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A CLOSED-FORM SOLUTION FOR THE HEALTH CAPITAL MODEL

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A Closed-form Solution for the Health Capital Model*

Holger Strulik†

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Abstract. This paper provides a closed-form solution for the health capital model of health demand. The results are exploited in order to prove analytically the comparative dynamics of the model. Results are derived for the so called pure investment model, the pure consumption model and a combination of both types of models. Given the plausible assumptions that (i) health declines with age and that (ii) the health capital stock at death is lower than the health capital stock needed for eternal life, it is shown that the optimal solution always implies eternal life. This outcome occurs independently from the initial stock of health, the impact of health on productivity, and the importance of health for utility and it is robust against the introduction of a finite age-dependent rate of health depreciation.

Keywords: Longevity, Health, Health Care Demand.

JEL: D91, J17, J26, I12.

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1. Introduction

This paper provides for the first time a closed-form solution of the health capital model of health demand. The model is also known as the Grossman model, named after the seminal paper of Grossman (1972), which developed its main ingredients. In its long history the Grossman model has been criticized for various shortcomings and counterfactual predictions. Several of these (alleged) shortcomings have been addressed by further developments of the original model. The core mechanics of the Grossman model is the conventional paradigm in the economics of health demand and remained, until recently, basically unchallenged by the development of an alternative theory. Empirically, the Grossman model is the inspiration if not the foundation of many reduced-form and structural models of health demand.

The core mechanics of the Grossman model arise from the assumption that individuals accumulate health capital H in a similar fashion as they accumulate human capital in form of education. In any period or, in continuous time, in any instant of time, health capital depreciates and is potentially augmented by health investment. The health capital stock of an individual of age t thus evolves, in continuous time, according to $\dot{H}(t) = f(I(t)) - \delta(t)H(t)$, in which I is investment, f is a positive function, and δ is the depreciation rate. The key assumption is that the loss of health capital through depreciation is an increasing function of its stock. This means that of two individuals of the same age t, the one in better health, i.e. the one with the greater health stock $H(t)$ loses more health capital in the next instant, since health depreciation $\delta(t)H(t)$ is increasing in $H(t)$. Notice that this basic assumption is imposed independently from whether δ is considered to be constant or age-dependent.[1]

The notion of health capital accumulation according to the Grossman model contradicts basic insights from modern gerontology. There, the human life course is understood as "intrinsic, cumulative, progressive, and deleterious loss of function that eventually culminates in death." (Arking, 2006, Masoro, 2006). Evidence from gerontology supports the reverse of the Grossman assumption. The accumulation of health deficits is found to be a positive function of the health deficits that are already present in an individual. Of two individuals of the same age the unhealthier one is predicted to lose more health (accumulate more health deficits) in the next instant. This law of health deficit accumulation has a micro-foundation in reliability theory and

[1] This paper is not the first one that observes this potentially problematic assumption of the Grossman model, see, for example, Case and Deaton (2005), McFadden (2008).

it is a very strong predictor of mortality (Mitnitski et al., 2002a, 2002b, 2005, 2006).

In defense of the Grossman model one could argue, based on Friedman (1953), that a theory's assumptions should not matter as long as its predictive quality is good. Generating testable predictions from the Grossman model, however, is a tough task. In order to appreciate this fact, notice that even the simplest version of the Grossman model generates two differential equations (or in discrete time two difference equations): one equation of motion for the health capital stock and one equation of motion generated from the first order conditions for optimal health investment. The latter could be expressed as equation of motion for the shadow price of health, or health investment, or consumption. The solution is thus expressed as a trajectory in a two-dimensional phase space. The problem is that there are *infinitely many* trajectories fulfilling the first order conditions, usually pointing in all possible directions in the phase space. In other words, based solely on the first order conditions and the equation of motion for the state variable (i.e. health capital), the solution is indeterminate. The unique optimal solution of the Grossman model is identified by the transversality condition. This unique optimal solution allows to derive testable predictions of the model.

It is perhaps fair to say that most of the problems that the literature had with solving the Grossman model originated from an inappropriate use of the transversality condition. Grossman (1972) and some followers (e.g. Jacobsen, 2000) just ignored the transversality condition, others had problems of applying it appropriately because they stated the health demand problem in discrete time (Ried, 1998). Neglecting the transversality condition is particularly worrying when reduced-form or structural equations for empirical estimation are derived. Many applications derive these equations for health care demand from solving simplified versions of the first order conditions and the equation motion (Muurinen, 1982; Wagstaff, 1986; Grossman, 2000). But since there are infinitely many trajectories fulfilling the first order conditions, any structural form obtained by ignoring the transversality condition is a result from (unwarranted) simplifications.[2]

Some other studies suggested to reformulate the original Grossman model in order to reduce the difficulties involved with identification. The original Grossman model assumes that death is

[2] For example, Muurinen (1982) assumes that $\dot{H}/H$ is constant, i.e. an exponential decline (or increase) of health with age is assumed rather than derived. Muurinen actually states the transversality condition but then ignores it in the derivation of health care demand. Similarly, Wagstaff (1986) accurately states a problem of free terminal time but never invokes the transversality condition when solving for the structural form. Instead he records carefully the steps of simplifying assumption which distil from the infinitely many solution of the first order conditions one particular set of estimation equations.

a free terminal condition. Death occurs when a minimum state of health is reached and health investment and the state of health influence the decline of health and thus the age at death T. For this problem, identification requires to solve the associated Hamiltonian function at the yet to be determined time of death. This difficult task is circumvented by assuming alternatively that individuals face a predetermined time of death, which occurs irrespective of their health, and then optimally chose the state of health $H(T)$ that they want to experience when they die (e.g. Eisenring, 1999; Kuhn et al., 2012; van Kippersluis and Galama, 2014). Clearly, an approach based on a predetermined time of death cannot lead to an informative reasoning about human aging and longevity. A rigorous analysis and critique of the effects of the different (non-) treatments of the transversality condition is provided by Forster (2001). Yet, even studies investigating the original Grossman model and stating one potentially appropriate transversality condition tend to ignore the full solution space because they assume at the outset that life ends at a finite T (Ehrlich and Chuma, 1990; Forster, 2001). As will be discussed below, the Grossman model usually allows for eternal life. This requires a different transversality condition to hold, which is usually fulfilled by the Grossman model.[3]

So far, comparative statics of the Grossman model have been derived by phase diagram analysis or numerical methods. Clearly it is not possible to use these methods in order to derive (structural) equations for an estimation of the model. This paper, proposes a different approach. It obtains a closed-form solution by imposing certain (iso-elastic) functional forms and a particular parametrization of the model. This provides non-simplified structural equations for empirical testing and allows to prove analytically not only the comparative statics but also the comparative dynamics of the model. Because the closed-form solution allows for an explicit verification of the transversality condition, it provides a theoretical identification of the optimal health-for-age trajectory and its determinants.

The closed-form solution is obtained for a particular value of the curvature parameter of the utility function σ, where $1/\sigma$ is known as the elasticity of intertemporal substitution. Given a plausible parametrization of the model, σ is between 1.5 and 2.5, depending on how much health

[3] Ehrlich and Chuma (1990) briefly discuss infinite life but then dismiss it for being unfeasible. Similarly, Laporte and Ferguson (2007), identify convergence toward eternal life as the optimal solution but then dismiss it by imposing a predetermined finite life. An early study coming to the same conclusion as the present paper is Cropper (1977). However, after acknowledging that a finite life requires that the fixed point for health capital lies below the minimum health needed for survival, the paper continues without debating the potential logical inconsistency involved in this assumption.

matters for utility and for productivity. Fortunately, a value of σ in this range is supported by many empirical studies. A recent meta-analysis of 2735 published estimates of the intertemporal elasticity of substitution found the world average at 2.0 (Havranek, 2013).

Nevertheless, the question may arise how general the obtained results are. In order to address this problem, I furthermore show that the steady state of health is independent from σ. This means that for any value of σ individual health behavior becomes more and more similar to the closed form solution as individuals age. The closed-form value of σ provides a threshold value that identifies whether health care investment increases or declines as individuals age and their health capital deteriorates. Health care expenditure increases if and only if the "true" σ lies below the threshold value. I show that this result has an intuitive explanation. Most importantly, however, phase diagram analysis reveals that, aside from the slope of the health expenditure trajectory, nothing is "special" about the threshold σ and the closed-form solution. Convergence towards the fixed point of eternal life is the unique optimal solution for any value of σ, any positive income level, any positive power of health investment on health, any finite impact of age on the health depreciation rate, and any initial state of health.

The paper also provides an identification of the cause of this potentially troubling implications of the Grossman model. It is the core mechanism assuming that health depreciation $\delta(t)H(t)$ is large when the state of health $H(t)$ is good and small when the state of health is bad. This creates an equilibrating force that allows individuals to use health investments in order to converge towards a fixed point of constant health.

The only possibility to choke off convergence to immortality is to assume that individuals die at a level of the health capital stock that is higher than the health level needed to live forever. While this ad hoc assumption formally "solves" the troublesome prediction of global convergence towards immortality, it leaves a lingering feeling of logical inconsistency. An analogous assumption in economics would be that firms go bankrupt at an equity level that is higher than the equity level needed for their perpetual viability. In the conclusion I briefly discuss an alternative way out of this dilemma. It consists of the replacement of the core mechanism of the Grossman model by a physiologically founded mechanism of health deficit accumulation.

2. The Model

In order to derive a closed-form solution we need to assume that the utility function and the production function are iso-elastic. Let the instantaneous utility from goods consumption C and health capital H be given by

$$U(C,H) = \frac{\left(C^{\beta}H^{1-\beta}\right)^{1-\sigma} - 1}{1-\sigma} \tag{1}$$

with $\sigma > 0$ and $\sigma \neq 1$. The parameter β reflects the relative weight of goods consumption in utility. We assume that goods consumption provides always utility and that health may or may not enter the utility function, $0 < \beta \leq 1$. The parameter σ reflects the inverse of the elasticity of intertemporal substitution. We assume that consumption is scaled appropriately in order to avoid negative utility, which would lead to the degenerate outcome that life-time utility is decreasing in the length of life such that individuals would prefer immediate death (see Hall and Jones, 2009, for an extensive discussion of this property). Furthermore $U(C,H)$ is assumed to display decreasing marginal utility, the usual assumption for a meaningful maximization problem to exist.[4]

Additionally, health expenditure may exert a positive effect on productivity. In Grossman's original version productivity is a function of an individual's production of healthy time, which is a function of health capital. For simplicity we consider here a "reduced form" approach according to which productivity, and thus income Y, is a strictly concave function of an individual's health status. We could also introduce an upper bound above which health does not improve productivity. These modifications would not change the basic mechanics of the model because the first order conditions are structurally identical in both cases.[5]

$$Y = \theta H^{\alpha}. \tag{2}$$

The parameter α controls the return to health in terms of productivity, which is assumed to be non-negative and strictly smaller than unity, $0 \leq \alpha < 1$. The model thus includes two special

[4] For later purpose we note that a negative second derivative, $U_{CC} = -\beta\left[1-\beta(1-\sigma)\right]C^{\beta(1-\sigma)-2}H^{(1-\beta)(1-\sigma)}$, requires $1-\beta(1-\sigma) > 0$, which is always true under the parameter restrictions made.

[5] To see this explicitly, suppose income is a function of exogenous productivity and healthy time h spent at work, $Y = \theta h$. Assume that individuals have at most $\bar{H}$ healthy time at their proposal (i.e. a working life without any illness). Assume that healthy time is produced via a concave function from health capital, such that $h = \min\left\{\bar{H}, \phi H^{-\epsilon}\right\}$. Then the interior solution is structurally identical to the one obtained below.

cases, which are frequently discussed in isolation in the literature (see e.g. Grossman, 2000):

- pure investment model: $\alpha > 0$, $\beta = 1$
- pure consumption model: $\alpha = 0$, $\beta < 1$.

In the latter case the individual receives a constant income stream θ.

Income is spent on goods consumption C and health investment (health care) I:

$$Y = C + I. \tag{3}$$

Without loss of generality we normalize the price of both items to unity.

The central assumption of the Grossman model is that individuals accumulate health capital more or less in the same fashion as human capital in the form of education is accumulated in many economic models of human capital accumulation. Specifically health capital H evolves according to

$$\dot{H} = AI - \delta H, \tag{4}$$

in which δ is the rate of depreciation of health capital. The parameter $A > 0$ captures the state of the medical technology. As most of the literature we focus on linear returns to health investment. Allowing for decreasing returns would add more realism to the model but would undo the possibility of a closed-form solution and it would not change the qualitative features of the model. Specifically, as demonstrated below, a linear function does *not* lead to a bang-bang solution, a feature of which the original Grossman model has been criticized for (Ehrlich and Chuma, 1990; Galama and Kapteyn, 2011). The optimal solution is smooth and interior for the linear case as well. The original Grossman model additionally assumes that the production of health needs also a time input beyond health expenditure. This adds more realism but is unessential for the model's basic mechanics.

Individuals are endowed with an initial stock of health capital $H(0) = H_0$ and survival requires that the health stock exceeds $H_{min} \geq 0$. In other words, individuals die at age T when health deteriorates to the level $H(T) = H_{min}$. In order to develop the solution we begin with assuming a constant health depreciation rate δ and discuss increasing depreciation later. In any case the crucial feature of the Grossman is that the loss of health at any age δH is greater when the stock of health is large, that is when individuals are relatively young. Formally, this can be seen from $\partial \dot{H}/\partial H < 0$. Ceteris paribus, individuals age at a high rate when they are young and healthy and at relatively slow rate when they are old. This behavior is a distinctive feature of

the Grossman model irrespective of whether health depreciation is constant or increasing with age.

Individuals maximize life-time utility

$$V = \int_0^T U(C, H)\mathrm{e}^{-\rho t}\mathrm{d}t, \tag{5}$$

in which t is age, ρ is the discount rate of future consumption, and T is the yet to be determined age of death. In contrast to the available literature, we do not impose a finite T. In principle, $T = \infty$. Of course, we expect from a plausible model of human aging that it is capable of generating a finite life, for example because the state of medical technological knowledge is not (yet) sufficiently advanced to life forever. In any case, however, mere logical consistency requires the following assumption about the size-ordering of health capital stocks.

Assumption 1. *The health capital stock at death is smaller than the health capital stock that would guarantee eternal life, $H_{min} < H^*$.*

Individuals are assumed to chose optimal health expenditure over the life course by maximizing (4) subject to (1) - (3) given initial health H_0 and the boundary condition $H \geq H_{min}$. Using (3) we can eliminate either C or I. It turns out, however, that it is more convenient to formulate the problem in the health-consumption-space. Eliminating I, the associated current value Hamiltonian is given by

$$J = \frac{\left(C^{\beta}H^{1-\beta}\right)^{1-\sigma} - 1}{1-\sigma} + \lambda\left[A\left(\theta H^{\alpha} - C\right) - \delta H\right], \tag{6}$$

in which λ denotes the costate variable, i.e. the shadow price of health. The associated first order condition and costate equation are:

$$\frac{\partial J}{\partial C} = \frac{\beta\left(C^{\beta}H^{1-\beta}\right)^{1-\sigma}}{C} - \lambda A = 0 \tag{7}$$

$$\frac{\partial J}{\partial H} = \frac{(1-\beta)\left(C^{\beta}H^{1-\beta}\right)^{1-\sigma}}{H} + \lambda\left[A\theta\alpha H^{\alpha-1} - \delta\right] = \lambda\rho - \dot{\lambda} \tag{8}$$

The optimal solution moreover fulfills the transversality condition (see e.g. Acemoglu (2009, Theorem 7.1):

$$J(C(T), H(T), \lambda(T)) = 0 \text{ for finite } T \tag{9a}$$

$$\lim_{T\to\infty} J(C(T),H(T),\lambda(T))\mathrm{e}^{-\rho T} = 0 \text{ otherwise.} \tag{9b}$$

If a fixed point exists such that $\lim_{T\to\infty} H(T) = H^*$, condition (9b) simplifies to

$$\lim_{T\to\infty} \lambda(T)H(T)\mathrm{e}^{-\rho T} = 0. \tag{9c}$$

As discussed in the introduction, many studies neglect (9b)-(9c). However, the reasoning that the economic and technical constraints of the Grossman model already exclude an infinite life is not well-founded, as shown in the next section.

3. The Solution

Equations (7) and (8) can be condensed in one equation of motion for optimal consumption (10) and using (2) and (3) the equation of motion for health is given by (11).

$$\frac{\dot{C}}{C} = \frac{1}{1-\beta(1-\sigma)}\left\{\frac{(1-\beta)A}{\beta}\frac{C}{H} + A\theta\alpha H^{\alpha-1} - (\delta+\rho) + (1-\beta)(1-\sigma)\frac{\dot{H}}{H}\right\} \tag{10}$$

$$\dot{H} = A(\theta H^{\alpha} - C) - \delta H. \tag{11}$$

The system (10)-(11) and the transversality (9) condition determine the optimal solution.

In order to derive the closed-form solution consider the expenditure share of consumption $x \equiv C/Y$. It evolves according to $(\dot{x}/x) = (\dot{C}/C) - (\dot{Y}/Y) = (\dot{C}/C) - \alpha(\dot{H}/H)$. Using (10) and (11) and noting that $Y/H = \theta H^{\alpha-1}$ this can be written as:

$$\begin{aligned}\frac{\dot{x}}{x} = {} & \frac{1}{1-\beta(1-\sigma)}\left\{\frac{(1-\beta)A}{\beta}x\theta H^{\alpha-1} + A\theta\alpha H^{\alpha-1} - (\delta+\rho)\right\} \\ & + \left[\frac{(1+\beta)(1-\sigma)}{1-\beta(1-\sigma)} - \alpha\right]\left[A\theta H^{\alpha-1} - Ax\theta H^{\alpha-1} - \delta\right].\end{aligned} \tag{12}$$

The expression looks cumbersome but for a special constellation of parameters it reduces to a neat solution. To see this solve (12) for $\dot{x}/x = 0$, that is

$$\begin{aligned}0 = {} & \left[(1-\beta)/\beta - (1-\sigma)(1-\beta+\alpha\beta) + \alpha\right]x + (1-\sigma)(1-\beta+\alpha\beta) \\ & - \left\{\rho + \delta\left[(1-\sigma)(1-\beta-\alpha\beta)\right] - \alpha\right\}\frac{H^{1-\alpha}}{\theta A}.\end{aligned} \tag{13}$$

Now consider the case where

$$\sigma = \tilde{\sigma} \equiv \frac{\rho + \delta\left[2-\alpha-\beta+\alpha\beta\right]}{\left[1-(1-\alpha)\beta\right]\delta}. \tag{14}$$

In this case the last term in (13) disappears and we get a simple solution for the expenditure share:

$$x = \frac{\beta\left[\rho + \delta(1-\alpha)\right]}{\delta + \beta\rho}. \tag{15}$$

In other words, given (14), which is assumed until Section 5, individuals prefer a constant consumption share and thus a constant share of health care expenditure throughout their life. Notice from (14) that $\sigma > 1$. As mentioned in the Introduction, many empirical studies suggest a value of σ around 2. In the present case we have, for example, $\tilde{\sigma} = 2.37$ for $\alpha = 1/3$, $\beta = 1/2$, $\rho = 0.02$ and $\delta = 0.08$. For $\alpha = 2/3$ and $\beta = 1$ we obtain $\tilde{\sigma} = 1.87$. This means that the explicit solution does not require an implausible assumption about the value of σ. For later purpose notice that $\tilde{\sigma}$ depends negatively on the rate of health depreciation δ and that it converges towards a positive lower bound for $\delta \to \infty$. For example σ converges to 1.5 for $\alpha = 2/3$ and $\beta = 1$ (a pure investment model) and it converges to 2.25 for $\alpha = 0$ and $\beta = 0.2$ (a pure consumption model). Likewise, optimal consumption expenditure depends negatively on δ. As shown in (15), x converges towards $\alpha(1-\beta)$ for $\delta \to \infty$. In other words, the optimal solution remains interior when the rate of health depreciation increases.

PROPOSITION 1 (Comparative Statics). *The consumption share x rises (the health expenditure share declines) when the time preference ρ rises, the health depreciation rate δ declines, the income elasticity of health α, declines, and the weight of consumption in utility β rises.*

These results are verified by taking the derivatives of (15) with respect to α, β, δ, and ρ. They are immediately intuitive.

Inserting x from (15) into (11) the equation of motion for health can be written as

$$\frac{\dot{H}}{H} = (1-x)\theta A H^{\alpha-1} - \delta, \tag{16}$$

in which $(1-x)$ is the constant health expenditure share. Equation (16) is a Bernoulli differential equation, a rare case of a non-linear differential equation for which there exists an exact solution. In order to obtain it, set $z = H^{1-\alpha}$. We thus have $\dot{z}/z = (1-\alpha)\dot{H}/H$, that is

$$\dot{z} = (1-\alpha)(1-x)\theta A - (1-\alpha)\delta z. \tag{17}$$

Equation (17) is a linear differential equation, which can be solved straightforwardly. Using the

initial condition $z(0) = z_0 = H_0^{1-\alpha}$ and resubstituting x from (15) we obtain:

$$z(t) = a + \left(H_0^{1-\alpha} - a\right) e^{-bt}$$
$$a \equiv \frac{[1-(1-\alpha)\beta]\,\theta A}{\delta + \rho\beta}, \qquad b \equiv (1-\alpha)\delta, \qquad H(t) \equiv z(t)^{\frac{1}{1-\alpha}}, \tag{18}$$

in which the last expression results from a retransformation of variables. This concludes the solution of the Grossman model.

4. Comparative Dynamics

Proposition 2 (Health and Health Care). *Initially healthier people are healthier at any given age t. Unless health has no affect on productivity, healthier people spend more on health care, implying that initially healthier people spend more on health care throughout life.*

For the proof notice from (18) that $H(t)$ is a positive function of H_0. From (2) we see that healthier people are wealthier unless $\alpha = 0$. Since the health care share $1 - x$ is constant, wealthier people spend more on health. This result has already been derived in alternative approaches to the Grossman model and its counterfactual implications have been noted in the literature (see e.g. Wagstaff, 1986; Case and Deaton, 2005).

Proposition 3 (Steady State). *As people age, their health capital converges towards the steady state*

$$H^* = \left\{ \frac{[1-(1-\alpha)\beta]\,\theta A}{\delta + \rho\beta} \right\}^{\frac{1}{1-\alpha}}. \tag{19}$$

For the proof notice from (18) that $z(t) = a$ for $t \to \infty$ and that $H(t) = z(t)^{1/(1-\alpha)}$.

Proposition 4 (Aging). *As individuals age their health capital stock is declining if their initial health is larger than H^* and rising if their initial health is lower than H^*.*

For the proof notice from (18) that $\partial z/\partial t < 0$ for $(H_0^{1-\alpha} > a$ that is for $H_0 > H^*$ and that $\partial z/\partial t \geq 0$ vice versa. In the following we realistically assume that humans age, i.e. that $H_0 > H^*$. Path $\mathcal{A}$ in Figure 1 shows an example lifetime trajectory (we discuss path $\mathcal{B}$ later).

Proposition 5 (Income and Medical Technology). *Health improves at any age with rising productivity θ and better medical technology A.*

PROPOSITION 6 (Indulgence and Time Preference). *A larger weight of consumption in utility β and a higher time preference rate ρ lead to lower health at any age.*

PROPOSITION 7 (Health Returns in Productivity). *A larger return of health in productivity leads to better health at any age.*

The proof for Propositions 5-7 inspects in (18) the derivatives of a with respect to k, $k \in \{\theta, A, \beta, \rho, \alpha\}$ and notices that $\partial z(t)/\partial a > 0$. Let b, the speed at which health capital adjusts towards its steady state, be called the rate of aging.

PROPOSITION 8 (Rate of Aging). *The rate of aging is independent from productivity, medical technology, time preference, and the weight of health in utility. It declines with increasing rate of depreciation δ.*

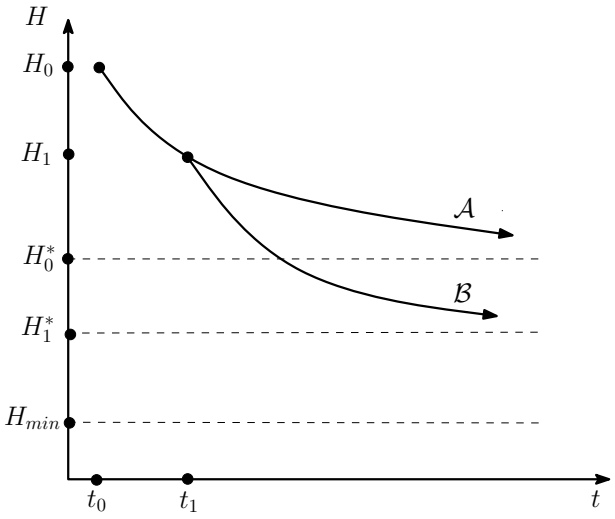
The proof notices from (18) that b is independent of θ, A, ρ, and β and that it depends negatively on δ. The results from Proposition 5-8 are intuitive and empirically plausible. However, the Grossman model has also a dark side to which we turn next.

PROPOSITION 9 (Eternal Life). *Irrespective of the power of medical technology, the weight of health in utility, and income, eternal life is the optimal solution and it is approached from everywhere, i.e. for any state of initial health.*

The proof starts with the observation that the health capital stock is constant at the steady state H^*. Since health does not deteriorate, individuals live forever. Furthermore, since health is constant, consumption is constant and thus the shadow price of health λ is constant as well, see equation (7). Since a steady state exists, transversality condition (9c) is the relevant one. Since $H \to H^*$ and $\lambda \to \lambda^*$, it simplifies to $\lim_{T\to\infty} e^{-\rho T} = 0$, which is true. Living forever, is thus not only feasible but also optimal, according to the Grossman model. Notice from (18) that H^* is approached from any initial condition. By Assumption 1 individuals do not die at a state of health that is better than the one needed for eternal life. But individuals could want to let their health erode below H^*. In the Appendix I show that it is not optimal to let health erode thus far. The only remaining optimal solution is to live forever.

The striking finding of Proposition 9 is not so much that eternal life is a possibility. It is rather that immortality is inescapable. It is approached independently from the initial state of health, income θ, and the power of medical technology A. Individuals simply refuse to die. A

Figure 1: Aging According to the Grossman Model



reasonable model of aging would allow for death at least at some low levels of productivity θ and at some low states of medical technology A.

As a remedy of these troubling results it has been suggested that health depreciation increases with age. An undesirable side-effect of age-dependent health depreciation is that the comparative statics can no longer be assessed qualitatively. Qualitative phase diagram analysis is basically impossible in three dimensional space and Oniki's (1973) method of comparative statics can no longer be applied. Consequently, the available discussion of the comparative statics of the Grossman model has focussed on models with constant δ (Eisenring, 1999; Meier, 2000; Forster, 2001).[6]

More importantly the introduction of age-dependent health depreciation only seemingly solves the problem of inescapable eternal life. In order to see this conveniently it is helpful to imagine the increase of δ in discrete steps (say, a yearly deterioration of the depreciation rate). This means that as the individual ages the fixed point H^* declines. However, as long as health depreciation is finite, the fixed point continues to exist (see above). Only an infinite depreciation rate would "solve" the problem by killing people off immediately but it would no lead to a

[6] Ehrlich and Chuma (1990) did not mention that they made this simplifying assumption in order to derive the comparative statics of their model (Table 3). But Oniki's method, which they apparently apply, requires the reduction to a two-dimensional system; see also Eisenring (1999).

meaningful understanding of aging and longevity.[7]

In Figure 1 (path $\mathcal{B}$) the individual experiences an increase of the depreciation rate at age t_1. As a consequence, the steady state moves from H_0^* to H_1^* and the rate of aging b, increases. As proven in connection with Proposition 9 it is not optimal to die with a health stock below H^*. In terms of Figure 1, applications of the Grossman model frequently assume that individuals die when $H(T)$ reaches H_{min}. However, for aging to occur, this means that H^* lies below H_{min} as well, since $H(T) > H^*$ (cf. Proposition 4). This outcome can only be reached by violating Assumption 1, which requires that individuals die at a state of health lower than the one that would enable eternal life.

5. Generalization

Since a closed-form solution exists only for a special parametrization the question naturally occurs how general these results are. The following proposition establishes that the qualitative features regarding the steady state of immortality are universal.

Proposition 10 (Eternal Life is Universal). *(i) For the Grossman model there exists always a unique positive steady state of eternal life H^*. (ii) The steady state is independent from the choice of σ. (iii) The steady state is approached from everywhere, i.e. from any initial state of health H_0, for any level of productivity $\theta > 0$ and for any power of medical technology $A > 0$.*

The proof is based on phase diagram analysis. It begins with obtaining the $\dot{x} = 0$–isocline from (12):

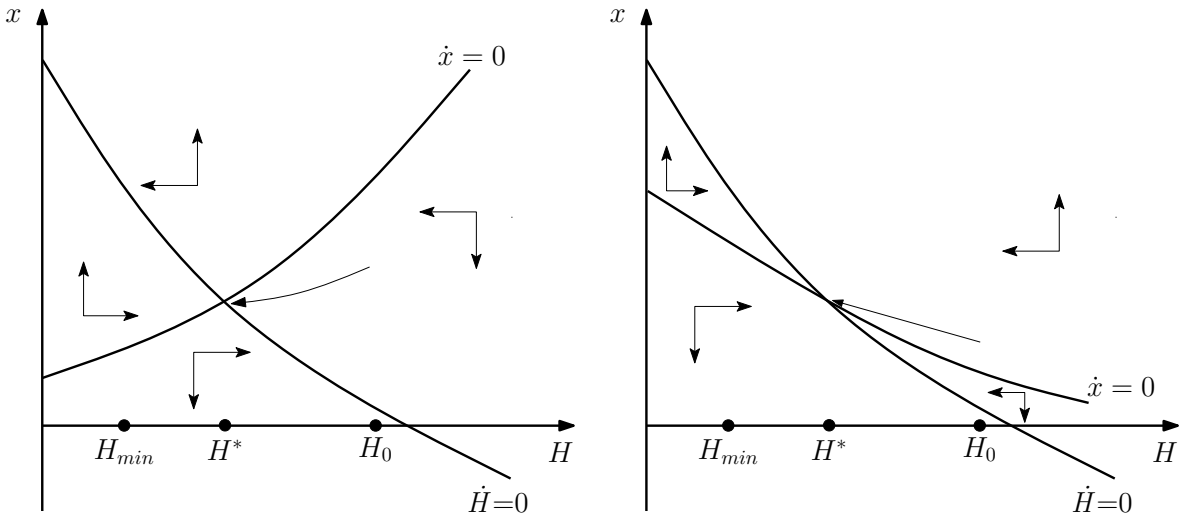
$$x = \frac{\beta(\sigma - 1)}{1 + \beta(\sigma - 1)} + s(\sigma)\frac{H^{1-\alpha}}{\theta A}, \qquad s(\sigma) \equiv \frac{\delta + 2\alpha\beta\delta + \rho\beta}{[1 - (1 - \alpha)\beta]\,[1 - \beta(1 - \sigma)]} - \delta. \tag{20}$$

Recall that the slope parameter $s(\sigma)$ is zero for $\sigma = \tilde{\sigma}$. Observe from (20) that $s' < 0$. This means that the slope of $\dot{x} = 0$–isocline is positive for $\sigma < \tilde{\sigma}$ and negative for $\sigma > \tilde{\sigma}$. Observe from (12) that $\partial\dot{x}/\partial x > 0$ where $\dot{H} = 0$. This means that the arrows of motion point away from the $\dot{x} = 0$ isocline. Next, obtain the $\dot{H} = 0$–isocline from (16):

$$x = 1 - (\delta/\theta A)H^{1-\alpha}. \tag{21}$$

[7] From the perspective of gerontology it makes sense to assume a minimum state of health below which life is untenable. Assuming an *age* at which individuals die at an infinite rate, however, makes no sense. Formally, the age-dependent mortality is well described by the Gompertz-Makeham law, the slope of which turns never infinite and, if anything, it declines for very old ages (Arking, 2006; Gavrilov and Gavrilova, 1991). In words, "no matter how old one is, the probability to die on the next day is never equal to one" (Jacquard, 1982).

Figure 2: Phase Diagrams: Left: $\sigma < \tilde{\sigma}$. Right: $\sigma > \tilde{\sigma}$



It is a negatively sloped curve originating from 1. Notice from (16) that $\partial(\dot{H}/H)/\partial H < 0$. The arrows motion point towards the $\dot{H} = 0$–isocline. From (20) and (21) we obtain the unique positive fixed point at

$$H^* = \left\{ \frac{[1-(1-\alpha)\beta]\,\theta A}{\delta + \rho\beta} \right\}^{\frac{1}{1-\alpha}},$$

which coincides with the solution in (19). The special case and the general case share the same steady state. The steady state is independent from σ and exists always.

Figure 2 shows the phase diagrams. The panel on the left hand side shows the case for $\sigma < \tilde{\sigma}$, i.e. for a positively sloped $\dot{x} = 0$–isocline. The steady state is a saddle point. It can be approached from everywhere, irrespective of the initial health condition H_0. Since the steady state is the same as before, it fulfils the transversality condition and the trajectory leading to it is identified as optimal. Analogous reasoning applies for the case of $\sigma > \tilde{\sigma}$, which is shown at the right hand side of Figure 2. This completes the proof.

Inspection of the phase diagrams is also useful in order to understand the role of σ for health expenditure. Coming from a low σ, the $x = 0$–isocline is turned clockwise around its steady state and the $\dot{H} = 0$–isocline remains unchanged. The not-drawn special case where $\sigma = \tilde{\sigma}$, is reached when the $\dot{x} = 0$–isocline is horizontal and coincides with the stable saddlepath. For $\sigma < \tilde{\sigma}$, x is declining as H declines, as shown in the left panel. Since the $1 - x$ is the health expenditure share, this means that the aging individual spends a larger fraction of his or her

income on health. For $\sigma > \tilde{\sigma}$ the opposite holds true, x rises as H declines, as shown in the panel on the right hand side of Figure 2. The aging individual spends a smaller share of income on health care. These observations verify the last proposition.

PROPOSITION 11. *The expenditure share of health care increases with age and deteriorating health capital stock if and only if* $\sigma < \tilde{\sigma}$.

To get the intuition for this result, obtain the cross-derivative U_{CH} from (1), $U_{CH} = (1 - \sigma)(1 - \beta)\beta C^{\beta(1-\sigma)-1} H^{(1-\beta)(1-\sigma)-1}$. A positive cross derivative U_{CH} is obtained for $\sigma < 1$. It means that individuals prefer to consume a lot when they are in a healthy state. Consequently, as shown in the left panel of Figure 2, the consumption expenditure share is high initially and declines as the individual ages. For $\sigma > 1$, the cross derivative is negative and individuals prefer to "substitute health by consumption", i.e. to consume a lot at higher ages when health has deteriorated. However, $\sigma > 1$ is not sufficient for an increasing consumption share. We know from the analysis above that a flat consumption profile is preferred for $\sigma = \tilde{\sigma} > 1$. For an increasing consumption profile, we need a σ larger than $\tilde{\sigma}$ because of two countervailing mechanisms: time preference and declining returns. To see this formally, consider $\rho \to 0$ (no time preference) and $\alpha \to 1$ (no declining returns) in (14) and observe that then $\tilde{\sigma} \to 1$ from above. A positive rate of time preference implies that individuals want to consume more in young age, declining returns of health in productivity imply that individuals need to invest more in health when the health capital stock is low, in order to prevent income from declining "too fast". Both mechanisms cause the individual to allocate relative more consumption to young ages and thus $\sigma > \tilde{\sigma} > 1$ is needed for consumption to increase with age.

The phase diagrams are also useful in order to identify the cause of global convergence toward the fixed point of eternal life. It is the core mechanism of the Grossman model, which assumes that health depreciation $\delta H(t)$ is large when the state of health $H(t)$ is good and small in bad health. Diagrammatically, this is expressed by the arrows of motion pointing towards the $\dot{H} = 0$–isocline, along which health does not change. This equilibrating force occurs independently from whether δ is age-dependent. It originates from the assumption that for any given age, healthy types lose a lot of health capital while health capital of unhealthy types depreciates relatively little.

6. Conclusion

This paper has provided an analytical closed-form solution of the Grossman model. The results turned out to be useful to reconsider earlier conclusions from the Grossman model, particularly with respect to their application of the transversality condition. One key result is that the Grossman model generally predicts immortality. It exhibits a unique saddlepath-stable fixed point at which health does not deteriorate. Convergence towards the fixed point is feasible and optimal for any initial health conditions and any parameters (determining, for example, the level of income and the power of medical technology). Global convergence towards immortality is a troubling prediction. It questions the suitability of the model to address real problems of aging, longevity, and the demand for health.

An ad hoc solution within the "Grossman paradigm" seems to be to require that individuals die at a level of health capital higher than the one needed for eternal life. But, as discussed in the Introduction, the assumption leaves a lingering feeling of logical inconsistency. An alternative solution would be to abandon the Grossman paradigm and search for an alternative core mechanism of human aging that does not imply these counterfactual predictions. Such a mechanism has been proposed by the Dalgaard and Strulik (2014) model of health deficit accumulation. It turns the Grossman mechanism upside down by assuming that unhealthy persons, ceteris paribus, develop more health deficits in the next period. This assumption has a micro-foundation in modern gerontology up to the precise estimation of its underlying parameters. With the present paper at hand it is easy to see how it reverts the equilibrating forces of the Grossman model. Since health depreciation of unhealthy individuals is greater, the arrows of motion point away from the situation of constant health deficits. A fixed point, if it exists at all, cannot be reached. Individuals are predicted to age by developing health deficits at an increasing speed and then to die in finite time when an upper boundary of viable health deficits has been reached. The new approach solves also the measurement problems that plagued the empirical literature by replacing the latent variable "health capital stock" by an observable variable "health deficits". Because of its gerontological foundation the model of health deficit accumulation is straightforwardly calibrated with real data. It has already been utilized for a novel analysis of the nexus between health demand and income (Dalgaad and Strulik, 2010, 2014), health and education (Strulik, 2012), and health and retirement (Dalgaard and Strulik, 2013) and the gates for many fruitful future applications are wide open.

Appendix

Part 2 of Proposition 9. It remains to show that dying at a state of health below H^* is not optimal. When life is finite, the transversality condition (9a) applies. Inserting λ from (7) into (6) we obtain:

$$J = \frac{\left(C^{\beta}H^{1-\beta}\right)^{1-\sigma}}{1-\sigma} + \frac{1}{\sigma-1} + \frac{\beta\left(C^{\beta}H^{1-\beta}\right)^{1-\sigma}}{AC}\left[A\left(\theta H^{\alpha}-C\right)-\delta H\right], \tag{A.1}$$

In the following I show that $J(T)$ is positive for any $H(T) < H^*$. Since $\sigma > 1$ it is sufficient to show that

$$\tilde{J} = \beta C^{\beta(1-\sigma)-1}H^{(1-\beta)(1-\sigma)}\left(\frac{C}{\beta(1-\sigma)} + \theta H^{\alpha} - C - \frac{\delta}{A}H\right) \tag{A.2}$$

is positive. Since the first term is positive for positive health and positive consumption, it sufficient to show that the second term is positive, i.e. that

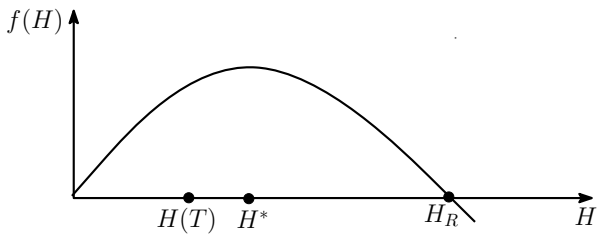
$$\bar{J} = \frac{1-\beta(\sigma-1)}{\beta(\sigma-1)}C + \beta\left[\theta H^{\alpha} - \frac{\delta}{A}H\right] \tag{A.3}$$

is positive. Notice that the first term is positive since $\sigma > 1$. A sufficient, not necessary condition for the Hamiltonian to be positive is thus that $f(H) = \theta H^{\alpha} - (\delta/A)H$ is positive at the time of death. The function f comes out of the origin, is concave and has another root at H_R, as depicted in Figure A.1. The root is found at $H_R = (\theta A/\delta)^{1/(1-\alpha)}$. Since $\delta + \rho\beta > \delta - \delta(1-\alpha)\beta$, we have

$$\left(\frac{\theta A}{\delta}\right)^{\frac{1}{1-\alpha}} > \left(\frac{[1-(1-\alpha)\beta]\theta A}{\delta+\rho\beta}\right)^{\frac{1}{1-\alpha}} \qquad \Rightarrow \qquad H_R > H^*. \tag{A.4}$$

This implies $f(H^*) > 0$ and thus $f(H(T)) > 0$ for any $H(T) < H^*$. A positive Hamiltonian at death means that the transversality condition is violated. It is not optimal to die.

Figure A.1: The Curve $f(H)$



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