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Article
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Econometrics

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Suggested Citation: McAleer, Michael; Hafner, Christian M. (2014) : A one line derivation of EGARCH, Econometrics, ISSN 2225-1146, MDPI, Basel, Vol. 2, Iss. 2, pp. 92-97, http://dx.doi.org/10.3390/econometrics2020092

This Version is available at:
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A One Line Derivation of EGARCH

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Received: 16 June 2014; in revised form: 19 June 2014 / Accepted: 20 June 2014 / Published: 23 June 2014

Abstract: One of the most popular univariate asymmetric conditional volatility models is the exponential GARCH (or EGARCH) specification. In addition to asymmetry, which captures the different effects on conditional volatility of positive and negative effects of equal magnitude, EGARCH can also accommodate leverage, which is the negative correlation between returns shocks and subsequent shocks to volatility. However, the statistical properties of the (quasi-) maximum likelihood estimator of the EGARCH parameters are not available under general conditions, but rather only for special cases under highly restrictive and unverifiable conditions. It is often argued heuristically that the reason for the lack of general statistical properties arises from the presence in the model of an absolute value of a function of the parameters, which does not permit analytical derivatives, and hence does not permit (quasi-) maximum likelihood estimation. It is shown in this paper for the non-leverage case that: (1) the EGARCH model can be derived from a random coefficient complex nonlinear moving average (RCCNMA) process; and (2) the reason for the lack of statistical properties of the estimators of EGARCH under general conditions is that the stationarity and invertibility conditions for the RCCNMA process are not known.
1. Introduction

In the world of univariate conditional volatility models, the ARCH model of Engle [1] and the
generalization to the GARCH model by Bollerslev [2] are the two most widely estimated symmetric
models of time-varying conditional volatility, where symmetry refers to the identical effects on
volatility of positive and negative shocks of equal magnitude.

The asymmetric effects on conditional volatility of positive and negative shocks of equal magnitude
can be captured in different ways by the exponential GARCH (or EGARCH) model of Nelson [3,4],
and the GJR (alternatively, asymmetric or threshold) model of Glosten, Jagannathan and Runkle [5].
These are the two most widely estimated asymmetric univariate models of conditional volatility.

A special case of asymmetry is that of leverage. As defined by Black [6] in terms of the
debt-to-equity ratio, leverage is associated with increases in volatility for negative shocks to returns
and decreases in volatility for positive shocks to returns. In short, leverage captures the negative
correlation between returns shocks and subsequent shocks to volatility. The EGARCH model is
capable of capturing leverage, depending on appropriate restrictions on the parameters of the model.

Although the GJR model can capture asymmetry, leverage is not possible, unless the short run
persistence effect (that is, the ARCH parameter) is negative. Such a restriction is not consistent with
the standard sufficient condition for conditional volatility to be positive.

The univariate GARCH model has been extended to its multivariate counterpart in, for example, the
BEKK model of Baba et al. [7] and Engle and Kroner [8], and the VARMA-GARCH model of Ling
and McAleer [8]. The GJR model has a multivariate counterpart in the VARMA-AGARCH model of
McAleer, Hoti and Chan [10].

However, the EGARCH model has not yet been developed formally for multivariate processes, with
appropriate regularity conditions. Kawakatsu [11] examined a matrix exponential GARCH model and
its estimation under alternative multivariate distributions, but did not provide any asymptotic theory.

Although it is not essential, leverage is an attractive feature of the EGARCH model. Since
EGARCH is derived as a discrete-time approximation to a continuous-time stochastic volatility
process, and is expressed in logarithms, the exponential operator is required to obtain conditional
volatility, which is guaranteed to be positive. Therefore, no restrictions are required for conditional
volatility to be positive.

However, the statistical properties for the (quasi-) maximum likelihood estimator of the EGARCH
parameters are not available under general conditions, but rather only for special cases under highly
restrictive and unverifiable conditions. It is often argued heuristically that the reason for the lack of
general statistical properties arises from the presence in the model of an absolute value of a function of
the parameters, which does not permit analytical derivatives, and hence does not permit (quasi-)
maximum likelihood estimation.

Some specific statistical results are available under highly restrictive assumptions that cannot readily
be verified. For example, Straumann and Mikosch [12] assume that conditional volatility
models are solutions to stochastic recurrence equations which satisfy a contraction property, and derive
some asymptotic results for the EGARCH(1,0), or EARCH(1) model. However, their regularity
conditions are difficult to interpret or verify.

Wintenberger [13] proves consistency using continuous invertibility, which is assumed to hold for
the EGARCH(1,1) model, and shows asymptotic normality for the quasi-maximum likelihood estimator
of EGARCH(1,1) under “non-verifiable” conditions, and asymptotic normality for the stable
quasi-maximum likelihood estimator of an invertible EGARCH(1,1) model. However, the sufficient
conditions for invertibility seem to be restrictive on the admissible parameter space, and cannot be verified.

Demos and Kyriakopoulou [14] present sufficient conditions under which the EGARCH(1,1)
processes have bounded first- and second-order variance derivatives, and the expectation of the
supremum norm of the second order log-likelihood derivative is finite. These sufficient conditions for
asymptotic normality also restrict the admissible parameter space, and are difficult to verify.

It is shown in this paper for the non-leverage case that the EGARCH model can be derived from a
random coefficient complex nonlinear moving average (RCCNMA) process, and that the reason for
the lack of statistical properties of the estimators of EGARCH under general conditions is that the
stationarity and invertibility conditions for the RCCNMA process are not known.

The remainder of the paper is organized as follows. In Section 2, the EGARCH model is discussed.
Section 3 presents a RCCNMA process, from which EGARCH is derived in Section 4. Some
concluding comments are given in Section 5.

2. EGARCH

Let the conditional mean of financial returns be given as:

\[ y_t = \mathbb{E}(y_t | I_{t-1}) + \varepsilon_t \]  

(1)

where \( y_t = \Delta \log P_t \) represents the log-difference in stock prices \( (P_t) \), \( I_{t-1} \) is the information set at
time \( t-1 \), and \( \varepsilon_t \) is conditionally heteroskedastic.

The EGARCH specification of Nelson [3,4] is given as:

\[ \log h_t = \omega + \alpha | \eta_{t-1} | + \gamma \eta_{t-1} + \beta \log h_{t-1}, \ | \beta | < 1 \]  

(2)

where the standardized shocks, \( \eta_t \), are given by \( \eta_t = \frac{\varepsilon_t}{\sqrt{h_t}}, \ \eta_t \sim iid (0, \omega), \) and \( | \beta | < 1 \) is the
stability condition when \( \log h_{t-1} \) is included in the model. Asymmetry exists if \( \gamma \neq 0 \), while leverage
exists if \( \gamma < 0 \) and \( \gamma < \alpha < -\gamma \). The specification in Equation (2) is EGARCH(1,1), but this can easily
be extended to EGARCH(\( p,q) \).

As \( \eta_{t-1} \) is a function of both \( \varepsilon_t \) and \( \sqrt{h_t} \), each of which is a function of the parameters through
Equations (1) and (2), it is clear that quasi-maximum likelihood estimation of EGARCH is problematic
as \( | \eta_{t-1} | \) is not differentiable with respect to the parameters. Moreover, invertibility of EGARCH is
also problematic because of the presence of the logarithmic transformation as well as the absolute value function.

3. Random Coefficient Complex Nonlinear Moving Average Process

Consider a random coefficient complex nonlinear moving average (RCCNMA) process given by:

\[ \varepsilon_t = \varphi_t \sqrt{|\eta_{t-1}|} + \psi_t \sqrt{|\eta_{t-1}|} + \eta_t \]  \hspace{1cm} (3)

where \( \sqrt{|\eta_{t-1}|} \) is a complex-valued function of \( \eta_{t-1} \), \( \varphi_t \sim iid (0, \alpha) \), and \( \psi_t \sim iid (0, \gamma) \).

If \( \eta_t \) is a standard normal variable, then \( E|\eta_t| = c \), which is a known constant. Moreover:

\[ E\sqrt{|\eta_t|} = E \sqrt{|\eta_t|} \times I(\eta_t \geq 0) + E \sqrt{|\eta_t|} \times I(\eta_t < 0) = c / 2 + ic / 2 = (1+i)c / 2 \]

where \( i = \sqrt{-1} \) and \( E[I(\eta_t \geq 0)] = E[I(\eta_t < 0)] = 0.5 \) are the expectations of two indicator functions.

As the mean of the complex-valued function is a finite constant, it follows that both the unconditional and conditional means of \( \varepsilon_t \) in Equation (3) are zero.

As the RCCNMA process given in Equation (3) is not in the class of random coefficient linear moving average models examined in Marek [15], the sufficient conditions for stationarity and invertibility of the RCCNMA process are not known.

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It follows from Equation (3) that:

\[ h_t = E(\varepsilon_t^2 | I_{t-1}) = \omega + \alpha |\eta_{t-1}| + \gamma \eta_{t-1} \] \hspace{1cm} (4)

The use of an infinite lag for the RCCNMA process in Equation (3) would yield the EGARCH model in Equation (2).

It is worth noting that the transformation of \( h_t \) in Equation (4) is not logarithmic, but the approximation given by:

\[ \log h_t = \log(1 + (h_t - 1)) \approx h_t - 1 \]

can be used to replace \( h_t \) in Equation (4) with \( 1 + \log h_t \).

The interpretation of leverage, whereby \( \gamma < 0 \) and \( \gamma < \alpha < -\gamma \), is lost in the above derivation because both \( \alpha \) and \( \gamma \) are required to be positive.

More importantly, in terms of interpreting the model, the derivation of EGARCH, albeit without the logarithmic transformation, in Equation (4) shows that the statistical properties of the (quasi-) maximum likelihood estimator of the EGARCH parameters do not exist under general conditions because the model is based on a RCCNMA process, for which the stationarity and invertibility conditions are not known.
5. Conclusion

The paper was concerned with one of the most popular univariate asymmetric conditional volatility models, namely the exponential GARCH (or EGARCH) specification. The EGARCH model is popular, among other reasons, because it can capture both asymmetry, namely the different effects on conditional volatility of positive and negative effects of equal magnitude, and leverage, which is the negative correlation between returns shocks and subsequent shocks to volatility.

The statistical properties for the (quasi-) maximum likelihood estimator of the EGARCH parameters are not available under general conditions, but rather only for special cases under highly restrictive and unverifiable conditions. It is often argued heuristically that the reason for the lack of general statistical properties arises from the presence in the model of an absolute value of a function of the parameters, which does not permit analytical derivatives, and hence does not permit (quasi-) maximum likelihood estimation.

It was shown in the paper for the non-leverage case that the EGARCH model could be derived from a random coefficient complex nonlinear moving average (RCCNMA) process, and that the reason for the lack of statistical properties of the estimators of EGARCH under general conditions is that the stationarity and invertibility conditions for the RCCNMA process are not known.

Acknowledgments

The authors are grateful to the Editor, Kerry Patterson, Alistair Freeland, and three referees for helpful comments and suggestions. For financial support, the first author wishes to acknowledge the Australian Research Council and the National Science Council, Taiwan.

Author Contributions

The authors contributed jointly to the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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