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**Working Paper**

## The competition of user networks: Ergodicity, lock-ins, and metastability

Tübinger Diskussionsbeiträge, No. 127

**Provided in Cooperation with:**

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*Suggested Citation:* Woekener, Bernd W. (1998) : The competition of user networks: Ergodicity, lock-ins, and metastability, Tübinger Diskussionsbeiträge, No. 127, Eberhard Karls Universität Tübingen, Wirtschaftswissenschaftliche Fakultät, Tübingen

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**Wirtschaftswissenschaftliche Fakultät  
der Eberhard-Karls-Universität Tübingen**

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Diskussionsbeitrag Nr. 127  
März 1998

Wirtschaftswissenschaftliches Seminar  
Mohlstraße 36, D-72074 Tübingen

# The competition of user networks: ergodicity, lock-ins, and metastability

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## Abstract

This paper models the competition of user networks as a continuous-time Markov process. It presents a dynamic version of the Discrete Choice Analysis with state-dependent choice probabilities. Among other things, we show that the network competition can be characterized by the coexistence of lock-in regimes and a ‘metastable’ state – i.e. a state which is a probability maximum for a finite period of time. Then, unlike in the case of ergodicity or of simple lock-in scenarios, the networks can coexist for a considerable period of time, although the market is a natural monopoly.

*JEL classification:* D11, L10

*Keywords:* Ergodicity; Lock-in; Metastability; Network effects; Networks

# 1. Introduction

Often the surplus from the use of a good depends positively on the total number of users of this good. In this case, the totality of users of this good constitute a user network. Obvious examples are communication systems such as fax systems and e-mail systems. Here a growth in the total number of users directly induces positive network effects, because then each user can contact and be contacted by more other users. Furthermore, most information processing systems and a lot of consumer electronics systems are characterized by network effects as well. Clear examples are hardware-software systems such as computer systems or home video systems (with the recorder as 'hardware' and the prerecorded tapes as 'software'). Here the network effects are market-mediated: an increasing total number of users leads to a rising variety of software, and this in turn results in a higher surplus of each user. A further example is (world-wide-) web browsers, where the network effects are due to the fact that, with a more popular browser, a user has access to more web servers. In the following, we present an analysis of the competition of two incompatible user networks,  $A$  and  $B$ , which focuses on those cases where the choice of a network is not binding and where each user can review and revise her decision whenever she wants. We have in mind a user who, for example, has installed on her PC the newest versions of two (partly) incompatible web browsers and whose choice varies from case to case depending on the concrete circumstances. Or imagine a user who owns both a console for video game cartridges and a PC with a CD drive, and who has to choose whether she next buys a video game or a PC game.

In our model, the current network membership of a user is determined by her last (software) choice. Hence, a network's current share of the last choices of all users is its (standardized) network size and current market share. A user's decisions are modelled as a sequence of discrete choices which are, due to the network effects, state-dependent. We follow the Discrete Choice Approach in assuming that the basic willingness to pay, i.e. that part of the willingness to pay which does not stem from the network effects, is subject to exogenous shocks. These shocks are caused by new features of a network, for example a new video game or an innovative software update, and/or by new information about the characteristics of a network. Hence, the individual choices are guided by state-dependent choice probabilities. In order to take into account that the typical user does not make her decisions at discrete points in time but can revise her last choice whenever she wants, we treat time as a continuous variable, i.e. in modelling the evolution of the probability distribution of the network sizes, we follow a Master Equation Approach.

Our analysis will show that depending, among other things, on the strength of the network effects and on the significance of the exogenous shocks, four qualitatively different cases can be distinguished. In the first case, the probability distribution converges to a unique continuous stationary distribution with positive probabilities

of all states; i.e. the stochastic processes are 'ergodic'. In the second case, one of the two boundary states 'all users are in network *A*' and 'all users are in network *B*' is a unique probability mass 'absorbing' state. Hence, every realization of a stochastic process is sooner or later 'locked in' to the respective network. In the third case, both boundary states are absorbing states and are not accompanied by a 'metastable' state – i.e. a state which is a long-lived probability maximum. Here there is a strong tendency towards the realization of a monopoly, and it is a priori an open question as to which network will prevail. Finally, there is a fourth case where both lock-in regimes coexist with a metastable state. Unlike the other three cases, this can only happen if it is taken into account that network effects typically decrease with increasing network size.<sup>1</sup> This fourth case can explain why competing networks can coexist for a considerable period of time, although the market is a natural monopoly.

Pioneering work in the modelling of user (or consumer) behavior as a stochastic process with state dependence is Smallwood/Conlisk (1979) and Arthur/Ermoliev/Kaniowski (1987). There, however, discrete-time processes are analyzed.<sup>2</sup> Furthermore, the nonlinear Polya processes popularized by Arthur *et al.* are restricted to the modelling of irreversible choices. Our continuous-time approach with endogenous review rates follows the Master Equation Approach of Weidlich *et al.*; see for example Weidlich/Haag (1983) and Weidlich/Braun (1992).<sup>3</sup> However, whereas usually the transition rates of the stochastic processes are specified by behavioral assumptions, we present a version of the Master Equation Approach in which they are microfounded along the lines of the Discrete Choice Approach [see Anderson/de Palma/Thisse (1992), especially pp. 257ff, and de Palma/Lefevre (1983), for the Discrete Choice Approach]. In this version, the functional form of the transition rates can be derived from the distribution of the users' basic willingness to pay and from the specification of the 'network effect function', i.e. that part of the willingness to pay which is due to the existence of network effects. Against the background of the static Discrete Choice Analysis, the following approach can be seen as a dynamization in continuous time. The second aim of the paper is to introduce the concept of metastability into economic analysis [for this concept, see van Kampen (1992), pp. 326ff]. We see this concept as central for the understanding of the evolution of markets for network effect goods, because it is the only way to explain the often observed long-term coexistence of competing user networks against the background

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<sup>1</sup>That means, for example, that in most user networks, ten thousand additional users increase the individual surplus of a user in the case of a network size of fifty thousand users by a higher amount than they do in the case of a network size of a million users.

<sup>2</sup>See also Conlisk (1976) and the related work of Kirman (1993) and Orléan (1995).

<sup>3</sup>Recent economic applications of this approach are Woeckener (1993), Aoki (1996) and Witt (1997), pp. 763ff.

that almost every user network has to exceed a critical mass, i.e. that in most cases both lock-in regimes exist.

In the next section, first the choice probabilities of our model are derived and then aggregated to a master equation for the evolution of the probability distribution of the network sizes. In the third section, the model is discussed under the assumption that the network effects are approximately constant. There we discuss the cases of ergodicity and of lock-ins in the absence of metastable states. We will demonstrate how nonlinear stochastic continuous-time processes can be analyzed with the help of the corresponding Fokker-Planck equation and by means of the so-called ‘corresponding deterministic model’. Finally, in section 4, we allow for the fact that network effects are typically decreasing and discuss the concept of metastability.

## 2. The model

### 2.1 *The individual decisions: choice probabilities*

We assume that the competing user networks are horizontally differentiated à la Hotelling with network  $A$  at the left-end point and network  $B$  at the right-end point of the unit line; i.e. network  $A$  has the ‘address’  $i = 0$  and network  $B$  has the ‘address’  $i = 1$ . At each point in time, the users’ basic willingnesses to pay for being a member of network  $A$  and for being a member of network  $B$  are uniformly distributed along the unit line. Furthermore, it is supposed that the total number of users is very high and that they cannot coordinate their choices.<sup>4</sup> For simplicity, the total ‘mass’ of users is normalized to one, so that the total size of a network is identical with its market share. We presume that users do not differ in their valuation of the network effects and that each user’s surplus is increasing in network size. The network effect functions are specified as  $nx^c$  and  $n(1-x)^c$  with  $x$  and  $1-x$  as the size (market share) of network  $A$  and network  $B$ , respectively, and  $n$  as a measure of (the general level of) the network effect strength. As for the curvature of the network effect function, it is assumed that  $0 < c \leq 1$  holds, where the case of constant network effects ( $c = 1$ ) is considered as an approximation for only modestly decreasing network effects. With  $h$  as a measure of the extent of the horizontal differentiation, the current surplus of a user with address  $0 \leq i \leq 1$  from being a member of network  $A$  or  $B$  can be formulated as

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<sup>4</sup>For a model with two groups of users with conflicting preferences where the users of each group can perfectly coordinate their choices, see Woekener (1997), pp. 400ff.

$$s_{iA} = a_A + n x^c - h i \quad \text{and} \quad (1)$$

$$s_{iB} = a_B + n (1 - x)^c - h (1 - i), \quad (2)$$

respectively. Here,  $a_A$  and  $a_B$  are the differences between the address-independent parts of the basic willingnesses to pay and the current user costs. We assume that  $a_A \geq a_B$  holds, i.e. network  $A$  can have a ‘systematic basic advantage’  $b = a_A - a_B > 0$ . This  $i$ - and  $x$ -independent systematic basic advantage is due to a quality and/or a cost advantage [or a quality (cost) advantage which is not overcompensated by a cost (quality) disadvantage]. It is the systematic part of  $A$ ’s ‘basic advantage’  $b_i = b + h (1 - 2 i)$  (which can be negative).

In line with the Discrete Choice Analysis, we take into account that a user’s address  $i$  is subject to exogenous shocks. In our model, these shocks are information shocks and/or technology shocks affecting the product characteristics of network components. We assume that  $i$  is a random variable which is uniformly distributed along the unit line.<sup>5</sup> Together with the assumption of a very high total number of users, this results in the equivalent distribution of users at each point in time presumed above. Hence,  $h$  can be re-interpreted as a measure for the significance of the exogenous shocks. A user with current address  $i$  chooses  $A$  whenever  $a_A + n x^c - h i > a_B + n (1 - x)^c - h (1 - i)$  holds. This condition can be reformulated as  $-b - h (1 - 2 i) \leq n [x^c - (1 - x)^c]$ ; i.e. network  $A$  is chosen whenever  $B$ ’s basic advantage  $-b_i$  (which can be negative) is smaller than (or equal to)  $A$ ’s ‘network size advantage’ (which can be negative as well). As  $i$  is uniformly distributed between zero and one,  $B$ ’s basic advantage is uniformly distributed between  $-b - h$  and  $-b + h$ . Hence, its cumulative distribution is

$$K(-b_i) = \begin{cases} 0 & \text{if } -b_i \leq -b - h \\ 0.5 + \frac{b - b_i}{2h} & \text{otherwise} \\ 1 & \text{if } -b_i \geq -b + h. \end{cases} \quad (3)$$

By substituting  $A$ ’s network size advantage for  $B$ ’s basic advantage, we obtain the choice probabilities of network  $A$  as

$$\alpha(x) = \begin{cases} 0 & \text{if } x^c - (1 - x)^c \leq -(h + b)/n \\ 0.5 + \frac{b + n [x^c - (1 - x)^c]}{2h} & \text{otherwise} \\ 1 & \text{if } x^c - (1 - x)^c \geq (h - b)/n. \end{cases} \quad (4)$$

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<sup>5</sup>The decisive feature of the assumed uniform distribution is its truncatedness. This contrasts with most Discrete Choice Models, which presume a logistic or a normal distribution. In these models, shocks of any extent are admitted, so that lock-ins cannot occur when choices are reversible.



The choice probabilities of network  $B$  are  $\beta(x) = 1 - \alpha(x)$ . These choice probabilities are the probabilities that network  $A$  or  $B$  is chosen given that a user reviews her last choice. They are the ‘individual’ transition probabilities of the process. Due to the network effects, they are state-dependent. Hence, the individual choice process is a Markov process. For the special case of constant network effects ( $c = 1$ ), we obtain

$$\alpha(x) = \begin{cases} 0 & \text{if } x \leq 0.5 - (h + b)/(2n) \\ 0.5 + \frac{b - n}{2h} + \frac{n}{h} x & \text{otherwise} \\ 1 & \text{if } x \geq 0.5 + (h - b)/(2n). \end{cases}$$

## 2.2 The evolution of networks: master equations

In order to derive the equations of motion for the probability distribution of  $A$ 's network size  $x$ , let us suppose for a moment that time is a discrete variable and that in fixed time intervals  $\tau$  one of the users reviews her choice (for example, on the occasion of a software purchase). Let us further assume that the probability of doing so is of the same value for each user (at a given point in time), and let  $\epsilon$  be the inverse of the total number of users. Then if at a point in time  $t$  the state  $x$  is realized, the probability that at time  $t + \tau$  the state  $x + \epsilon$  is realized amounts to  $\mathcal{A}(x) = (1 - x)\alpha(x)$ : a member of network  $B$  reviews her choice (appears at the market) and chooses network  $A$ . Analogously, the ‘total’ one-period transition probability for transitions from state  $x$  to state  $x - \epsilon$  (after having been in  $x$  at time  $t$ ) can be derived as  $\mathcal{B} = x\beta(x)$ . If at time  $t$  one of the two neighboring states  $x - \epsilon$  and  $x + \epsilon$  is realized, the one-period transition probabilities for transitions to state  $x$  are  $\mathcal{A}(x - \epsilon) = [1 - (x - \epsilon)]\alpha(x - \epsilon)$  and  $\mathcal{B}(x + \epsilon) = (x + \epsilon)\beta(x + \epsilon)$ , respectively. Hence, the Chapman-Kolmogorov forward equation for this discrete-time Markov process is

$$P(x; t + \tau) - P(x; t) = \mathcal{A}(x - \epsilon) P(x - \epsilon; t) + \mathcal{B}(x + \epsilon) P(x + \epsilon; t) - [\mathcal{A}(x) + \mathcal{B}(x)] P(x; t).$$

So far the time span  $\tau$  which elapses until a review takes place is constant and exogenously given. As we want to endogenize it, we pass over to continuous-time processes. This requires an expansion of the discrete-time transition probabilities in a Taylor series around a point in time and the subsequent calculation of its limit for  $\tau \rightarrow 0$  in order to obtain the continuous-time transition rates (hazard rates). Here we presume that the probability that two users revise their last choices exactly at the same point in time is negligible.<sup>6</sup> Then the short-term behavior of a realization of the process is characterized by phases of constant  $x$  which are interrupted by jumps

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<sup>6</sup>Given that the users cannot coordinate their choices, this is not a strong assumption.

of size  $\epsilon$  (with  $\epsilon$  near to zero). In such a case, a Taylor expansion results, for example, in  $\mathcal{A}(x)\tau + \mathcal{O}(\tau)$  with  $\lim_{\tau \rightarrow 0} \mathcal{O}(\tau)/\tau = 0$ , where  $\mathcal{O}(\tau)$  are the higher-order terms [see Honerkamp (1990), p. 73f]. This means that transition probabilities and transition rates are of the same form.<sup>7</sup> Hence, the master equation of the continuous-time Markov process is

$$\begin{aligned} \dot{P}(x; t) &= (1 - x + \epsilon) \alpha(x - \epsilon) P(x - \epsilon; t) + (x + \epsilon) \beta(x + \epsilon) P(x + \epsilon; t) \\ &\quad - [(1 - x) \alpha(x) + x \beta(x)] P(x; t) \end{aligned} \quad (5)$$

for all  $0 \leq x \leq 1$  and with  $\dot{P}(x; t) = \partial P(x; t) / \partial t$ . This master equation can be seen as a probability flux balance, where the first line yields the probability influx into a state  $x$ , and the second line yields the probability outflux from this state.<sup>8</sup>

In the absence of network effects, i.e. with state-independent choice rates  $\alpha$  and  $\beta$ , the master equation would be linear in  $x$  and a closed differential equation system for the motion of all moments of the probability distribution could be derived from it via the generating function [see van Kampen (1992), pp. 149ff]. For  $n > 0$ , i.e. in the case of nonlinearity, no such explicit dynamic solution is feasible. However, an approximate solution can be obtained insofar as  $x$  is a quasi-continuous variable and as long as the transition rates are continuously differentiable. Then the master equation can be approximated by a state-continuous Fokker-Planck equation, from which we can derive a differential equation system for the moments. While the first condition is, due to our assumption of a very high total number of users, always fulfilled, the second condition requires that the choice rates do not take on the values zero and one [see equation (4)]. If the latter is not fulfilled, the nonlinear process can be analyzed by means of its ‘corresponding deterministic model’. In the next section, both the approximate solution via the Fokker-Planck equation and the analysis via the corresponding deterministic model is carried out for the case of (approximately) constant network effects.

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<sup>7</sup>Of course, when passing over to rates, they must be appropriately rescaled. In the following, for simplicity we have normalized the velocity parameter of the master equation to one.

<sup>8</sup>Alternatively, it can be interpreted as a generalized Poisson process; see Cox/Miller (1965), pp. 146ff.

### 3. Network competition with constant network effects: ergodicity vs. lock-ins

#### 3.1 Ergodicity

In the case of constant network effects, the choice rate of network  $B$  remains positive even with  $x = 1$  for  $n < h - b$  [and the choice rate of network  $A$  remains positive even with  $x = 0$  for  $n < h + b$ ; see equation (4) with  $c = 1$ ]. In this case, the transition rates are continuously differentiable and the master equation can be approximated by a state-continuous Fokker-Planck equation. The condition  $n < h - b$  is met if both  $n < h$  and  $b < h - n$  holds, i.e. if the network effects are ‘relatively weak’ (compared to the extent of the horizontal differentiation/exogenous shocks) and  $A$ ’s systematic basic advantage is ‘relatively low’ (compared to the difference between the extent of the horizontal differentiation/exogenous shocks and the network effect strength). In this case, sooner or later all states have a positive probability of being realized (which, however, can be very small for some states). Hence, every process is sooner or later characterized by a continuous probability distribution which covers the whole state line.

As an example, figure 1 shows three transient probability distributions which are generated by a master equation with  $h = 1$ ,  $n = 0.5$  and  $b = 0.2$  for an initial condition  $x(t = 0) = x_0 = 0$ . Then, in the first stage, the whole distribution moves towards higher  $x$ -values, and sooner or later, due to  $b > 0$ , the most probable  $x$ -value is higher than 0.5. Obviously, during the whole process, the distribution remains unimodal, relatively sharply peaked and more or less symmetric. Hence, the evolution of the mean value of the distribution is a good approximation for the most probable trajectory of a realization of the process. For  $t \rightarrow \infty$ , the transient distributions converge to the stationary distribution shown in figure 2 (dotted). Here the ‘detailed-balance condition’ holds: the probability flux into each state is equal to the probability flux out of it [see van Kampen (1992), p. 142]. Once the stationary probability distribution has been reached, the probability of a state is equivalent to its relative frequency realized in the course of time; this is the so-called ‘ergodic property’ [see van Kampen (1992), p. 93]. Therefore, the moments of the stationary probability distribution are of the same value as the moments calculated from a realized trajectory (provided that it is long enough). Obviously, ergodicity of the stochastic process means coexistence of competing user networks, where, among other things, the systematic basic advantage is decisive for the evolution of the network sizes.

In order to obtain some general results, we derive the state-continuous Fokker-Planck equation by expanding the master equation as a Taylor series up to the

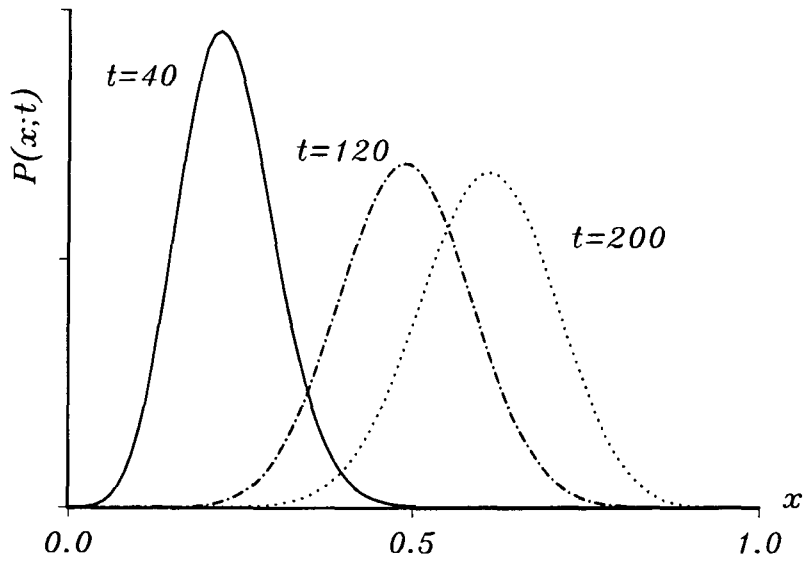


Fig. 1. Transient probability distributions of an ergodic process for  $x_0 = 0$   
 ( $c = 1, h = 1, n = 0.5, b = 0.2$ )

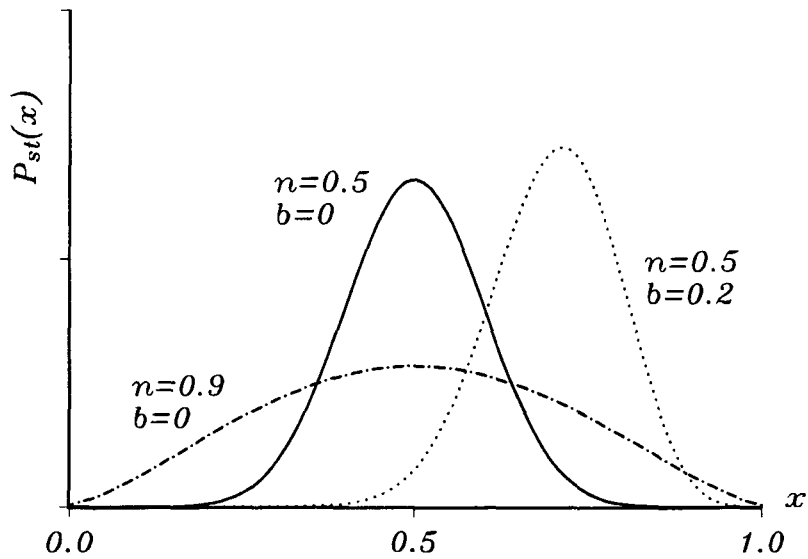


Fig. 2. Stationary probability distributions of three ergodic processes  
 ( $c = 1, h = 1$ )

second-order term.<sup>9</sup> This leads to

$$\dot{P}(x; t) = -\frac{\partial}{\partial x}[D(x) P(x; t)] + \frac{\epsilon}{2} \frac{\partial^2}{\partial x^2}[F(x) P(x; t)] \quad (6)$$

with

$$D(x) = \mathcal{A}(x) - \mathcal{B}(x) = \alpha(x) - x = 0.5 + \frac{b-n}{2h} + \left(\frac{n}{h} - 1\right) x \quad (7)$$

and

$$F(x) = \mathcal{A}(x) + \mathcal{B}(x) = (1-x)\alpha(x) + x\beta(x) = 0.5 + \frac{b-n}{2h} + \frac{2n-b}{h}x - \frac{2n}{h}x^2. \quad (8)$$

Here the drift  $D(x)$  can be interpreted as the systematic part of the process, and the fluctuation term  $F(x)$  as the added influence of the random shocks. By multiplication of the Fokker-Planck equation with  $x$  or  $x^2$  and subsequent partial integration, we can obtain the equations of motion for the mean value and for the second moment, respectively. Here we assume that the probabilities of the boundary states  $x = 0$  and  $x = 1$  remain negligible. Then the dynamic mean-value equation turns out to be

$$\langle \dot{x} \rangle = \langle D(x) \rangle, \quad (9)$$

and the differential equation for the second moment results as  $\langle \dot{x}^2 \rangle = 2 \langle x D(x) \rangle + \epsilon \langle F(x) \rangle$ . Hence, the differential equation for the variance is

$$\langle \langle \dot{x}^2 \rangle \rangle = 2 \langle x D(x) \rangle + \epsilon \langle F(x) \rangle - 2 \langle x \rangle \langle D(x) \rangle. \quad (10)$$

From equation (9), it becomes clear that the drift of the stochastic process gives the best possible approximation for the most probable trajectory of a realization. With  $\langle \dot{x} \rangle = 0$ , we obtain (the best approximation of) the most probable state after the stationary distribution has been reached:

$$\langle x \rangle^* = 0.5 + \frac{b}{2(h-n)}. \quad (11)$$

According to equation (10), the variance of the stationary distribution is

$$\langle \langle x^2 \rangle \rangle^* = \frac{\epsilon h}{4[h-n(1-\epsilon)]} \left[ 1 - \left( \frac{b}{h-n} \right)^2 \right]. \quad (12)$$

Hence, a higher systematic basic advantage  $b$  (with  $b < h - n$ ) of network  $A$ , for example due to lower software prices, leads on average to a higher network size

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<sup>9</sup>For the general procedure of the derivation of a Fokker-Planck equation and its moment equations, which are the best possible approximations of the moment equations of the master equation, see Weidlich/Haag (1983), pp. 22ff.

of  $A$  as well as to a lower variance. The two stationary distributions for  $n = 0.5$  depicted in figure 2 provide an example. From equation (11), we can deduce that network effects work like a multiplier of a systematic basic advantage and that a higher network effect strength (with  $n < h$ ) results on average in a larger size of network  $A$  (given  $b > 0$ ). Moreover, as long as  $b$  is not too high, a higher  $n$  means a higher variance, i.e. stronger network effects make the network competition more uncertain. Figure 2 provides an example for  $b = 0$ . With  $b$  near to  $h - n$  ( $\langle x \rangle^*$  near to one), however, the contrary is true.

The drift  $D(x)$  serves not only as the approximate mean-value equation but also as the corresponding deterministic model  $\dot{x} = \alpha(x) - x$  of the process. Its equilibria  $x^*$  result from equation (11) and, as long as the process is ergodic, the following correspondence holds [cf. van Kampen (1992), pp. 254ff]: stable equilibria of the deterministic model correspond with (local) maxima of the stationary distribution of the stochastic process, and unstable equilibria of the deterministic model correspond with local minima of the stationary distribution of the stochastic process. If the process is not ergodic, but rather absorbing states exist, the whole probability mass is absorbed by these states for  $t \rightarrow \infty$ . Here stable equilibria of the deterministic model correspond with states which are (local) probability maxima for a finite period of time. We will return to this case of metastability in section 4.

### 3.2 Lock-ins

If the ergodicity condition  $n < h - b$  is not fulfilled, two qualitatively different cases have to be examined. For  $h - b < n < h + b$ ,  $B$ 's choice rate takes on a value of zero if  $x$  is high, but  $A$ 's choice rate remains positive even for  $x = 0$ . In contrast, if  $n > h + b$  holds, both choice rates can become zero. The first case is characterized by a 'relatively high' systematic basic advantage of user network  $A$ : if the network effects are relatively weak ( $n < h$ ),  $b > h - n$  holds, and if the network effects are relatively strong ( $n > h$ ),  $b > n - h$  holds. Here, due to the fact that choices are reversible, sooner or later each realization of the process must be locked in to  $A$ . The interior boundary state of the lock-in regime can be calculated as  $x_{\ell,A} = 0.5 + (h - b)/(2n)$ . Once a realization of the process has entered the lock-in regime  $x_{\ell,A} \leq x \leq 1$ , the only question is whether a member of  $A$  appears at the market and chooses  $A$  again or whether a member of  $B$  appears and switches to  $A$ . However, sooner or later all users are members of network  $A$ , i.e.  $x = 1$  is an absorbing state.

Whereas in this first case the final market outcome is always an  $A$ -monopoly, the second case is characterized by the fact that it is a priori uncertain which network will prevail. Here the network effects are relatively strong ( $n > h$ ), and the systematic

basic advantage is relatively low ( $b < n - h$ ). In this case, both lock-in regimes exist, and the interior boundary state for lock-ins to network  $B$  is  $x_{\ell,B} = 0.5 - (h+b)/(2n)$ . The two states  $x_{\ell,A}$  and  $x_{\ell,B}$  are the critical masses of network  $B$  and  $A$ , respectively, which must be exceeded in order to enter the market. As in reality most user networks possess a critical mass, we concentrate in the following on processes with both lock-in regimes.

Using equation (7) and taking into account the lock-in regimes, we obtain the corresponding deterministic model of the second case as

$$\dot{x} = \begin{cases} -x & \text{if } x \leq 0.5 - (h+b)/(2n) \\ 0.5 + \frac{b-n}{2h} + \left(\frac{n}{h} - 1\right)x & \text{otherwise} \\ 1-x & \text{if } x \geq 0.5 + (h-b)/(2n). \end{cases} \quad (13)$$

Here this deterministic model cannot be interpreted as an approximation of the mean-value equation, but it reflects some qualitative features of the stochastic process. It has three equilibria:  $x = 0$ ,  $x = 1$  and an interior equilibrium  $x^*$  which can be calculated from equation (11). The two boundary equilibria are stable (the eigenvalues are  $\partial\dot{x}/\partial x = -1$ ) and correspond with the absorbing states of the stochastic process. In contrast, the interior equilibrium is unstable ( $\partial\dot{x}/\partial x = n/h - 1 > 0$  due to  $n > h$ ). This means that every realization of the process with an interior initial condition  $x_{\ell,B} < x_0 < x_{\ell,A}$  more or less quickly enters one of the lock-in regimes.

In figure 3, this is demonstrated for a stochastic process with  $h = 1$ ,  $n = 2$  and  $b = 0.6$ . There are two large lock-in regimes with  $x_{\ell,B} = 0.1$  and  $x_{\ell,A} = 0.6$ , and the unstable interior equilibrium amounts to  $x^* = 0.2$ . Starting the process with  $x_0 = 0.24$ , initially the distributions have a unique maximum near this state. But due to the competition of the absorbing boundary states for probability mass, they soon become bimodal. In the first stage, there are two local interior maxima inside the lock-in regimes, but the both absorbing states become most probable relatively quickly. As the process progresses, they absorb all probability mass before  $t = 500$  is reached.

Figure 4 depicts the influence of the initial condition, the network effect strength and of the systematic basic advantage on the outcome of the network competition. There we have depicted the numerically calculated relationship between the stationary lock-in-to- $A$  probability and the initial condition for  $n = 2$  and  $b = 0$  as well as for  $n = 1.1$  and  $b = 0.06$  (and  $h = 1$  in both cases). It turns out that these relationships are functions with turning points at the interior equilibria (which are  $x^* = 0.5$  and  $x^* = 0.2$ , respectively). Of course, a higher systematic basic advantage of network  $A$  due to lower prices and/or higher quality shifts the whole function to the left. Moreover, the significance of the initial condition on the success probabilities of network  $A$  is higher, the stronger the network effects are. In other words, stronger network effects make the process more deterministic.

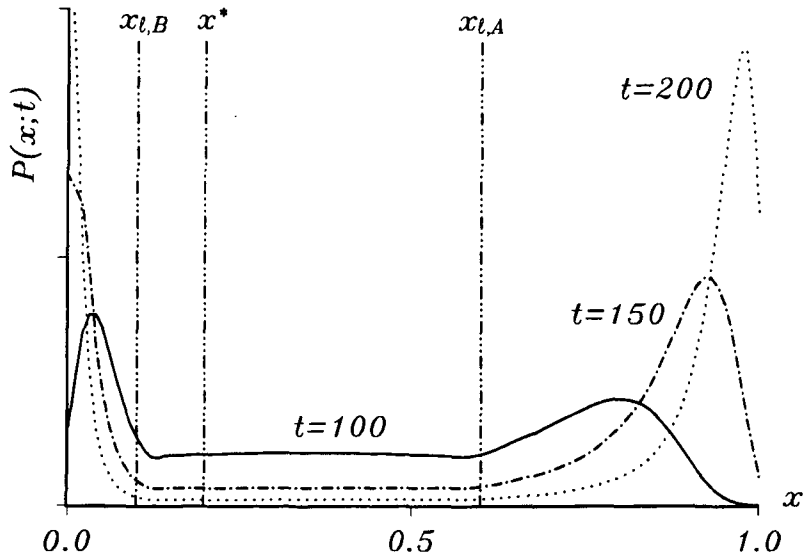


Fig. 3. Transient probability distributions of a process with both lock-in regimes for  $x_0 = 0.24$  ( $c = 1, h = 1, n = 2, b = 0.6$ )

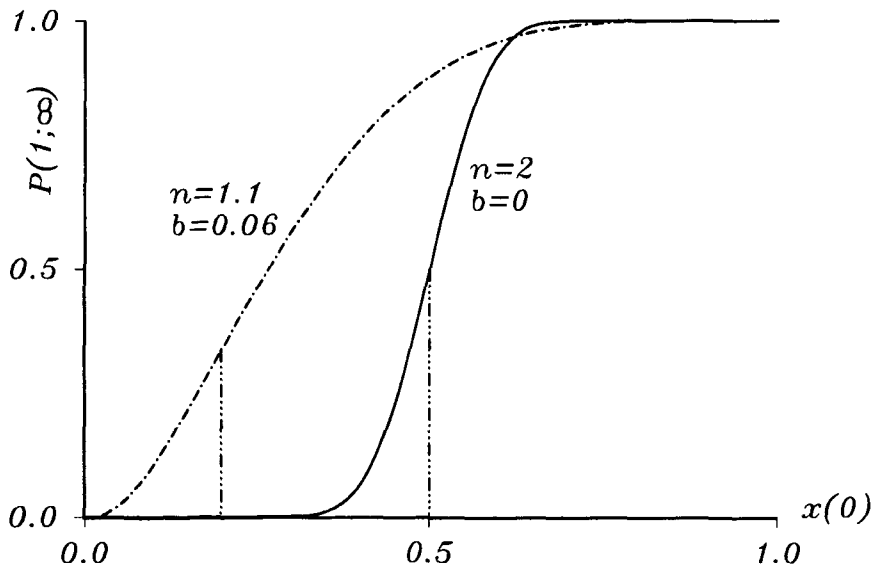


Fig. 4. Stationary probabilities of a lock-in to A for two processes with both lock-in regimes ( $c = 1, h = 1$ )



#### 4. Network competition with decreasing network effects: metastability

Usually, network effects become noticeably weaker with increasing network size. If this fact is taken into account, we can explain why competing user networks often coexist for a considerably long time, although one or both networks have a critical mass – i.e. although the market is a natural monopoly. The key for explaining this phenomenon is the concept of metastability: one or both lock-in regimes can coexist with an interior (local) probability maximum which can be arbitrarily long-lived [cf. van Kampen (1992), pp. 328f].

Returning to the choice probabilities [equation (4)] with  $c < 1$  and examining the conditions for the existence of lock-in regimes, it becomes clear that they are the same as with  $c = 1$ : for  $x \rightarrow 1$ , we have  $x^c - (1 - x)^c \rightarrow 1$ , i.e.  $(h - b)/n < 1$  results in the existence of a lock-in-to- $A$  regime; and for  $x \rightarrow 0$ , we have  $x^c - (1 - x)^c \rightarrow -1$ , i.e.  $(h + b)/n < 1$  results in the existence of a lock-in-to- $B$  regime.<sup>10</sup> In the case of ergodicity, it makes no qualitative difference whether the network effects are (approximately) constant or (noticeably) decreasing. In the case of one or both lock-in regimes, however, the existence of a metastable state changes the characteristics of a stochastic process considerably. As in the previous section, we focus on processes with both lock-in regimes.

If the network effects are constant, the eigenvalues of the corresponding deterministic models amount to  $n/h - 1$  (as long as  $x_{\ell,B} < x < x_{\ell,A}$ ), i.e. they are  $x$ -independent. In contrast, if the network effects are decreasing, not only the value of the eigenvalues is state-dependent, but also their sign can be state-dependent. From  $\dot{x} = \alpha(x) - x$  with  $\alpha(x)$  according to equation (4), we obtain

$$\frac{\partial \dot{x}}{\partial x} = \frac{cn[x^{c-1} + (1-x)^{c-1}]}{2h} - 1. \quad (14)$$

This is a parabolic function with its minimum at  $x = 0.5$ , where the minimum eigenvalue is  $\partial \dot{x} / \partial x = cn2^{1-c}/h - 1$ . Furthermore,  $\lim_{x \rightarrow 0} \partial \dot{x} / \partial x = +\infty$  and  $\lim_{x \rightarrow 1} \partial \dot{x} / \partial x = +\infty$  hold, i.e. for (very) low as well as for (very) high  $x$ -values, the eigenvalues are always positive. Hence, there are two parameter regimes:  
– For  $n/h > 0.5^{1-c}/c$ , the minimum eigenvalue is positive. Then  $\dot{x}$  is increasing over the entire range  $x_{\ell,B} < x < x_{\ell,A}$ , so that there is either a unique unstable interior equilibrium or no interior equilibrium at all. Therefore, either one or both lock-in regimes exit, and the stochastic processes do not differ qualitatively from those discussed in subsection 3.2.

<sup>10</sup>In the special case of  $c = 0.5$ , the interior boundary states of the lock-in regimes can be calculated as  $x_{\ell,A} = 0.5 + [(h - b)/(2n)]\sqrt{2 - [(h - b)/n]^2}$  and  $x_{\ell,B} = 0.5 - [(h + b)/(2n)]\sqrt{2 - [(h + b)/n]^2}$ .

– For  $n/h < 0.5^{1-c}/c$ , the minimum eigenvalue is negative. Here  $\dot{x}$  is increasing for low as well as for high  $x$ -values but decreasing in between. In this case, the deterministic model can have three interior equilibria: two unstable ones and between them a central stable equilibrium. The latter corresponds with a metastable state of the stochastic process.

Figure 5 shows the corresponding deterministic models of three stochastic processes with  $c = 0.5$  and  $h = 1$ . Two of them are symmetric ( $b = 0$ ), so that  $x = 0.5$  is the stable equilibrium. While  $n = 1.15$  results in an unstable equilibrium at  $x = 0.0275$  as well as at  $x = 0.9725$  and has lock-in regimes with  $x_{\ell,B} = 0.015$  and  $x_{\ell,A} = 0.985$ , a higher network effect strength ( $n = 1.25$ ) leads to larger lock-in regimes (see the blips of the functions) and to unstable equilibria which lie further inside. In the third example, network  $A$  has a systematic basic advantage of  $b = 0.04$ , so that the stable equilibrium amounts to  $x = 0.72$ . From the equilibria of these deterministic models, we can infer the qualitative properties of the stochastic processes and, thus, of a typical realization. Let us assume, for example, that  $A$ 's network size is initially a little bit higher than its critical mass. Then, in the first stage, there is a considerably high probability that a realization will be locked in to network  $B$ . If, however, network  $A$  ‘survives’ this critical stage and succeeds in exceeding the state which corresponds to an unstable equilibrium of the deterministic model, it becomes increasingly probable that a realization fluctuates around the metastable state for a very long time.

Figure 6 provides an example for  $n = 1.15$  and  $x_0 = 0.2$ , i.e. with an initial state to the right of the left-hand unstable equilibrium. In this case, a long-term coexistence of both user networks is more probable than an early lock-in to  $B$  right from the start. In the first stage, however, the escape probabilities for switches from interior states to the lock-in-to- $B$  regime are comparatively high. This becomes clear from the fact that, for  $t = 200$ , the minimum probability between states near  $A$ 's critical mass and those further to the right is comparatively high. In the second stage, these escape probabilities diminish dramatically, see the distributions for  $t = 600$  and  $t = 1000$ . Here the unimodal sub-distributions along the interior states center (more or less) symmetrically around the metastable state, which gains probability until ca.  $t = 800$ . In the third stage from ca.  $t = 800$  onwards, this interior sub-distribution is depleted very slowly in favor of the two absorbing boundary states. Of course, for  $t \rightarrow \infty$ , the whole probability mass is absorbed by these two states. This, however, can take arbitrarily long.

How strong the metastability is, i.e. how long competing user networks can coexist although the market is a natural monopoly, depends, among other things, on the general strength of the network effects. This is demonstrated in figure 7 via the comparison of the probability distributions for  $t = 1000$  and  $x_0 = 0.2$  of the processes with  $n = 1.15$  and with  $n = 1.25$ . Obviously, stronger network effects mean

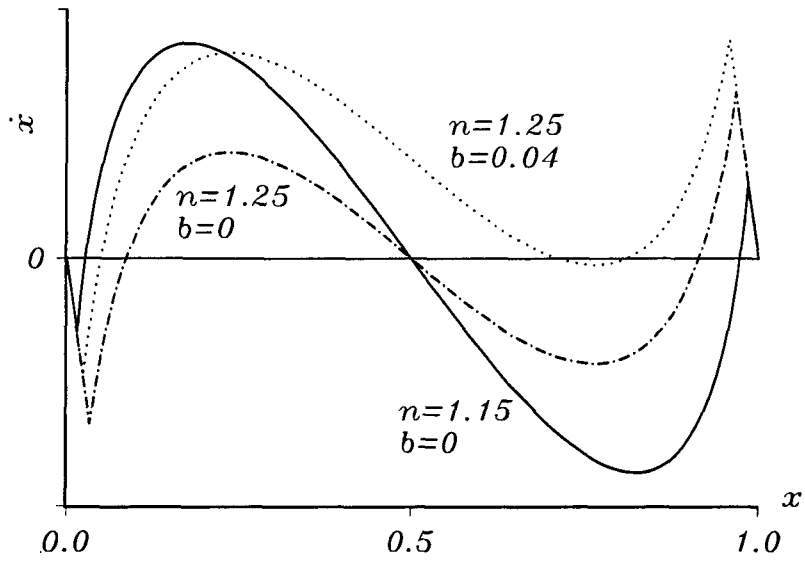


Fig. 5. Corresponding deterministic models of three processes with a metastable state ( $c = 0.5, h = 1$ )

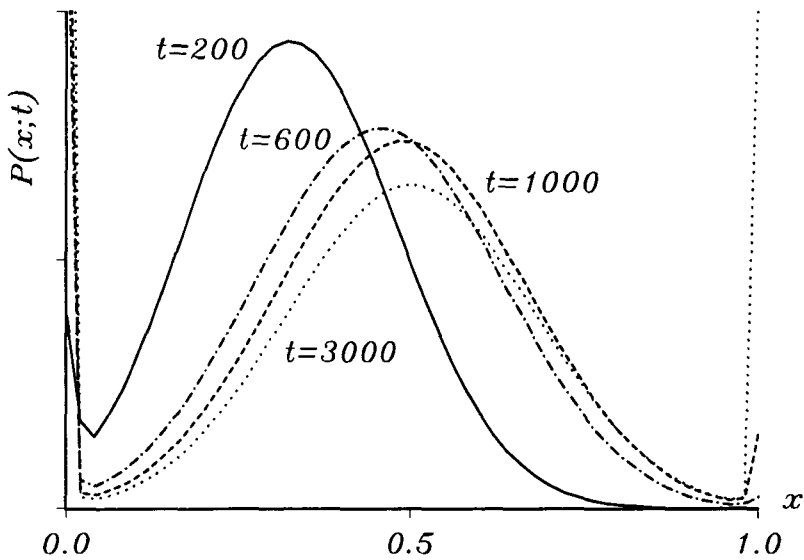


Fig. 6. Transient probability distributions of a process with a metastable state for  $x_0 = 0.2$  ( $c = 0.5, h = 1, n = 1.15, b = 0$ )

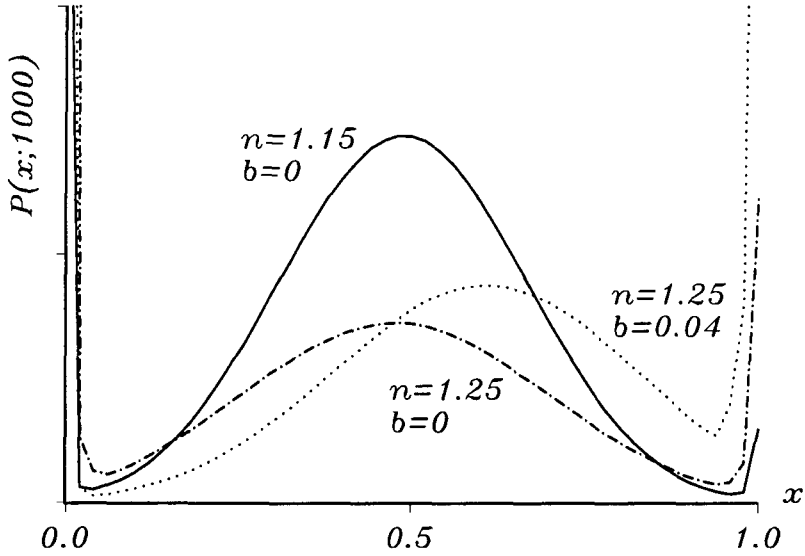


Fig. 7. Transient probability distributions of three processes with a metastable state for  $t = 1000$  and  $x_0 = 0.2$  ( $c = 0.5$ ,  $h = 1$ )

a stronger tendency towards the driving out of one of the networks: the probabilities for being locked in (for being in the neighborhood of the metastable state) are higher (lower), the higher  $n$  is. This can also be deduced from the corresponding deterministic models depicted in figure 5: a higher  $n$  means larger lock-in regimes and a lower eigenvalue (in absolute terms) of the stable equilibrium. Finally, in order to illustrate the effect of the existence of a systematic basic advantage of network  $A$ , figure 7 shows the probability distribution for  $t = 1000$  and  $x_0 = 0.2$  which results from  $b = 0.04$ . Here the peak of the interior distribution lies to the right of  $x = 0.5$ , and the probabilities for a lock-in to  $A$  ( $B$ ) are comparatively high (low). Of special importance for realizations of this process are the relatively high escape probabilities for switches to  $x = 1$ . These are due to the fact that the distance between the metastable state ( $x = 0.72$ ) and the state which corresponds to the right-hand unstable equilibrium of the deterministic model ( $x = 0.81$ ) is relatively small.

## 5. Conclusions

This paper applies a dynamic version of the Discrete Choice Model to the problem of competing user networks. In order to determine the evolution of the probability distribution of the network sizes, we have employed a continuous-time Master Equation Approach with state-dependent choice rates. We have shown how these nonlinear stochastic continuous-time processes can be analyzed by means of the moment equations of the Fokker-Planck equation and/or with the help of a corresponding deterministic model. It turns out that, depending on the strength of the network effects, the extent of the horizontal differentiation (the importance of the exogenous shocks), the significance of a systematic basic advantage and on the curvature of the network effect function, four cases can be distinguished:

- If the systematic basic advantage is relatively high, a unique absorbing state exists, and, sooner or later, every realization of a stochastic process is locked in to the network with the systematic advantage.
- If both the systematic basic advantage and the network effect strength are relatively low, the stochastic processes are ergodic; i.e. every realization of a stochastic process results in a coexistence of networks.
- If the systematic basic advantage is relatively low and the network effects are considerably strong (given the curvature of the network effect function), both lock-in regimes but no metastable state exist. Hence, there is a strong tendency towards a quick driving out of one of both networks, and it is a priori an open question as to which network this will be.
- If the systematic basic advantage is relatively low and the network effects are relatively strong but not considerably strong, the two lock-in regimes could coexist with a metastable state. Here if both networks have exceeded sufficiently their critical masses, a long-lasting coexistence becomes probable, although the market is a natural monopoly.

The latter can only happen when the fact that network effects are decreasing is taken into account. Unlike the second case, it can explain the long-lasting coexistence of user networks in the presence of critical masses.

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