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**Working Paper**

## Interest rate swap credit valuation adjustment

IES Working Paper, No. 16/2014

**Provided in Cooperation with:**

Charles University, Institute of Economic Studies (IES)

*Suggested Citation:* Černý, Jakub; Witzany, Jiří (2014) : Interest rate swap credit valuation adjustment, IES Working Paper, No. 16/2014, Charles University in Prague, Institute of Economic Studies (IES), Prague

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<https://hdl.handle.net/10419/102579>

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**Bibliographic information:**

Černý J., Witzany J. (2014). “Interest Rate Swap Credit Valuation Adjustment” IES Working Paper 16/2014. IES FSV. Charles University.

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# Interest Rate Swap Credit Valuation Adjustment

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May 2014

## **Abstract:**

The credit valuation adjustment (CVA) of OTC derivatives is an important part of the Basel III credit risk capital requirements and current accounting rules. Its calculation is not an easy task - not only it is necessary to model the future value of the derivative, but also the probability of default of a counterparty. Another complication arises in the calculation when the exposure to a counterparty is adversely correlated with the credit quality of that counterparty, i.e. when it is needed to incorporate the wrong-way risk. A semi-analytical CVA formula simplifying the interest rate swap (IRS) valuation with the counterparty credit risk including the wrong-way risk is derived and analyzed in the paper. The formula is based on the fact that the CVA of an IRS can be expressed using swaption prices. The link between the interest rates and the default time is represented by a

Gaussian copula with constant correlation coefficient. Finally, the results of the semi-analytical approach are compared with the results of a complex simulation study.

**Keywords:** Counterparty Credit Risk, Credit Valuation Adjustment, Wrong-way Risk, Risky Swaption Price, Semi-analytical Formula, Interest Rate Swap Price

**JEL:** C63, G12, G13, G32

**Acknowledgements:**

This research has been supported by the Czech Science Foundation Grant P402/12/G097 "Dynamical Models in Economics" and by SVV grant No. SVV-2014-260105.

## 1 Introduction

An important part of the new banking regulation, Basel III, is the credit valuation adjustment (CVA) of OTC derivatives. CVA concerns credit risk capital requirements calculation and also current accounting rules.

Although the principle of the CVA is known and applied by some banks more than twenty years, it is coming to the forefront nowadays with the new Basel III regulation. Basel III regulation became effective in 2010 but its implementation has been phased into the years 2013-2019.

In this paper we consider counterparty credit risk (CCR) for interest rate swap (IRS) in presence of the so called wrong-way risk. The wrong-way risk can be defined as an adverse dependence between exposure and credit quality of the counterparty. The use of the simulation approach is usually computationally and time consuming, but we present in particular a semi-analytical formula for interest rate swap credit valuation adjustment (IRS CVA) calculation which significantly simplifies whole computation process.

From [7] we know that a swap price including CVA without the wrong-way risk can be calculated as a sum of swaption prices weighted by risk-neutral probabilities of default. The IRS CVA formula is an analogy of the risky option price from [4] to IRS pricing with CVA including the wrong-way risk.

We show that CCR has a relevant impact on the IRS prices and that the wrong-way risk has a relevant impact on the CVA. We analyze the results of our semi-analytical formula and compare them with the results of a simulation study from [2]. In [2] Brigo and Pallavicini assume the G2++ model for interest rates development and CIR++ for hazard rate development which require quite complex modeling. It turns out that for zero correlation between interest rates and the default time (resp. hazard rate) the results of the semi-analytical formula and the simulation approach are similar. For nonzero correlation the results vary which is caused by different concepts of correlation. The correlation in [2] represents dependence between the instantaneous differences of the interest rates and default intensity, i.e. instantaneous correlation, while in our semi-analytical formula the correlation is between levels of the interest rates and the time of default. In our view, higher interest rates economically may, for example, cause more corporate defaults over time and so the correlation between the levels of the interest rates and the default time is economically better interpreted than the correlation between instantaneous differences of the interest rates and default intensities. We discuss the results, findings and possible challenges in more detail in the conclusions.

## 2 Unilateral CVA formula

Throughout the text we will consider OTC derivatives where only one counterparty is subject to credit risk. We denote the discounted risk-neutral value of the financial

derivative at time  $t$  expiring at time  $T$  as  $V(t, T)$  and the discounted value of the derivative including CCR as  $V^*(t, T)$ . Assume  $\tau$  is the default time of the counterparty,  $\mathbb{Q}$  is the risk-neutral measure with respect the money market account and  $RR$  is a recovery rate on any defaulted amount then

$$V^*(t, T) = V(t, T) - CVA(t, T), \quad (2.1)$$

where

$$CVA(t, T) = \mathbb{E}^{\mathbb{Q}} [(1 - RR) \mathbb{1}_{[\tau \leq T]} \cdot \max(V(\tau, T), 0) D(t, \tau)]$$

and  $D(t, \tau) = \exp \left\{ - \int_t^\tau r(s) ds \right\}$  is the value of the money market account at time  $\tau$  starting with a unit value at time  $t$ .  $RR$  stands for the (constant or stochastic) recovery rate of the derivative and  $(1 - RR) \equiv LGD$  is the fractional loss given default. So the CVA of the derivative value is the expectation of the irrecoverable part of the exposure in the risk-neutral world. The expectation may be rewritten into the following form

$$\mathbb{E}^{\mathbb{Q}} [LGD \cdot \mathbb{1}_{[\tau \leq T]} D(t, \tau) V(\tau, T)^+] = -\mathbb{E}^{\mathbb{Q}} \left[ \int_t^T LGD \cdot D(t, u) V(u, T)^+ dS(u) \right] \quad (2.2)$$

where  $S(t) = \mathbb{Q}(\tau > t) = 1 - \mathbb{Q}(\tau \leq t)$  is the survival function of the counterparty. One of possible approaches how to calculate the default probability is to use the default intensity or hazard rate. It means probability of default occurring in "infinitesimally small" time step. If the distribution function  $F(t) = 1 - S(t)$  is absolutely continuous then

$$\lim_{\Delta t \rightarrow 0} \mathbb{Q}(t < \tau < t + \Delta t | \tau \geq t) = f(t) dt \frac{1}{S(t)} = -S'(t) dt \frac{1}{S(t)} = \lambda(t) dt, \quad (2.3)$$

where  $\lambda(t)$  is the hazard rate and  $f$  is the density function of  $\tau$  regarding Lebesgue measure. We can use the hazard rate to simplify CVA expression into the form

$$-\mathbb{E}^{\mathbb{Q}} \int_t^T LGD \cdot D(t, u) V(u, T)^+ dS(u) = \mathbb{E}^{\mathbb{Q}} \int_t^T LGD \cdot D(t, u) V(u, T)^+ \lambda(u) S(u) du.$$

In particular, we will use exponential distribution of the default time which is the most important (and the simplest) distribution in the survival theory. The main property of the exponential distribution is its memoryless and constant hazard rate (intensity). Of course it is also possible to use other distributions, e.g. Weibull, Lognormal, Loglogistic (see [8]), or nonparametric approach which would be very difficult to implement for the lack of data.

### 3 Model-free linear dependence

Before we present our main results concerning the calculation of IRS CVA, we show that it is possible to simply calculate the CVA of any financial derivative with

inclusion of wrong-way risk given a correlation between the indicator of default time and the exposure, their variances, and the CVA excluding wrong-way risk.

Let us assume that the dependence between the exposure and the default time indicator (wrong-way risk) is linear. Then the correlation coefficient can be used to calculate CVA. The model-free CVA formula has following form

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} [\text{LGD} \cdot \mathbb{1}_{[\tau \leq T]} D(t, \tau) V(\tau, T)^+] &= \text{LGD} (\text{cov}^{\mathbb{Q}} [\mathbb{1}_{[\tau \leq T]}, D(t, \tau) V(\tau, T)^+] \\ &\quad + \mathbb{E}^{\mathbb{Q}} [\mathbb{1}_{[\tau \leq T]}] \cdot \mathbb{E}^{\mathbb{Q}} [D(t, \tau) V(\tau, T)^+]) \end{aligned}$$

assuming that LGD is constant and deterministic. The second part of the formula including LGD is the known CVA without wrong-way risk ( $\text{CVA}_{\text{NoWrong}}$ ), so we have

$$\text{CVA}(t, T) = \text{LGD} \cdot \text{cov}^{\mathbb{Q}} [\mathbb{1}_{[\tau \leq T]}, D(t, \tau) V(\tau, T)^+] + \text{CVA}_{\text{NoWrong}}(t, T), \quad (3.1)$$

The covariance of the exposure and the default time indicator can be expressed by a correlation coefficient  $\rho \in [-1, 1]$  as

$$\text{cov}^{\mathbb{Q}} [\mathbb{1}_{[\tau \leq T]}, D(t, \tau) V(\tau, T)^+] = \rho \sqrt{\text{var}^{\mathbb{Q}} [\mathbb{1}_{[\tau \leq T]}] \text{var}^{\mathbb{Q}} [D(t, \tau) V(\tau, T)^+]}. \quad (3.2)$$

The default time indicator has a Bernoulli distribution with probability  $q$  equal to the survival probability <sup>1</sup>

$$\text{var}^{\mathbb{Q}} [\mathbb{1}_{[\tau \leq T]}] = (1 - q)q. \quad (3.3)$$

Assuming exponential distribution of the default time with constant parameter, the variance can be expressed as

$$\text{var}^{\mathbb{Q}} [\mathbb{1}_{[\tau \leq T]}] = e^{-\lambda T} (1 - e^{-\lambda T}).$$

The second part of the covariance depends on the derivative properties, and so it cannot be easily simplified as the variance of the indicator function.

However, in practice it is difficult to calculate these characteristics and, therefore, in this paper we are going to develop a semi-analytical formula that approximates, under certain assumptions, the IRS CVA with the wrong-way risk.

## 4 Risky Swaption Price

To calculate the IRS CVA with inclusion of the wrong-way risk, it is necessary to derive a formula for the risky swaption price, i.e. swaption price with CCR. A series of such swaptions with different expirations and fixed tenor gives the final IRS CVA formula.

First, we recall the well-known swaption price formula based on the Black's model and then we will focus on its extended risky version.

---

<sup>1</sup>For exponential distribution with constant parameter  $\lambda$  the survival probability is  $q = S(T) = \mathbb{Q}(\mathbb{1}_{[\tau > T]} = 1) = e^{-\lambda T}$



Let us consider a swaption with nominal value  $L$  of a swap starting at the exercise date  $T_0$  with fixed coupon rate  $s_K$  paid at times  $T_1, \dots, T_n$ . As noted in [9], based on the Black's model, given the forward swap rate  $s_t$ , and the log-variance  $\sigma^2(T_0 - t)$  of the future swap rate  $s_{T_0}$ , the payer swaption price at time  $t$  is

$$V_{\text{pay}}(t, T_0, T_n) = X(t, T_1, T_n) \cdot L \cdot (s_t \Phi(d_1) - s_K \Phi(d_2))$$

where

$$\begin{aligned} d_1 &= \frac{\log(s_t/s_K) + \sigma^2(T_0 - t)/2}{\sigma\sqrt{T_0 - t}}, \\ d_2 &= d_1 - \sigma\sqrt{T_0 - t}, \end{aligned}$$

and  $X$  is the annuity. Swaption price is calculated as the discounted expectation of the payoff. But zero coupon bond discounting is replaced by another one, so called annuity, defined as

$$X(t, T_1, T_n) = \sum_{i=0}^{n-1} \delta_{i+1} P(t, T_{i+1}) \quad (4.1)$$

where the second and third argument are the first and the last swap payments,  $P(t, T_{i+1})$  is the zero coupon bond value with maturity at  $T_{i+1}$  and  $\delta_{i+1}$  is the time factor related to the period  $(T_i, T_{i+1})$ .

Our first partial result is the semi-analytical formula for the risky swaption price, i.e. the swaption is settled only if the option seller does not default and we admit a nonzero probability of default. We assume that the random component of the interest rate and default time can be decomposed into a common systematic and different specific factors. These factors are independent with each other, however, we admit the dependence between the default time and interest rate, which is expressed by the constant correlation coefficient. This result is summarized in the following theorem.

**Theorem 1.** *Suppose that in the risk-neutral world swap rate  $s_{T_0}$ ,  $T_0 > 0$ , development follows the Black's model with a constant volatility  $\sigma > 0$ , i.e.*

$$s_{T_0} = s_0 \exp \left\{ -\sigma^2 T_0 / 2 + \sigma \sqrt{T_0} Y \right\}, \quad Y \sim N(0, 1) \quad (4.2)$$

where  $Y$  is decomposed into a systematic and an idiosyncratic factor

$$Y = aU + \sqrt{1 - a^2} \varepsilon_1, \quad a \in [-1, 1].$$

In addition, the default time is defined as

$$\tau = S^{-1}(1 - \Phi(Z)), \quad (4.3)$$

where  $S(t) = e^{-ht}$  is the constant hazard rate  $h$  exponential survival probability function (with respect to the annuity risk-neutral measure) and  $Z$  is again decomposed into the systematic and an idiosyncratic factor

$$Z = bU + \sqrt{1 - b^2} \varepsilon_2, \quad b \in [-1, 1].$$

We assume that  $U$ ,  $\varepsilon_1$  and  $\varepsilon_2$  are independent standard Gaussian random variables. Then the risky payer, resp. receiver, swaption price with strike rate  $s_K$ , no recovery,  $0 < \tilde{T} \leq T_1$ , annuity numeraire satisfying (4.1) and with the payoff function

$$V_{RS}(T_0, \tilde{T}, T_0, T_1, T_n) = \begin{cases} X(T_0, T_1, T_n) \cdot \mathbb{1}_{[\tau > \tilde{T}]} (s_{T_0} - s_K)^+ & \text{for the payer swaption} \\ X(T_0, T_1, T_n) \cdot \mathbb{1}_{[\tau > \tilde{T}]} (s_K - s_{T_0})^+ & \text{for the receiver swaption} \end{cases}$$

is for the payer swaption

$$V_{RS}(0, \tilde{T}, T_0, T_1, T_n) = L \cdot X(0, T_1, T_n) \cdot (s_0 \cdot A_1 - s_K \cdot A_2)$$

resp. for the receiver swaption

$$V_{RS}(0, \tilde{T}, T_0, T_1, T_n) = L \cdot X(0, T_1, T_n) \cdot (s_K \cdot A_{-2} - s_0 \cdot A_{-1}),$$

where

$$A_{\pm 1} = \int_{-\infty}^{\infty} \exp \left\{ au\sigma\sqrt{T_0} - a^2\sigma^2 T_0/2 \right\} \Phi \left( \pm \frac{d_1 + au - a^2\sigma\sqrt{T_0}}{\sqrt{1-a^2}} \right) \cdot \Phi \left( \frac{bu - \Phi^{-1}(1 - S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du,$$

$$A_{\pm 2} = \int_{-\infty}^{\infty} \Phi \left( \pm \frac{d_2 + au}{\sqrt{1-a^2}} \right) \Phi \left( \frac{bu - \Phi^{-1}(1 - S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du,$$

$$d_1 = \frac{\log(s_0/s_K) + \sigma^2 T_0/2}{\sigma\sqrt{T_0}},$$

$$d_2 = d_1 - \sigma\sqrt{T_0},$$

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, u \in \mathbb{R}.$$

*Proof.* The present price of the payer swaption is calculated as the discounted expectation of the future payoff, i.e.

$$\begin{aligned} V_{RS}(0, \tilde{T}, T_0, T_1, T_n) &= L \cdot X(0, T_1, T_n) \cdot \mathbb{E}^{\mathbb{Q}} \left[ \frac{V(T_0, \tilde{T}, T_0, T_1, T_n)}{X(T_0, T_1, T_n)} \right] \\ &= L \cdot X(0, T_1, T_n) \cdot \mathbb{E}^{\mathbb{Q}} \left[ \mathbb{1}_{[\tau > \tilde{T}]} (s_{T_0} - s_K)^+ \right] \end{aligned}$$

where  $\mathbb{Q}$  is the annuity risk-neutral measure. The default time  $\tau$  can be expressed by

the standard Gaussian variables  $U$  and  $\varepsilon_1$ , so

$$\begin{aligned}
 \mathbb{E} \left[ (s_{T_0} - s_K)^+ \mathbb{1}_{[\tau > \tilde{T}]} \right] &= \mathbb{E} \left[ \mathbb{E} \left[ (s_{T_0} - s_K)^+ \mathbb{1}_{[\tau > \tilde{T}]} \middle| U \right] \right] \\
 &= \int_{-\infty}^{\infty} \mathbb{E} \left[ (s_{T_0} - s_K)^+ \mathbb{1}_{[Z > \Phi^{-1}(1-S(\tilde{T}))]} \middle| U = u \right] \varphi(u) du \\
 &= \int_{-\infty}^{\infty} \mathbb{E} \left[ (s_{T_0} - s_K)^+ \mathbb{1}_{\left[ \varepsilon_2 > \frac{\Phi^{-1}(1-S(\tilde{T})) - bu}{\sqrt{1-b^2}} \right]} \middle| U = u \right] \varphi(u) du.
 \end{aligned}$$

The stochastic part of the swap rate  $s_{T_0}$  with given  $U = u$  is equal  $\varepsilon_1$ . The independence of  $\varepsilon_1$  and  $\varepsilon_2$  implies

$$\begin{aligned}
 &\int_{-\infty}^{\infty} \mathbb{E} \left[ (s_{T_0} - s_K)^+ \mathbb{1}_{\left[ \varepsilon_2 > \frac{\Phi^{-1}(1-S(\tilde{T})) - bu}{\sqrt{1-b^2}} \right]} \middle| U = u \right] \varphi(u) du = \\
 &= \int_{-\infty}^{\infty} \mathbb{E} \left[ (s_{T_0} - s_K)^+ \middle| U = u \right] \mathbb{E} \left[ \mathbb{1}_{\left[ \varepsilon_2 > \frac{\Phi^{-1}(1-S(\tilde{T})) - bu}{\sqrt{1-b^2}} \right]} \middle| U = u \right] \varphi(u) du \\
 &= \int_{-\infty}^{\infty} \mathbb{E} \left[ (s_{T_0} - s_K)^+ \middle| U = u \right] \int_{-\infty}^{\infty} \mathbb{1}_{\left[ x_2 > \frac{\Phi^{-1}(1-S(\tilde{T})) - bu}{\sqrt{1-b^2}} \right]} \varphi(x_2) dx_2 \varphi(u) du \\
 &= \int_{-\infty}^{\infty} \mathbb{E} \left[ (s_{T_0} - s_K)^+ \middle| U = u \right] \Phi \left( \frac{bu - \Phi^{-1}(1-S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du \\
 &= \int_{-\infty}^{\infty} \int_{\frac{\log(s_K) - m}{w}}^{\infty} (e^{x_1 w + m} - s_K) \varphi(x_1) dx_1 \Phi \left( \frac{bu - \Phi^{-1}(1-S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du,
 \end{aligned}$$

where

$$\begin{aligned}
 m &= \log(s_0) - \sigma^2 T_0 / 2 + au\sigma\sqrt{T_0}, \\
 w &= \sqrt{1 - a^2} \sigma \sqrt{T_0}.
 \end{aligned}$$

Now we can separate the integral in two parts. First, we will integrate the part with

the strike rate  $s_K$

$$\begin{aligned}
 & -s_K \int_{-\infty}^{\infty} \int_{\frac{\log(s_K)-m}{w}}^{\infty} \varphi(x_1) dx_1 \Phi \left( \frac{bu - \Phi^{-1}(1 - S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du = \\
 & = -s_K \int_{-\infty}^{\infty} \Phi \left( \frac{m - \log(s_K)}{w} \right) \Phi \left( \frac{bu - \Phi^{-1}(1 - S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du = \\
 & = -s_K \int_{-\infty}^{\infty} \Phi \left( \frac{\log(s_0/s_K) - \sigma^2 T_0/2 + au\sigma\sqrt{T_0}}{\sqrt{1-a^2\sigma\sqrt{T_0}}} \right) \Phi \left( \frac{bu - \Phi^{-1}(1 - S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du = \\
 & = -s_K \int_{-\infty}^{\infty} \Phi \left( \frac{d_2 + au}{\sqrt{1-a^2}} \right) \Phi \left( \frac{bu - \Phi^{-1}(1 - S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du = -s_K \cdot A_2.
 \end{aligned}$$

then the part with the swap rate  $s_0$

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \int_{\frac{\log(s_K)-m}{w}}^{\infty} e^{x_1 w + m} \varphi(x_1) dx_1 \Phi \left( \frac{bu - \Phi^{-1}(1 - S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du = \\
 & = \int_{-\infty}^{\infty} \int_{\frac{\log(s_K)-m}{w} - w}^{\infty} e^{m+w^2/2} \varphi(x_1) dx_1 \Phi \left( \frac{bu - \Phi^{-1}(1 - S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du \\
 & = \int_{-\infty}^{\infty} e^{m+w^2/2} \Phi \left( w - \frac{\log(s_K) - m}{w} \right) \Phi \left( \frac{bu - \Phi^{-1}(1 - S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du \\
 & = s_0 \int_{-\infty}^{\infty} \exp \left\{ au\sigma\sqrt{T_0} - a^2\sigma^2 T_0/2 \right\} \Phi \left( \frac{d_1 + au - \sigma\sqrt{T_0}a^2}{\sqrt{1-a^2}} \right) \cdot \\
 & \quad \Phi \left( \frac{bu - \Phi^{-1}(1 - S(\tilde{T}))}{\sqrt{1-b^2}} \right) \varphi(u) du \\
 & = s_0 \cdot A_1.
 \end{aligned}$$

The proof of the receiver swaption price is analogous.

□

This result gives us the formula for evaluation of the swaption price with the CCR. In the next section we will use this formula to calculate the IRS CVA.

*Remark 1.* Note that the swaption start date can be any time  $\tilde{T} \leq T_1$  where  $T_1$  is the time of the first payment. We will use the result for  $\tilde{T} = T_0$  where  $T_0$  is the standard swap start date, but also for the case where  $\tilde{T} = T_1$ .

*Remark 2.* The correlation coefficient between the interest rates and the default time is  $\rho = ab$ , and therefore  $\rho \in [-1, 1]$ . Furthermore we assume that  $|a| = |b|$ , i.e. the correlation  $\rho$  is positive if  $a$  and  $b$  have the same sign, negative if  $a$  and  $b$  have different sign and  $\rho = 0$  otherwise.

## 5 Interest Rate Swap CVA

The CVA of an interest rate swap (hereinafter "CVA<sub>IRS</sub>") in case of no wrong-way risk can be approximated as a sum of interest rate swaption prices weighted by survival probabilities, as noted in [7]. The formula presented in [3] is in form

$$\text{CVA}_{IRS} \approx \sum_{i=0}^{n-1} (S(T_i - t) - S(T_{i+1} - t))V(t, T_{i+1}, T). \quad (5.1)$$

where  $LGD = 1$ ,  $n$  is the number of swap payments,  $S(T_i - t)$  are the (risk neutral) survival probabilities, and  $V(t, T_{i+1}, T)$  is the present swaption price with option expiration at time  $T_{i+1}$  and swap maturity at time  $T$ .

Even in the presence of the wrong-way risk we can use the swaption price to evaluate the CVA<sub>IRS</sub> with  $n$  swap payments ( $t = T_0 < T_1 < \dots < T_n = T$ ) as

$$\text{CVA}_{IRS}(t, T) = \sum_{i=0}^{n-1} \text{CVA}_{IRS}(t, T_i, T_{i+1}) \quad (5.2)$$

where  $\text{CVA}_{IRS}(t, T_i, T_{i+1})$  is the expected value of the loss if the counterparty defaults between the times  $T_i$  and  $T_{i+1}$ . More rigorously, for  $i = 0, \dots, n - 1$

$$\begin{aligned} \text{CVA}_{IRS}(t, T_i, T_{i+1}) &= \mathbb{E}^{\mathbb{Q}} \left[ \mathbb{1}_{[T_i < \tau \leq T_{i+1}]} V(\tau, T_{i+1}, T)^+ D(t, \tau) \right] \\ &\approx \mathbb{E}^{\mathbb{Q}} \left[ \mathbb{1}_{[T_i < \tau \leq T_{i+1}]} V(T_{i+1}, T_{i+1}, T)^+ D(t, T_{i+1}) \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \mathbb{1}_{[\tau > T_i]} V(T_{i+1}, T_{i+1}, T)^+ D(t, T_{i+1}) \right. \\ &\quad \left. - \mathbb{1}_{[\tau > T_{i+1}]} V(T_{i+1}, T_{i+1}, T)^+ D(t, T_{i+1}) \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \mathbb{1}_{[\tau > T_i]} V(T_{i+1}, T_{i+1}, T)^+ D(t, T_{i+1}) \right] \\ &\quad - \mathbb{E}^{\mathbb{Q}} \left[ \mathbb{1}_{[\tau > T_{i+1}]} V(T_{i+1}, T_{i+1}, T)^+ D(t, T_{i+1}) \right] \\ &= V_{RS}(t, T_i, T_{i+1}, T_{i+1}, T) - V_{RS}(t, T_{i+1}, T_{i+1}, T_{i+1}, T). \end{aligned} \quad (5.3)$$

In other words, the CVA<sub>IRS</sub> at time  $t$  from the time  $T_i$  up to time  $T_{i+1}$  is approximated by the difference of the risky swaption prices with expiration at  $T_{i+1}$ , payments starting at  $T_{i+1}$  but with different default time process starting date.

If we combine this with previous results we obtain the following theorem, which is the main contribution of this paper.

**Theorem 2.** *Let the assumptions of the Theorem 1 are satisfied. Moreover, we assume that the default time is postponed to the following swap payment moment in case of default. Then  $CVA_{IRS}$  of the fix rate payer, resp. receiver, IRS including wrong-way risk can be calculated as*

$$CVA_{IRS}(t, T) \approx L \cdot \sum_{i=0}^{n-1} X(t, T_{i+1}, T) [s_{t,i}(A_{1,i} - B_{1,i}) - s_K(A_{2,i} - B_{2,i})],$$

resp.

$$CVA_{IRS}(t, T) \approx L \cdot \sum_{i=0}^{n-1} X(t, T_{i+1}, T) [s_K(A_{-2,i} - B_{-2,i}) - s_{t,i}(A_{-1,i} - B_{-1,i})],$$

where

$$A_{\pm 1,i} = \int_{-\infty}^{\infty} \exp \left\{ a_i u \sigma \sqrt{T_{i+1} - t} - a_i^2 \sigma^2 (T_{i+1} - t) / 2 \right\} \Phi \left( \pm \frac{d_{1,i} + a_i u - a_i^2 \sigma \sqrt{T_{i+1} - t}}{\sqrt{1 - a_i^2}} \right) \cdot \Phi \left( \frac{b_i u - \Phi^{-1}(1 - S_i(T_i - t))}{\sqrt{1 - b_i^2}} \right) \varphi(u) du,$$

$$B_{\pm 1,i} = \int_{-\infty}^{\infty} \exp \left\{ a_i u \sigma \sqrt{T_{i+1} - t} - a_i^2 \sigma^2 (T_{i+1} - t) / 2 \right\} \Phi \left( \pm \frac{d_{1,i} + a_i u - a_i^2 \sigma \sqrt{T_{i+1} - t}}{\sqrt{1 - a_i^2}} \right) \cdot \Phi \left( \frac{b_i u - \Phi^{-1}(1 - S_i(T_{i+1} - t))}{\sqrt{1 - b_i^2}} \right) \varphi(u) du,$$

$$A_{\pm 2,i} = \int_{-\infty}^{\infty} \Phi \left( \pm \frac{d_{2,i} + a_i u}{\sqrt{1 - a_i^2}} \right) \Phi \left( \frac{b_i u - \Phi^{-1}(1 - S_i(T_i - t))}{\sqrt{1 - b_i^2}} \right) \varphi(u) du,$$

$$B_{\pm 2,i} = \int_{-\infty}^{\infty} \Phi \left( \pm \frac{d_{2,i} + a_i u}{\sqrt{1 - a_i^2}} \right) \Phi \left( \frac{b_i u - \Phi^{-1}(1 - S_i(T_{i+1} - t))}{\sqrt{1 - b_i^2}} \right) \varphi(u) du,$$

$$d_{1,i} = \frac{\log(s_{t,i}/s_K) + \sigma^2(T_{i+1} - t)/2}{\sigma \sqrt{T_{i+1} - t}},$$

$$d_{2,i} = d_{1,i} - \sigma \sqrt{T_{i+1} - t}.$$

and the swap rate  $s_{t,i}$  is the forward swap rate of the swap starting at time  $T_{i+1}$ .

The proof of this theorem follows from the Theorem 1 and formulas (5.2) and (5.3).

Note that, in general, the survival functions  $S_i$  and correlations  $\rho_i = a_i b_i$  (again  $|a_i| = |b_i|$  for all  $i$ ) change with respect to the numeraire. The survival function  $S$  and correlation coefficient  $\rho$  without subscript correspond to the unified risk-neutral measure. In terms of practical use we neglect the change of numeraire and approximate  $S_i = S$  and  $\rho_i = \rho$  for all  $i = 0, \dots, n - 1$ .

## 6 Numerical study

We are going to price a plain-vanilla at-the-money fix-receiver 10Y IRS (with swap rate 4.05 %) on the EUR market where the fixed leg pays annually a 30E/360 strike rate and the floating leg pays semi-annually LIBOR. The recovery rate is equal zero, i.e.  $LGD = 1$ , and the volatility  $\sigma$  is equal 12 %. Remaining inputs of the model are the zero-bond spot rates which are shown in the Appendix.

The wrong-way risk occurs when the default time and the interest rates are decreasing, so we consider only positive correlation coefficient. The correlation coefficient describes the dependence between levels of the interest rate and default time (not between their instantaneous changes). The following table shows the CVA of the IRS including CCR using semi-analytical formula for different correlation coefficients and hazard rate  $h = 5\%$ .

$\rho$	$\mathbf{CVA}_{\text{IRS}}$
0	0.343 %
0.1	0.394 %
0.3	0.501 %
0.5	0.619 %
0.7	0.751 %
0.9	0.914 %
1	1.028 %

Table 1:  $\mathbf{CVA}_{\text{IRS}}$  values as a percentage of the notional amount with hazard rate equal 5 %

The impact of the wrong-way risk is certainly not negligible. In case of perfect correlation between the interest rate and the default time the wrong-way risk is about 0.69 % of the nominal value.

Previously presented CVA was related to the risk-neutral hazard rate equal 5 % which corresponds to worse rated countries or well-rated companies. A corporate risk-neutral hazard rate will be typically about 10 % (see Moody's rating B in [5]). The figure below illustrates the behavior of the CVA of the IRS with respect to correlation coefficient and also the hazard rate.

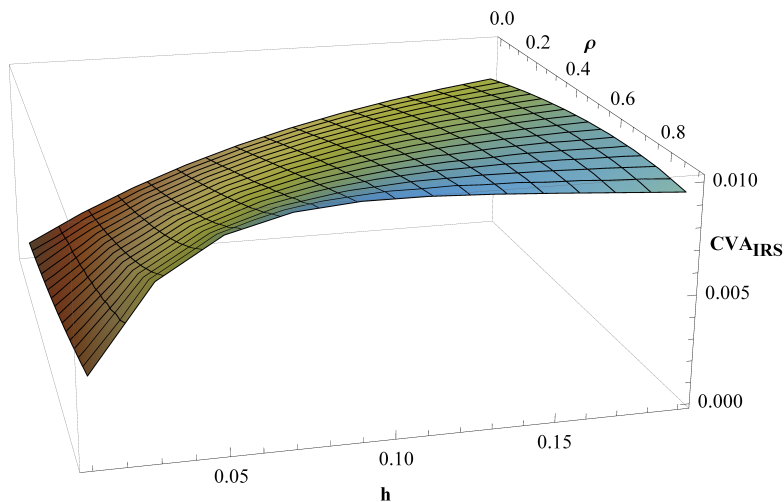


Figure 1:  $CVA_{\text{IRS}}$  as a function of hazard rate  $h$  and correlation  $\rho$

We may see that from a certain values of the hazard rate and correlation the CVA tends to go down. This is caused by the high probability of the early default, i.e. the IRS price will not change much during short time interval neither does the exposure.

## 7 Comparison with simulation study

We compared the prices of IRS including CCR calculated by the semi-analytical approach presented in previous section with the simulation study provided in [2].

In [2] the interest rate development is described by the two-factor G2++ model, i.e. short-rate process under the risk-neutral measure is given by

$$r(t) = x(t) + z(t) + \phi(t, \alpha), \quad r(0) = r_0$$

where  $\alpha$  is a parameter vector, with  $\phi$  is a deterministic function and the processes  $x$  and  $z$  are  $\mathcal{F}_t$  adapted and satisfy

$$\begin{aligned} dx(t) &= -ax(t)dt + \sigma dW_1(t), \quad x(0) = 0 \\ dy(t) &= -by(t)dt + \eta dW_2(t), \quad z(0) = 0 \end{aligned}$$

where  $(W_1, W_2)$  is a two-dimensional Brownian motion with instantaneous correlation  $\rho_{1,2}$ . The elements of the parameter vector  $\alpha = (a, b, \sigma, \eta, \rho_{1,2})$  are positive constants. The stochastic hazard rate is governed by the CIR++ model, i.e.

$$\begin{aligned} \lambda_t &= y_t + \psi(t, \beta), \quad t \geq 0, \\ dy_t &= \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW_3(t) \end{aligned}$$



where the parameter vector is  $\beta = (\kappa, \mu, \nu, y_0)$ , with  $\kappa, \mu, \nu, y_0$  positive deterministic constants.  $W_3$  is a standard Brownian motion process under the risk neutral measure.

The instantaneous correlation between the short rate and the hazard rate, i.e. the instantaneous interest-rate / credit-spread correlation is

$$\bar{\rho} = \text{Corr}(dr_t, d\lambda_t). \tag{7.1}$$

The description and properties of mentioned interest rate and stochastic hazard rate models can be also found in [1]. The calibration of these models and the calibration results are in detail described in [2].

Following table contains results of this comparison without and with wrong-way risk. The hazard rate  $h$ , resp. initial hazard rate  $\gamma$  from [2], is fixed. The results of the semi-analytical approach are in column  $\text{CVA}_{\text{CW}}$  and column  $\text{CVA}_{\text{Brigo}}$  corresponds to the results of the simulation study. Please note that the correlation used in [2] is an instantaneous correlation of the interest rate and the hazard rate denoted by  $\bar{\rho}$  (see equation (7.1)) whilst we are using the correlation

$$\rho = ab = \text{Corr}(Y, Z) \tag{7.2}$$

between the levels of the interest rate and the default time, resp. correlation between the random drivers of the interest rate and the default time given by (4.2) and (4.3).

$\text{Corr}_{\text{CW}}$	$\text{Corr}_{\text{Brigo}}$	$h, \gamma$	$\text{CVA}_{\text{CW}}$	$\text{CVA}_{\text{Brigo}}$
$\rho = 0$	$\bar{\rho} = 0$	3 %	0.222 %	0.22 %
		5 %	0.343 %	0.34 %
		7 %	0.447 %	0.44 %
$\rho = 1$	$\bar{\rho} = -1$	3 %	0.841 %	0.36 %
		5 %	1.028 %	0.46 %
		7 %	1.074 %	0.54 %

Table 2: IRS prices including CCR with and without wrong-way risk

The results without wrong-way risk (zero correlation) are almost the same which we have expected. The results with the wrong-way risk are different which was also expected because the correlations does not express the same dependence. Although our approach provides more conservative results, it can be expected that after the correlation coefficients calibration on the same market data, both methods give similar results.

## 8 Conclusion

In this paper we have noted that the CVA of the OTC derivative including wrong-way risk can be expressed as a difference between the covariance and the CVA without

wrong-way risk. It is a general formula without any additional assumptions on the model of the default time nor the underlying asset. But it can be simplified assuming linear dependence represented by the constant correlation coefficient.

Main part of this paper deals with the development of the semi-analytical formula for the IRS price calculation including CCR. The formula uses the constant correlation Gaussian copula dependence of the default time and interest rate based on the [4]. Using this formula we also found that wrong-way risk has relevant impact on the OTC IRS price with lower hazard rate, resp. probability of default, which has been analyzed in the numerical study where we evaluate 10Y plain vanilla IRS.

We compared the semi-analytical formula with the simulation study presented in [2]. In case of no wrong-way risk both approaches give almost the same results. But if we include the wrong-way risk the results vary. This difference is caused by different calculation of the correlation coefficient. In [2] the correlation is instantaneous between the hazard rate and interest rates changes, in this paper the correlation is between levels of the default time and the interest rate.

The correlation between levels gives, in our view, better information of the dependence than the instantaneous correlation. On the other hand the calibration of the instantaneous correlation is much easier because the hazard rates are observable on the market from the CDS spreads. The calibration of the correlation between the default time and the interest rate (or other underlying assets) is a subject of the further research.

Other possible disadvantages of the semi-analytical formula are constant LGD, Gaussian copula dependence and constant correlation coefficient assumptions. For more accurate calculation of the CVA the LGD should be stochastic taking into account economic cycles, especially downturn periods. LGD has typically U-shaped distribution which is mostly modeled by the beta distribution. The use of the Gaussian copula does not correspond to the situation on the market where heavy-tailed distribution should be assumed (see [6]). In general the correlation should be stochastic so the constant correlation assumption is not realistic. As a compromise we can assume the term structure of the correlation which leaves the formula almost unchanged (except the correlation).

Despite all these weaknesses of the formula it is very quick and simple method to implement and to calculate the IRS CVA with and without wrong-way risk.

## Appendix

Maturity Date	Rate	Maturity Date	Rate	Maturity Date	Rate
26-Jun-06	2.83%	20-Sep-07	3.46%	27-Jun-16	4.19%
27-Jun-06	2.83%	19-Dec-07	3.52%	27-Jun-17	4.23%
28-Jun-06	2.83%	19-Mar-08	3.57%	27-Jun-18	4.27%
04-Jul-06	2.87%	19-Jun-08	3.61%	27-Jun-19	4.31%
11-Jul-06	2.87%	18-Sep-08	3.65%	29-Jun-20	4.35%
18-Jul-06	2.87%	29-Jun-09	3.75%	28-Jun-21	4.38%
27-Jul-06	2.88%	28-Jun-10	3.84%	27-Jun-22	4.41%
28-Aug-06	2.92%	27-Jun-11	3.91%	27-Jun-23	4.43%
20-Sep-06	2.96%	27-Jun-12	3.98%	27-Jun-24	4.45%
20-Dec-06	3.14%	27-Jun-13	4.03%	27-Jun-25	4.47%
20-Mar-07	3.27%	27-Jun-14	4.09%	29-Jun-26	4.48%
21-Jun-07	3.38%	29-Jun-15	4.14%	28-Jun-27	4.50%

Table 3: EUR zero-coupon continuously-compounded spot rates (ACT/360) observed on June,23 2006.

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