8.8]	.1399	5.2765	.1943	.347
std22 ($T = 68$) [b]	.0125 (.0239) [.5]	.0158 (.0162) [1.0]	.0157*** (.0027) [5.8]	.1119	3.9478	.2486	.349
std10 ($T = 68$) [b]	.0180*** (.0051) [3.6]	.1005*** (.0403) [2.5]	.0418*** (.0020) [20.8]	.1342	25.1297	.6612	.544
NoT ($T = 68$) [b]	.0271*** (.0011) [24.5]	.5890*** (.0922) [6.4]	.0297*** (.0030) [9.8]	.1646	147.2429	.4688	.058
			l = auto [a]				
VDAX ($T = 68$) [b] ($l = 2.00$) [a]	.0214*** (.0021) [10.3]	.0848*** (.0332) [2.6]	.0110*** (.0018) [6.1]	.1463	21.1998	.1734	.873
std33 ($T = 68$) [b] ($l = 1.00$) [a]	.0207** (.0097) [2.1]	.0260 (.0365) [.7]	.0128*** (.0021) [6.1]	.1438	6.5017	.2022	.388
std22 ($T = 68$) [b] ($l = .52$) [a]	.0175 (.0137) [1.3]	.0201 (.0269) [.7]	.0156*** (.0027) [5.8]	.1324	5.0326	.2471	.353
std10 ($T = 68$) [b] ($l = 1.08$) [a]	.0196*** (.0050) [3.9]	.0994* (.0629) [1.6]	.0416*** (.0061) [6.8]	.1399	24.8455	.6585	.535
NoT ($T = 68$) [b] ($l = 2.05$) [a]	.0268*** (.0012) [23.2]	.6191*** (.0879) [7.0]	.0323*** (.0030) [10.8]	.1637	154.7769	.5104	.018

Table 4.1: Parameters estimated from empirical time series.
The numbers in parenthesis give the estimated standard deviations of the estimated values, and the numbers in brackets are the t statistics.
*/**/***: significant at a 10%/5%/1%-level for a one-sided t test
[a] l: lag
[b] T: # of data points
[c] p-value of Hansen's χ^2 test statistic

	$\widehat{\theta}$	$\widehat{\kappa^*}$	$\widehat{\sigma^*}$	$\sqrt{\widehat{\theta}}$	$\widehat{\kappa}$	$\widehat{\sigma}$	p [c]
			l = 1 [a]				
VDAX ($T = 251$) [b]	.0206*** (.0012) [17.1]	.0698*** (.0223) [3.1]	.0093*** (.0008) [11.4]	.1434	17.4543	.1471	.020
std33 ($T = 251$) [b]	.0161*** (.0042) [3.9]	.0172 (.0201) [.9]	.0100*** (.0012) [8.1]	.1270	4.3057	.1577	.678
std22 ($T = 251$) [b]	.0157*** (.0033) [4.7]	.0285* (.0204) [1.4]	.0141*** (.0016) [8.9]	.1252	7.1223	.2236	.915
std10 ($T = 251$) [b]	.0170*** (.0034) [4.9]	.0705** (.0402) [1.8]	.0333*** (.0037) [9.0]	.1306	17.6331	.5268	.451
NoT ($T = 251$) [b]	.0228*** (.0011) [20.8]	.3679*** (.0586) [6.3]	.0435*** (.0025) [17.3]	.1511	91.9843	.6880	.542
			l = 9 [a]				
VDAX ($T = 251$) [b]	.0201*** (.0013) [15.6]	.0739*** (.0194) [3.8]	.0089*** (.0010) [9.2]	.1419	18.4777	.1413	.143
std33 ($T = 251$) [b]	.0166*** (.0046) [3.6]	.0200 (.0233) [.9]	.0102*** (.0010) [9.3]	.1287	5.0024	.1605	.604
std22 ($T = 251$) [b]	.0156*** (.0035) [4.4]	.0285* (.0174) [1.6]	.0141*** (.0016) [9.0]	.1248	7.1183	.2236	.914
std10 ($T = 251$) [b]	.0176*** (.0036) [4.9]	.0754** (.0323) [2.3]	.0342*** (.0030) [11.6]	.1326	18.8417	.5404	.385
NoT ($T = 251$) [b]	.0228*** (.0012) [18.5]	.3665*** (.0517) [7.1]	.0433*** (.0025) [17.2]	.1509	91.6358	.6844	.623
			l = auto [a]				
VDAX ($T = 251$) [b] ($l = 2.42$) [a]	.0205*** (.0012) [17.1]	.0721*** (.0197) [3.7]	.0092*** (.0009) [10.8]	.1431	18.0126	.1459	.041
std33 ($T = 251$) [b] ($l = 1.20$) [a]	.0161*** (.0042) [3.9]	.0172 (.0201) [.9]	.0100*** (.0012) [8.1]	.1270	4.3057	.1577	.678
std22 ($T = 251$) [b] ($l = .31$) [a]	.0157*** (.0033) [4.7]	.0286* (.0209) [1.4]	.0141*** (.0016) [8.6]	.1252	7.1418	.2237	.917
std10 ($T = 251$) [b] ($l = 1.51$) [a]	.0173*** (.0035) [5.0]	.0700** (.0395) [1.8]	.0335*** (.0035) [9.6]	.1314	17.4998	.5303	.429
NoT ($T = 251$) [b] ($l = 2.43$) [a]	.0228*** (.0011) [20.9]	.3670*** (.0535) [6.9]	.0435*** (.0025) [17.4]	.1511	91.7618	.6879	.566

Table 4.2: Parameters estimated from empirical time series.
The numbers in parenthesis give the estimated standard deviations of the estimated values, and the numbers in brackets are the t statistics.
*/**/***: significant at a 10%/5%/1%-level for a one-sided t test
[a] l: lag
[b] T: # of data points
[c] p-value of Hansen's χ^2 test statistic

	$\widehat{\theta}$	$\widehat{\kappa^*}$	$\widehat{\sigma^*}$	$\sqrt{\widehat{\theta}}$	$\widehat{\kappa}$	$\widehat{\sigma}$	p [c]
			l = 1[a]				
VDAX ($T = 1006$)[b]	.0278*** (.0051) [5.5]	.0155* (.0102) [1.5]	.0120*** (.0007) [17.1]	.1666	3.8721	.1892	.007
std33 ($T = 1006$)[b]	.0221*** (.0057) [3.9]	.0097* (.0065) [1.5]	.0112*** (.0008) [14.0]	.1486	2.4366	.1772	.263
std22 ($T = 1006$)[b]	.0213*** (.0044) [4.8]	.0175** (.0087) [2.0]	.0165*** (.0012) [13.8]	.1460	4.3848	.2605	.343
std10 ($T = 1006$)[b]	.0207*** (.0025) [8.3]	.0589*** (.0145) [4.1]	.0357*** (.0023) [15.5]	.1438	14.7321	.5640	.188
NoT ($T = 1006$)[b]	.0213*** (.0007) [30.4]	.2677*** (.0272) [9.8]	.0424*** (.0018) [23.6]	.1458	66.9245	.6704	.244
			l = 9[a]				
VDAX ($T = 1006$)[b]	.0255*** (.0029) [8.8]	.0209*** (.0080) [2.6]	.0118*** (.0008) [14.8]	.1597	5.2169	.1859	.035
std33 ($T = 1006$)[b]	.0228*** (.0064) [3.6]	.0102* (.0070) [1.5]	.0114*** (.0007) [16.3]	.1509	2.5584	.1798	.198
std22 ($T = 1006$)[b]	.0204*** (.0038) [5.4]	.0210*** (.0084) [2.5]	.0165*** (.0011) [15.0]	.1428	5.2399	.2605	.358
std10 ($T = 1006$)[b]	.0198*** (.0022) [9.0]	.0654*** (.0119) [5.5]	.0352*** (.0027) [13.0]	.1406	16.3570	.5567	.216
NoT ($T = 1006$)[b]	.0212*** (.0007) [30.3]	.2694*** (.0236) [11.4]	.0421*** (.0020) [21.1]	.1455	66.3527	.6651	.282
			l = auto[a]				
VDAX ($T = 1006$)[b] ($l = 3.49$)[a]	.0266*** (.0037) [7.1]	.0186** (.0094) [2.0]	.0119*** (.0007) [17.0]	.1632	4.6389	.1881	.013
std33 ($T = 1006$)[b] ($l = 2.13$)[a]	.0221*** (.0057) [3.9]	.0099* (.0066) [1.5]	.0112*** (.0008) [14.0]	.1485	2.4867	.1774	.257
std22 ($T = 1006$)[b] ($l = .34$)[a]	.0217*** (.0047) [4.6]	.0163* (.0088) [1.9]	.0165*** (.0012) [13.8]	.1473	4.0792	.2605	.325
std10 ($T = 1006$)[b] ($l = 3.60$)[a]	.0203*** (.0023) [8.8]	.0620*** (.0137) [4.5]	.0355*** (.0024) [14.8]	.1424	15.5076	.5617	.202
NoT ($T = 1006$)[b] ($l = 5.44$)[a]	.0212*** (.0007) [30.3]	.2692*** (.0235) [11.4]	.0422*** (.0019) [22.2]	.1455	67.3049	.6665	.265

Table 4.3: Parameters estimated from empirical time series.
The numbers in parenthesis give the estimated standard deviations of the estimated values, and the numbers in brackets are the t statistics.
*/**/***: significant at a 10%/5%/1%-level for a one-sided t test
[a] l: lag
[b] T: # of data points
[c] p-value of Hansen's χ^2 test statistic

	$\widehat{\theta}$	$\widehat{\kappa^*}$	$\widehat{\sigma^*}$	$\sqrt{\widehat{\theta}}$	$\widehat{\kappa}$	$\widehat{\sigma}$	p [c]
			l = 1[a]				
std33 ($T = 9034$)[b]	.0325** (.0161) [2.0]	.0049 (.0061) [.8]	.0230*** (.0030) [7.7]	.1803	1.2219	.3630	.044
std22 ($T = 9034$)[b]	.0308** (.0125) [2.5]	.0087 (.0093) [.9]	.0341*** (.0044) [7.8]	.1755	2.1713	.5392	.042
std10 ($T = 9034$)[b]	.0301*** (.0109) [2.8]	.0193 (.0154) [1.3]	.0744*** (.0097) [7.7]	.1735	4.8223	1.1767	.067
			l = 9[a]				
std33 ($T = 9034$)[b]	.0269*** (.0078) [3.4]	.0097 (.0098) [1.0]	.0228*** (.0031) [7.4]	.1641	2.4248	.3611	.067
std22 ($T = 9034$)[b]	.0265*** (.0070) [3.8]	.0148 (.0126) [1.2]	.0340*** (.0046) [7.4]	.1627	3.7067	.5376	.073
std10 ($T = 9034$)[b]	.0274*** (.0086) [3.2]	.0252* (.0165) [1.5]	.0732*** (.0101) [7.2]	.1656	6.2981	1.1572	.090
			l = auto[a]				
std33 ($T = 9034$)[b] ($l = 14.41$)[a]	.0257*** (.0060) [4.3]	.0122 (.0108) [1.1]	.0228*** (.0031) [7.4]	.1603	3.0586	.3608	.080
std22 ($T = 9034$)[b] ($l = 13.75$)[a]	.0256*** (.0060) [4.3]	.0170* (.0131) [1.3]	.0340*** (.0046) [7.4]	.1600	4.2500	.5373	.089
std10 ($T = 9034$)[b] ($l = 11.06$)[a]	.0262*** (.0070) [3.7]	.0271** (.0157) [1.7]	.0707*** (.0105) [6.7]	.1617	6.7821	1.1183	.091

Table 4.4: Parameters estimated from empirical time series.
The numbers in parenthesis give the estimated standard deviations of the estimated values, and the numbers in brackets are the t statistics.
*/**/***: significant at a 10%/5%/1%-level for a one-sided t test
[a] l: lag
[b] T: # of data points
[c] p-value of Hansen's χ^2 test statistic

In the case of the VDAX, shorter time series lead to larger significance. This is in contrast with the time series NoT. Here, we get the largest p-values for $T = 251$.

5. Conclusions

In this paper we examine whether it is possible to estimate the three parameters of the square-root process for the instantaneous variance rate of Heston's model using the GMM approach. We use Monte Carlo simulations and empirical data to address this question. The use of simulated time

series shows that the estimations of the parameters produce reliable results, although their quality depends heavily on the length of the time series. In general, the estimates of all three parameters, but especially those for $\widehat{\kappa^*}$, are better for longer time series. This result is not confirmed by long empirical time series. Compared with $T = 1006$, not only the estimated standard deviations are larger for $T = 9034$, but also the values of the estimates themselves differ across the samples. An indication for the stability of the parameters could be the corresponding t statistics, since the estimations should be easier for stable parameters, leading to smaller variances of the GMM estimators. Hence, a further examination of the time series NoT with its very high t statistics could be interesting with regard to the accuracy of the results.

It can be shown that good estimates of $\widehat{\kappa^*}$ have small t statistics whereas large t statistics indicate estimated values that are far away from their true value. For $\widehat{\sigma^*}$ and $\widehat{\theta}$ the opposite applies.

We do not find significant differences in the estimates, when changing the number of autocorrelations used in the estimation of the positive-definite symmetric weighting matrix $W_T(\varphi)$. Hence, in our opinion, it is of secondary importance to find an "optimal" lag-truncation parameter for this model.

Almost all empirical time series show p-values which support the hypothesis that the given square-root process can be used as a parsimonious specification. The only exceptions are the series NoT for $T = 68$ and VDAX for $T = 251$ and $T = 1006$. One reason for the specification problem with regard to the VDAX could be its calculation using implicit volatilities of the Black-Scholes formula. The values of the estimated parameters are all reasonable for our empirical data. That includes the results for the daily number of transactions in German stocks. In contrast to all other time series used in this study, the NoT series is the only one that is calculated using neither a window of historical data nor implicit volatilities. Hence, it is a good candidate when looking for a variable that includes volatility information of just one trading day. In contrast to the VDAX, it thereby covers the entire trading activities of a trading day, whereas the VDAX merely includes the information of one single moment of the trading day [16].

Based on these findings, our further research will be directed toward the reliable estimation of the complete parameter set of the model (1.1) and (1.2) using historical price and volume data.

[16] The VDAX is calculated using prices that are quoted at 1.30 p.m.

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