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## Assessing stop-loss and constant proportion portfolio insurance: The impact of transaction costs

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**Wirtschaftswissenschaftliche Fakultät  
der Eberhard-Karls-Universität Tübingen**

Assessing Stop-Loss and Constant Proportion  
Portfolio Insurance:  
The impact of transaction costs

Thomas Braun



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# **Assessing Stop-Loss and Constant Proportion Portfolio Insurance: The impact of transaction costs**

## **ABSTRACT**

*This paper deals with dynamic asset allocation strategies which guarantee that an investor's terminal wealth will not fall short of a predetermined amount of money. It focuses on two strategies which do not require any information other than the current market price of the involved risky asset. The key objective is to support decisions on the basis of theoretical probability distributions.*

## **1. Introduction**

This paper compares stop-loss (SL) and constant proportion (CP) portfolio strategies. The principal motivation for this choice is that both strategies do require no more than monitoring the performance of the underlying risky asset in order to be conducted.

Viewed from a theoretician's perspective CP strategies come with a clear advantage: Whereas SL strategies are neither utility maximizing nor efficient in dynamically complete markets<sup>1</sup> CP strategies may be utility maximizing when intertemporal consumption is introduced<sup>2</sup>. There is, however, considerable evidence in the literature that theoretical disadvantages of SL strategies tend to turn into comparative advantages<sup>3</sup> as more real-world restrictions are imposed on the more complex CP strategies.

Certainly transaction costs do belong to these influences that should be given cautious attention. This paper aims at a readily accessible explanation of how transaction costs do affect the relative performance of SL and CP strategies without resort to the highly sophisticated math that is usually employed to address this issue.

In the accomplishment of this aim we will proceed as follows. The next paragraph will determine the payoffs generated by SL and CP strategies and assess the respective disturbances caused by transaction costs. The payoffs generated by SL will be related to those originating from a specific barrier option. Based on an approach devised by BLACK/PEROLD (1992) the mechanics of CP will be stripped down to two effects steered by one and the same decision parameter, the so-called multiplier. One effect is the volatility cost effect which is preeminent in flat but oscillating markets because in this case CP implies buying at the high and selling at the low. The other effect unfolds in the amplification of positive and negative trends. As far as SL is concerned basic economic reasoning will be applied to show that transaction costs are not of

crucial importance. And as far as CP is concerned BLACK/PEROLD showed that the fundamental impact of transaction costs emerges as an increase in the volatility cost.

The third paragraph will indicate the conditions that warrant the existence of a CP strategy that is equivalent to a specific SL strategy in that it guarantees the same minimum terminal wealth and the same expected return.

The fourth paragraph will compare the probability distributions of terminal wealth generated by SL and its equivalent CP strategy with and without transaction costs. The subsequent conjecture about the impact of transaction costs rests on the assumption of a positive risk premium which means that the chance of positive trends is higher than that of negative trends. In this case the fundamental insight that the loss in expected returns caused by transaction costs can only be compensated for by reinforcing the amplifier effect leads to the conclusion that adapting to transaction costs comes down with increased risk as far as CP is concerned. This conjecture will be confirmed by an example which will be presented graphically in compliance with our objective to provide an analysis geared to practioners' needs for decision support. The paper finishes with a short conclusion.

## 2. Preliminaries

Applying SL means that at the start of the investment period the investor invests all his initial wealth  $W_0$  in some payout-protected risky asset. Whenever the market price of this investment falls to the then present value of the minimum terminal wealth  $F$  (floor) at the end of the investment period the investor will shift all his remaining wealth from risky asset to riskless assets. Thus, he can rely on ending up with a terminal wealth no less than  $F$  if sell-orders transact at the current market price.

Transaction costs are of minor significance for SL because the maximum number of in-between transactions is one and the influence of thereby incurred transaction costs on the investor's decision depends on his further plans for the time after the investment period. Therefore, they are ignored in connection with SL.

The formal exposition of a SL strategy simplifies considerably if interest rate effects are eliminated. This will be achieved by measuring prices and wealth in quantities of a numéraire zero-bond (referred to briefly as zerobond in what follows) with face value  $Z$ , maturity  $T$  and market price

$$(1) \quad Z_t = Z \exp(-\rho(T-t))$$

instead of currency units. The use of this measure will be indicated by italics. Thus, the number of zerobonds an investor could buy at time  $t \in [0, T]$  if he had invested all his wealth  $W_0$  over the period  $[0, t]$  in some payout-protected index

$$(2a) \quad I_t := I_0 \exp(i_t)$$

following geometric Brownian motion, where ( $g$  is a standard Gaussian random variable)

$$(2b) \quad i_t := \mu t + \sigma \sqrt{t} g,$$

will be denoted as

$$(3a) \quad W_t := \frac{W_t}{Z_t} = \frac{W_0 \exp(i_t)}{Z \exp(-\rho(T-t))} = \frac{W_0}{Z} \exp(i_t - \rho t) = W_0 \exp(i_t - \rho t)$$

As long as the investor is able to buy the quantity

$$(4) \quad F := \frac{F}{Z}$$

of zerobonds he can rule out that his terminal wealth at time  $T$  will fall below the amount  $F$  as this is the total redemption value of  $F$  zerobonds with face value  $Z$ . A SL strategy stipulates: Invest all your money in the index and reallocate it into zerobonds as soon as your wealth is just enough to buy  $F$  zerobonds, i.e. as soon as  $W_t$  falls to  $F$  for the first time. This suggests to denote the terminal wealth generated by a buy-and-hold strategy as

$$(5a) \quad W_t = F \exp(i_t),$$

where

$$(5b) \quad i_t := i_0 + i_t - \rho t = i_0 + (\mu - \rho) t + \sigma \sqrt{t} g$$

and

$$(5c) \quad i_0 := \ln\left(\frac{W_0}{F}\right),$$

and the terminal wealth originating from a stop-loss strategy as

$$(6a) \quad W_T^{SL} = F \exp(\hat{i}_T), \quad \hat{i}_T = \begin{cases} i_T & \text{if } i_t > 0 \quad \forall t \in [0, T] \\ 0 & \text{otherwise} \end{cases}$$

Equation (6a) might as well be denoted as

$$(6b) \quad W_T^{SL} = F + \begin{cases} W_T - F & \text{if } W_t > F \quad \forall t \in [0, T] \\ 0 & \text{otherwise} \end{cases}$$

$$= F + DOC(W_0, F, F, T) ,$$

where  $DOC(.....)$  represents a European down-and-out call on  $W_t$  with knock-out limit  $F$ , exercise price  $F$  and maturity  $T$ .

Eliminating interest rate effects also considerably simplifies the exposition of a CP strategy. CP is driven by the so-called cushion which is the difference between actual and indispensable wealth, i.e.

$$(7) \quad C_t = W_t^{CP} - F .$$

The exposure  $E_t$  is the investment in the risky asset. If there is no borrowing limit it is simply a constant multiple  $m$  ( $m \geq 1$ ) of the cushion, i.e.

$$(8) \quad E_t := m C_t, \quad m \geq 1$$

As  $F$  is constant we have

$$(9a) \quad dC = dW^{CP} = \frac{dW}{W} E = \frac{dW}{W} m C$$

implying

$$(9b) \quad \frac{dC}{C} = m \frac{dW}{W} .$$

Employing Itô's Lemma yields the following solution to (9b)

$$(10a) \quad C_t = a C_0 \left( \frac{W_t}{W_0} \right)^m ,$$

where

$$(10b) \quad a := a(m) = \exp\left(-\frac{1}{2} m(m-1) \sigma^2 t\right) < 1 \text{ for } m > 1 .$$

The latter equations readily allow the performance of a CP strategy to be compared ex-post to that of investing the monetary equivalent of  $C_0$  over the whole period  $[0, T]$  in the index<sup>†</sup>. Substituting  $m = 1$  into (10a) in connection with  $a(1) = 1$  reveals that there is ex-post no differ-

ence between a CP and a buy-and-hold strategy for  $m = 1$ . That  $m = 1$  actually is a buy-and-hold strategy may be confirmed by substituting  $m = 1$  which implies  $C = E$  into (9b) to observe that  $dE/E = dW/W$  must hold. This immediately reveals that adapting the exposure to fluctuations in wealth does not require any change in the asset allocation in this case.

Substituting  $W_T = W_0$  which means that the index performed precisely like the zerobond into (10a) yields  $C_T = a C_0 < C_0$  for  $m > 1$ . In that case the fraction  $1-a$  of the cushion must have been lost due to the random character of  $W_t$  which must have exhibited reversals. Reversals cost money because they induce the investor to allocate more money to zerobonds after the index performed relatively worse only to see the index perform relatively better afterwards and to allocate more money to the index after it performed relatively better only to see it perform relatively worse afterwards<sup>5</sup>. Thus, the fraction  $1-a$  will be referred to as volatility cost in what follows.

Contrary to the role they play in connection with SL strategies transaction costs are of crucial importance for CP strategies and therefore should be incorporated in the analysis. At this end, we will use the following approximate solution<sup>6</sup>

$$(11a) \quad C_t = a' C_0 \left( \frac{W_t}{W_0} \right)^{m'},$$

where

$$(11b) \quad a' := \kappa'^{\frac{n}{2}}$$

$$(11c) \quad m' := \frac{1}{2} \frac{\ln((1+mu')/(1-md'))}{\ln(1+u)}$$

$$(11e) \quad \kappa' := (1+mu')/(1-md')$$

$$(11f) \quad u' := (1+k)/(1+mk) u < u$$

$$(11g) \quad d' := (1-k)/(1-mk) d > d$$

$$(11h) \quad n := (\sigma^2/u^2) T$$

The basic conclusion from (11) is that the mechanics of CP remains the same as the permanent trading strategy that supports (10) is accommodated to sufficiently small proportional transaction costs  $k$  by the stipulation that the portfolio should not be rebalanced until the price ratio  $W_t$  has moved up (down) the fraction  $u$  ( $d$ ) of its value at the time of the last transaction. Thorough



inspection of the parameter definitions (11b-h) show that transaction costs preeminently increase the volatility cost<sup>7</sup>.

### 3. Determination of equivalent stop-loss and constant proportion strategies

This paragraph will deal with the conditions that warrant the existence of a CP strategy that is equivalent to a specific SL strategy in that it guarantees the same floor and the same expected return. Apparently the problem boils down to the determination of an equivalent multiplier

$$(12) \quad \hat{m} := \left\{ m \mid E(W_T^{SL}) = E(W_T^{CP}) \right\}.$$

The existence and uniqueness of  $\hat{m}$  is solely dependent on the impact of  $m$  on  $E(W_T^{CP})$ . This impact may be decomposed into a trend expansion effect and a volatility cost effect. In the case of zero transaction costs and continuous rebalancing we have

$$E(W_T^{CP}) = F + E(C_T) = F + a C_0 \exp\left(m(\mu - \rho)T + \frac{1}{2}m^2\sigma^2T\right)$$

Note that  $m > 1$  and that the risk premium for any buy-and-hold strategy is

$$\exp\left((\mu + \frac{1}{2}\sigma^2)T\right) - \exp(\rho T) = \exp(\rho T) \left(\frac{E(W_T)}{W_0} - 1\right)$$

and confirm that a positive risk premium for a buy-and-hold strategy is a sufficient (but not necessary) condition for the trend expansion effect

$$(13) \quad \frac{\partial E(W_T^{CP})}{\partial m} = ((\mu - \rho)T + m\sigma^2T) E(C_T)$$

to be positive. The volatility cost effect is always negative as can be seen from

$$(14) \quad \frac{\partial E(W_T^{CP})}{\partial a} \frac{da}{dm} = \left(\frac{1}{2} - m\right) \sigma^2 T E(C_T) < 0$$

Equations (13) and (14) combine to

$$(15) \quad \frac{dE(W_T^{CP})}{dm} = \frac{\partial E(W_T^{CP})}{\partial m} + \frac{\partial E(W_T^{CP})}{\partial a} \frac{da}{dm} = \ln\left(\frac{E(W_T)}{W_0}\right) E(C_T),$$

which is positive (negative) if and only if the risk premium for a buy-and-hold strategy is positive (negative). This implies that for positive risk premia there is a unique equivalent multiplier  $\hat{m} > 1$  as long as the expected return of the SL strategy is larger than that of the CP strategy

with  $m = 1$ . From (6b), (7) and (10a,b) it can be directly inferred that the latter condition is equivalent to

$$(16) \quad E(DOC(W_0, F, F, T)) > (W_0 - F) \frac{E(W_T)}{W_0}.$$

Condition (16) will always hold for sufficiently high positive risk premia and sufficiently low volatilities. In this case the knock-out feature loses its significance as the drift-driven risk premium increases so that the difference between a European down-and-out call  $DOC(W_0, F, F, T)$  and its ordinary analogue  $C(W_0, F, T)$  fades away as the risk premium approaches infinity<sup>8</sup>. Thus,

$$\lim_{\mu \rightarrow \infty} E(DOC(W_0, F, F, T)) = \lim_{\mu \rightarrow \infty} E(C(W_0, F, T)) = E(W_T) - F > (W_0 - F) \frac{E(W_T)}{W_0}$$

implying that  $\hat{m} > 1$  does exist.

From the above line of reasoning it is obvious that if compensation for the increase in volatility cost caused by transaction costs is possible at all it can only be achieved by a stronger trend expansion given that the probabilities for  $W_t$  are biased to positive trends. Thus, if  $\hat{m}$  exists in the case of positive transaction costs and discrete rebalancing it must be higher than in the case of zero transaction costs. Intuition suggests that compensation for increasing transaction costs will be possible as long as the risk premium is sufficiently high, although there is no guarantee for the offsetting effect to be strong enough once the impact of transaction costs and of discrete rebalancing on the trend expansion effect itself is explicitly accounted for.

#### 4. Comparison of the return distributions generated by equivalent stop-loss and constant proportion strategies

This paragraph will compare the return distributions of equivalent SL and CP strategies. Permanent rebalancing with zero transaction costs will serve as fundamental basis of the argument and it will be shown that a successful (in the sense of still being able to reach the same expected return as the SL strategy with the same floor) accommodation to transaction costs inevitably goes with increased risk. This conclusion will subsequently be confirmed by an example which will be presented graphically.

The analysis is based on the probabilities  $\text{Prob}(W_T > w)$  for passing certain levels of terminal wealth  $w > F$ , because their graphs allow risk and expected return of different strategies to be recognized and compared at the same time. This is due to the fact that the areas below the

respective curves resemble the mathematical expectation of the terminal wealth that is in excess of the floor, what is readily confirmed using integration by parts: Let  $X \in [\underline{x}, \bar{x}]$  be some random variable with density  $f(x)$  and respective cumulative  $F(x)$ , then

$$(17) \quad E(X) := \int_{\underline{x}}^{\bar{x}} x f(x) dx = \bar{x} - \int_{\underline{x}}^{\bar{x}} F(x) dx = \int_{\underline{x}}^{\bar{x}} 1 - F(x) dx .$$

As has been done before monetary quantities will be restated in quantities of the zerobond, i.e.  $\text{Prob}(W_T > w) = \text{Prob}(W_T > w)$ , where  $w := w/Z > F$ .

For SL strategies the probabilities of passing certain wealth levels may be derived from the observation that  $\hat{I}_T$  (cf. (6a)) represents Brownian motion absorbed at the origin with density<sup>9</sup>

$$(18) \quad p(\hat{I}_T) = \hat{p}(i_T) = \frac{1}{\sigma\sqrt{T}} \left( n(\phi(i_T - i_0)) - \exp\left(-\frac{2(\mu-\rho)}{\sigma^2} i_0\right) n(\phi(i_T + i_0)) \right)$$

for  $\hat{I}_T > 0$ , where  $n(\cdot)$  and  $N(\cdot)$  are the density and cumulative density of the standard normal distribution and

$$\phi(x) := \frac{x - (\mu - \rho)T}{\sigma\sqrt{T}} .$$

Thus, we have

$$(19) \quad \text{Prob}(W_T^{SL} > w) = \text{Prob}\left(\hat{I}_T > \ln\left(\frac{w}{F}\right)\right) = 1 - N(a) - \exp(-\lambda i_0) (1 - N(b)) ,$$

where

$$a := \phi\left(\ln\left(\frac{w}{F}\right) - i_0\right) = \phi\left(\ln\left(\frac{w}{W_0}\right)\right)$$

$$b := \phi\left(\ln\left(\frac{w}{F}\right) + i_0\right) = \phi\left(\ln\left(\frac{wW_0}{F^2}\right)\right)$$

and

$$\lambda := \frac{2(\mu-\rho)}{\sigma^2} .$$

As far as CP strategies are concerned note that  $\text{Prob}(W_T^{CP} > w) = \text{Prob}(C_T > c)$ , where  $c := w - F > 0$ . Thus, it follows immediately from (10a) that

$$(20) \quad \text{Prob}(C_T > c) = \text{Prob}\left(\left(\frac{W_T}{W_0}\right)^m > \frac{1}{a} \frac{c}{C_0}\right) = 1 - N(c) ,$$

where

$$c := \Phi\left(\frac{1}{m} \ln\left(\frac{1}{a} \frac{c}{C_0}\right)\right).$$

If “volatility cost” effect and “trend expansion” effect are no misnomers intuition tells us that an increase in the multiplier that is dedicated to keep expected returns constant must have a negative (positive) effect on the probabilities  $\text{Prob}(W_T^{CP} > w)$  for low (high) levels of  $w$ , because the trend expansion effect should be negative for low realisations (= negative ex-post trends) and positive for high realisations (= positive ex-post trends) and the volatility cost effect should be negative everywhere. The following derivatives do confirm this reasoning (recall that  $m > 1$ )

$$(21) \quad \frac{\partial \text{Prob}(C_T > c)}{\partial a} \frac{da}{dm} = \frac{n(c)}{ma\sigma\sqrt{T}} \left(\frac{1}{2} - m\right) a \sigma^2 T < 0$$

$$(22) \quad \frac{\partial \text{Prob}(C_T > c)}{\partial m} = \frac{n(c)}{m^2\sigma\sqrt{T}} \ln\left(\frac{1}{a} \frac{c}{C_0}\right) < (>) 0$$

$$\Leftrightarrow \frac{c}{C_0} < (>) a$$

Combining (21) and (22) leads to

$$(23) \quad \frac{d \text{Prob}(C_T > c)}{dm} = \frac{n(c)}{m^2\sigma\sqrt{T}} \left(\ln\left(\frac{c}{C_0}\right) - \frac{1}{2} m^2 \sigma^2 T\right) < (>) 0$$

$$\Leftrightarrow \ln\left(\frac{c}{C_0}\right) < (>) \frac{1}{2} m^2 \sigma^2 T$$

and proves that an increase in the equivalent multiplier which is dedicated to compensate for the deterioration of expected terminal wealth caused by transaction costs will necessarily increase risk<sup>10</sup>. It should be noted that the range over which the investor faces an adverse effect becomes larger as  $m$  and  $\sigma$  increase and that an increase in  $m$  lowers  $\text{Prob}(C_T > c)$  for any  $c \in [0, C_0]$  which means that it lowers  $\text{Prob}(W_T > w)$  for any level  $w \in [F, W_0 \exp(\rho T)]$ . It follows immediately from

$$\int_0^{C_0} (C_0 - c) f(c) dc = \int_0^{C_0} F(c) dc = C_0 - \int_0^{C_0} \text{Prob}(C_T > c) dc$$

that the expected value of foregone interest will always rise as the multiplier is increased.

In the remaining part of this paper the above reasoning will be verified by an example. This example compares the distributions generated by a SL strategy and its equivalent CP strategy

calculated under the assumptions of either zero transaction costs and permanent rebalancing or positive transaction costs and discrete rebalancing. The analysis is conducted for a low and a high volatility scenario. Figures 1 and 2 are based on the following set of parameters (all percentages except for  $k$  and  $u$  on an annual basis): initial wealth: 100; floor: 100; interest rate: 6%; risk premium: 3%; volatilities: 5% (low volatility scenario) and 10% (high volatility scenario) respectively; transaction costs:  $k = 0,5\%$ ; up-move tolerance  $u = 3\%$ .

Fig. 1: Probabilities for passing certain wealth levels with a stop-loss strategy and two constant proportion portfolio strategies which both generate the same expected return but differently account for transaction costs ( $\sigma = 5\%$ )

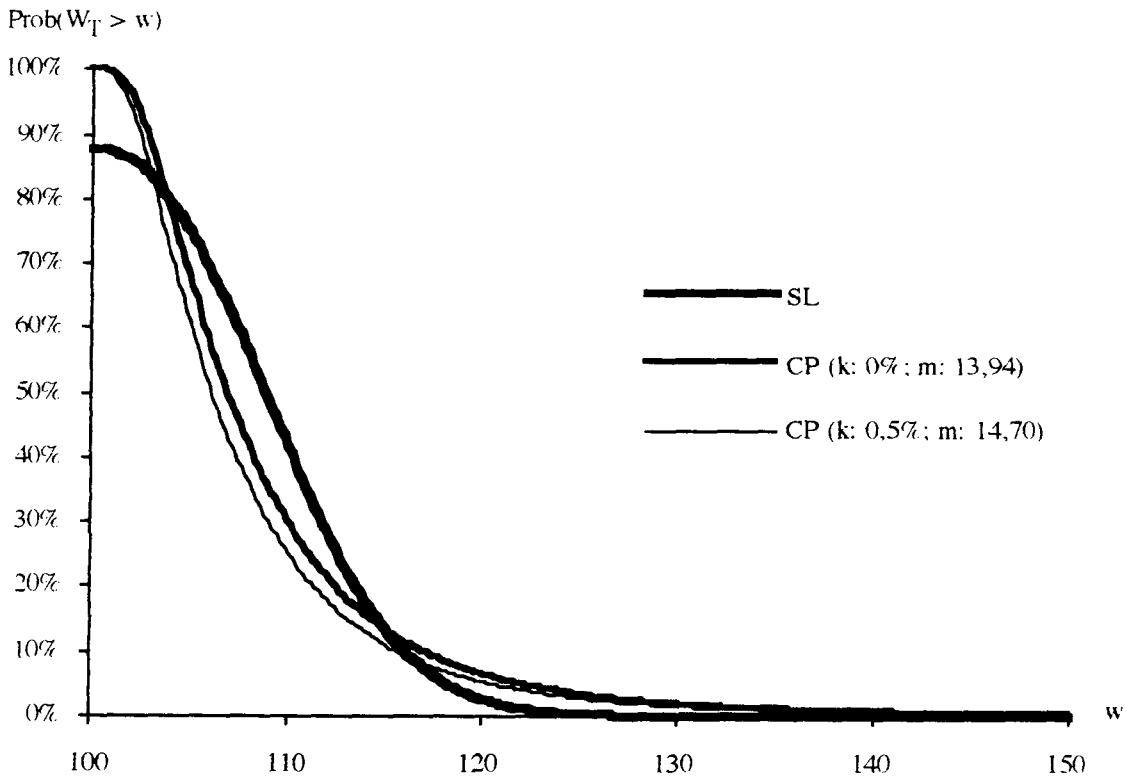
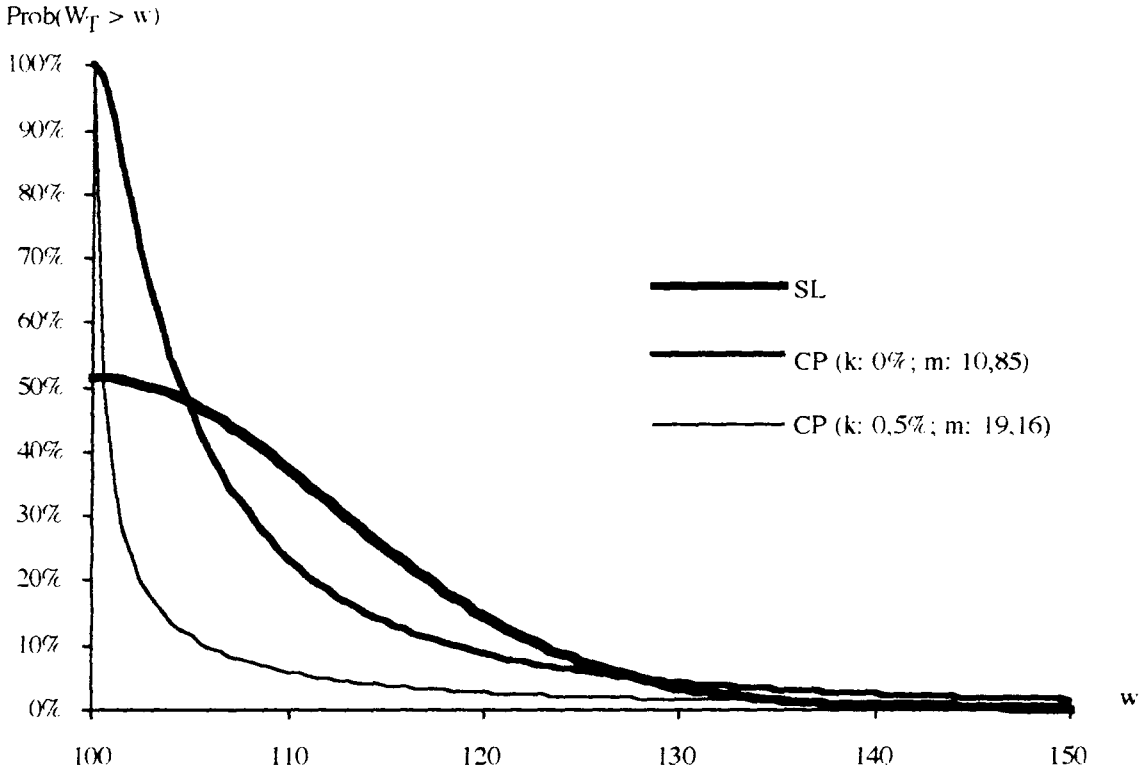


Fig. 2: Probabilities for passing certain wealth levels with a stop-loss strategy and two constant proportion portfolio strategies which both generate the same expected return but differently account for transaction costs ( $\sigma = 10\%$ )



In the absence of transaction costs and with permanent rebalancing the CP strategies display considerably higher probabilities of passing wealth levels slightly above the floor than the SL strategy regardless of the volatility. This is due to the fact that the probability for a complete erosion of the cushion is zero for CP as opposed to a non-neglectable probability of hitting the limit without any chance to get more than the floor for SL. Now figure 2 tells us that this range close to the floor and therefore of particular importance to risk averters may well shrink to neglectable size for moderate parameter choices if transaction costs are accounted for. For higher volatilities it was no longer possible to reach the expected wealth of the SL strategy with a CP strategy adapted to transaction costs in the way described above.

#### 4. Conclusion

The analysis presented in this paper adds more evidence to the findings of other researchers who (within completely different frameworks) came to the conclusion that the relatively simple SL strategies are no longer at a disadvantage in comparison to the theoretically more appealing CP strategies if assessed under assumptions that may be considered as reasonable approxima-

tions of real-world conditions. Concerning real-world disturbances this paper focused on transaction costs. Another real-world aspect that has not been addressed is the existence of borrowing limits. Taking these into consideration also leads to the conclusion that the presumable superiority of CP strategies may prove illusory. With borrowing limits both strategies may even become virtually identical. This is easily seen in the case of discrete trading and a sufficiently large multiplier when the borrowing limit prevents the exposure to be adjusted to up-moves thus putting the portfolio in a buy-and-hold state while down-moves destroy the cushion<sup>11</sup>.

<sup>1</sup> The intuition behind the proof that path dependence conflicts with utility maximizing behaviour dating back to COX/LELAND (1982) is that the additional uncertainty from path-dependence is uncompensated risk, see RUBINSTEIN (1985). The inefficiency of stop-loss strategies in dynamically complete markets has been shown by DYBVIK (1988).

<sup>2</sup> See BLACK/PEROLD (1992) and GROSSMAN/ZHOU (1993).

<sup>3</sup> See PELSSER/VORST (1993) and BENNINGA (1990).

<sup>4</sup> This is principally due to the fact that  $C_t$  is path independent. See BLACK/PEROLD (1992), appendix A.1, for an intuitively appealing explanation of the path independence result.

<sup>5</sup> Cf. PEROLD/SARPE (1988). They pointed out that performing poorly in flat but oscillating markets is a common feature of strategies which generate convex payoff curves.

<sup>6</sup> It is supported by an approach that has been devised by BLACK/PEROLD (1992). See appendix A of this paper and proposition 4 of BLACK/PEROLD (1992) for an in-depth treatment of the underlying discrete trading strategy.

<sup>7</sup> See BLACK/PEROLD (1992), 413.

<sup>8</sup> See the proof in appendix B of this paper.

<sup>9</sup> See COX/MILLER (1977), 220 - 221.

<sup>10</sup> See INGERSOLL (1987), 119 - 124, for the underlying definition of increasing risk that dates back to ROTHSCCHILD/STIGLITZ (1970).

<sup>11</sup> See BLACK/PEROLD (1992), 414 - 416. They address the issue in further depth showing that even with permanent rebalancing (and zero transaction costs) the expected payoff of a constant proportion strategy approaches that of a stop-loss strategy as the multiple becomes infinitely large.

## Appendix A

(11a) is based on a discrete trading strategy which requires the portfolio to be rebalanced not until up- and down-moves of the price ratio  $W_t$  reach certain percentage tolerances  $u$  and  $d$ . Assume that  $W_t$  starts to rise at time  $t = 0$ , reaches  $W_1 = (1+u) W_0$  at time  $t = 1$  then edges only little past this level and finally dives to  $W_2 = (1-d) W_1$  in  $t = 2$ . Up to  $t = 1$  the portfolio balance does not change, so  $C_1 - C_0 = uE_0 = u m C_0$ . If there are no transaction costs increasing the exposure leaves the cushion unchanged so at time  $t = 1$  the exposure can be increased as prescribed by the strategy rule to  $E_1 = m C_1$ . The thereby established portfolio balance holds until  $t = 2$ . Thus,  $C_2 - C_1 = -d E_1 = -d m C_1$  implying that for the cushion not to be destroyed the relation  $d < 1/m$  must hold. The two consecutive changes constitute  $C_2 = \kappa C_0$ , where

$$\kappa := (1+mu)(1-md).$$

Now, what is different with transaction costs? The main difference is that adopting the strategy rule does not leave the cushion unchanged. Net of transaction costs at  $t = 1$  the exposure can only be increased to  $E'_1$  instead of  $E_1$ , where  $(1+u) E_0 < E'_1 < E_1$ . If there are proportional transaction costs  $k$  for buying and selling the risky asset and no transaction costs for buying and selling zerobonds (the term “zerobond” is just a technical term for any reserve asset that yields the risk-free rate of return and does not need to be securitized) taking action according to the strategy rule incurs transaction costs  $k(E'_1 - (1+u) E_0) = k(m C'_1 - (1+u) m C_0)$ . Thus, the change in the cushion becomes  $C'_1 - C_0 = u m C_0 - k m (C'_1 - (1+u) C_0)$  leading to  $C'_1 = (1 + m u') C_0$ . Scaling down the exposure in  $t = 2$  incurs transaction costs  $k((1-d) E'_1 - E_2) = k((1-d) m C'_1 - m C_2)$  and analogously diminishes the cushion to  $C'_2 = (1 - m d') C'_1$ . Thus, accounting for transaction costs the two consecutive changes form

$$C'_2 = \kappa' C_0,$$

where  $\kappa' := (1+mu')(1-md') < \kappa$  as  $u' < u$  and  $d' > d$ .

Assuming that the tolerances are set to define a reversal, i.e.  $(1+u)(1-d) = 1$ , we have

$$\kappa' < \kappa := (1+mu)(1-md) = 1 - (m^2 - m) d u < 1$$

for  $m > 1$ .

After a total of  $n$  trades comprising  $i$  up-moves and  $j$  down-moves the cushion would be

$$C_n = (1+mu')^i (1-md')^j C_0$$



implying

$$\ln \left( \frac{C'_n}{C'_0} \right) = \frac{1}{2} n \ln(\kappa') + \left( \frac{1}{2} n - j \right) \ln \left( \frac{1+mu'}{1-md'} \right) .$$

Assuming that the  $n$ th trade is executed precisely at  $t = T$  and substituting

$$\ln \left( \frac{W'_T}{W'_0} \right) = \ln \left( \frac{W'_n}{W'_0} \right) = 2 \left( \frac{1}{2} n - j \right) \ln(1+u)$$

yields (11a).

(11h) is backed by the fact that the number of trades  $n$  which is a measure of ex-post volatility is deterministically linked to ex-ante volatility  $\sigma$  in the limit case  $u \rightarrow 0$  if  $W_t$  follows geometric Brownian motion (see BLACK/PEROLD (1992), appendix A.1, for a proof). It should be clear that (11h) is no more than an approximate relationship for small but strictly positive tolerances  $u$ . In the limit case of permanent rebalancing transaction costs would destroy the cushion but no more than the cushion.

## Appendix B

Integrating in a standard fashion yields

$$\begin{aligned} E(DOC(W_0, F, F, T)) &= \int_0^{\infty} (F \exp(i_T) - F) \hat{p}(i_T) di_T \\ &= E(W_T) N(d) - F N(d - \sigma\sqrt{T}) - \exp(-\lambda i_0) \left( \alpha E(W_T) N(d^*) - F N(d^* - \sigma\sqrt{T}) \right) \\ &= E(C(W_0, F, T)) - \exp(-\lambda i_0) E(C(\alpha W_0, F, T)) , \end{aligned}$$

where

$$\lambda := \frac{2(\mu - \rho)}{\sigma^2}$$

$$\alpha := \left( \frac{F}{W_0} \right)^2 = \exp(-2i_0)$$

$$d := \frac{(\mu - \rho)T + i_0}{\sigma\sqrt{T}} + \sigma\sqrt{T} = \frac{\ln\left(\frac{E(W_T)}{F}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

$$d^* := \frac{(\mu - \rho)T - i_0}{\sigma\sqrt{T}} + \sigma\sqrt{T} = \frac{\ln\left(\frac{\alpha E(W_T)}{F}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

And applying de l'Hospital's rule gives

$$\begin{aligned} & \lim_{\mu \rightarrow \infty} E(DOC(W_0, F, F, T)) \\ &= \lim_{\mu \rightarrow \infty} E(C(W_0, F, T)) - \frac{\sigma^2 T}{2i_0} F \lim_{\mu \rightarrow \infty} \exp\left(\left(\frac{2(\mu - \rho)}{\sigma^2} + 1\right) \left(\frac{1}{2}\sigma^2 T - i_0\right)\right) N(d^*) \end{aligned}$$

implying

$$\lim_{\mu \rightarrow \infty} E(DOC(W_0, F, F, T)) = \lim_{\mu \rightarrow \infty} E(C(W_0, F, T)) = E(W_T) - F$$

$$\text{for } \frac{1}{2}\sigma^2 T < i_0 := \ln\left(\frac{W_0}{F}\right).$$

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