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## Dynamic bivariate mixture models: Modeling the behavior of prices and trading volume

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**Wirtschaftswissenschaftliche Fakultät  
der Eberhard-Karls-Universität Tübingen**

**Dynamic Bivariate Mixture Models:  
Modeling the Behavior of Prices  
and Trading Volume**

Roman Liesenfeld



**Tübinger Diskussionsbeiträge**

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Dynamic Bivariate Mixture Models:  
Modeling the Behavior of Prices  
and Trading Volume       

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### **Abstract**

Bivariate mixture models have been used to explain the stochastic behavior of daily price changes and trading volume on financial markets. In this class of models price changes and volume follow a mixture of bivariate distributions with the unobservable number of price relevant information serving as the mixing variable. The time series behavior of this mixing variable determines the dynamics of the price-volume system. In this paper, bivariate mixture specifications with a serially correlated mixing variable are estimated by simulated maximum likelihood and analyzed concerning their ability to account for the observed dynamics on financial markets, especially the persistence in the variance of price changes. The results based on German stock market data reveal that the dynamic bivariate mixture models cannot account for the persistence in the price change variance.

# 1. INTRODUCTION

The mixture of distributions model introduced by Clark (1973) plays a prominent role in the empirical finance literature which is devoted to modeling the distribution of stock price changes. In this approach, the variance of daily price changes is directed by the random number of daily price relevant information, which serves as the mixing variable. Using trading volume as a proxy for the latent number of information, Clark (1973) found that this variable contains significant explanatory power with regard to the price change variance. But this inference is based on the assumption that trading volume is weakly exogenous, which is not adequate if price changes and trading volume are jointly determined. Tauchen and Pitts (1983) refined Clark's univariate mixture specification by including trading volume as an endogenous variable and proposed a bivariate mixture model in which the price change variance and the trading volume are simultaneously directed by the information arrival process as the common mixing variable. However, Clark (1973) as well as Tauchen and Pitts (1983) assumed that the information process is serially independent which cannot account for the well documented autoregressive behavior of the price change variance which is generally addressed by the autoregressive conditionally heteroskedastic (ARCH) model of Engle (1982) or Bollerslev's (1986) generalized ARCH (GARCH) model. Using Clark's univariate mixture model, Lamoureux and Lastrapes (1990) analyzed whether an autocorrelated number of daily information can be regarded as the source of the autocorrelation in the process of price change variance. They found that the persistence in the variance process disappears after controlling for the information arrival rate approximated by trading volume. Nonetheless, as the in study of Clark (1973), this analysis suffers from a possible simultaneity bias, induced by treating trading volume as an exogenous variable.

In this paper endogenous trading volume and an autocorrelated latent information arrival process are combined within a bivariate mixture framework. In addition to the bivariate mixture model of Tauchen and Pitts (1983), I consider a modified specification of this model recently proposed by Andersen (1996). For both versions a dynamic information arrival variable is explicitly specified. These dynamic specifications of a bivariate mixture system provide structural models for the co-movements of price change variance and trading volume in which the dynamics of both series are directed by the time series behavior of the common mixing variable. Given that these specifications are correct, trading volume provides information about the factor which generates the persistence of the price change variance. The main purpose of this paper is to investigate whether these bivariate mixture systems with a dynamic mixing variable capture the observed dynamic features of the price change and the volume series, especially the persistence

in the price change variance.

Since the latent mixing variable is serially correlated, the likelihood functions of the mixture models are given by high dimensional integrals which make a direct application of maximum likelihood (ML) infeasible. Hence, for estimation and inference, I use a strategy which is based on a simulated maximum likelihood (SML) approach developed by Danielsson and Richard (1993). There are alternative approaches which also allow for endogenous trading volume and a dynamic mixing variable. Richardson and Smith (1994) applied the generalized method of moments (GMM) procedure to test the distributional restrictions of the standard bivariate model of Tauchen and Pitts (1983). Also using GMM techniques, Andersen (1996) found that his modified mixture model outperforms the standard specification concerning the ability to account for the distributional properties of the data. Finally, Lamoureux and Lastrapes (1994) focused on the dynamic implications of a bivariate mixture model and employed a signal extraction procedure to estimate the time series of the unobservable information arrival rate. They analyzed whether the time series behavior of the estimated series can account for the observed persistence in the variance process. By using a different estimation and inference strategy this paper can be regarded as complementary to the studies mentioned above.

The remainder of this paper is organized as follows. In section 2 the bivariate mixture models are presented. Section 3 describes the SML estimator and its application to the bivariate mixture models. The statistical properties of the data and the estimation results are discussed in section 4. Section 5 summarizes the results and contains the conclusions.

## 2. BIVARIATE MIXTURE MODELS

According to the mixture model of Tauchen and Pitts (1983) the price change  $dp_t$  on day  $t$  and the corresponding trading volume  $v_t$  follow a bivariate normal distribution conditional on the daily number of price relevant information  $i_t$ :

$$dp_t \mid i_t \sim N(\mu_{dp}, \sigma_{dp}^2 i_t) \quad (1)$$

$$v_t \mid i_t \sim N(\mu_v, \sigma_v^2 i_t) \quad (2)$$

$$\text{with } \text{Cov}(dp_t, v_t \mid i_t) = 0,$$

where  $\sigma_{dp}$ ,  $\mu_v$  and  $\sigma_v$  are positive parameters. Assuming that the unobservable number of daily information arrivals  $i_t$  is a random variable, it follows that the unconditional joint distribution of  $dp_t$  and  $v_t$  is a mixture of independent bivariate normals with  $i_t$  serving as the mixing variable.

To obtain the mixture model (1) and (2), Tauchen and Pitts (1983) used a sequential Walrasian equilibrium framework in which trading volume is only caused by the arrival of information. In contrast to this, Andersen (1996) developed a market microstructure model with asymmetric information inspired by the models of Kyle (1985), and Glosten and Milgrom (1985). In the setting of Andersen (1996) trading volume is not only determined by the demand of informed traders which react to the arrival of new information, but also by the demand of liquidity traders. The bivariate distribution of price change and trading volume conditional on the daily number of information Andersen (1996) obtained is given by

$$dp_t \mid i_t \sim N(\mu_{dp}, \sigma_{dp}^2 i_t) \quad (3)$$

$$\frac{v_t}{c} \mid i_t \sim \text{Po}(m_0 + m_1 i_t) \quad (4)$$

$$\text{with } \text{Cov}(dp_t, v_t \mid i_t) = 0,$$

where  $\sigma_{dp}$ ,  $c$  and  $m_0$  are positive parameters. This modified mixture model differs from the standard Tauchen-Pitts specification by a Poisson distribution for the daily trading volume conditional on  $i_t$ , whereas the marginal price change distribution is left unchanged. The parameter  $c$  is an unknown proportional constant which must be introduced due to a scaling indeterminacy that arises if detrended volume data are used to estimate the model. The parameter  $m_0$  captures that part of daily volume which is generated by liquidity trading and which is therefore independent of the arrival of information. The remaining part of trading volume which is induced by new information is represented by  $m_1 i_t$ . Thus, by accounting for an information independent part of trading volume this modified version of the mixture model can be regarded as a generalization of the standard Tauchen-Pitts model.

The procedure to estimate the mixture models will be based on the SML technique, which relies on the full specification of the conditional density functions. However, in the modified mixture model the density of  $v_t/c$  is defined for integer-valued random variables only. Since  $c$  is an unknown parameter which must be estimated it is impossible to ensure that for an estimate  $\hat{c}$ , the series  $v_t/\hat{c}$  consists of integer values. Hence, for the SML procedure the Poisson distribution cannot be used directly and an approximation must be applied. If one assumes that  $(m_0 + m_1 i_t)$  is large enough due to a sufficiently large number of daily information the Poisson distribution can be approximated by a corresponding normal distribution:

$$\frac{v_t}{c} \mid i_t \sim N([m_0 + m_1 i_t], [m_0 + m_1 i_t]).$$

To estimate the modified bivariate mixture model, the volume specification in (4) is replaced by

the above approximation to the Poisson distribution.

The bivariate mixture models have many implications for the price-volume system. I focus on the implications with regard to the dynamics of price changes and trading volume, especially to the behavior of the variance of price changes over time. Since the conditional variance of  $dp_t$  is determined by the common mixing variable  $i_t$ , the dynamics in the variance process of price changes are solely due to the time series behavior of  $i_t$  which also affects the dynamics of trading volume. As noted by Lamoureux and Lastrapes (1990, 1994) a positive serial correlation in  $i_t$  leads to a serial correlation in the squared price changes and the observed persistence in the variance of  $dp_t$  may be the result of a persistence in the information arrival process. To incorporate such an autocorrelation in the information arrival process, I assume for  $\ln(i_t) = \lambda_t$  the following time series model:  $\lambda_t = \gamma + \delta\lambda_{t-1} + \nu\epsilon_t$  with  $\epsilon_t \sim iid N(0, 1)$ . This Gaussian AR(1)-process for the logarithm of the mixing variable generalizes the assumption of Tauchen and Pitts (1983) that this variable is serially independent and lognormally distributed.

Due to the latent character of the mixing variable, the bivariate mixture models are invariant with respect to scalar transformations of  $i_t$ . In order to normalize the models, the parameter  $\sigma_{dp}$  in the price change equations (1) and (3) is set equal to one. Thus combining the dynamic specification of the mixing variable with the bivariate systems, for the standard mixture model results:

$$dp_t | \lambda_t \sim N( \mu_{dp} , e^{\lambda_t} ) \quad (5)$$

$$v_t | \lambda_t \sim N( \mu_v e^{\lambda_t} , \sigma_v^2 e^{\lambda_t} ) \quad (6)$$

and

$$\lambda_t | \lambda_{t-1} \sim N(\gamma + \delta\lambda_{t-1} , \nu^2) . \quad (7)$$

For the modified mixture model Equation (6) is substituted by

$$\frac{v_t}{c} | \lambda_t \sim N( [m_0 + m_1 e^{\lambda_t}] , [m_0 + m_1 e^{\lambda_t}] ) . \quad (8)$$

Notice that these bivariate models imply a univariate specification for price changes (5) and (7), which corresponds to the stochastic volatility model used, for example, by Danielsson (1994), Jacquier, Polson and Rossi (1994) and Ruiz (1994). In accounting for the persistence in the variance process of price changes, this stochastic volatility model, where the persistence is measured by the parameter  $\delta$ , represents a successful alternative to (G)ARCH-specifications. However, given that a bivariate dynamic mixture system is adequate to model the joint dynamics of price changes and trading volume, the time series behavior of the common latent factor filters the



persistence in the variance of price changes and one can use the series of trading volume as an additional information to infer the latent process which directs the dynamics of the price change variance.

Due to the presence of a dynamic latent variable, the ML procedure cannot be applied directly to the estimation of the bivariate mixture specifications. In order to derive the marginal distribution of the observable variables the vector of the latent variable  $\Lambda = \{\lambda_t\}_{t=1}^T$  has to be integrated out of the trivariate joint probability function of  $dP = \{dp_t\}_{t=1}^T$ ,  $V = \{v_t\}_{t=1}^T$  and  $\Lambda$ :

$$f(dP, V | \theta) = \int_{\mathbb{R}^T} f(dP, V, \Lambda | \theta) d\Lambda, \quad (9)$$

where  $\theta$  denotes the vector of the parameters to be estimated. Thus the unconditional joint distribution of the observable variables  $f(dP, V | \theta)$  is given by an integral with a dimension determined by the number of observations  $T$ . The fundamental problem is that for this high dimensional integral which does not have a closed form solution no standard numerical integration methods can be applied to compute it.

To estimate a dynamic bivariate mixture model Richardson and Smith (1994), and Andersen (1996) proposed the GMM strategy which avoids the calculation of the high dimensional integral. In contrast to this approach, I use a SML method developed by Danielsson and Richard (1993) which is based on Monte Carlo (MC) integration to evaluate the integral (9). This estimation strategy has the following advantages over the GMM. First, applying SML avoids the problem of the GMM technique related to the appropriate choice of moment restrictions. Second, if the number of replications is large enough, using SML it is possible to adopt the standard instruments of inference developed for maximum likelihood methods. Finally, GMM estimators are only based on certain aspects of the distributional and dynamic assumptions in form of the selected moment restrictions, whereas SML retains the complete a priori information given by the structure of the model. So one can expect that the GMM estimator is less efficient. However, SML is a more computer intensive estimation procedure compared to GMM.

### 3. THE SIMULATED MAXIMUM LIKELIHOOD ESTIMATOR

Suppose one has a solution of the  $T$  dimensional integral given in (9), then the estimate of the unknown parameter vector  $\theta$  based on maximum likelihood is

$$\hat{\theta}_{\text{ML}} = \arg \max_{\theta} \ln[f(dP, V | \theta)]. \quad (10)$$

To evaluate the high dimensional integral  $f(dP, V | \theta)$ , I use the MC technique developed by Danielsson and Richard (1993). This MC technique is based on an importance sampling procedure and allows the integral to be estimated with a high degree of precision. Given such an estimate of the integral, the log-likelihood can be evaluated and maximized over the unknown parameter vector  $\theta$ .

To obtain such a MC estimate of  $f(dP, V | \theta)$ , the trivariate joint density  $f(dP, V, \Lambda | \theta)$  is factorized in an importance sampling function  $\mu(\Lambda | dP, V)$  and a remainder function  $h(\Lambda, dP, V)$  such that the following equation holds

$$f(dP, V, \Lambda | \theta) = h(\Lambda, dP, V) \mu(\Lambda | dP, V). \quad (11)$$

Since the importance sampling function is a density function of  $\Lambda$ , the integral can be expressed in terms of the expectation value of  $h(\Lambda, dP, V)$  evaluated over the distribution  $\mu(\Lambda | dP, V)$ :

$$\begin{aligned} f(dP, V | \theta) &= \int_{\mathbb{R}^T} f(dP, V, \Lambda | \theta) d\Lambda \\ &= \int_{\mathbb{R}^T} h(\Lambda, dP, V) \mu(\Lambda | dP, V) d\Lambda \\ &= E_{\mu}[h(\Lambda, dP, V)]. \end{aligned} \quad (12)$$

Because the sample mean is a natural estimator of an expectation value, a MC estimator of the integral can be constructed by

$$\hat{f}_N(dP, V | \theta) = \frac{1}{N} \sum_{n=1}^N h(\Lambda_n, dP, V), \quad (13)$$

where  $\{\Lambda_1, \dots, \Lambda_n, \dots, \Lambda_N\}$  represents a simulated random sample of size  $N$  drawn from the probability distribution  $\mu(\Lambda | dP, V)$ . To obtain the SML estimator of the parameter vector,  $f(dP, V, | \theta)$  in Equation (10) is replaced by its estimate (12).

Under weak assumptions the estimator  $\hat{f}_N(dP, V, | \theta)$  converges almost surely to  $f(dP, V | \theta)$ , see Danielsson and Richard (1993) and Geweke (1989). Furthermore, the precision of the MC estimate of the integral and therefore of the unknown parameter vector can be arbitrarily increased

by raising the simulation sample size  $N$ .

A natural factorization of the trivariate density  $f(dP, V, \Lambda | \theta)$  for the bivariate mixture models (5)-(8), according to the conditions (11) and (12), can be obtained as follows. The trivariate joint density is factorized in

$$f(dP, V, \Lambda | \theta) = f(dP, V | \Lambda) f(\Lambda) \quad (14)$$

$$\text{with} \quad f(dP, V | \Lambda) = \prod_{t=1}^T f(dp_t, v_t | \lambda_t) \quad \text{and} \quad f(\Lambda) = \prod_{t=1}^T f(\lambda_t | \lambda_{t-1}),$$

where  $f(dp_t, v_t | \lambda_t)$  represents the joint density of price change and volume in period  $t$  conditional on the corresponding value of the mixing variable. Since conditional on  $\lambda_t$ , the random variables  $dp_t$  and  $v_t$  are assumed to be independent,  $f(dp_t, v_t | \lambda_t)$  is given by the product of two normal distributions for the standard mixture model, and by the product of an approximation to a Poisson distribution and a normal distribution in the case of the modified one.  $f(\lambda_t | \lambda_{t-1})$  is the conditional distribution of the mixing variable given its past observation which according to (7) is a Gaussian distribution for both mixture models. So one can define the following initial sampling function and corresponding remainder function which preserves the conditions (11) and (12):

$$\mu_0(\Lambda | dP, V) \equiv \prod_{t=1}^T f(\lambda_t | \lambda_{t-1}) \quad (15)$$

$$h_0(\Lambda, dP, V) \equiv \prod_{t=1}^T f(dp_t, v_t | \lambda_t). \quad (16)$$

These initial remainder and importance functions can be used to construct a naive MC-estimate of the marginal distribution  $f(dP, V | \theta)$  for a given value of the parameter vector  $\theta$ . Therefore, a simulated sample  $\{\Lambda_{0,n}\}_{n=1}^N$  is drawn from the initial remainder  $\mu_0$ , where each  $\Lambda_{0,n}$  is a vector of length  $T$  given by  $(\lambda_{0,n,1}, \dots, \lambda_{0,n,t}, \dots, \lambda_{0,n,T})$ . Then, the remainder function is evaluated for every  $\Lambda_{0,n}$  and the resulting arithmetic mean of  $\{h_0(\Lambda_{0,n}, dP, V)\}_{n=1}^N$  serves according to Equation (13) as a naive MC-estimate of  $f(dP, V | \theta)$ .

Danielsson and Richard (1993) demonstrated that this initial choice for the remainder and importance function is not efficient in the sense that the resulting MC sampling variance of the estimator for the integral increases dramatically with the sample size  $T$ , and therefore with the dimension of the integral. If  $T$  is greater than 200 a prohibitively high simulation sample size  $N$  would be necessary to achieve a reasonable degree of accuracy for the MC estimate of the integral. But since the factorization of  $f(dP, V, \Lambda | \theta)$  in a remainder and an importance function, which preserves the conditions (11) and (12), is not unique Danielsson and Richard (1993) proposed

an acceleration method, called Accelerated Gaussian Importance Sampling (AGIS). This AGIS method transforms the initial remainder and importance function  $\mu_0$  and  $h_0$  in such a way that the MC sampling variance is minimized while retaining conditions (11) and (12). In this way, the AGIS method searches for an importance function which minimizes the MC sampling variance of the corresponding remainder function:

$$\min_{\mu} \text{Var}_{\mu}[h(dP, V, \Lambda)] \quad (17)$$

$$\text{with} \quad \text{Var}_{\mu}[h(dP, V, \Lambda)] = \int_{\mathbb{R}^T} \left[ \frac{f(dP, V, \Lambda | \theta)}{\mu(\Lambda | dP, V)} - f(dP, V | \theta) \right]^2 \mu(\Lambda | dP, V) d\Lambda .$$

Because this minimizing problem has no analytical solution numerical and iterative procedures are used. Therefore, an auxiliary variance reduction function  $\xi(\Lambda, Q)$  is defined where  $Q$  is a matrix of parameters.

With such an auxiliary function, one can construct a new pair of remainder and importance functions by transforming the initial pair of functions in as follows:

$$\mu(\Lambda | dP, V) = \frac{\mu_0(\Lambda | dP, V) \xi(\Lambda, Q)}{\kappa(Q)} \quad (18)$$

$$h(\Lambda, dP, V) = \frac{h_0(\Lambda, dP, V) \kappa(Q)}{\xi(\Lambda, Q)} . \quad (19)$$

$\kappa(Q)$  represents the integration constant which ensures that the new importance function is indeed a proper probability distribution function and is given by:

$$\kappa(Q) = \int_{\mathbb{R}^T} \mu_0(\Lambda | dP, V) \xi(\Lambda, Q) d\Lambda . \quad (20)$$

With the new remainder (19) and the new importance function (18) the integral to be estimated can be re-written as:

$$\begin{aligned} f(dP, V | \theta) &= \int_{\mathbb{R}^T} \left( \frac{h_0(\Lambda, dP, V) \kappa(Q)}{\xi(\Lambda, Q)} \right) \left( \frac{\mu_0(\Lambda | dP, V) \xi(\Lambda, Q)}{\kappa(Q)} \right) d\Lambda \\ &= \int_{\mathbb{R}^T} h(\Lambda, dP, V) \mu(\Lambda | dP, V) d\Lambda . \end{aligned} \quad (21)$$

Thus, the minimizing problem given in (17) is transformed into searching for an auxiliary function  $\xi(\Lambda, Q)$  which minimizes the MC sampling variance of the remainder function  $h(\Lambda, dP, V)$ . In principle, an infinitely large number of functional forms for  $\xi(\Lambda, Q)$  are possible. But for analytical and computational reasons the AGIS method of Danielsson and Richard (1993) restricts

the search for a function  $\xi(\Lambda, Q)$  to that class of functions for which the new importance sampler in (18) is a  $T$ -dimensional multivariate normal distribution. Furthermore, instead of minimizing the *total* MC sampling variance of the remainder function *itself*, the variances of the *individual components* of the *logarithm of the remainder* are minimized for each time period *separately*.

These modifications lead to an auxiliary function which is a product over  $T$  exponential quadratic functions:

$$\xi(\Lambda, Q) = \prod_{t=1}^T \xi(\lambda_t, Q_t) \quad (22)$$

$$\text{with} \quad \xi(\lambda_t, Q_t) = \exp\{-\frac{1}{2}\eta'_t Q_t \eta_t\} \quad \text{and} \quad \eta'_t = (\lambda_t, \lambda_{t-1}, 1).$$

To determine  $Q = \{Q_t\}_{t=1}^T$ , which is a sequence of  $(3 \times 3)$  matrixes, an iterative estimation procedure with  $k$  steps is applied. Starting with the initial sampler  $\mu_0$ , an initial simulated sample  $\{\Lambda_{0,n}\}_{n=1}^N$  is drawn and used to run the following auxiliary regression for every time period  $t = 1, \dots, T$ :

$$\ln h_0(\lambda_{0,n,t}) = a_{1,t} + b_{1,t} \lambda_{0,n,t} + c_{1,t} \lambda_{0,n,t}^2 + \text{residual} \quad , \quad n = 1, \dots, N. \quad (23)$$

The OLS-estimates of the coefficients  $a_{1,t}$ ,  $b_{1,t}$  and  $c_{1,t}$  can be used to construct for every  $t$  the matrix

$$\hat{Q}_{1,t} = \begin{pmatrix} -2\hat{c}_{1,t} & 0 & -\hat{b}_{1,t} \\ 0 & 0 & 0 \\ -\hat{b}_{1,t} & 0 & -2\hat{a}_{1,t} \end{pmatrix} \quad t = 1, \dots, T. \quad (24)$$

With  $\hat{Q}_1 = \{\hat{Q}_{1,t}\}_{t=1}^T$ , a first new sampler is given by  $\mu_1(\Lambda | dP, V) = \mu_0(\Lambda | dP, V) \xi(\Lambda, \hat{Q}_1) / \kappa(\hat{Q}_1)$ . A second step sampler  $\mu_2(\Lambda | dP, V)$  is constructed in the same fashion by drawing a random sample from  $\mu_1(\Lambda | dP, V)$  and regressing  $\ln h_0(\lambda_{1,n,t})$  on a constant,  $\lambda_{1,n,t}$  and  $\lambda_{1,n,t}^2$ . With the resulting sequence of matrixes  $\hat{Q}_2$  the sampler  $\mu_2(\Lambda | dP, V)$  is determined. This procedure is repeated until  $\hat{Q}_k$  is sufficiently close to the matrixes  $\hat{Q}_{k-1}$ . Danielsson and Richard (1993) showed that the convergence is reached very quickly, typically after less than 5 iterations (see the Appendix for the implementation of the complete AGIS algorithm).

Finally, the simulated sample  $\{\Lambda_{k,n}\}_{n=1}^N$  from  $\mu_k(\Lambda | dP, V)$  is used to calculate the  $k$ th step AGIS estimate of the integral:

$$\hat{f}_{N,k}(dP, V | \theta) = \frac{1}{N} \sum_{n=1}^N \frac{h_0(\Lambda_{k,n}, dP, V) \kappa(\hat{Q}_k)}{\xi(\Lambda_{k,n}, \hat{Q}_k)}. \quad (25)$$

To obtain the SML estimate of  $\theta$ , this  $k$ th step AGIS estimate of the marginal distribution is maximized with respect to  $\theta$ .

Clearly, with the modifications noted above, the AGIS algorithm does not ensure that one obtains an importance function with the smallest possible MC sampling variance of the corresponding remainder. But Danielsson and Richard (1993) showed that for a univariate mixture model the reduction of the MC sampling variance which is reached by the AGIS algorithm is so significant that it can be regarded as sufficient. This holds especially if one considers the trade-off between computing costs and variance reduction. So the experiences of Danielsson and Richard (1993) showed that the application of the AGIS algorithm to estimate a univariate mixture model for  $T = 2022$  observations reduces the necessary sample size of simulated random vectors to  $N = 50$  or less.

## 4. EMPIRICAL RESULTS

### 4.1 Data

The empirical results are based on a dataset consisting of daily closing prices and daily number of shares traded for the four major German stocks listed in the DAX: Siemens (SIE), Daimler-Benz (DAI), Volkswagen (VOW), and Deutsche Bank (DBK). The data were obtained from the Karlsruher Kapitalmarktdatenbank (KKMDB) and are adjusted for effects of dividends and capital changes. The sample period starts on January 2, 1990 and ends on May 31, 1994 which gives a sample size of roughly 1100. The daily closing prices  $p_t$  are transformed to price changes measured in continuously compounded rates:  $dp_t = 100 \cdot \ln(p_t/p_{t-1})$ . In order to make the volume series stationary, the volume data are adjusted by an exponential time trend which is estimated by regressing  $\ln(v_t)$  on a constant and on time  $t = 1, \dots, T$ . The exponential function of the residuals of this regression are then linearly transformed in such a way that the raw data and the detrended data have the same mean and variance. For all the following results, the detrended volume series is multiplied by  $10^{-6}$ .

The statistical properties of the price change data and the detrended volume series are summarized in Table 1. As can be seen from Table 1, the empirical distributions of price changes are not normal. The excess kurtosis exceeds the value of zero for all stocks. In order to detect serial dependencies the Ljung-Box statistics including 20 lags are calculated for the price changes  $LB_{dp}(20)$  and for the squared data  $LB_{dp^2}(20)$ . Under the null hypotheses that all autocorrelation coefficients are zero,  $LB_{dp}(20)$  and  $LB_{dp^2}(20)$  are  $\chi^2_{(20)}$  distributed. The results demonstrate that the price change series appear to be serially uncorrelated, whereas the squared data exhibit

a significant autocorrelation which indicates the presence of ARCH-effects. This is confirmed by the Engle (1982) test statistic  $E_{dp}(20)$ , which is  $\chi^2_{(20)}$  distributed under the null hypotheses of no ARCH-effects. The Ljung-Box statistic  $LB_v(20)$  displayed in Table 1 shows that there is a high autocorrelation in the series of trading volume for all stocks. Note that the autocorrelation in trading volume is substantially higher than that in the squared price changes, as can be inferred from the comparison of  $LB_{dp^2}(20)$  and  $LB_v(20)$ . This fact is consistent with the predictions of the bivariate mixture models, see Harris (1987).

## 4.2 Estimation Results

### 4.2.1 Univariate Mixture Model for Price Changes

This section will be devoted to the estimation of the univariate mixture model for the price change series resulting from the bivariate systems and which is given by the Equations (5) and (7). Since this specification is compatible with the stochastic volatility model, it can be used to prove whether the price change data of the four analyzed stocks exhibit the properties similar to those generally reported for financial markets or not. The SML estimation of the univariate model is carried out by using a simulation sample size of  $N = 500$  and four iterations for the AGIS algorithm.

The SML results are given in Table 2. In order to assess the precision of the SML method the estimated MC sampling standard deviations of the maximized log-likelihoods are given in brackets below the log-likelihood values. These estimated standard deviations are calculated by estimating a hundred times the likelihood for the given SML estimates of the parameters and computing the standard deviation of the logarithm of these likelihood estimates. Since these MC standard deviations are about 0.003% for all stocks relative to the corresponding log-likelihood values, the SML estimates seems to be reasonably precise. The SML estimates of the parameter  $\delta$  are always highly significant and lie between 0.91 and 0.96, indicating a high persistence of variance shocks which is a typical result for financial price change series.

### 4.2.2 Bivariate Mixture Models

In this section the SML estimates of the joint price change-volume specifications are presented. The results for the standard bivariate mixture model given by the Equations (5), (6), and (7)

are displayed in Table 3 and raise the following remarks. The estimated MC standard deviations of the maximized log-likelihood show that the MC sampling errors are much smaller than in the univariate specification, although the same simulation sample size and the same number of AGIS iterations are used for both. The MC standard deviations for the bivariate setting vary from 0.00001% (DBK) to 0.0003% (DAI) relative to the log-likelihood values, indicating a very high precision of the SML results. Furthermore, even if the persistence parameter  $\delta$  is highly significant in all cases, the estimates which lie between 0.60 (SIE) and 0.65 (DAI) are obviously lower for all stocks than those obtained in the univariate price change model, whereas its estimated standard deviations for both specifications are equally small. These significant differences between the estimates of  $\delta$  resulting from the univariate and the bivariate setting cast doubt on the validity of this bivariate specification. If the bivariate mixture model were correctly specified, then the estimator of the parameter  $\delta$  would be consistent in the univariate as well as in the bivariate setting, and the estimates of  $\delta$  resulting from the univariate and the bivariate specification should be of the same order of magnitude. However, the sharp decrease of  $\delta$  if trading volume is included indicates that the bivariate mixture model fails to capture the high persistence in the variance process of price changes. I shall return to this issue shortly.

Table 4 displays the results of the modified bivariate mixture model given by Equations (5), (7), and (8). The MC standard deviations of the log-likelihood can be regarded as sufficiently small, even if they are generally higher than those obtained for the standard model in Table 3. Of special interest are the estimation results concerning the parameter  $c \cdot m_0$ , which captures that part of daily trading volume which is independent of the information arrival and which is not included in the standard mixture model. This parameter is significant at the 1% level for all analyzed stocks. This result is consistent with the findings of Andersen (1996), whose estimates for the US stock market are based on GMM techniques. To compare the modified model with the standard bivariate mixture model directly, the classical testing procedure cannot be applied, since the specifications are non-nested. Hence the comparison is based on the information criterion (SIC) of Schwarz (1978). With the exception of DAI, SIC favours the modified bivariate specification over the standard bivariate model. But even if the modified version of the mixture model is generally the preferred specification, Table 4 shows that the estimates of the persistence parameter  $\delta$  remain significantly lower than those obtained in the univariate price change model.

Hence, both versions of the bivariate mixture model in which the dynamics of the price change variance and of trading volume are solely driven by the information arrival rate, seem to be



inadequate in accounting for the high persistence in the variance process of price changes. To prove this by a formal statistical test, I allow for a separate source of persistence in the variance of price changes, in addition to the persistence captured by the common mixing variable  $i_t = \exp(\lambda_t)$ . Therefore, in the bivariate models the price change specification (5) is generalized as follows:

$$dp_t \mid \lambda_t, h_t \sim N(\mu_{dp}, e^{\lambda_t + h_t}) \quad (26)$$

$$\text{with} \quad h_t = \sum_{j=1}^k \alpha_j |dp_{t-j} - \mu_{dp}|,$$

whereas the volume specifications are left unchanged. The additional term  $h_t$  is a function of lagged absolute residuals  $|dp_{t-j} - \mu_{dp}|$ , which represent lagged variances of  $dp_t$ . Under the hypotheses that the time series behavior of the common mixing variable is the only source for the persistence in the variance of price changes, the lagged absolute residuals should not have any explanatory power with regard to the contemporaneous variance of  $dp_t$ , and the coefficients  $\alpha_j$  should be small and statistically insignificant. Otherwise, the bivariate mixture model with one common factor must be rejected. To test this hypotheses by a standard likelihood ratio test, I use the lag length  $k = 5$ . Under the null hypotheses that the additional source of persistence captured by the  $h_t$  is nil, the likelihood ratio statistic is  $\chi^2_{(5)}$  distributed.

The SML estimates of the standard bivariate model with the separate persistence given by the Equations (6), (7), and (26) are summarized in Table 5. It shows that the coefficients  $\alpha_j$  are significantly positive in almost all cases. Furthermore, the statistics of the likelihood ratio test of the hypotheses  $H_0 : \alpha_1 = \dots = \alpha_5 = 0$  are significant at any level for all stocks and vary between 50 (VOW) and 146 (DAI). This result reveals that an additional source of persistence in the price change variance exists which is not captured by the common mixing variable. Table 6 contains the SML results for the modified bivariate mixture model with the generalized price change specification (26). As for the standard model, the lagged absolute price change residuals contain a significant explanatory power concerning the contemporaneous price change variance. Thus, I conclude that the restriction of both versions of the bivariate mixture model, that the dynamics in the price change variance are solely due to the time series behavior of the information arrival process which also determines the dynamics of trading volume, have to be rejected. This empirical conclusion is consistent with the findings of the studies of Lamoureux and Lastrapes (1994) and Andersen (1996). Lamoureux and Lastrapes (1994) found that in their signal extraction approach, the estimated series of the latent information arrival variable does not account for the persistence in the price change variance; Andersen (1996), who used GMM, also

found a reduction of the measure of variance persistence if trading volume is included.

A possible interpretation of this sort of mis-specification of the bivariate mixture models could be that more than one information process exists which directs trading volume and price change variance differently. As can be inferred from the empirical autocorrelation functions, not reported here, the shocks in the variance process of price change have a longer memory compared to the shocks in the volume process. Thus it seems to be necessary to decompose the information arrival process in a long-run component which has a greater impact on price change variance than on trading volume, and a short-run component which affects trading volume more than the variance of price change. The results concerning the price change and volume dynamics presented here indicate that future research should be along this line.

## 5. SUMMARY AND CONCLUSION

This paper analyzes the bivariate mixture models of Tauchen and Pitts (1983) and Andersen (1996) concerning their abilities to explain the dynamic features observed in financial markets. In these models the joint distribution of daily price changes and daily trading volume are given by a mixture of conditional bivariate distributions. The unobservable number of daily price relevant information serves as the conditioning variable, the so-called mixing variable. In order to account for the observed persistence of volatility shocks, a serially correlated mixing variable is specified. Due to the serially correlated mixing variable, the likelihood function consists of a high dimensional integral which makes standard maximum likelihood techniques infeasible. Hence, a simulated maximum likelihood (SML) approach developed by Danielsson and Richard (1993) is used to estimate the dynamic bivariate mixture models. This SML approach uses Monte Carlo techniques based on an importance sampling procedure to evaluate the high dimensional integral which constitutes the likelihood function.

The estimation results using the four major German stocks can be summarized as follows: First, the dynamic bivariate mixture models can be estimated by SML with a high degree of precision. Second, for both bivariate mixture models the implied persistence of variance shocks are significantly smaller than the persistence generally reported for stochastic volatility models for price changes only. Finally, allowing for a separate persistence in the price change variance, in addition to the persistence implied by the common dynamic mixing variable, reveals that the common mixing variable does not eliminate the high persistence in the price change variance.

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# APPENDIX: THE ACCELERATED GAUSSIAN IMPORTANCE SAMPLER

## Step (0): Initial Sampler

- (i) Simulate a set of  $N$  independent random vectors  $\{U_n\}_{n=1}^N$ , each of which is drawn from a  $T$ -dimensional standardized multivariate normal distribution. These  $(T \times N)$  simulated random variables are used in all steps of the AGIS algorithm and constitute the so-called common random numbers.
- (ii) Use these common random numbers to generate a first set of  $\{\Lambda_{0,n}\}_{n=1}^N$  according to the initial sampling function given in Equation (16), where the value of  $\lambda_t$  in time period  $t = 0$  is set equal to zero.
- (iii) The initial sampling function is the product of conditional univariate normal densities:

$$\mu_0(\Lambda | dP, V) = \prod_{t=1}^T \mu_{0,t}(\lambda_t | \lambda_{t-1}) \equiv \prod_{t=1}^T f(\lambda_t | \lambda_{t-1}).$$

Defining the vectors  $\eta'_t = (\lambda_t, \lambda_{t-1}, 1)$  and  $z'_t = (\lambda_{t-1}, 1)$ , the individual components of the initial sampler are written as:

$$\mu_{0,t}(\lambda_t | \lambda_{t-1}) = c_{0,t}(z_t) \exp\left\{-\frac{\eta'_t M_{0,t} \eta_t}{2}\right\},$$

where

$$M_{0,t} = \begin{pmatrix} H_{0,t} & H_{0,t} B_{0,t} \\ B'_{0,t} H_{0,t} & D_{0,t} + B'_{0,t} H_{0,t} B_{0,t} \end{pmatrix},$$

$$c_{0,t}(z_t) = \frac{1}{\sqrt{2\pi}} \sqrt{H_{0,t}} \exp\left\{\frac{z'_t D_{0,t} z_t}{2}\right\}.$$

With  $E_0(\lambda_t | \lambda_{t-1}) = -B_{0,t} z_t$  and  $\text{Var}_0(\lambda_t | \lambda_{t-1}) = H_{0,t}^{-1}$  as the sequential, conditional means and variances of the initial sampler, which is characterized by the parameters given in Equation (7), the following matrixes are identified:

$$M_{0,t} = \frac{1}{\nu^2} \begin{pmatrix} 1 & -\delta & -\gamma \\ -\delta & \delta^2 & \delta\gamma \\ -\gamma & \delta\gamma & \gamma^2 \end{pmatrix}, \quad D_{0,t} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_{0,t} = (-\delta, -\gamma), \quad H_{0,t} = \frac{1}{\nu^2}.$$

**Step (k):  $k$ th step AGIS Sampler and  $k$ th step AGIS estimate.**

(i) Use the set of simulated random vectors of the previous step  $k-1$ :  $\{\Lambda_{k-1,n}\}_{n=1}^N$  to run the auxiliary regressions described in Equation (23) and calculate the matrixes  $\hat{Q}_k = \{\hat{Q}_{k,t}\}_{t=1}^T$  according to (24).

(ii) In the same fashion as for the initial sampler the  $k$ th step AGIS sampler can be expressed as:

$$\mu_k(\Lambda | dP, V) = \prod_{t=1}^T \mu_{k,t}(\lambda_t | \lambda_{t-1}) ,$$

where its components are given by:

$$\mu_{k,t}(\lambda_t | \lambda_{t-1}) = c_{k,t}(z_t) \exp\left\{ -\frac{\eta_t' M_{k,t} \eta_t}{2} \right\} ,$$

with:

$$M_{k,t} = \begin{pmatrix} H_{k,t} & H_{k,t} B_{k,t} \\ B_{k,t}' H_{k,t} & D_{k,t} + B_{k,t}' H_{k,t} B_{k,t} \end{pmatrix} ,$$

$$c_{k,t}(z_t) = \frac{1}{\sqrt{2\pi}} \sqrt{H_{k,t}} \exp\left\{ \frac{z_t' D_{k,t} z_t}{2} \right\} .$$

The sequential conditional moments of the  $k$ th step AGIS sampler are  $E_k(\lambda_t | \lambda_{t-1}) = -B_{k,t} z_t$  and  $\text{Var}_k(\lambda_t | \lambda_{t-1}) = H_{k,t}^{-1}$ . Defining the selection matrix

$$S_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad t = 2, \dots, T+1 ,$$

the matrixes  $\{M_{k,t}\}_{t=1}^T$  are calculated by backward recursion, beginning in  $t = T$  and ending with  $t = 1$ :

$$M_{k,t} = M_{0,t} + \hat{Q}_{k,t} + S_{t+1}' P_{k,t+1} S_{t+1} , \quad t = 1, \dots, T ,$$

where:

$$P_{k,t+1} = \begin{cases} D_{k,t+1} - D_{0,t+1} & \text{if } t = 1, \dots, T-1 \\ 0 & \text{if } t = T \end{cases} .$$

Once the matrixes  $\{M_{k,t}\}_{t=1}^T$  have been calculated the sequential, conditional moments of the  $k$ th step AGIS sampler are determined and used to generate a new set of random vectors  $\{\Lambda_{k,n}\}_{n=1}^N$

(iii) The  $k$ th step AGIS estimate of  $f(dP, V | \theta)$  is calculated according to Equation (25) by:

$$\hat{f}_{N,k}(dP, V | \theta) = \frac{1}{N} \sum_{n=1}^N \frac{h_0(\Lambda_{k,n}, dP, V) \kappa(\hat{Q}_k)}{\xi(\Lambda_{k,n}, \hat{Q}_k)} \quad ,$$

where the integration constant is given with:

$$\kappa(\hat{Q}_k) = \left( \prod_{t=1}^T \sqrt{\frac{H_{0,t}}{H_{k,t}}} \right) \exp \left\{ - \frac{P_{k,1}}{2} \right\} .$$

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Table 1. Statistical properties of the price changes and the  
detrended volume data

Statistics	SIE	DAI	VOW	DBK
Price changes				
Mean	.004	.005	-.001	.000
Standard deviation	1.198	1.531	1.798	1.245
Excess kurtosis	7.587	4.243	5.394	6.541
$LB_{dp}(20)$	22.55 (.3112)	17.64 (.6110)	7.96 (.9921)	23.05 (.2862)
$LB_{dp^2}(20)$	102.93 (.0000)	278.21 (.0000)	42.29 (.0025)	97.72 (.0000)
Trading volume				
Mean	.694	.685	.662	.642
Standard deviation	.298	.314	.326	.324
$LB_v(20)$	637.61 (.0000)	667.27 (.0000)	829.29 (.0000)	620.27 (.0000)

NOTE: Sample size is 1095 for SIE, DAI, VOW and 1096 for DBK.

Marginal significance levels are in parentheses.

Table 2. SML estimation of the univariate mixture model for price changes

Parameter	SIE	DAI	VOW	DBK
$\mu_{dp}$	-.0010 (.0250)	.0034 (.0347)	.0236 (.0385)	.0235 (.0249)
$\gamma$	-.0014 (.0079)	.0211 (.0103)	.0841 (.0263)	.0006 (.0074)
$\delta$	.9526 (.0165)	.9535 (.0142)	.9055 (.0285)	.9629 (.0127)
$\nu$	.2488 (.0408)	.2715 (.0391)	.3031 (.0492)	.2408 (.0359)
Log-likelihood	-1612.9 [.0512]	-1877.5 [.0604]	-2117.9 [.0611]	-1641.7 [.0595]

NOTE: The estimated model is described in Equations (5) and (7). Standard errors are in parentheses and estimated MC sampling standard deviations are in brackets. The estimation are based on a simulation sample size  $N = 500$  and four AGIS iterations.



Table 3. SML estimation of the standard bivariate model

Parameter	SIE	DAI	VOW	DBK
$\mu_{dp}$	-.0569 (.0315)	-.0683 (.0368)	-.1063 (.0416)	-.0495 (.0012)
$\mu_v$	.5276 (.0217)	.3298 (.0132)	.2194 (.0091)	.4655 (.0086)
$\sigma_v$	.0590 (.0117)	.0690 (.0082)	.0324 (.0059)	.0118 (.0002)
$\gamma$	.0791 (.0205)	.2193 (.0275)	.3864 (.0303)	.0853 (.0025)
$\delta$	.5986 (.0293)	.6538 (.0282)	.6117 (.0206)	.6012 (.0128)
$\nu$	.3143 (.0109)	.3467 (.0135)	.3770 (.0095)	.3712 (.0082)
Log-likelihood	-1545.6 [.0037]	-1919.2 [.0073]	-2063.8 [.0031]	-1544.4 [.0002]
SIC	3133.1	3880.5	4169.6	3130.7

NOTE: The estimated model is described in Equations (5), (6), and (7). Standard errors are in parentheses and estimated MC sampling standard deviations are in brackets. The estimation are based on a simulation sample size  $N = 500$  and four AGIS iterations.

Table 4. SML estimation of the modified bivariate model

Parameter	SIE	DAI	VOW	DBK
$\mu_{dp}$	-.0650 (.0273)	-.0719 (.0230)	-.1117 (.0231)	-.0519 (.0305)
$c$	.0174 (.0026)	.0229 (.0041)	.0113 (.0031)	.0145 (.0026)
$c \cdot m_0$	.2115 (.0304)	.1176 (.0427)	.0906 (.0297)	.1858 (.0229)
$c \cdot m_1$	.3589 (.0307)	.2716 (.0254)	.1860 (.0139)	.3219 (.0235)
$\gamma$	.0459 (.0196)	.1770 (.0268)	.3378 (.0341)	.0418 (.0202)
$\delta$	.6850 (.0301)	.7041 (.0314)	.6543 (.0277)	.7057 (.0282)
$\nu$	.4091 (.0242)	.3944 (.0262)	.4215 (.0217)	.4729 (.0232)
Log-likelihood	-1532.6 [.1806]	-1916.5 [.0697]	-2060.2 [.0533]	-1524.6 [.2086]
SIC	3114.1	3882.0	4169.3	3098.1

NOTE: The estimated model is described in Equations (5), (7), and (8). Standard errors are in parentheses and estimated MC sampling standard deviations are in brackets. The estimation are based on a simulation sample size  $N = 500$  and four AGIS iterations.

Table 5. SML estimation of the generalized standard mixture model

Parameter	SIE	DAI	VOW	DBK
$\mu_{dp}$	-.0616 (.0262)	-.0354 (.0256)	-.1068 (.0224)	-.0195 (.0093)
$\mu_v$	1.2628 (.0248)	.7136 (.0221)	.3602 (.0256)	.9297 (.0460)
$\sigma_v$	.0753 (.0217)	.1008 (.0107)	.0426 (.0105)	.0259 (.0110)
$\gamma$	-.2811 (.0227)	-.0484 (.0156)	.1917 (.0318)	-.1892 (.0220)
$\delta$	.5856 (.0312)	.6527 (.0284)	.6158 (.0266)	.6044 (.0220)
$\nu$	.3212 (.0120)	.3464 (.0135)	.3753 (.0121)	.3692 (.0083)
$\alpha_1$	.0531 (.0244)	-.0213 (.0245)	-.0331 (.0211)	-.0383 (.0118)
$\alpha_2$	.1119 (.0246)	.1270 (.0246)	.0882 (.0216)	.0837 (.0184)
$\alpha_3$	.3453 (.0249)	.2403 (.0246)	.1525 (.0257)	.3235 (.0290)
$\alpha_4$	.1456 (.0260)	.2010 (.0227)	.0664 (.0246)	.2274 (.0267)
$\alpha_5$	.2251 (.0250)	.0811 (.0224)	.0710 (.0206)	.1027 (.0334)
Log-likelihood	-1473.4 [.0023]	-1864.2 [.0068]	-2038.9 [.0033]	-1493.2 [.0005]
SIC	3023.7	3805.4	4154.8	3063.3
Likelihood ratio test	144.2	146.0	49.8	102.4

NOTE: The estimated model is described in Equations (6), (7), and (26). Standard errors are in parentheses and estimated MC sampling standard deviations are in brackets. The estimation are based on a simulation sample size  $N = 500$  and four AGIS iterations.

Table 6. SML estimation of the generalized modified mixture model:

Parameter	SIE	DAI	VOW	DBK
$\mu_{dp}$	-.0618 (.0261)	-.0361 (.0233)	-.1089 (.0247)	-.0192 (.0255)
$c$	.0124 (.0027)	.0152 (.0038)	.0100 (.0030)	.0117 (.0024)
$c \cdot m_0$	.1421 (.0337)	.0145 (.0355)	.0713 (.0293)	.1494 (.0235)
$c \cdot m_1$	.9699 (.1009)	.6958 (.0674)	.3149 (.0306)	.6770 (.0604)
$\gamma$	-.2413 (.0327)	-.0475 (.0282)	.1735 (.0296)	-.1564 (.0229)
$\delta$	.6471 (.0254)	.6590 (.0299)	.6478 (.0289)	.6852 (.0248)
$\nu$	.3759 (.0210)	.3516 (.0183)	.4094 (.0209)	.4461 (.0215)
$\alpha_1$	.0429 (.0363)	-.0224 (.0281)	-.0364 (.0224)	-.0571 (.0288)
$\alpha_2$	.1059 (.0326)	.1261 (.0264)	.0866 (.0223)	.0714 (.0281)
$\alpha_3$	.3480 (.0293)	.2396 (.0259)	.1500 (.0245)	.3095 (.0424)
$\alpha_4$	.1395 (.0303)	.2008 (.0246)	.0666 (.0228)	.2223 (.0413)
$\alpha_5$	.2284 (.0344)	.0816 (.0240)	.0735 (.0229)	.1213 (.0383)
Log-likelihood	-1467.8 [.0845]	-1864.1 [.0096]	-2036.6 [.0389]	-1480.2 [.1610]
SIC	3019.5	3812.2	4157.1	3044.3
Likelihood ratio test	129.6	104.8	47.2	88.8

NOTE: The estimated model is described in Equations (7), (8), and (26). Standard errors are in parentheses and estimated MC sampling standard deviations are in brackets. The estimation are based on a simulation sample size  $N=500$  and four AGIS iterations.