

Reiß, Ariane

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## Option replication with large transactions costs

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**Wirtschaftswissenschaftliche Fakultät  
der Eberhard-Karls-Universität Tübingen**

Option Replication With Large  
Transactions Costs

Ariane Reiß



**Tübinger Diskussionsbeiträge**

Wirtschaftswissenschaftliche Fakultät  
der Eberhard-Karls-Universität Tübingen

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Transactions Costs

Ariane Reiß

Universität Tübingen  
Lehrstuhl für betriebliche Finanzwirtschaft  
Mohlstraße 36  
72074 Tübingen  
Germany  
Tel.: (0) 7071 / 29-78204  
Fax.: (0) 7071 / 55 06 22  
E-Mail: [ariane.reiss@uni-tuebingen.de](mailto:ariane.reiss@uni-tuebingen.de)

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## 1. Introduction

Preference-free option pricing theory relies on the redundancy of the payoff of the option: Whenever it is possible to form a self-financing strategy in the underlying stock and a riskfree bond which exactly replicates the payoff of the option contract at maturity, the initial endowment required must be the value of the option in the absence of arbitrage opportunities. If proportional transactions costs are incurred each time shares of the stock are traded the replicating strategy no longer is self-financing. The value of the portfolio must suffice to buy the appropriate replicating portfolio the following trading dates regardless of the stock price evolution and to cover the transactions costs incurred in the restructuring.

In a binomial framework, the stock price either moves up or down, so that a European option only takes two different values the next trading date. Hence, it is possible to construct a self-financing portfolio inclusive of transactions costs if the stock price follows a binomial process (cf. Boyle and Vorst (1990)) and not a continuous-time process (cf. Leland (1985) and Lott (1993)). However, in a market with frictions perfect replication need not be an efficient strategy (cf. Bensaid, Lesne, Pages and Scheinkman (1992) and Edirisinghe, Naik and Uppal (1993)).<sup>1</sup> In a binomial framework the stock price at maturity is bounded from above and from below. Therefore it may be optimal to refrain from rebalancing in a few states of nature such that on the one hand the option payoff is at least guaranteed if not exceeded and on the other hand transactions costs are saved. This so called super-replicating strategy may dominate an exact replicating strategy in that it not only has a lower initial endowment but also leads to a larger payoff at maturity.

The analysis presented here extends the existing literature in several ways. Like Bensaid, Lesne, Pages and Scheinkman (1992) we give conditions so that the super-replicating strategy is dominant. We model a different transactions cost structure and give an economic explanation for the condition of dominance. Furthermore, we prove that super-replication is never optimal if every investor faces the same magnitude of transactions costs. In addition, we consider the case of differential transactions costs. We present a closed-form solution for the price of a Short European call option and show how the price of the Long call option can be solved computationally easily by determining the time after which no more

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<sup>1</sup>Avellaneda and Paras (1994), Clark and Davis (1994), and Soner, Shreve and Cvitanic (1995) analyze super-replicating strategies in continuous time. The least-cost super-replicating strategy for a Long option consists in one share.

transactions take place.

## 2. The Model

In this section we characterize the price system that does not admit arbitrage opportunities and rules out the possibility of dominance between the stock and the riskless security. This price system differs from the one in a frictionless market if transactions costs with rate  $k$  must be paid each time shares of the risky asset are traded.

Short-selling of the stock and purchasing after one period does not lead to arbitrage opportunities if the ask price in the up-state  $Su(1+k)$  exceeds the return of riskfree investing the bid price  $(1-k)S$  with interest rate per period  $r$ . Going long one share and selling it after one period does not lead to an arbitrage opportunity if the ask price in the down-state  $Sd(1-k)$  is below the amount required to pay off the credit for purchasing the share. The set of restrictions for the price system that prevents dominance of the stock price process can thus be written as

$$u(1+k) > (1-k)(1+r) \quad (1+k)(1+r) > d(1-k). \quad (2.1)$$

If we assume that  $ud = 1$  the first condition implies the second one. This condition is less stringent than the set of restrictions in a market without frictions  $u > (1+r) > d$ . Consequently, the stock price process needs not be arbitrage-free in a frictionless model.

To rule out dominance opportunities between the stock and the riskless security a necessary condition is that going long one share of stock and selling it the next trading date leads to a larger return than the riskless security if the stock moves up and that short-selling one share and putting the proceeds in the bank account leads to a profit as long as the stock is purchased in the down-state the next trading date. Hence, the following set of restrictions prevents dominance of the riskfree security:<sup>2</sup>

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<sup>2</sup>Merton (1990), p. 433, considers only the right-hand-side condition to preclude dominance opportunities.

$$u \frac{1-k}{1+k} > 1+r > d \frac{1+k}{1-k}. \quad (2.2)$$

Assuming  $ud = 1$  we only have to deal with the left-hand-side of (2.2). This restriction is more stringent than the set of restrictions in the frictionless model. If this restriction is not satisfied, neither going short nor going long for one period is favorable. While inequalities (2.1) become less stringent with increasing transactions costs factor, (2.2) becomes more stringent. If (2.2) is satisfied, the stock price process is arbitrage-free in the frictionless model as well as in our model.

### 3. Efficient Replicating Strategies

In order to find efficient strategies to replicate the payoff of a European option contract at maturity we formulate the traditional binomial model as a nonlinear programming problem where the objective is to find the least-cost initial portfolio that meets the target payoff and is self-financing. Let  $R_t(\varphi_{t-1}, \omega^t)$ ,  $t \in [1, T]$  be the least amount of cash needed at  $t$  in order to synthesize the option payoff  $C_T$  at the terminal date  $T$  with  $\varphi_{t-1}$  being the number of shares inherited from time  $t-1$ .  $\omega^t = (\omega_1, \dots, \omega_t)$  with  $(\omega_s = u, d \mid s = 1, \dots, t)$  denotes the current state.

At the terminal date, the minimal amount of cash required to replicate  $C_T$  corresponds to the least required riskless bond position  $\beta_T$  with par value 1 when the number of stocks inherited from time  $T-1$  is  $\varphi_{T-1}$  and no rebalancing is needed at  $T$ :

$$C_T(S_T(\omega^T)) \leq \varphi_{T-1} S_T(\omega^T) + \beta_T(\omega^T)$$

There is no need to revise at the terminal date if option replication only means having a portfolio with the same value as the option regardless of the composition of the portfolio, i. e. one is not obliged to be long one share of stock in case a Long call option ends in the money. If the value of the portfolio matches the option payoff perfectly we have exact replication and if the value exceeds the option payoff we have super-replication.

In all periods prior to  $T$  the amount of cash is minimized with the requirement that the cash is sufficient to cover the cost of the restructuring and to cover the discounted amount of money needed the next trading date. The required amount

of cash the following date either is larger when the stock price moves up with factor  $u$  or when it moves down with factor  $d$ . Given the requirement that the cash covers the maximum amount of cash needed the next trading date and the cost of rebalancing we derive the following sequential problem<sup>3</sup>

$$R_t(\varphi_{t-1}, \omega^t) = \min_{\varphi_t} R_{t+1}^{\text{Max}}(\varphi_t, \omega^{t+1}) + S_t(\omega^t) [(\varphi_t(\omega^t) - \varphi_{t-1}) + k |\varphi_t(\omega^t) - \varphi_{t-1}|] \quad (3.1)$$

subject to the terminal constraint

$$R_T(\varphi_{T-1}, \omega^T) = C_T (S_T(\omega^T)) - \varphi_{T-1} S_T(\omega^T) \quad (3.2)$$

and to the initial constraint

$$R_0 = \min_{\varphi_0} R_1^{\text{Max}}(\varphi_0) + \varphi_0 S_0,$$

where  $R_{t+1}^{\text{Max}}(\varphi_t, \omega^{t+1})$  is defined as

$$R_{t+1}^{\text{Max}}(\varphi_t, \omega^{t+1}) = \frac{1}{1+r} \text{Max} [R_{t+1}(\varphi_t, (\omega^t, u)), R_{t+1}(\varphi_t, (\omega^t, d))] . \quad (3.3)$$

In order to eliminate the dependence of the solution upon the initial endowment in the stock we assume zero transactions costs in establishing the initial portfolio at  $t = 0$ . The initial endowment to replicate the option payoff with a self-financing strategy is simply the value of the option  $C_0$ . The strategy is self-financing in the sense that the proceeds of the sale of the assets are entirely reinvested in the portfolio after the payment of transactions costs.

A necessary and sufficient condition of optimality for this nonlinear program is

$$\frac{\partial_- R_t}{\partial \varphi_t} \leq 0 \leq \frac{\partial_+ R_t}{\partial \varphi_t} \quad (3.4)$$

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<sup>3</sup>Cf. Bensaid, Lesne, Pages and Scheinkman (1992), p. 73, with a different transactions cost structure.



at point  $\varphi_t = \varphi_t^{opt}$ ,<sup>4</sup> where  $\varphi_t^{opt}$  denotes the optimal number of shares.  $\frac{\partial_- R_t}{\partial \varphi_t}$  denotes the right-hand derivative and  $\frac{\partial_+ R_t}{\partial \varphi_t}$  the left-hand derivative. This leads to the following conditions of optimality of the strategy at point  $\varphi_t = \varphi_t^{opt}$  for all<sup>5</sup> states  $\omega^t$

<u>Purchase (<math>\varphi_t &gt; \varphi_{t-1}</math>)</u>	<u>Sale (<math>\varphi_t &lt; \varphi_{t-1}</math>)</u>
$\frac{\partial_- R_{t+1}^{Max}(\varphi_t)}{\partial \varphi_t} + S_t(1+k) \leq 0$	$\frac{\partial_- R_{t+1}^{Max}(\varphi_t)}{\partial \varphi_t} + S_t(1-k) \leq 0$
$\frac{\partial_+ R_{t+1}^{Max}(\varphi_t)}{\partial \varphi_t} + S_t(1+k) \geq 0$	$\frac{\partial_+ R_{t+1}^{Max}(\varphi_t)}{\partial \varphi_t} + S_t(1-k) \geq 0$

We now check these conditions for every trading date in order to find the optimal policy. At time  $T$  the necessary amount of cash is apparent from (3.2). Hence, the maximum required cash at  $T - 1$  (3.3) amounts to

$$R_{T-1}^{Max}(\varphi_{T-1}) = \frac{1}{1+r} \text{Max} [C_T(S_{T-1}u) - \varphi_{T-1} S_{T-1}u, \tag{3.5}$$

$$C_T(S_{T-1}d) - \varphi_{T-1} S_{T-1}d].$$

The optimality criteria are satisfied at time  $T - 1$  for all states  $\omega^{T-1}$  because of the condition of dominance (2.2).

<u>Purchase (<math>\varphi_{T-1} &gt; \varphi_{T-2}</math>)</u>	<u>Sale (<math>\varphi_{T-1} &lt; \varphi_{T-2}</math>)</u>
$-\frac{S_{T-1}u}{1+r} + S_{T-1}(1+k) < 0$	$-\frac{S_{T-1}u}{1+r} + S_{T-1}(1-k) < 0$
$-\frac{S_{T-1}d}{1+r} + S_{T-1}(1+k) \geq 0$	$-\frac{S_{T-1}d}{1+r} + S_{T-1}(1-k) \geq 0$

<sup>4</sup>Cf. Bensaid, Lesne, Pages and Scheinkman (1992), p. 73.

<sup>5</sup>In the following we refrain from explicitly stating that the number of stocks, bonds and the required cash have the state variable as an argument if this is clear.

The least number of stocks is derived by equating the amount of cash needed in the up-state with the amount needed in the down-state. This leads to the optimal number of shares  $\varphi_{T-1}^{opt}$  and bonds  $\beta_{T-1}^{opt}$ . These numbers correspond to the numbers in a frictionless market. Neither the optimal amount of shares nor the optimal amount of bonds depends on the inherited stock position. Consequently, the strategy is path independent.

The maximum amount of cash needed at  $T-1$  is for all  $\omega^{T-2}$

$$\begin{aligned}
R_{T-1}^{Max}(\varphi_{T-2}) &= \frac{1}{1+r} \text{Max} [\beta_{T-1}^{opt}(u) + (1+k) (\varphi_{T-1}^{opt}(u) - \varphi_{T-2}) S_{T-2}u, \\
&\quad \beta_{T-1}^{opt}(u) + (1-k) (\varphi_{T-1}^{opt}(u) - \varphi_{T-2}) S_{T-2}u, \\
&\quad \beta_{T-1}^{opt}(d) + (1+k) (\varphi_{T-1}^{opt}(d) - \varphi_{T-2}) S_{T-2}d, \\
&\quad \beta_{T-1}^{opt}(d) + (1-k) (\varphi_{T-1}^{opt}(d) - \varphi_{T-2}) S_{T-2}d],
\end{aligned} \tag{3.6}$$

where  $\beta_{T-1}^{opt}(u)$ ,  $\beta_{T-1}^{opt}(d)$  is defined as the optimal bond position and  $\varphi_{T-1}^{opt}(u)$ ,  $\varphi_{T-1}^{opt}(d)$  is defined as the optimal stock position in the up-state and down-state at  $T-1$ . The required cash at  $T-1$  is conditioned on the inherited stock and bond position and therefore either is largest when purchasing or selling in the up-state or when purchasing or selling in the down-state. The optimal stock position depends on when the amount of cash needed is largest.

Replicating a Long European call option means synthesizing a derivative with a convex payoff. In this case the optimal number of bonds decreases with the stock price and the optimal number of shares increases with the stock price. The required cash is either maximum when purchasing in the up-state or when selling in the down-state. Thus, only the first and the fourth term of (3.6) are relevant. In this case the conditions of optimality are satisfied for purchasing and for selling at  $T-2$  and the stock and bond position is derived by equating the amount of cash needed when purchasing after an up-move with the amount required when selling after a down-move. The optimal numbers are path independent so that the solution is independent of the history of the stock price.

When replicating a Short European call option, i.e. a concave payoff, the amount required may be maximum in all four cases since the optimal bond position

is increasing with the stock price and the optimal number of shares is decreasing with the stock price and because of the inequality  $u(1 - k) > d(1 + k)$  implied by (2.2). The optimality criteria are satisfied when selling in the up-state and purchasing in the down-state. The optimal number of stocks and bonds in this case is thus path independent, too.

It can be shown by induction that the optimal strategy for replicating Long and Short call options, respectively, is unique and path independent for all  $t < T$ . The optimal bond and stock position in all prior periods can be derived in an analogous way. The value of the option at time  $t = 0$  is simply the least initial cost of the replicating strategy.

The optimal number of shares  $\varphi^S$  and bonds  $\beta^S$  when replicating a Short call option differs from the optimal strategy  $\varphi^L$  and  $\beta^L$  when replicating a Long call option. Thus the value of a Short call option  $C_0^S$  differs from the value of a Long call option  $C_0^L$ . The value of the Long option increases with increasing transactions costs because more costs are incurred by duplicating the desired payoff with traded assets. The value of the Short call option is below the value of the Long option and decreases with increasing transactions costs because the return from the desired payoff becomes less with larger transactions costs. We obtain a range of option prices  $[C_0^S, C_0^L]$  that is consistent with no arbitrage. If an investor sells an option for  $C_0^L$  closing the position costs  $C_0^L$ . Consequently, it is not possible to gain riskless arbitrage profits.

The nonlinear optimization problem yields an optimal strategy that requires revising in every trading date. The sequentially optimal strategy therefore is entirely independent of the history of the stock price.<sup>6</sup> The least-cost strategy replicates the given payoff exactly. This result is entirely due to the condition of dominance (2.2) that makes sure that the optimality criteria are satisfied in case of purchasing as well as selling stock. If the condition of dominance is not satisfied no investor will go short or long stock for just one period since the riskless asset dominates the stock over one period. In this case a few up-moves are required so that the discounted bid price exceeds the current ask price and the investor is compensated for the transactions costs and sells the stock. On the other hand a few down-moves are necessary for the current ask price to be lower than the discounted bid price so that the investor is compensated for the transactions costs when he purchases stock. Thus the conditions of dominance are necessary to ensure trade at every trading date.

With an increasingly finer partition of the binomial tree the value of the Short

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<sup>6</sup>Boyle and Vorst (1990) derive the same strategy by using a recursive binomial tree approach.

option decreases and the value of the Long option increases. The finer the partition the lower is the magnitude of the up-move  $u$  and the down-move  $d$  in one period. The condition of dominance (2.2) thus becomes more stringent.

#### 4. Differential Transactions Costs

The last section established that the only efficient replicating strategy is an exact replicating strategy with trading in every period. This solution is due to the condition of dominance. This condition states that trading stock may be optimal at every trading date depending on the evolution of the stock. However, in real life it is often experienced that holding stock for only one period is neither optimal in case of a down-move nor in case of an up-move and that selling stock and purchasing it the next trading date is not optimal irrespective of the evolution of stock. This results from large transactions costs. It is impossible for the condition of dominance not to be satisfied by every investor since stock is traded in every period. However, if different investors face different transactions costs, the condition must only be fulfilled by those investors with the lowest transactions costs.<sup>7</sup>

For those investors the value of the option is derived by determining the initial cost of an exact replicating strategy. Those investors duplicate an option with the least cost. Thus, they offer the least price and they charge the highest price when selling the option. Hence, these investors determine the upper and lower bounds for the market prices of options. Those investors with large transactions costs have to determine their individual writing (buying) price. This price is the least endowment that guarantees that they are not worse off being short (long) the option than having no option at all. Once they have determined this price they are prepared to join negotiations about the option price. The buying and the writing price, respectively must not be the least cost of a perfect replicating strategy since the condition of dominance must not be fulfilled by those investors. If the condition of dominance is satisfied the price corresponds to the value of an exact replicating portfolio.<sup>8</sup> If the condition is not satisfied the riskless asset

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<sup>7</sup>Cf. Gould and Galai (1974): „The efficiency of a market is presumably determined by the set of individuals with the lowest transactions costs”. (p. 110).

<sup>8</sup>Bensaid, Lesne, Pages and Scheinkman (1992) give conditions under which exact replication is optimal. The resulting upper boundary for their proportional transactions cost factor corresponds to our condition of dominance. Cf. Bensaid, Lesne, Pages and Scheinkman (1992), Lemma 3.2, p. 74.

dominates the risky asset over one period. Exact replication requires revising in every trading date. This may lead to waste of money. Establishing a larger position initially and reduced trading in later periods may result in terminal cash flows that exceed those of the option and requires less initial wealth. Thus a so-called super-replicating strategy may be optimal. We will analyze this case below.

#### 4.1. Valuation by Investors with Large Transactions Costs

In this section we compute the minimum amount investor  $i$  has to invest initially in order to obtain terminal proceeds at least as valuable as the required payoff in case a transactions cost factor  $k_i$  is charged investor  $i$  every time shares are traded.  $k_i$  is larger than permitted by (2.2). Therefore investor  $i$  is better off holding bonds than stock regardless of the evolution of the stock when selling after one period:

$$u(1 - k_i) < (1 + k_i)(1 + r). \quad (4.1)$$

With  $ud = 1$  it is still possible that the condition of dominance is satisfied by the down-move. However, here we consider the case that the condition of dominance is violated by the down-move as well, so that short-selling stock over one period is never favorable to investor  $i$ :

$$d(1 + k_i) > (1 + r)(1 - k_i). \quad (4.2)$$

We further assume that

$$u < (1 + r)(1 + k_i) \text{ and } d > (1 + r)(1 - k_i) \quad (4.3)$$

holds for investor  $i$ . These conditions are more stringent than (4.1) and (4.2) and state that lending the ask price and borrowing the bid price dominates trading stock even if selling and purchasing stock the next trading date is free of

transactions charge. Inequalities (4.3) imply

$$d(1 + k_i) > u(1 - k_i).$$

Since (2.2) must be satisfied by those investors with the least transactions costs, i.e. by the investors for whom exact replication is optimal, the following constraint is necessary:

$$u > 1 + r > d$$

The constraints of the sequentially minimizing problem (3.1) for investor  $i$  require riskfree securities at time  $T-1$  which at least amount to  $R_T$  with accrued interest regardless of the evolution of stock. Due to inequality (4.3) we obtain the following gradients of the minimizing function (3.1) at  $T-1$ :

$$\begin{array}{ll} \text{Purchase } (\varphi_{T-1}^i > \varphi_{T-2}^i) & \text{Sale } (\varphi_{T-1}^i < \varphi_{T-2}^i) \\ -\frac{S_{T-1}u}{1+r} + S_{T-1}(1+k_i) \geq 0 & -\frac{S_{T-1}u}{1+r} + S_{T-1}(1-k_i) < 0 \\ -\frac{S_{T-1}d}{1+r} + S_{T-1}(1+k_i) \geq 0 & -\frac{S_{T-1}d}{1+r} + S_{T-1}(1-k_i) < 0 \end{array} \quad (4.4)$$

The conditions of optimality (3.4) at  $T-1$  are therefore not satisfied irrespectively of whether stock is purchased or sold. However, due to  $u > 1 + r > d$  they are satisfied if no transactions occur. Consequently, the optimal number of shares  $\varphi_{T-1}^i$  at time  $T-1$  equals the number of shares  $\varphi_{T-2}^i$  at time  $T-2$ .

At time  $T-2$  the following amount of money is necessary

$$R_{T-1}^{i\text{Max}}(\varphi_{T-2}^i, \omega^{T-1}) = \frac{1}{1+r} \text{MAX} \left\{ R_{T-1}^i(\varphi_{T-2}^i, (\omega^{T-2}, u)), R_{T-1}^i(\varphi_{T-2}^i, (\omega^{T-2}, d)) \right\}, \quad (4.5)$$

where  $R_{T-1}^i$  is apparent from (3.5) with  $\varphi_{T-1}$  replaced by  $\varphi_{T-2}^i$ .

Let  $n \in \mathbb{N}$  be the least number of up-moves required so that the stock price exceeds the return from riskfree investment of the ask price over  $n$  periods. Furthermore, let  $n' \in \mathbb{N}$  be the least number of down-moves required so that the

stock price is less than the repayment from riskfree borrowing of the bid price over  $n'$  periods:

$$\frac{u^n}{(1+r)^n} > 1 + k_i > \frac{u^{n-1}}{(1+r)^{n-1}} \quad (4.6)$$

and

$$\frac{d^{n'}}{(1+r)^{n'}} < 1 - k_i < \frac{d^{n'-1}}{(1+r)^{n'-1}}. \quad (4.7)$$

It is assumed without loss of generality that  $n = n'$ , so that  $n$  represents the number of holding periods after which neither purchasing nor selling stock is dominated by the riskfree security.

The gradient of the objective function at time  $T-2$  is negative for selling and positive for purchasing for every possible maximum amount of cash (4.5) if  $T > n > 2$ . Therefore it is optimal not to trade at all at this point of time.

Accordingly from time  $T-n$  till  $T$  restructuring means more loss in transactions costs than improvement of the accuracy of replication. Consequently, investor  $i$  refrains from transacting during this period. At time  $T-n$  we obtain the following required amount of cash

$$R_{T-n}^{i\text{Max}}(\varphi_{T-n}^i) = \frac{1}{(1+r)^n} \text{MAX} \{ C_T (S_{T-n} u^{n-j} d^j) - \varphi_{T-n}^i S_{T-n} u^{n-j} d^j; \\ j \in \mathbb{N}; j \leq n \}. \quad (4.8)$$

Given  $T > n$  the gradients of the objective function are<sup>9</sup>

$$\underline{\text{Purchase } (\varphi_{T-n}^i > \varphi_{T-n-1}^i)} \quad \underline{\text{Sale } (\varphi_{T-n}^i < \varphi_{T-n-1}^i)} \quad (4.9)$$

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<sup>9</sup>Since  $u > 1+r > d$  the conditions of optimality are fulfilled in  $t = 0$  if  $T < n$ . The optimal number of shares and bonds equal (4.12) and (4.13) with  $n$  replaced by  $T$ .

$$\begin{aligned}
-S_{T-n} \frac{u^n}{(1+r)^n} + S_{T-n} (1+k_i) &< 0 & -S_{T-n} \frac{u^n}{(1+r)^n} + S_{T-n} (1-k_i) &< 0 \\
-S_{T-n} \frac{d^n}{(1+r)^n} + S_{T-n} (1+k_i) &\geq 0 & -S_{T-n} \frac{d^n}{(1+r)^n} + S_{T-n} (1-k_i) &\geq 0.
\end{aligned} \tag{4.10}$$

Inequalities (4.6) and (4.7) yield for all  $n > j \geq 1$

$$\begin{array}{cc}
\underline{\text{Purchase } (\varphi_{T-n}^i > \varphi_{T-n-1}^i)} & \underline{\text{Sale } (\varphi_{T-n}^i < \varphi_{T-n-1}^i)} \\
-S_{T-n} \frac{d^j u^{n-j}}{(1+r)^n} + S_{T-n} (1+k_i) > 0 & -S_{T-n} \frac{d^j u^{n-j}}{(1+r)^n} + S_{T-n} (1-k_i) < 0.
\end{array} \tag{4.11}$$

It now depends on the desired payoff at time  $T$  of how many shares the optimal strategy consists since the particular payoff determines which term of (4.8) is possibly largest. In the next section we have a look at the payoff of a Long call option and thereafter we consider Short call options.

#### 4.1.1. Valuation of a Long Call Option

The replication of a Long Call option requires the replication of a convex payoff. This convexity results in the required amount of money (see 4.8) being largest either after mere up-moves or after mere down-moves.<sup>10</sup>

Inequalities (4.9) and (4.10) imply that the conditions of optimality are satisfied at time  $T-n$  both for purchase and sale so that the replicating portfolio is unique as well as path independent. The least amount of stocks is derived by equating the required amount of cash in case of mere up-moves with the required amount in case of mere down-moves. For all states  $\omega^{T-n}$  we obtain the following optimal replicating portfolio

$$\varphi_{T-n}^{i\text{opt}} = \frac{C_T(S_{T-n}u^n) - C_T(S_{T-n}d^n)}{(u^n - d^n) S_{T-n}} \tag{4.12}$$

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<sup>10</sup>Cf. Bensaid, Lesne, Pages and Scheinkman (1992), p. 84.



$$\beta_{T-n}^{i\text{opt}} = (1+r)^{-n} \frac{C_T(S_{T-n}d^n)u^n - C_T(S_{T-n}u^n)d^n}{u^n - d^n}. \quad (4.13)$$

The required amount of cash at the previous date  $T-n-1$  either is maximum when purchasing or selling after an up-move or when purchasing or selling after a down-move for the purpose of adjusting to the perfect replicating portfolio.

Since the option payoff is a convex function of the stock price the optimal amount of riskfree securities at time  $T-n$  in the up-state  $\beta_{T-n}^{i\text{opt}}(u)$  either is smaller and the optimal amount of shares  $\varphi_{T-n}^{i\text{opt}}(u)$  larger than in the down-state or if the stock position in the up-state is of the same quantity as in the down-state then the bond position is the same regardless of the realization of the stock price process as well.

The required amount of cash at time  $T-n-1$  therefore either is maximum when purchasing after an up-move at time  $T-n$  or when selling after a down-move at time  $T-n$ . In this case the optimality criteria are fulfilled regardless of whether purchasing or selling at time  $T-n-1$ . In all periods prior to  $T-n$  we obtain a unique optimal stock position which does not depend on the inherited stock position.

Accordingly if  $n < T$  investor  $i$  adjusts his portfolio permanently till  $T-n$  and refrains from restructuring thereafter because the losses incurred by transactions costs are larger than the gains in the accuracy of the replication.

The date which rings in the period of no transacting equals the date from which on the riskfree security dominates stock. Consequently, this date marks the date where an investment in stock is inferior. Again, the analysis is connected to the condition of dominance.

Investor  $i$  owns a portfolio at time  $T-n$  that leads to a terminal cash flow which perfectly matches the option payoff only in case of mere up-moves or mere down-moves. Up-moves alternating with down-moves result in super-replication. Hence, investor  $i$  pursues a strategy that exactly replicates the option in some states of nature and super-replicates the option in others. The initial value of this strategy is below the initial value of an exact replicating strategy. Thus, given the violation of the condition of dominance super-replication dominates exact replication.

### 4.1.2. Valuation of a Short Call Option

If the payoff to be replicated is not convex then the required amount of cash in  $T-n$  not necessarily either is largest in case of mere up-moves or mere down-moves. Hence, the optimal stock position is not necessarily unique and path independent up to time  $T-n$ . Assuming concavity of the payoff as in a Short call option leads to entirely different conclusions.

Similar to a Long call option no trading takes place after  $T-n$ . At time  $T-n$  required cash is maximum given  $\varphi_{T-n}^i \in (-\infty, -1]$  if mere up-moves occur, and given  $\varphi_{T-n}^i \in [0, \infty)$  if mere down-moves occur. Given  $\varphi_{T-n}^i \in (-1, 0)$  cash requirement is either largest in case of mere up-moves if the stock price after mere up-moves is below the exercise price or in case of mere down-moves if the stock price after mere down-moves exceeds the exercise price. If the exercise price lies between the stock price at  $T$  after mere up-moves and the stock price after mere down-moves, then cash requirement is maximum in case of the stock price at  $T$  matching the striking price perfectly. If there is no node of the binomial tree at  $T$  so that the stock price exactly matches the striking price then the maximum cash requirement either occurs when the stock price just exceeds the striking price ( $S_T^+$ ) or is just below the striking price ( $S_T^-$ ).

If the required amount either is maximum in case of  $n$  up-moves or of  $n$  down-moves then the conditions of optimality (See (4.9) and (4.10)) are fulfilled. If the stock price after  $n$  down-moves exceeds the striking price the optimal strategy consists of one share short and riskfree bonds maturing at time  $T$  with a face value of the striking price  $K$ . If the striking price exceeds the stock price after  $n$  up-moves having neither bonds nor stock is optimum. If the striking price lies between the stock price after  $n$  up- and  $n$  down-moves then a strategy with two control barriers is optimal. Inequalities (4.10) and (4.11) imply that the conditions of optimality are fulfilled for selling at  $T-n$ . On the other hand inequalities (4.9) and (4.11) imply that the conditions are satisfied for purchasing at time  $T-n$ . However, the optimal stock position when purchasing (-1) differs from the optimal position when selling (0). Hence, stock is only purchased if the inherited amount is below the lower bound (-1) and stock is only sold if the upper bound (0) is exceeded. The optimal strategy is a strategy with two boundaries which means that revising only takes place if the inherited stock position  $\varphi_{T-n-1}^i$  either exceeds the upper bound  $\varphi_{T-n}^{i \max}$  or is below the lower bound  $\varphi_{T-n}^{i \min}$ . Then the investor adjusts his portfolio just back to  $\varphi_{T-n}^{i \max}$  and  $\varphi_{T-n}^{i \min}$ , respectively. For a stock position in the range  $[\varphi_{T-n}^{i \min}, \varphi_{T-n}^{i \max}]$  the optimal strategy is to keep the portfolio unchanged because rebalancing involves transactions costs that are larger than the increment

in accuracy of replication. The optimality criteria are satisfied for purchasing if  $\varphi_{T-n-1}^i \leq \varphi_{T-n}^{i \min}$  and for selling if  $\varphi_{T-n-1}^i \geq \varphi_{T-n}^{i \max}$ .

At time  $T-n-1$  we have five scenarios concerning the maximum required amount of cash at time  $T-n$ .

1. If regardless of the realization of the stock price process the optimal strategy consists in -1 shares or in no shares then required cash is maximum when purchasing in the up-state or when selling in the down-state. The conditions of optimality are satisfied and the optimal strategy is path independent. The number of shares amounts to -1 and 0, respectively.
2. If -1 shares are optimal in the up-state and in the down-state restructuring only takes place if the inherited stock position is out of the range  $[-1,0]$  then we obtain the following required cash

$$R_{T-n-1}^{i \max}(\varphi_{T-n-1}^i) = \frac{1}{(1+r)} \text{MAX} \left\{ \frac{K}{(1+r)^n} + (-1 - \varphi_{T-n-1}^i)(1+k_i)S_{T-n-1}u, \right. \\ \left. -\varphi_{T-n-1}^i(1-k_i)S_{T-n-1}d, \beta_{T-n}^K \right\}, \quad (4.14)$$

where  $\beta_{T-n}^K$  denotes the required amount of cash in case the option is at the money or very little out or in the money at time  $T$  and there is no revising at time  $T-n$ . If there is a stock price that exactly matches the striking price  $K$  at  $T$  then we obtain

$$\beta_{T-n}^K = \frac{-\varphi_{T-n-1}^i K}{(1+r)^n}. \quad (4.15)$$

Otherwise we derive

$$\beta_{T-n}^K = \text{MAX} \left\{ \frac{-\varphi_{T-n-1}^i S_T^-}{(1+r)^n}, \frac{-(S_T^+ - K) - \varphi_{T-n-1}^i S_T^+}{(1+r)^n} \right\}, \quad (4.16)$$

where  $S_T^+$  denotes the stock price that is just above the exercise price such that  $(S_T^+ - K)$  is the least possible profit from the option, and  $S_T^-$  is the largest stock price such that one still refrains from exercising the option.

Since (4.16) is more general than (4.15), we only analyze this case.

Depending on whether both  $S_T^+(1+r)^{-(T-n)}$  and  $S_T^-(1+r)^{-(T-n)}$  are below the bid price (case *d*) or both are between the bid and the ask price (case *a*) or whether  $S_T^+(1+r)^{-(T-n)}$  is in the bid-ask-price-range but  $S_T^-(1+r)^{-(T-n)}$  is below the bid price (case *e*), we attain different optimal stock positions. The optimal strategy in case *d* consists of -1 shares, and in case *a* a strategy with control range  $[-1,0]$  is optimal. Case *e* leads to a strategy with control range  $[-1, \bar{\varphi}]$ , where

$$\bar{\varphi} = -\frac{S_T^+ - K}{S_T^+ - S_T^-}.$$

3. If the optimal strategy in the down-state consists of no shares and in the up-state a strategy with control range  $[-1,0]$  is optimal, then again (4.14) reflects possible levels of required cash. The optimal strategy depends on the relation of  $S_T^-, S_T^+$  and  $S$ . In case  $S_T^-$  discounted to  $T-n$  exceeds the ask price (case *c*) no shares are optimum. Case *a* results in a strategy with control barriers  $[-1,0]$ . If  $S_T^+$  discounted to  $T-n$  exceeds the ask price and  $S_T^-$  discounted to  $T-n$  is in the bid-ask-price-range (case *b*) we derive a strategy with control at  $[\bar{\varphi},0]$ .
4. The fifth scenario occurs if a strategy with control is optimal regardless of the realization of the stock price process at  $T-n$ . (4.14) applies and the same strategy is optimal at the previous date.

We only obtain five different optimal strategies at time  $T-n-1$  with respect to the optimal strategy at  $T-n$ . At all prior dates we have to deal with eleven possible scenarios because the optimal strategy in the up-state may differ from the optimal strategy in the down-state.

Some scenarios are ruled out by assumption  $d(1+k_i) > u(1-k_i)$  and by the conditions that must be fulfilled so that the strategy the following date is optimal. First we determine the maximum required cash with respect to the inherited amount of stock. Afterwards we check the conditions of optimality.

To clarify, we present the optimal strategy at time  $t-1$  with respect to the strategy at time  $t$  in tabular form. (The left value in parentheses denotes the

lower bound and the right value the upper bound  $[\varphi_t^{i\min}, \varphi_t^{i\max}]$ . If the lower and the upper bound coincide, the result is a unique optimal number of stocks  $\varphi_t^{i\text{opt}}$ .)

		Strategy at $t$ for $\omega_t = d$				
		$\varphi_t^{i\text{opt}}$	0	$[\bar{\varphi}, 0]$	$[-1, 0]$	$[-1, \bar{\varphi}]$
Strategy at $t$ for $\omega_t = u$	0	0	-	-	-	-
	$[\bar{\varphi}, 0]$	$[\bar{\varphi}, 0]^b$ $0^c$	$[\bar{\varphi}, 0]$	-	-	-
	$[-1, 0]$	$[-1, 0]^a$ $[\bar{\varphi}, 0]^b$ $0^c$	$[-1, 0]^a$ $[\bar{\varphi}, 0]^b$	$[-1, 0]$	-	-
	$[-1, \bar{\varphi}]$	-	$[-1, 0]^a$ $[\bar{\varphi}, 0]^b$	$[-1, \bar{\varphi}]^e$ $[-1, 0]^a$	$[-1, \bar{\varphi}]$	-
	-1	-	-	$-1^d$ $[-1, \bar{\varphi}]^e$ $[-1, 0]^a$	$-1^d$ $[-1, \bar{\varphi}]^e$	-1

Table 1: Strategy at  $t - 1$  with respect to the optimal strategy at  $t$  in the up-state  $\omega_t = u$  and in the down-state  $\omega_t = d$

$$a \text{ for } (1 + k_i)S_{t-1} > \frac{S_T^+}{(1+r)^{T-t+1}} > \frac{S_T^-}{(1+r)^{T-t+1}} > (1 - k_i)S_{t-1}$$

$$b \text{ for } \frac{S_T^+}{(1+r)^{T-t+1}} > (1 + k_i)S_{t-1} > \frac{S_T^-}{(1+r)^{T-t+1}} > (1 - k_i)S_{t-1}$$

$$c \text{ for } \frac{S_T^+}{(1+r)^{T-t+1}} > \frac{S_T^-}{(1+r)^{T-t+1}} > (1 + k_i)S_{t-1}^B > (1 - k_i)S_{t-1}$$

$$d \text{ for } (1 + k_i)S_{t-1} > (1 - k_i)S_{t-1} > \frac{S_T^+}{(1+r)^{T-t+1}} > \frac{S_T^-}{(1+r)^{T-t+1}}$$

$$e \text{ for } (1 + k_i)S_{t-1} > \frac{S_T^+}{(1+r)^{T-t+1}} > (1 - k_i)S_{t-1} > \frac{S_T^-}{(1+r)^{T-t+1}}$$

Hence, in all periods prior to  $T - n$  the investor who is replicating optimally either owns no shares, is one share short or  $-\bar{\varphi}$  shares short. The conditions of optimality lead to the following optimal stock and bond position at  $t = 0$ :

$$(\varphi_0^{iopt}, \beta_0^{iopt}) = \begin{cases} (0, 0) & \text{for } \frac{S_T^+}{(1+r)^T} > \frac{S_T^-}{(1+r)^T} > S_0 \\ \left( \bar{\varphi}, -\bar{\varphi} \frac{S_T^-}{(1+r)^T} \right) & \text{for } \frac{S_T^+}{(1+r)^T} > S_0 > \frac{S_T^-}{(1+r)^T} \\ \left( -1, \frac{K}{(1+r)^T} \right) & \text{for } S_0 > \frac{S_T^+}{(1+r)^T} > \frac{S_T^-}{(1+r)^T}. \end{cases} \quad (4.17)$$

While investor  $i$  either does not establish any portfolio or goes short one share and invests the present value of  $K$  in riskfree bonds maturing at  $T$  if there exists a node of the binomial tree at time  $T$  such that  $S$  equals  $K$ , otherwise an additional policy may be optimal: Purchase exactly  $\bar{\varphi}$  shares and invest in riskfree bonds maturing at  $T$  with face value  $-\bar{\varphi}S_T^-$ .

The writing price of the option is

$$C_0^{iS} = \begin{cases} -\bar{\varphi} \left( S_0 - \frac{S_T^-}{(1+r)^T} \right) & \text{for } \frac{S_T^+}{(1+r)^T} > S_0 > \frac{S_T^-}{(1+r)^T} \\ \max \left( S_0 - \frac{K}{(1+r)^T}, 0 \right) & \text{otherwise.} \end{cases} \quad (4.18)$$

This price is independent of  $k_i$  and corresponds to the preference-free lower bound for European call options in a frictionless market.<sup>11</sup> Thus, the writer only asks for the so-called intrinsic value. The time value which reflects the right to only exercise in case of a favorable stock price movement is not paid for. In view of this interesting result, we have a closer look at the strategy for super-replicating a Short option.

If the replicating portfolio consist of no assets at  $t = 0$  and the investor neither buys nor sells stock till  $T$ , then the portfolio value at  $T$  either matches the payoff of the option (option ends out-of-the-money) or exceeds the payoff (option ends in-the-money) and vice versa if the investor goes short one share and puts the present value of  $K$  in the bank account at  $t = 0$ . If  $-\bar{\varphi}$  shares are sold short

<sup>11</sup>Merton (1973), corallary 1, p. 144 derives this preference-free lower boundary.

and bonds with face value  $-\bar{\varphi}S_T^-$  are purchased at  $t = 0$  and if furthermore no adjusting of the portfolio takes place the portfolio is worth  $K - S_T$  at  $T$  if  $S_T = S_T^+ > K$ , and the option is exactly replicated. However,  $S_T > S_T^+$  leads to super-replication due to

$$-\bar{\varphi}S_T^- + \bar{\varphi}S_T = (K - S_T^+) \frac{S_T - S_T^-}{S_T^+ - S_T^-} > K - S_T.$$

$S_T \leq S_T^- < K$  either leads to super-replication or to exact replication:

$$\frac{S_T^+ - K}{S_T^+ - S_T^-} (S_T^- - S_T) \geq 0.$$

Hence, even a static strategy results in exact replication in some states of nature and super-replication in others. The strategy with dynamic restructuring stated in the table leads to the same results but with a portfolio payoff in excess of the required payoff that is distributed more evenly.

While the writer's price is derived in a preference-free manner using only dominance properties, the strategy to be pursued when super-replication is permitted, buy-and-hold or dynamic restructuring, depends on individual preferences and on individual probability beliefs.

Pursuing the static strategy means ignoring the right not to exercise the option. In a frictionless market the time value of the option which is paid the seller reflects this right. Here the right is not paid for because the transactions costs necessary for a better replication do not exceed the benefits of better replication. Nobody will sell for less than the intrinsic value because one is able to obtain this value by pursuing a static strategy which even yields a better payoff. Thus, the lower bound for European call options derived by Merton (1973) stays valid even if the stock price process is dominated by the riskfree bond price process as long as the replicating portfolio is only required to be of the same value as the option.

## 4.2. Super-Replicating Strategies

In this section we analyze the result that duplication of a Long option demands permanent restructuring in the first periods and no rebalancing thereafter whereas

duplication of a Short call option is possible by following a static strategy. When replicating a Long option the lower bound for the terminal value of the replicating portfolio increases with the stock price and the lower bound in absolute terms is zero. When replicating a Short option the lower bound for the value of the replicating portfolio at maturity decreases with increasing stock price and the maximum lower bound is zero. When replicating a Long call option the only strategy that is free of transactions costs, and thus the only buy-and-hold strategy, with a terminal value that does not fall below the increasing lower bound consists of one share long. This results in super-replication in every state at maturity. If the investor revises his portfolio in order to achieve better replication he pursues a strategy that exactly replicates the option in a few states of nature. Rebalancing requires transactions costs, however, they are lower than the benefits of better replication.

On the other hand if the replicating portfolio of a Short call option does not violate the lower bound at the time of establishing the portfolio the investor needs not rebalance in order to satisfy the lower bound till maturity. Furthermore this initial portfolio leads to exact replication in some states of nature. Better replication inevitably means trading at every node. This is inefficient, though, due to the transactions cost factor that violates inequality (2.2). Rebalancing in order to achieve exacter replication thus just means waste of money.

We derived super-replicating strategies whose initial costs are lower than the initial costs of exact replicating strategies. Nobody will buy the option for more than the determined value of a Long option because this is the initial endowment in order to achieve a terminal payoff that dominates the payoff of the option. Conversely, the proceeds from the super-replicating strategy exceed the proceeds from exact replication when considering a Short call option. Consequently, nobody will sell for less than the derived Short option value since the commitment to deliver the asset at maturity for  $K$  leads to proceeds of this magnitude if the above super-replicating strategy is pursued.

We now study the case that exact replication is an absolute necessity in the presence of large transactions costs. Exact replication combined with no transactions at maturity lead to a unique path independent replicating strategy at time  $T-1$ .

When considering a Long option perfect replication and thus restructuring at all prior trading dates is optimal. Due to the additional transactions costs the above requirement leads to a higher option value. With Short options things are different. If super-replication of the portfolio value at  $T-1$  were possible a path



dependent strategy would be derived. Since this path dependent super-replicating strategy dominates the exact replicating strategy the portfolio value at  $T-1$  would be super-replicated. The excess payoff gained by following the super-replicating instead of the exact replicating strategy up to time  $T-1$  could be given to the intermediary through purchasing and selling simultaneously in the intermediate periods. Obviously, such a strategy is absurd. If we exclude selling and purchasing at the same time explicitly, we introduce one more constraint to the minimizing problem. This leads to an even larger objective value. Hence, if not only exact replication of a Short option in the presence of large transactions costs is required but giving away money is permitted, then the result is an even lower option value. Pursuing an exact replicating strategy sometimes means being initially more than one share short. Consequently, stock must be repurchased regardless of the stock price evolution. The proceed may even vanish due to transactions costs, so that we derive a negative option price.

## **5. Bid-Ask Market Values Versus Writing and Buying Prices**

Investors with transactions cost factors of the same magnitude may reach a settlement about the option price because the writer demands a lower price than the buyer is prepared to pay. They agree on a range of option prices. This spread is consistent with no arbitrage. An arbitrage opportunity does not arise if the writer receives more than the initial value of the super-replicating strategy because hedging the payoff of the option requires an initial value that equals the initial endowment to super-replicate a Long call option. This endowment exceeds the writer's price and reflects the upper bound of the spread. Conversely, when establishing a long position in the option and paying less than the buying price hedging the payoff leads to proceeds of the same magnitude as the writing price. Thus no free lunch is available.

Investors with different transactions cost levels may reach a settlement about the option price as well.

The writer with the lowest level of transactions costs asks for the largest price and thus is not willing to accept a price as low as the writing price of an investor with larger transactions costs. Both buyers with a large transactions cost factor and with the least factor bid prices which exceed the highest price asked for.

However, unlike the case with only one level of transactions costs prices in the range from the lowest ask price to the largest bid price do not necessarily rule out arbitrage. In order to hedge their positions writers with the lowest cost

factor require  $C_0^L$  initially. This denotes the market value of a Long call option and thus is the least initial endowment to replicate a Long option when one is faced with the lowest transactions costs. Investors with larger costs offer higher prices. Consequently, a writer with the lowest costs is able to gain arbitrage profits when selling the option to an investor with larger costs. Conversely, hedging a long position when faced with the least cost factor yields  $C_0^S$ , which denotes the market value of a Short call option. Purchasing the option from an investor with larger costs and thus for a price below  $C_0^S$  creates arbitrage profits.

Hence, a free lunch may exist for investors who face the lowest cost factor when dealing with investors with larger costs. On the other hand, note that the option is not redundant from the point of view of investors with a cost factor that violates the condition of dominance. A terminal value which in some states exceeds the option payoff can be obtained for a lower price. These investors, of course, try to get a better deal by negotiating.

## 6. Summary

In this paper we addressed the problem of determining the optimal replicating strategy for a European call option under differential transactions costs. We derived an upper boundary for the cost factor in a market where all investors face the same factor. This upper boundary ensures the efficiency of the stock price as well as the bond price process. It turned out that exact replication is optimal in the presence of only one transactions cost factor. Hence, the option is redundant. Nevertheless the bid price is below the ask price with prices in this range being arbitrage-free. With an increasingly finer partition of the binomial tree the condition of dominance becomes more stringent, so that the cost factor must be rather small. Furthermore, the bid price increases and the ask price decreases with a finer partition.

In the presence of differential cost factors, the condition of dominance must only hold for the lowest factor. Investors who do not face the lowest costs have to calculate their individual buying or writing price. These prices do not equal the market values of the option, because the range of market values is determined by investors with the least transactions costs. The violation of the condition of dominance results in a super-replicating strategy being the optimal strategy. Investors faced with large costs thus do not regard the option as a redundant security. The buying price can only be determined numerically. However, we are able to compute the date from which on it is optimal to refrain from transacting.

Furthermore, we have a closed-form solution for the optimal strategy (see (4.13) and (4.12)) at this date. The optimal strategy at all prior dates and thus the option value can now be determined recursively. Due to the path independence of the optimal strategy this is computationally easy. We derived an analytical solution for the individual writing price. It turned out that the replicating strategy of a Short call option is preference dependent and may be path dependent in the presence of large transactions costs. Hence, the initial endowment of the investor in the risky asset affects the strategy. With an increasingly finer partition of the binomial tree the individual buying price settles down at a determined value. Since investors establish a larger position initially and transact less later on, the price does not necessarily increase with the number of trading periods. The writing price does not decrease with an increasingly finer partition beyond a small number of periods.

We determined a range of possible option prices in the presence of differential transactions costs. The lower bound of this range is given by the intrinsic value. Prices in this range may create arbitrage profits for those investors with the least transactions costs.

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