

Chen, Ke; Poon, Ser-Huang

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Working Paper Series

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Ke Chen
Ser-Huang Poon

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Manchester Business School
The University of Manchester
Booth Street West
Manchester M15 6PB

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Consistent Pricing and Hedging Volatility Derivatives with Two Volatility Surfaces

Ke Chen and Ser-Huang Poon*

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*Ke Chen (ke.chen@postgrad.mbs.ac.uk) and Ser-Huang Poon (ser-huang.poon@mbs.ac.uk) are both at the Manchester Business School, University of Manchester. We would like to thank Simon Acomb, Michael Brennan, Peter Carr, Stephan Taylor, and participants at Quant Congress London for many helpful comments and suggestions.

Consistent Pricing and Hedging Volatility Derivatives with Two Volatility Surfaces

Abstract

Using the joint characteristic function of equity price and state variables, we can price contingent claims on both equity and VIX consistently. Based on linear approximation of jump size, we show that one factor models implies all VIX future contract of different maturities are perfectly correlated in contrast to market observations. In the examples of multi-factor model, we demonstrate how to calculate the optimal hedging ratio for VIX future to hedge VIX option. We derived the unconditional correlation term structure of VIX future implied by the model based on the stationary distribution of state variables. We show multifactor models that are calibrated to the two volatility surfaces will produce very different hedge ratios for VIX options.

JEL Classification: G12, G13

Keywords: SPX Volatility Surface, VIX Volatility Surface, VIX Futures, VIX Options, Hedge Ratio

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1 Introduction

In 1993, the Chicago Board Options Exchange (CBOE) introduced VIX volatility index, and since then VIX has been accepted universally as the stock market volatility. In 2003, CBOE switched to use a model free approach to calculate VIX based on prices of S&P 500 (SPX) options. The new method makes it possible, for the first time, to replicate VIX index value, which led to huge public interests in trading and hedging volatility. As expected, VIX futures contracts were launched in 2004, followed by VIX options in 2006. Exchange and OTC traded derivatives on VIX and its variants become increasingly popular; the average daily trading volume of VIX options and futures has increased 20 folds between 2006 and 2011. VIX derivatives initially created as an hedging instrument soon became a new asset class for investment and speculation. Such a change in market practice also reflects in the development of pricing methods for volatility derivatives.

Earlier papers treat volatility as a separate underlying and price volatility derivatives in isolation from the stock price process. Grunbichler and Longstaff (1996), for example, assume volatility follows a CIR (Cox, Ingersoll and Ross) process and derive closed form solutions for pricing volatility futures and options. Psychoyios et al. (2010) model spot VIX dynamic directly as a CIR process with jumps. The addition of jumps improves the calibration to VIX option prices significantly, since the CIR process itself cannot produce enough positive skewness to match the implied volatility of VIX options. The cost of adding jumps is that there is no closed form for the distribution of spot VIX and the prices of VIX futures and options have to be derived via the characteristic function of VIX.

A second, more recent, approach of modelling VIX is to use the volatility dynamics of SPX as the starting point and derive an expression for VIX. Solutions for VIX derivatives are then derived thereof. For example, Zhang and Zhu (2006) derive the dynamics of VIX future based on a Heston

process for SPX. Sepp (2008) extends this work by adding jumps to the stochastic variance for pricing VIX futures and options. Both Zhang and Zhu (2006) and Sepp (2008) are classified as one factor model for volatility. Lu and Zhu (2009), on the other hand, use multi-factor model to study the term structure of VIX futures. This second approach of modelling VIX has become more popular in recent years. Since this second approach uses a single set of dynamics to price SPX and VIX derivatives, one can measure the sensitivity of SPX derivatives with respect to changes in VIX, and *vice versa*. We call this type of joint modelling and pricing method, the consistent pricing approach.

In this paper, we adopt the consistent pricing approach, but unlike previous studies which focus on static calibration of VIX surface,¹ we focus on the term structure of VIX future dynamics which is a key aspect of hedging. In addition, we also show how to derive the exposure of VIX option with respect to VIX future and use it to derive hedge ratio. We study three affine models, viz. a one-factor model and two two-factor models for volatility as well as the models' implied correlation term structure for VIX futures conditional on the state variables. We show that the correlation term structure implied by all one factor models is unrealistic. Finally, we also show how to derive the unconditional correlation term structure for the multi-factor models. This unconditional correlation can be used to match the market data in model calibration.

2 Consistent Pricing Method

Contingent claims are priced by taking expectation under the risk neutral measure. Since characteristic function always exists and is the only way to represent the distribution of some random variable when the density function does not exist, it is now common to price contingent claims by using the characteristic function. To do this, we must first derive the Fourier transform for the payoff function. Sepp (2003) describes the general approach for deriving the characteristic function for affine models and how it is used in pricing options. This general pricing framework does

¹In the rest part of this paper, we use VIX surface to denote the implied volatility surface of VIX options

not always work for non-affine models. For example, the 3/2 model in Carr and Sun (2007) is not affine; the derivation of the characteristic function is still possible but is more complicated.

In the following subsections, we will first briefly explain how to use the characteristic function to price SPX derivatives and then extend the pricing approach to price VIX futures and options.

2.1 Derivatives on Equity

The price of a European option is the discounted expectation of future payoff under the risk neutral measure,

$$\begin{aligned} f(X_t, t) &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[f(X_T, T) | \mathcal{F}_t] \\ &= e^{-r(T-t)} \int_{\mathbb{R}^n} f(X_T, T) p(X_T | \mathcal{F}_t) dX_T \end{aligned} \quad (1)$$

where $X_t = \ln S_t$. Let us denote $G(\omega, X_t, T)$ as the characteristic function of the transition distribution $p(X_T | X_t)$, which is defined by the generalised Fourier Transform,

$$\begin{aligned} G(\omega, X_t, T) &= \int_{-\infty}^{\infty} e^{\omega X_T} p(X_T | \mathcal{F}_t) dX_T \\ p(X_T | \mathcal{F}_t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\omega X_T} G(\omega, X_t, T) d\Im(\omega) \end{aligned}$$

where ω is a complex number and $\Im(\cdot)$ denotes the imaginary part.

So if we define $\hat{f}(\omega)$ as the Fourier transform of the payoff function $f(X_T, T)$, or alternatively $f(X_T, T)$ is the inverse Fourier transform of $\hat{f}(\omega)$, then equation (1) can be rewritten as,

$$\begin{aligned} f(X_t, t) &= \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-\omega X_T} d\Im(\omega) p(X_T | \mathcal{F}_t) dX_T \\ &= \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \int_{\mathbb{R}} e^{-\omega X_T} p(X_T | \mathcal{F}_t) dX_T d\Im(\omega) \\ &= \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) G(-\omega, X_t, T) d\Im(\omega) \end{aligned} \quad (2)$$

Since the expression of $\hat{f}(\omega)$ and $G(-\omega, X_t, T)$ are normally known, we can evaluate the integral in equation (2) numerically. Many efficient methods are developed to perform this integration efficiently, for example, *Fast Fourier Transform* by Carr and Madan (1999) and *Cosine Fourier Transform* by Fang and Oosterlee (2008).

2.2 Pricing of VIX Future and Option

Since the payoff of VIX derivatives is a function of the expectation of the accumulated variance; the pricing of VIX derivatives is not as straightforward as the pricing of SPX derivatives. The model for SPX typically has instantaneous variance, not the accumulated variance, as one of the state variables. To price both SPX and VIX derivatives consistently, we need to derive the expected accumulated variance as a function of the state variables.

Assuming the dynamic of equity price with time varying variance and jump intensity is as follows,

$$d \ln S_t = \left(r - \frac{v_t}{2} - \lambda m_1\right) dt + \sqrt{v_t} dW_t + \int_{\mathbb{R} \times \mathbb{R}_+} J_x \mu(dJ_x, dt)$$

where $m_1 = \mathbb{E}[e^{J_x} - 1]$ and $\mu(dJ_x, dt)$ is the random measure for the compound Poisson jump J_x with jump intensity λ_t . So the accumulated variance over the period from t to $t + \Delta t$ is given by,

$$V = \int_t^{t+\Delta t} \left(v_s ds + \int_{-\infty}^{\infty} J_x^2 \mu(dJ_x, ds) \right) \quad (3)$$

where Δt is usually set as 30 days for contracts like variance swap and VIX future.

By assuming no dependence between jump timing and jump size, we have $\mu(dJ_x, dt) = \mu(dJ_x) \lambda_t dt$, where $\mu(dJ_x)$ is the distribution of jump size. Taking expectation on equation (3) and annualising, we have

$$\begin{aligned}
\bar{V}(v_t, \lambda_t) &= \frac{1}{\Delta t} \mathbb{E}[V | \mathcal{F}_t] \\
&= \frac{1}{\Delta t} \mathbb{E} \left[\int_t^{t+\Delta t} \left(v_s + \lambda_s \int_{-\infty}^{\infty} J_x^2 \mu(dJ_x) \right) ds \right] \\
&= \frac{1}{\Delta t} \mathbb{E} \left[\int_t^{t+\Delta t} (v_s + \lambda_s \mathbb{E}[J_x^2]) ds \right]
\end{aligned} \tag{4}$$

$\bar{V}(v_t, \lambda_t)$ is the time t fair price for an annualized Δt spot variance swap. For many stochastic variance and stochastic jump intensity models, e.g. Heston and Bates, there are close form solutions for (4). Based on the above definition, one can further define VIX future and option as

$$F_{VIX}(t, T) = \mathbb{E}^{\mathbb{Q}} \left[\sqrt{\bar{V}(v_T, \lambda_T)} \middle| \mathcal{F}_t \right] \tag{5}$$

$$C_{VIX}(t, T) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[\left(\sqrt{\bar{V}(v_T, \lambda_T)} - K \right)^+ \middle| \mathcal{F}_t \right] \tag{6}$$

which are priced at time t , but settled at time T .² Therefore, we can derive the VIX future and option as the expectation of the respective payoff functions under the joint distribution $p(v_T, \lambda_T | \mathcal{F}_t)$. Following the steps in Section 2.1, we need to first derive the Fourier Transform of $\sqrt{\bar{V}(v_T, \lambda_T)}$ and $\left(\sqrt{\bar{V}(v_T, \lambda_T)} - K \right)^+$, and also the characteristic function of $\bar{V}(v_T, \lambda_T)$.

Before we proceed, it is important to point out a subtlety of the VIX definition. VIX, as reported by CBOE, is based on a replicating strategy using SPX options. Under the assumption of no jump, the replication is perfect. However, when there are jumps, the replicated variance is different from equation (3). As shown in Carr and Wu (2009), the VIX constructed by CBOE is actually

$$\bar{V} = \frac{1}{\Delta t} \mathbb{E} \left[\int_t^{t+\Delta t} \left(v_s + \lambda_s \int_{-\infty}^{\infty} 2(e^{J_x} - 1 - J_x) v(dJ_x) \right) ds \right] \tag{7}$$

²Noting that for VIX future, the volatility reference period is from T to $T + \Delta t$ with expectation taken at time T .

The difference between equation (7) and equation (3) is of the third order of jump size, since³

$$\int_{-\infty}^{\infty} 2(e^{J_x} - 1 - J_x) \nu(dJ_x) = 2\mathbb{E}[e^{J_x} - 1 - J_x] = \mathbb{E}[J_x^2] + O(J_x^3)$$

Although $O(J_x^3)$ is small and trivial compared with $\mathbb{E}[J_x^2]$, we will use equation (7) as the definition of VIX for the rest of this paper to be consistent with the CBOE definition. We denote $m_2 = 2\mathbb{E}[e^{J_x} - 1 - J_x]$.

3 Characteristic Function of Equity Price and Variance

In this section, we derive the closed form solutions for derivative prices for three models; a single factor model with jumps in both stock price and variance, a two-factor model with stochastic variance and stochastic jump intensity where the jump intensity follows a self exciting Hawkes process, and another two-factor model with two Heston type stochastic variance, one of which shares the jump timing with the stock price. The number of factors here refer to the factors that drive the stock price variance. We solve the Partial Integral Differential Equation (PIDE) to derive the characteristic functions of the state variables and equity price.

3.1 One-Factor Model

As pointed out by Sepp (2008) and Psychoyios et al. (2010), a positive jump is needed in the variance process to generate enough positive skewness observed in the VIX volatility surface. So we use the following specification,

³Using Taylor expansion, $e^x = 1 + x + \frac{x^2}{2} + O(x^3)$.

$$\begin{aligned}
d\ln S_t &= \left(r - \frac{v_t}{2} - \lambda m_1\right)dt + \sqrt{v_t}dW_t^1 + \int J_x \mu(dJ_x, dJ_v, dt) \\
dv_t &= \kappa(\theta - v_t)dt + \sigma \sqrt{v_t}dW_t^2 + \int J_v \mu(dJ_x, dJ_v, dt) \\
J_x &\sim N(\varepsilon + \rho_J J_v, \delta^2) \\
J_v &\sim \text{Exp}(\eta)
\end{aligned} \tag{8}$$

where $m_1 = \mathbb{E}[e^{J_x} - 1] = \frac{1}{1 - \rho_J \eta} e^{\varepsilon + \delta^2/2} - 1$, $\mu(dJ_x, dJ_v, dt)$ is the random measure for jumps in both price and volatility which share the same jump timing, λ is the jump intensity, and ρ_J defines the dependence between jumps in log price and jumps in variance. The model (SVJJ) in (8) appears in in Bates (2000), Pan (2002), and Eraker (2004), and nests many other well known stochastic volatility models. For example, it is the Heston model if there are no jumps. If jumps are only allowed in equity price but not in variance, the model becomes Bates (1996). However, we should note that despite the generality of SVJJ, it is still a one factor model for volatility.

From equations (5) and (6), the prices of VIX future and option can be written as,

$$F_{VIX}(t, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{\pi}}{2(-\omega)^{3/2}} \phi_{\bar{V}}(-\omega, t, T) d\Im(\omega) \tag{9}$$

$$C_{VIX}(t, T) = \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \text{erf}(K\sqrt{-\omega})}{2(-\omega)^{3/2}} \phi_{\bar{V}}(-\omega, t, T) d\Im(\omega) \tag{10}$$

where $\phi_{\bar{V}}(-\omega, t, T)$ is the characteristic function of the expectation of accumulated variance $\bar{V}(v_T)$, erf is the error function, and $\frac{\sqrt{\pi}}{2(-\omega)^{3/2}}$ and $\frac{1 - \text{erf}(K\sqrt{-\omega})}{2(-\omega)^{3/2}}$ are the characteristic functions of the payoff of VIX future and option respectively, which are directly derived from the definition of characteristic function. To derive $\phi_{\bar{V}}(-\omega, t, T)$, we write \bar{V} as a function of v_T . Given equation (7), we have,

$$\bar{V}(v_T) = \left(\frac{e^{-\kappa\Delta t} - 1}{\kappa\Delta t} \left(\frac{\kappa\theta + \lambda\eta}{\kappa} - v_T \right) + \frac{\kappa\theta + \lambda\eta}{\kappa} \right) + \lambda m_2 = C_0 + C_1 v_T$$

where $m_2 = 2\mathbb{E}^{\mathbb{Q}}[e^{J_x} - 1 - J_x] = 2 \left(\frac{1}{1 - \rho_J \eta} e^{\varepsilon + \delta^2/2} - 1 - \varepsilon - \rho_J \eta \right)$ and $C_0 = \lambda m_2 + \frac{\kappa\theta + \lambda\eta}{\kappa\Delta t} \left(\frac{e^{-\kappa\Delta t} - 1}{\kappa} + \Delta t \right)$, $C_1 = -\frac{(e^{-\kappa\Delta t} - 1)}{\kappa\Delta t}$. We notice that the expectation of accumulated variance is an affine function of v_T . If we know the characteristic function of v_T , it is straightforward to derive that of \bar{V} . On the other hand, because we are interested in consistent pricing of derivatives on both equity and variance, we need to derive the joint characteristic function for S_T and v_T . The characteristic function of the joint distribution of states is defined as $G(\omega_0, \omega_1, t, T) = \mathbb{E}[e^{\ln S_T \omega_0 + v_T \omega_1} | \mathcal{F}_t]$, and due to Feynman-Kac formula, it is the solution to the PIDE,

$$\begin{aligned} 0 = & G_t + \left(r - \frac{v_t}{2} - \lambda m_1 \right) G_x + \frac{1}{2} v_t G_{xx} + \kappa(\theta - v_t) G_v + \frac{1}{2} \sigma^2 v_t G_{vv} + \rho \sigma v_t G_{xv} \\ & + \lambda \int_{\mathbb{R} \times \mathbb{R}^+} \delta G v(dJ_x, dJ_v) \end{aligned} \quad (11)$$

where $x = \ln S_t$, $\delta G = G(x + J_x, v + J_v) - G(x, v)$ and $v(dJ_x, dJ_v)\lambda dt$ is the compensator for the random measure $\mu(dJ_x, dJ_v, dt)$, and the initial condition at $t = T$ is given by,

$$G(\omega_0, \omega_1, T, T) = e^{\omega_0 \ln S_T + \omega_1 v_T} \quad (12)$$

As shown in Sepp (2003), the solution for the PIDE (11) takes the form of $\exp(\omega_0 \ln S_t + r(T - t)\omega_0 + A(\omega_0, \omega_1, t, T) + B(\omega_0, \omega_1, t, T)v_t)$. So the integral in the last term of equation (11) becomes,

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{R}^+} \delta G v(dJ_x, dJ_v) &= \int_{\mathbb{R} \times \mathbb{R}^+} (G(x + J_x, v + J_v) - G(x, v)) v(dJ_x, dJ_v) \\ &= G(x, v) \int (e^{\omega_0 J_x + B J_v} - 1) v(dJ_x, dJ_v) \\ &= G(x, v) \left(\frac{e^{\varepsilon \omega_0 + \delta^2 \omega_0^2/2}}{1 - \eta(B + \rho_J \omega_0)} - 1 \right) \end{aligned}$$

Substituting the proposed solution form into the PDE, we have the following ODE set,

$$\dot{A}_t = -\kappa\theta B_t - \lambda \left(\frac{e^{\varepsilon\omega_0 + \delta^2\omega_0^2/2}}{1 - \eta(B_t + \rho_J\omega_0)} - 1 \right) - \omega_0\lambda(e^{\varepsilon + \delta^2/2} - 1) \quad (13)$$

$$\dot{B}_t = \frac{1}{2}(\omega_0 - \omega_0^2) + (\kappa - \rho\sigma\omega_0)B_t - \frac{1}{2}\sigma^2 B_t^2 \quad (14)$$

where $\dot{\cdot}$ denotes the derivative w.r.t to time t . The initial conditions at $t = T$ are,

$$A_T = 0 \quad (15)$$

$$B_T = \omega_1 \quad (16)$$

which is derived from equation (12).

To price derivatives on equity price and variance, the joint characteristic function is not necessary, since the payoff only depends on the final state of either equity price or instantaneous variance. Therefore, we can solve the characteristic functions for $\ln S_T$ and v_T separately, which is much easier. If we force $\omega_1 = 0$, we have the solution for $A(\omega_0, 0, t, T)$ and $B(\omega_0, 0, t, T)$, which is the solution for the characteristic function of equity price. The solution as shown below can be found in Pan (2002) and Sepp (2003),

$$A(\omega_0, 0, t, T) = -\frac{\kappa\theta}{\sigma^2} \left(\psi_+(T-t) + 2 \ln \frac{\psi_- + \psi_+ e^{-\zeta(T-t)}}{2\zeta} \right) - \lambda(T-t) \left(\omega_0 \left(\frac{e^{\varepsilon + \delta^2/2}}{1 - \rho_J\eta} - 1 \right) + 1 \right) \quad (17)$$

$$+ \lambda e^{\varepsilon\omega_0 + \frac{\delta^2\omega_0^2}{2}} \left\{ \frac{\psi_-(T-t)}{\psi_- L + \eta U} - \frac{2\eta U}{(\zeta L)^2 - (ML + \eta U)^2} \ln \left[1 - \frac{\psi_+ - \eta U}{2\zeta L} \left(1 - e^{-\zeta(T-t)} \right) \right] \right\}$$

$$B(\omega_0, 0, t, T) = -(\omega_0 - \omega_0^2) \frac{1 - e^{-\zeta(T-t)}}{\psi_- + \psi_+ e^{-\zeta(T-t)}} \quad (18)$$

$$\zeta = \sqrt{(\kappa - \rho\sigma\omega_0)^2 + \sigma^2(\omega_0 - \omega_0^2)}$$

$$\psi_{\pm} = \zeta \mp (\kappa - \rho\sigma\omega_0)$$

$$\xi = \sqrt{\kappa_{\lambda}^2 - 2\sigma_{\lambda}^2\Lambda}$$

$$\chi_{\pm} = \xi \mp \kappa_{\lambda}$$

To solve the characteristic function for v_T , we can set $\omega_0 = 0$, and the solution for ODE (13) and (14) is given by,

$$A(0, \omega_1, t, T) = \frac{2}{\sigma^2} \kappa \theta \ln\left(\frac{2\kappa}{\sigma^2 \omega_1 (e^{-\kappa(T-t)} - 1) + 2\kappa}\right) + \frac{2\eta\lambda}{2\eta\kappa - \sigma^2} \ln\left(\frac{\omega_1 (\sigma^2 - 2\eta\kappa) e^{-\kappa(T-t)} + 2\kappa - \omega\sigma^2}{2\kappa(1 - \omega\eta)}\right) \quad (19)$$

$$B(0, \omega_1, t, T) = \frac{2\kappa\omega_1 e^{-\kappa(T-t)}}{\sigma^2 \omega_1 (e^{-\kappa(T-t)} - 1) + 2\kappa} \quad (20)$$

Similar results can be found in Sepp (2008). Since $\bar{V}(v_T) = C_0 + C_1 v_T$, we have the characteristic function for the expectation of accumulated variance,

$$\begin{aligned} \phi_{\bar{V}}(v, t, T) &= \int_{-\infty}^{\infty} e^{\bar{V}v} p(\bar{V} | \mathcal{F}_t) dX_T \\ &= \int_{-\infty}^{\infty} e^{(C_0 + C_1 v_T)v} p(v_T | \mathcal{F}_t) dX_T \\ &= e^{vC_0} \int_{-\infty}^{\infty} e^{vTC_1 v} p(v_T | \mathcal{F}_t) dX_T \\ &= e^{vC_0} G(0, C_1 v, t, T) \end{aligned} \quad (21)$$

So we can apply $G(-\omega_0, 0, t, T)$ to equation (2) to price Europe option on the SPX index, and substitute equation (21) to equations (9) and (10) to price VIX future and VIX option. In this manner, the pricing of options on SPX and VIX are consistent.

3.2 Multi-factor Models

In this section, we will study two multi-factor Models. We derive the characteristic function for both stock price and variance.

3.2.1 Stochastic Variance Self-Exciting Jump

The first model we propose here is a Self Exciting jump model,

$$\begin{aligned}
d\ln S_t &= (r - \frac{v_t}{2} - m_1 \lambda_t)dt + \sqrt{v_t}dW_t + \int J_x \mu(dJ_x, dJ_\lambda, dt) \\
dv_t &= \kappa_1(\theta_1 - v_t)dt + \sigma_1 \sqrt{v_t}dZ_t \\
d\lambda_t &= \kappa_2(\theta_2 - \lambda_t)dt + \int_{\mathbb{R} \times \mathbb{R}^+} 1_{J_x < 0} J_\lambda \mu(dJ_x, dJ_\lambda, dt) \\
J_x &\sim DE(\eta_u, \eta_d, p) \\
J_v &\sim Exp(\eta)
\end{aligned}$$

where $DE(\eta_u, \eta_d, p)$ denotes double exponential distribution, which is introduced by Kou (2002), η_u and η_d are the means of positive jump and negative jump respectively and p is the probability of positive jump. The two factors of the model are stochastic variance and stochastic jump intensity. The stochastic variance process is driven by a CIR process, as in the Heston model. The stochastic jump intensity is driven by negative realized jump in equity price. Once there is a negative jump in equity price, there is a corresponding jump in jump intensity. We assume jump size in equity price and that in jump intensity are independent. So the corresponding PIDE for the characteristic function for joint distribution of v_T and λ_T is given as,

$$\begin{aligned}
0 &= G_t + (r - \frac{1}{2}v_t - \lambda_t m_1)G_x + \frac{1}{2}v_t G_{xx} + \kappa(\theta - v_t)G_v \\
&\quad + \frac{1}{2}\sigma^2 v_t G_{vv} + \rho \sigma v_t G_{xv} + \kappa_\lambda(\theta_\lambda - \lambda_t)G_\lambda
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\quad + \lambda_t \int [G(\omega_0, \omega_1, \omega_2, x + J_x, v_t, \lambda_t + 1_{J < 0} J_\lambda, t, T) \\
&\quad - G(\omega_0, \omega_1, \omega_2, x, v_t, \lambda_t, t, T)] v(dJ_x, dJ_\lambda) \\
G_T &= e^{\omega_0 x_T + \omega_1 v_T + \omega_2 \lambda_T}
\end{aligned} \tag{23}$$

The solution form is $G = e^{\omega_0 x_t + r(T-t)\omega_0 + A + Bv_t + C\lambda_t}$. By substituting it into the PIDE (22), the

corresponding ODE set is,

$$\begin{aligned}
\dot{A}_t &= \kappa_1 \theta_1 B_t + \kappa_2 \theta_2 C_t \\
\dot{B}_t &= \frac{1}{2}(\omega_0 - \omega_0^2) + (\kappa_1 - \rho \sigma \omega_0) B_t - \frac{1}{2} \sigma^2 B_t^2 \\
\dot{C}_t &= -\kappa_2 C_t + \Lambda(\omega_0, C_t)
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
\Lambda(\omega_0, C_t) &= \int e^{\omega_0 J_x + 1_{J_x < 0} J_\lambda C_t} \pi(J_x, J_\lambda) dJ_x dJ_\lambda - 1 \\
&= \left(\frac{p}{1 - \omega_0 \eta_u} + \frac{1-p}{1 + \omega_0 \eta_d} \frac{1}{1 - C_t \eta} \right) - 1
\end{aligned}$$

and initial conditions at time T are,

$$\begin{aligned}
A_T &= 0 \\
B_T &= \omega_1 \\
C_T &= \omega_2
\end{aligned}$$

which are derived from equation (23).

Since the model is the sum of a Heston stochastic variance and Self Exciting jump and there is no dependence between them, we can find the solutions for B and C separately. It is easy to see that the solution of B is just the degenerate case of equations (18) and (20). However, there is no close form solution for C , and it has to be solved using numerical method, e.g. Runge Kutta method.

Finally, we derive the characteristic function for the expectation of accumulated variance. Following the steps in the previous section, we first write the expectation of accumulated variance starting at time T as a function of state variables, v_T and λ_T ,

$$\begin{aligned}
\bar{V}(v_T, \lambda_T) &= C_0(\Delta T) + C_1(\Delta T)v_T + C_2(\Delta T)\lambda_T \\
C_0 &= \theta_1(1 - C_1) + \theta_2(1 - C_2) \\
C_1 &= \frac{1 - e^{-\kappa_1 \Delta T}}{\kappa_1 \Delta T} \\
C_2 &= m \frac{1 - e^{-\kappa_2 \Delta T}}{\kappa_2 \Delta T}
\end{aligned}$$

Given the independence of v_T and λ_T , we have,

$$\phi_{\bar{V}}(v, t, T) = e^{vC_0} \phi_v(C_1 v, t, T) \phi_{\lambda}(C_2 v, t, T) \quad (25)$$

for the characteristic function of the expectation of accumulated variance settled at time T , where $\phi_v(v, t, T) = G(0, v, 0, x_t, v_t, \lambda_t, t, T)$ and $\phi_{\lambda}(v, t, T) = G(0, 0, v, x_t, v_t, \lambda_t, t, T)$ are the characteristic functions of v_T and λ_T respectively.

3.2.2 Double Heston with Jumps

The second multi-factor model is the double Heston model with jumps in one of the two stochastic variance processes.

$$\begin{aligned}
d\ln S_t &= \left(r - \frac{v_{1,t} + v_{2,t}}{2} - m_1 \lambda\right)dt + \sqrt{v_{1,t}}dW_{1,t} + \sqrt{v_{2,t}}dW_{2,t} \\
&\quad + \int J_x \mu(dJ_x, dJ_v, dt) \\
dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1 \sqrt{v_{1,t}}dZ_{1,t} + \int J_v \mu(dJ_x, dJ_v, dt) \\
dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2 \sqrt{v_{2,t}}dZ_{2,t} \\
[dW_{1,t}, dZ_{1,t}] &= \rho_1 dt \\
[dW_{2,t}, dZ_{2,t}] &= \rho_2 dt \\
J_x &\sim N(\varepsilon + \rho_J J_v, \delta^2) \\
J_v &\sim \text{Exp}(\eta)
\end{aligned}$$

This model is equivalent to the one factor model in Section 3.1 plus another CIR stochastic variance process. The corresponding PIDE is,

$$\begin{aligned}
0 &= G_t + \left(r - \frac{1}{2}(v_{1,t} + v_{2,t}) - \lambda m_1\right)G_x + \frac{1}{2}v_{1,t}G_{xx} + \frac{1}{2}v_{2,t}G_{xx} \\
&\quad + \kappa_1(\theta_1 - v_{1,t})G_{v_1} + \frac{1}{2}\sigma_1^2 v_{1,t}G_{v_1 v_1} + \rho_1 \sigma_1 v_{1,t}G_{xv_1} \\
&\quad + \kappa_2(\theta_2 - v_{2,t})G_{v_2} + \frac{1}{2}\sigma_2^2 v_{2,t}G_{v_2 v_2} + \rho_2 \sigma_2 v_{2,t}G_{xv_2} \\
&\quad + \lambda \int [G(\omega_0, \omega_1, \omega_2, x + J_x, v_t + J_v, t, T) \\
&\quad - G(\omega_0, \omega_1, \omega_2, x, v_{1,t}, t, T)] \nu(dJ_x, dJ_v) \\
&= 0
\end{aligned} \tag{26}$$

$$G_T = e^{\omega_0 x_T + \omega_1 v_{1,T} + \omega_2 v_{2,T}} \tag{27}$$

By substituting the solution form $G = e^{\omega_0 x_t + r(T-t)\omega_0 + A + B_1 v_{1,t} + B_2 v_{2,t}}$, we have the following ODE set,

$$\dot{A}_t = -\kappa_1 \theta_1 B_{1,t} - \kappa_2 \theta_2 B_{2,t} \quad (28)$$

$$-\lambda \left(\frac{e^{\varepsilon \omega_0 + \delta^2 \omega_0^2 / 2}}{1 - \eta(B_{1,t} + \rho_J \omega_0)} - 1 \right) - \omega_0 \lambda (e^{\varepsilon + \delta^2 / 2} - 1)$$

$$\dot{B}_{1,t} = \frac{1}{2}(\omega_0 - \omega_0^2) + (\kappa_1 - \rho_1 \sigma_1 \omega_0) B_{1,t} - \frac{1}{2} \sigma_1^2 B_{1,t}^2 \quad (29)$$

$$\dot{B}_{2,t} = \frac{1}{2}(\omega_0 - \omega_0^2) + (\kappa_2 - \rho_1 \sigma_2 \omega_0) B_{2,t} - \frac{1}{2} \sigma_2^2 B_{2,t}^2 \quad (30)$$

and the initial conditions at time T are,

$$A_T = 0$$

$$B_{1,T} = \omega_1$$

$$B_{2,T} = \omega_2$$

which are derived from equation (27).

The solution form for B_1 are the same as equation (18) and (20), and B_2 is the degenerating case of B_1 as λ goes to zero. Due to the independence of $v_{1,T}$ and $v_{2,T}$, the expectation of accumulated variance is,

$$\begin{aligned} \bar{V}(v_{1,T}, v_{2,T}) &= C_0 + C_1 v_{1,T} + C_2 v_{2,T} \\ C_0 &= \lambda \left(m_2 + \frac{\eta(e^{-\kappa_1 \Delta T} + \kappa_1 \Delta T - 1)}{\kappa_1^2 \Delta t} \right) \\ &\quad + \theta_1(1 - C_1) + \theta_2(1 - C_2) \\ C_1 &= \frac{1 - e^{-\kappa_1 \Delta T}}{\kappa_1 \Delta T} \\ C_2 &= \frac{1 - e^{-\kappa_2 \Delta T}}{\kappa_2 \Delta T} \end{aligned}$$

Similar to the case of SVSEJ model, because of the independence of two state variables, the

characteristic function of the expectation of accumulated variance is,

$$\phi_{\tilde{V}}(v, t, T) = e^{vC_0} \phi_{v_1}(C_1 v, t, T) \phi_{v_2}(C_2 v, t, T) \quad (31)$$

where $\phi_{v_1}(v, t, T) = G(0, v, 0, x_t, v_t, \lambda_t, t, T)$ and $\phi_{v_2}(v, t, T) = G(0, 0, v, x_t, v_t, \lambda_t, t, T)$. Applying equations (9) and (10), we can calculate the price for VIX future and VIX option.

4 Correlation and Hedging

In this section, we analyze the instantaneous correlation term structure for VIX futures implied by the models from the previous section. We first show that the one factor model implies an unrealistic constant correlation term structure for VIX futures, contrary to the empirical observation.

4.1 Flaw of Single Factor Models

Single factor stochastic variance models are very popular for pricing equity and FX derivatives. However, Single-Factor models have some implications that are undesirable. For example, if stochastic variance is the only driving factor for the volatility, the correlation term structure of variance derivatives is always one. We use the SVJJ model as example to illustrate this pitfall below. For the sake of simplicity, from now on, we use Vix_t^T instead of $F_{VIX}(t, T)$ to denote VIX future price observe at t and settled at T . Given the SVJJ specification in (8), we can derive the SDE for VIX future price, which is observed at t and settled at T ,

$$dVix_t^T = \frac{\partial Vix_t^T}{\partial v_t} \sqrt{v_t} \sigma dZ_t + \int_{\mathbb{R} \times \mathbb{R}^+} \delta Vix_t^T (\mu(dJ_v, dt) - v(dJ_v, dt))$$

where δVix_t^T is due to the jump in variance process, and $\delta Vix_t^T = Vix_t^T(v_t + J_v) - Vix_t^T(v_t)$.⁴ Given the above dynamics for VIX future, the correlation between VIX futures of different maturities is

⁴Note that the dynamics of VIX future price is a martingale.

$$\begin{aligned}
\rho_t &= \frac{\langle dVix_t^{T_1}, dVix_t^{T_2} \rangle}{\sqrt{\langle dVix_t^{T_1}, dVix_t^{T_1} \rangle \langle dVix_t^{T_2}, dVix_t^{T_2} \rangle}} \\
&= \frac{\frac{\partial Vix_t^{T_1}}{\partial v_t} \frac{\partial Vix_t^{T_2}}{\partial v_t} \sigma^2 v_t + \lambda \int_{\mathbb{R}} \delta Vix_t^{T_1} \delta Vix_t^{T_2} v(dJ_v)}{\sqrt{\prod_{i=1}^2 \left(\left(\frac{\partial Vix_t^{T_i}}{\partial v_t} \right)^2 \sigma^2 v_t + \lambda \int_{\mathbb{R}} \left(\delta Vix_t^{T_i} \right)^2 v(dJ_v) \right)}}
\end{aligned} \tag{32}$$

Substituting the characteristic function (21) into equation (9) and taking derivative, we have

$$\frac{\partial Vix_t^T}{\partial v_t} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B(C_1 v, t, T) \phi_{\bar{V}}(v, t, T) \right] dv$$

where $\phi_{\bar{V}}(v, t, T)$ is given by (21). For the change due to jump, we use the linear approximation as follows,

$$\begin{aligned}
\delta Vix_t^T &= Vix_t^T(v_t + J_v) - Vix_t^T(v_t) \\
&= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} (e^{BJ_v} - 1) \phi_{\bar{V}}(v, t, T) \right] dv \\
&\approx \frac{J_v}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B(C_1 v, t, T) \phi_{\bar{V}}(v, t, T) \right] dv
\end{aligned}$$

Therefore, under the linear approximation for jump, the correlation is,

$$\begin{aligned}
\rho_t &\approx \frac{D_1 D_2 \sigma^2 v_t + 2 D_1 D_2 \eta^2 \lambda}{\sqrt{(D_1^2 \sigma^2 v_t + 2 D_1^2 \eta^2 \lambda) (D_2^2 \sigma^2 v_t + 2 D_2^2 \eta^2 \lambda)}} \\
&= 1
\end{aligned}$$

where $D_i = \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B(C_1 v, t, T_i) \phi_{\bar{V}}(v, t, T_i) \right] dv$. Therefore, for two VIX futures with maturities T_1 and T_2 , the correlation is always one, contradicting to the market observation. For the one factor model, if the price of one VIX future with certain maturity is given, the state variable can

be derived and therefore price of the second VIX futures with a different maturity is also known. However, this is not the case for the multi-factor models.

4.2 VIX Future Dynamics and Correlation implied by Multiple Factor Model

4.2.1 Stochastic Variance Self-Exciting Jump

First we looked at the dynamics of VIX future under the Self-Exciting jump model, where the dynamics on the VIX future is derived as follows,

$$\begin{aligned}
dVix_t^T &= \bullet dt + \frac{\partial Vix_t^T}{\partial v_t} dv_t + \int_{\mathbb{R} \times \mathbb{R}^+} \delta Vix_t^T \mu^* \\
&= \bullet dt + \frac{dv_t}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} e^{ivC_0} B(C_1) \phi_v(C_1) \phi_\lambda(C_2) \right] dv \\
&\quad + \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R} \times \mathbb{R}^+} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} e^{ivC_0} \phi_v(C_1) (\phi_{\lambda+J_\lambda}(C_2) - \phi_\lambda(C_2)) \right] dv \times \mu^* \\
&= \frac{\sqrt{v_t} \sigma}{2\sqrt{\pi}} dW_t \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B(C_1) \phi_{\bar{v}}(v, t, T) \right] dv \\
&\quad + \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R} \times \mathbb{R}^+} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} (e^{J_\lambda C(C_2)} - 1) \phi_{\bar{v}}(v, t, T) \right] dv \times \mu^*
\end{aligned}$$

where $\mu^* = \mu(dJ_x, dJ_\lambda, dt) - v(dJ_x, dJ_\lambda, dt)$ is the compensated random measure for jump, and we write $B(C_i) = B(C_i v, t, T)$, $C(C_i) = C(C_i v, t, T)$, and $\phi(C_i) = \phi(C_i v, t, T)$ for short notation. $\bullet dt$ is some drift which will cancel the drift from other terms, since it is a future contract and total drift is zero. For the difference of jump, we have

$$\begin{aligned}
\delta Vix_t^T &= Vix_t^T(\lambda_t + J_v) - Vix_t^T(\lambda_t) \\
&= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} e^{ivC_0} \phi_v(C_1) \phi_\lambda(C_2) (e^{CJ_\lambda} - 1) \right] dv \\
&\approx \frac{J_\lambda}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} C(C_2) \phi_{\bar{v}}(v, t, T) \right] dv
\end{aligned}$$

in which the last equality is due to linear approximation. Therefore, the correlation can be calculated as,

$$\begin{aligned}
\rho_t &= \frac{\langle dVix_t^{T_1}, dVix_t^{T_2} \rangle}{\sqrt{\langle dVix_t^{T_1}, dVix_t^{T_1} \rangle \langle dVix_t^{T_2}, dVix_t^{T_2} \rangle}} \\
&= \frac{D_1 D_2 \sigma^2 v_t + 2E_1 E_2 P(J_x < 0) \eta^2 \lambda_t}{\sqrt{(D_1^2 \sigma^2 v_t + 2E_1^2 \eta^2 \lambda_t) (D_2^2 \sigma^2 v_t + 2E_2^2 \eta^2 \lambda_t)}} \\
&= \frac{\sigma^2 v_t + \frac{2E_1 E_2}{D_1 D_2} \eta^2 P(J_x < 0) \lambda_t}{\sqrt{\Pi_{i=1}^2 \left(\sigma^2 v_t + 2 \left(\frac{E_i}{D_i} \right)^2 \eta^2 P(J_x < 0) \lambda_t \right)}}
\end{aligned}$$

where

$$\begin{aligned}
D_i &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B(C_1 v, t, T_i) \phi_{\bar{V}}(v, t, T_i) \right] dv \\
E_i &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} C(C_2 v, t, T_i) \phi_{\bar{V}}(v, t, T_i) \right] dv
\end{aligned}$$

and B and C are solutions to the ODE set (24). B has the same solution forms as equation (20), and there is no close form for C , which can be solved by ODE solver numerically. $\phi_{\bar{V}}$ is given by equation (25).

4.2.2 Double Heston with Jumps

For the Double Heston with Jumps model, the dynamics for VIX future is,

$$\begin{aligned}
dVix_t^T &= \bullet dt + \frac{\partial Vix_t^T}{\partial v_{1,t}} dv_{1,t} + \frac{\partial Vix_t^T}{\partial v_{2,t}} dv_{2,t} \\
&= \frac{dv_{1,t}}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} e^{ivC_0} B(C_1) \phi_{v_1}(C_1) \phi_{v_2}(C_2) \right] dv \\
&\quad + \frac{dv_{2,t}}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} e^{ivC_0} B(C_2) \phi_{v_1}(C_1) \phi_{v_2}(C_2) \right] dv \\
&\quad + \int_{\mathbb{R} \times \mathbb{R}^+} \delta Vix_t^T (\mu(dJ_v, dt) - v(dJ_v, dt)) + \bullet dt \\
&= \frac{\sqrt{v_{1,t}} \sigma_1}{2\sqrt{\pi}} dZ_{1,t} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B(C_1) \phi_{\bar{V}}(v, t, T) \right] dv \\
&\quad + \frac{\sqrt{v_{2,t}} \sigma_2}{2\sqrt{\pi}} dZ_{2,t} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B(C_2) \phi_{\bar{V}}(v, t, T) \right] dv \\
&\quad + \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R} \times \mathbb{R}^+} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} \left(e^{J_v C(C_2)} - 1 \right) \phi_{\bar{V}}(v, t, T) \right] dv \times \mu^*
\end{aligned}$$

where $\mu^* = \mu(dJ_x, dJ_v, dt) - v(dJ_x, dJ_v, dt)$ is the compensated random measure for jump. As before, the total drift for a future contract is zero. Again, we use linear approximation for the jump part on stochastic variance,

$$\begin{aligned}
\delta Vix_t^T &= Vix_t^T(v_t + J_v) - Vix_t^T(v_t) \\
&= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} (e^{BJ_v} - 1) \phi_{\bar{V}}(v, t, T) \right] dv \\
&\approx \frac{J_v}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B(\omega, t, T) \phi_{\bar{V}}(v, t, T) \right] dv
\end{aligned}$$

As before, the correlation can be calculated as,

$$\begin{aligned}
\rho_t &= \frac{\langle dVix_t^{T_1}, dVix_t^{T_2} \rangle}{\sqrt{\langle dVix_t^{T_1}, dVix_t^{T_1} \rangle \langle dVix_t^{T_2}, dVix_t^{T_2} \rangle}} \\
&= \frac{D_1 D_2 (\sigma^2 v_{1,t} + 2\eta^2 \lambda) + E_1 E_2 \sigma^2 v_{2,t}}{\sqrt{\Pi_{i=1}^2 (D_i^2 (\sigma^2 v_{1,t} + 2\eta^2 \lambda) + E_i^2 \sigma^2 v_{2,t})}}
\end{aligned}$$

where for $i = 1, 2$

$$\begin{aligned}
D_i &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B_1(C_1 v, t, T_i) \phi_{\bar{V}}(v, t, T_i) \right] dv \\
E_i &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B_2(C_2 v, t, T_i) \phi_{\bar{V}}(v, t, T_i) \right] dv
\end{aligned}$$

and B_1 and B_2 are solutions to the ODE set (28), (29), and (30). The solution forms are exactly the same as equation (20) with corresponding parameters, and $\phi_{\bar{V}}$ is given by equation (31).

4.3 Calibration

Here, we calibrate the two multi-factor models proposed in the previous section using European style options on SPX and VIX on 25/04/2008. There are 5 maturities, May, June, July, August, and November, that have liquid options traded. The option data is downloaded from OptionMetrics and we have excluded all the options without trading volume. We only use put option on equity and call option on VIX to capture the negative skew for SPX surface and the positive skew for VIX surface. For risk free interest rate, we assume a flat 2% for all maturities.

The calibration is done by minimizing the mean square error between market implied volatility and model implied volatility. There are a few reasons for not matching option prices instead. First, VIX future price is an output of the model, which we need to take into account in the calibration. Hence, we need to assign weights on option price and future price, which has a big influence in the calibration. Moreover, since the option prices for SPX and VIX are not comparable, their pricing

Table 1: Calibrated Parameters of Self Exciting Jump Model

Parameter Estimates	
κ_1	1.4791
θ_1	0.0364
σ_1	0.5389
ρ_1	-0.8454
κ_2	34.3965
θ_2	6.2696
η	22.4183
η_u	0.0362
η_d	0.0422
p	0.3686

This table reports the calibrated parameters of Self Exciting Jump model. The calibration is done by min square error of implied volatility between market and model.

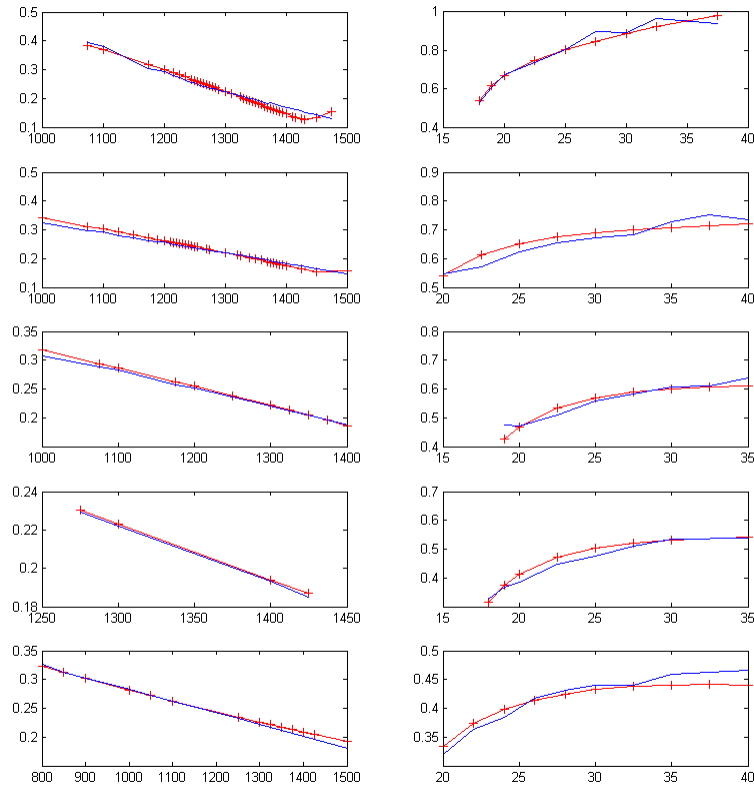
errors will have an unbalanced influence on the calibration. By using implied volatility, we avoid any such issue since the implied volatilities for equity and VIX are of the same scale and it is reasonable to put equal weights on the two surfaces in the calibration.

The calibration results for the Self-Exciting jump model is reported in Table 1 and the fit of volatility surface is plotted in Figure 1. The calibration for Double Heston with Jumps model using the same data is reported in Table 2 and the fit of volatility surface is plotted in Figure 2. Given the parameter values, we can calculate the instantaneous correlation term structure conditional on the state variables for the two models. The result is plotted in Figure 3. The results show that both models calibrated well jointly to the two surfaces. However, they produce a different correlation term structure for the VIX futures. The SVSEJ model has a faster decay while Double Heston with Jumps is more linear and decays slower.

4.4 Unconditional Correlation Term Structure of VIX Future

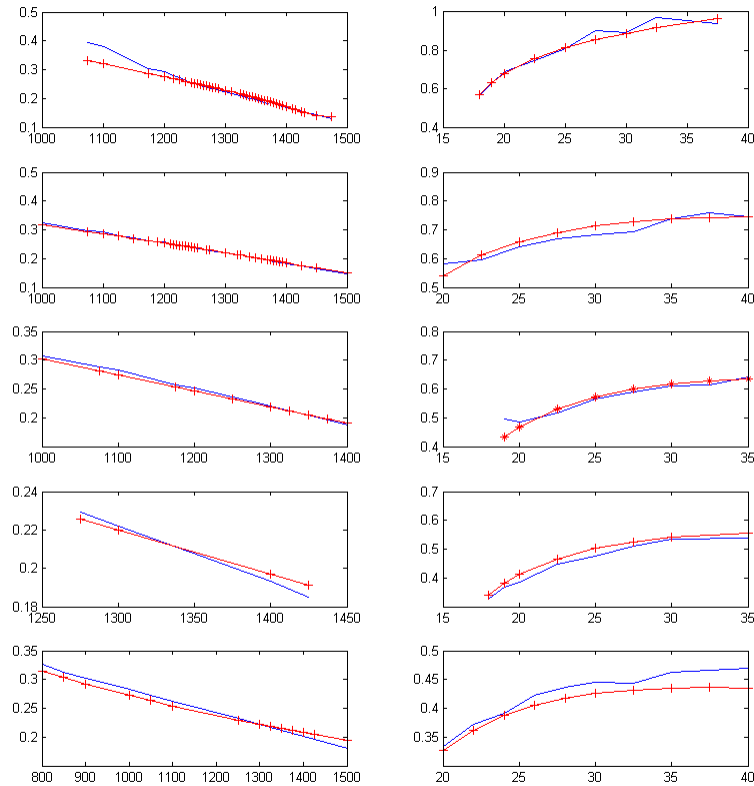
We have shown how to derive the conditional correlation of VIX futures in the previous sections. Since the conditional correlation term structure of VIX future is conditional on state variables and change over time and unobservable, the unconditional correlation might be of more interest, because one can calculate it from historical VIX future data. VIX future can be expressed as,

Figure 1: Fit of Implied Volatility by Self Exciting Jump Model



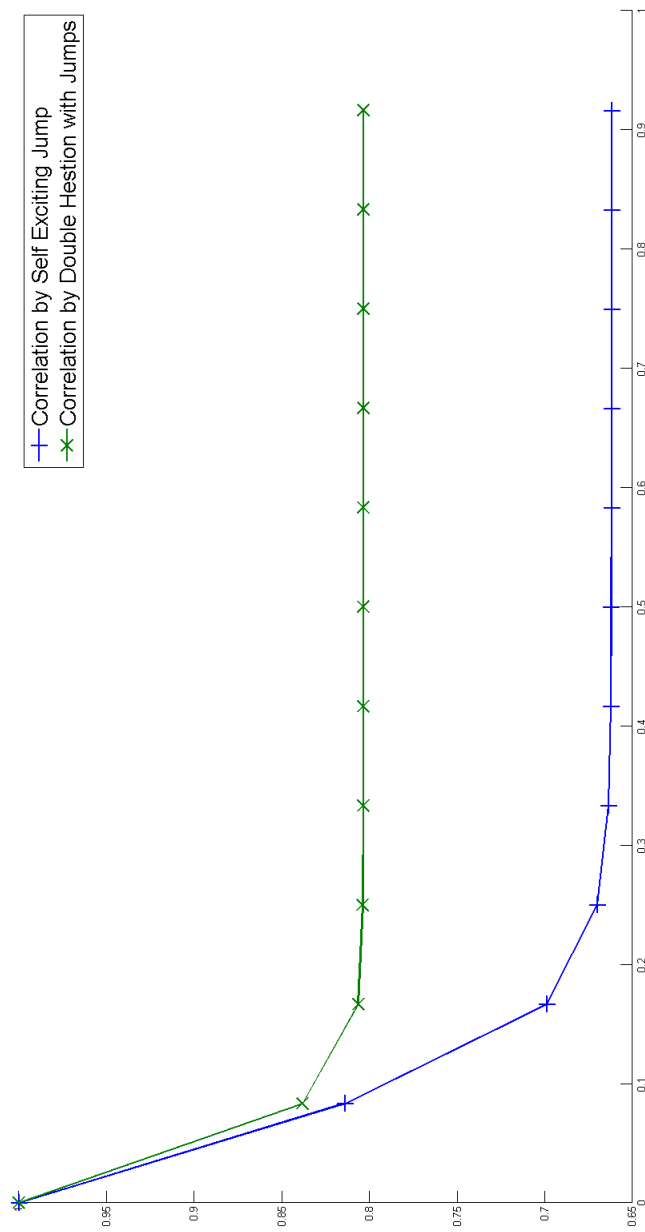
This graph plots the fit of implied volatility on 25/04/2008 with 5 maturities. The red line with crosses is the implied volatility by model and the blue line the implied volatility by market price. The left column is the fit of the implied vol of Equity put options; the right column is the fit of the implied vol of VIX call options. From top to bottom are the maturities in May, June, July, August, and November in 2008 respectively.

Figure 2: Fit of Implied Volatility by Double Heston with Jump Model



This graph plots the fit of implied volatility on 25/04/2008 with 5 maturities. The red line with crosses is the implied volatility by model and the blue line the implied volatility by market price. The left column is the fit of the implied vol of Equity put options; the right column is the fit of the implied vol of VIX call options. From top to bottom are the maturities in May, June, July, August, and November in 2008 respectively.

Figure 3: Conditional Instantaneous Correlation Term Structure between VIX Spot and VIX Future



This graph plots the conditional instantaneous correlation term structure of VIX future on 25/4/2008.

Table 2: Calibrated Parameters of Double Heston with Jump Model

Parameter Estimates	
κ_1	3.6721
θ_1	0.0195
σ_1	0.6981
ρ_1	-0.8562
η	0.0410
ρ_J	1.0395
κ_2	34.2975
θ_2	0.0234
σ_2	1.3896
ρ_2	-0.9999
λ	1.1576
ε	-0.0667
δ	0.0522

This table reports the calibrated parameters of Double Heston with Jump model. The calibration is done by min square error of implied volatility between market and model.

$$\rho = \mathbb{E}[\rho_t] = \int \rho_t p(f_1, f_2) df_1 df_2 \quad (33)$$

where f_1 and f_2 are the state variables. Since the SVSEJ and Double Heston with Jumps models are Markov models, the state variables have stationary distributions under some trivial assumptions. We derived the stationary distribution for the state variables for these two models in Appendix A. The following is a summary of the stationary distributions used in our study.

1. For the CIR process in Heston model, the stationary distribution for variance is a Gamma distribution.
2. For the Heston with jump in variance process, the stationary distribution is not any known distribution. However, when the jump size in variance is small, specifically, $\eta < \frac{\sigma^2}{2\kappa}$, the distribution can be approximated by another Gamma distribution. When the jump size is big, the distribution of variance is a sum of two independent Gamma distributions.
3. The stationary distribution for the jump intensity of the self exciting jump process is also a

Gamma distribution, but with a shift in its base line level θ .

Since the volatility factors are independent in both models, the joint distribution of the factors are just the product of the marginal distributions. Using equation (33), we can calculate the unconditional correlation, given the model parameters and stationary distribution functions.

Figure 4 plots the unconditional term structure for VIX future. Not surprisingly, two models produce very different term structures. Unlike the conditional correlation, the term structure of unconditional correlation for VIX future can be calculated from market data. It will be interesting to compare with the correlation implied by model with that observed in the market to see which model fit the market better. But it should be noted that the unconditional distribution of factors, $p(f_1, f_2)$, is supposed to be under the \mathbb{P} measure. In our study, we simply use the model parameters under the \mathbb{Q} measure as a demonstration. It is possible to calibrate the model under both measures with appropriate transforms from \mathbb{P} measure to \mathbb{Q} measure.

4.5 Hedging VIX Options

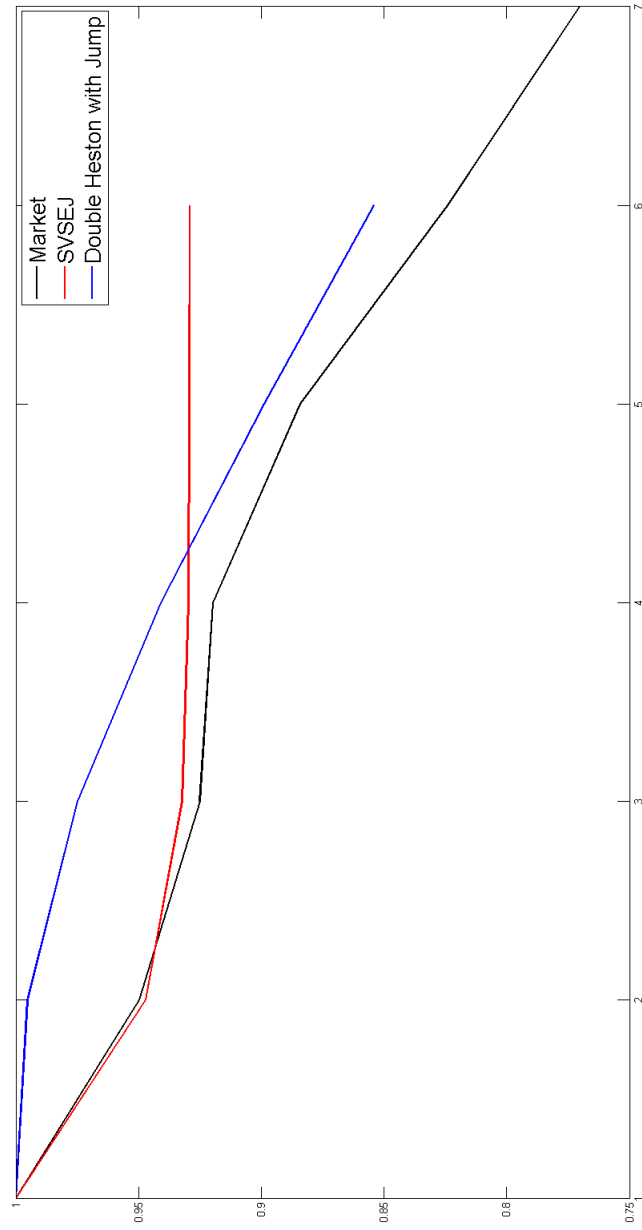
Here we compare the hedge ratio for VIX option w.r.t VIX future using the model studied in the previous section. In the one factor model without jumps, one can achieve perfect hedge, and the dynamical optimal hedge ratio is,

$$h_t = \frac{\frac{\partial C}{\partial v_t}}{\frac{\partial F}{\partial v_t}}$$

where C is the derivative, i.e., VIX option, to be hedged and F is the hedging instrument, i.e., VIX future. For the multi-factor models, perfect hedging is impossible. The hedge ratio or delta which gives minimal hedging variance is,

$$h(K) = \frac{\langle dC_t^T(K), dF_t^T \rangle}{\langle dF_t^T, dF_t^T \rangle}$$

Figure 4: Unconditional Instantaneous Correlation Term Structure between VIX Spot and VIX Future



This graph plots the unconditional instantaneous correlation term structure of VIX future implied by two models.

In SVSEJ, the dynamics of VIX future or option is given by,

$$d\bullet \approx \frac{\partial \bullet}{\partial v_t} \sqrt{v_t} \sigma dZ_t + \frac{\partial \bullet}{\partial \lambda_t} \int_{\mathbb{R} \times \mathbb{R}^+} 1_{J_x < 0} J_\lambda (\mu(dJ_x, dJ_\lambda, dt) - v(dJ_x, dJ_\lambda, dt))$$

whereas in Double Heston with Jumps model model, we have

$$\begin{aligned} d\bullet &\approx \frac{\partial \bullet}{\partial v_{1,t}} \left(\sqrt{v_{1,t}} \sigma dZ_{1,t} + \int_{\mathbb{R} \times \mathbb{R}^+} J_v (\mu(dJ_x, dJ_v, dt) - v(dJ_x, dJ_v, dt)) \right) \\ &+ \frac{\partial \bullet}{\partial v_{2,t}} \sqrt{v_{2,t}} \sigma dZ_{2,t} \end{aligned}$$

$\frac{\partial \bullet}{\partial v_t}$ can be derived given the close form of characteristic function as shown in Section 4. By following the same procedure in Section 4.2, we have the optimal hedge ratio for SVSEJ model as,

$$h(K) = \frac{D_F D_C \sigma^2 v_t + 2E_F E_C \eta^2 P(J_x < 0) \lambda_t}{\sqrt{(D_F^2 \sigma^2 v_t + E_F^2 \eta^2 P(J_x < 0) \lambda_t)}}$$

where

$$\begin{aligned} D_F &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B(C_1 v, t, T_1) \phi_{\bar{v}}(v, t, T_1) \right] dv \\ D_C &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\frac{1 - \operatorname{erf}(K\sqrt{-v})}{(iv)^{3/2}} B(C_1 v, t, T_2) \phi_{\bar{v}}(v, t, T_2) \right] dv \\ E_F &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} C(C_2 v, t, T_1) \phi_{\bar{v}}(v, t, T_1) \right] dv \\ E_C &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\frac{1 - \operatorname{erf}(K\sqrt{-v})}{(-iv)^{3/2}} C(C_2 v, t, T_2) \phi_{\bar{v}}(v, t, T_2) \right] dv \end{aligned}$$

The optimal hedge ratio for Double Heston with Jumps model is given by,

$$h(K) = \frac{D_1 D_2 (\sigma^2 v_{1,t} + 2\eta^2 \lambda) + E_1 E_2 \sigma^2 v_{2,t}}{\sqrt{(D_1^2 (\sigma^2 v_{1,t} + 2\eta^2 \lambda) + E_1^2 \sigma^2 v_{2,t})}}$$

where

$$\begin{aligned} D_F &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B_1(C_1 v, t, T_1) \phi_{\bar{V}}(v, t, T_1) \right] dv \\ D_C &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\frac{1 - \operatorname{erf}(K\sqrt{-v})}{(iv)^{3/2}} B_1(C_1 v, t, T_2) \phi_{\bar{V}}(v, t, T_2) \right] dv \\ E_F &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\left(\frac{1}{iv} \right)^{\frac{3}{2}} B_2(C_2 v, t, T_1) \phi_{\bar{V}}(v, t, T_1) \right] dv \\ E_C &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \Re \left[\frac{1 - \operatorname{erf}(K\sqrt{-v})}{(-iv)^{3/2}} B_2(C_2 v, t, T_2) \phi_{\bar{V}}(v, t, T_2) \right] dv \end{aligned}$$

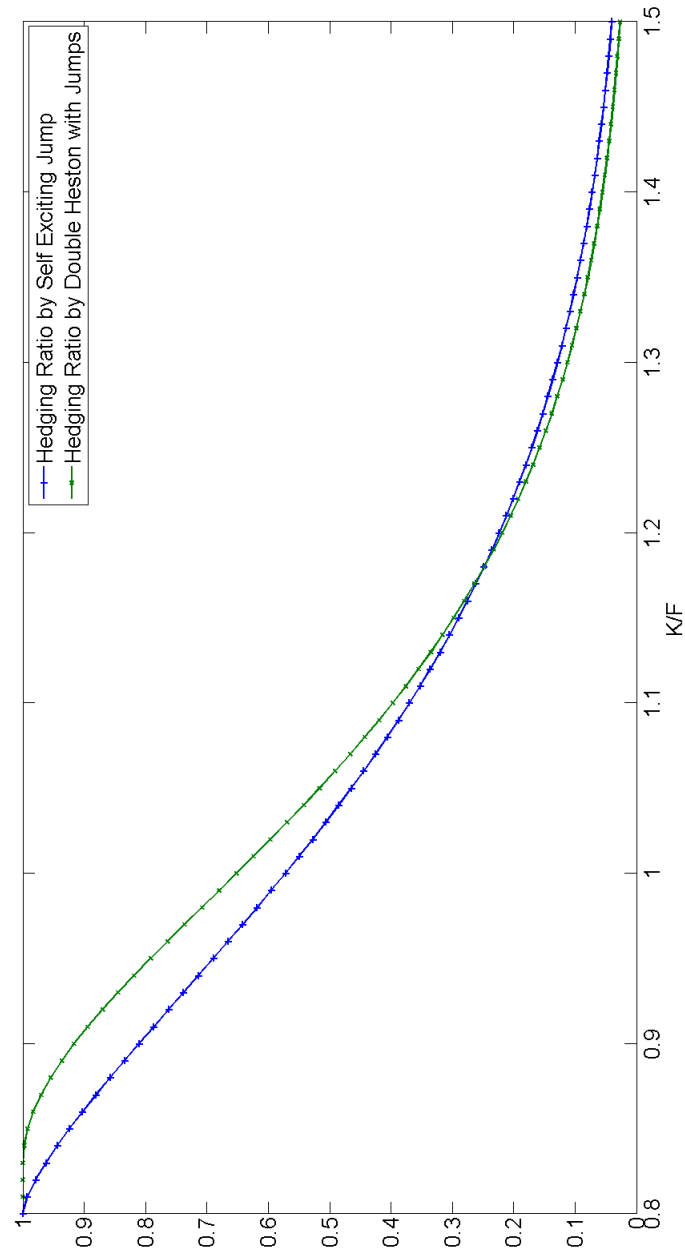
Figure 5 plots the hedge ratio implied by the two models based on the parameters estimated in the previous section. For ATM VIX option, the delta for the SVSEJ model is smaller than that for the Double Heston with Jumps model, suggesting a less exposure of VIX option to its underlying under SVSEJ model.

5 Conclusion

In this study, we provide a framework for pricing derivatives on SPX and VIX consistently, given the characteristic function of equity price and the factors that drive variance. This method can be applied to any affine model such as Heston model with jumps, since the characteristic functions of affine models are easy to derive.

In this paper, we are particular interested in the correlation term structure of VIX futures of different time to maturity. We studied two categories of affine models, a one-factor model and two multi-factor models. Based on a linear approximation of the jump size, we show that one-factor models always imply a perfect correlation between VIX futures of different maturities. However,

Figure 5: Optimal Hedge Ratio



The graph plots the hedge ratio of VIX options using VIX future on 25/4/2008. Both maturities are one month.

we know that it is not the case in the market. So even one factor model can fit volatility surfaces well, it still implies a wrong dynamics for variance.

Next, we show that two different multi-factor models are able to fit the implied volatility surfaces of SPX option and VIX option well. Based on the calibrated model parameters, however, they produce very different correlation term structure. We also derive the unconditional correlation so that the correlation implied by models can be compared with that calculated from the market historical data.

To calculate the unconditional correlation term structure, we derive the stationary distribution of the factors, based on the Kolmogorov forward equation. By taking expectation of conditional correlation, we can calculate the unconditional correlation term structure. However, to compare with the correlation observed in the market, we need the stationary distributions of the factors under the \mathbb{P} measure. In this study, we only use the parameters calibrated under \mathbb{Q} measure to calculate the unconditional correlation of the factors as a demonstration.

In the last part, we demonstrated how to calculate the optimal hedge ratio for hedging VIX option with VIX future. Since two models implies different dynamics for VIX future, it is not surprised to see that hedge ratio are also different.

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Appendix A.1 Stationary distribution of stochastic variance with jump

The CIR process with jumps is as follows,

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t + \int J_v\mu(dJ_v, dt)$$

The forward equation is,

$$\frac{\partial f(v, t)}{\partial t} = -\frac{\partial}{\partial v}\kappa(\theta - v)f(v, t) + \frac{1}{2}\frac{\partial^2}{\partial v^2}\sigma^2vf(v, t) - \lambda f(v, t) + \lambda \int f(v - J_v, t)v(dJ_v)$$

The stationary distribution is the solution of

$$0 = -\frac{\partial}{\partial v}\kappa(\theta - v)f(v) + \frac{1}{2}\frac{\partial^2}{\partial v^2}\sigma^2vf(v) - \lambda f(v) + \lambda \int f(v - J_v)v(dJ_v)$$

where $\frac{\partial f(v, t)}{\partial t} = 0$, since it is stationary. The difficulty arises due to the integral. We apply Fourier transform to solve this problem. The Fourier transform is defined as,

$$F(\omega) = \mathcal{F}[f(x)] = \int e^{\omega x}f(x)dx$$

where ω is complex number.

So the OIDE becomes,

$$\begin{aligned} 0 &= \omega [\kappa\theta F(\omega) - \kappa F'(\omega)] + \frac{1}{2}\omega^2\sigma^2F'(\omega) - \lambda F(\omega) + \lambda F(\omega) \int e^{J_v\omega}v(dJ_v) \\ &= (\omega\kappa\theta + \lambda \frac{\eta\omega}{1 - \eta\omega})F(\omega) + (\frac{1}{2}\omega^2\sigma^2 - \omega\kappa)F'(\omega) \end{aligned}$$

Therefore, we have,

$$(\omega\kappa - \frac{1}{2}\omega^2\sigma^2)F'(\omega) = (\omega\kappa\theta + \lambda\frac{\eta\omega}{1-\eta\omega})F(\omega)$$

The condition of determined solution is,

$$F(0) = 1$$

The solution for $F(\omega)$, which is the characteristic function of the stationary distribution is,

$$F(\omega) = (1 - \frac{\sigma^2}{2\kappa}\omega)^{-2\kappa\theta/\sigma^2} \left(\frac{1 - \eta\omega}{1 - \frac{\sigma^2}{2\kappa}\omega} \right)^{-2\eta\lambda/(2\eta\kappa - \sigma^2)}$$

When $\eta > \frac{\sigma^2}{2\kappa}$ and $\lambda < \kappa\theta\frac{2\eta\kappa - \sigma^2}{\sigma^2\eta}$, we can rearrange the expression as,

$$F(\omega) = (1 - \frac{\sigma^2}{2\kappa}\omega)^{2\eta\lambda/(2\eta\kappa - \sigma^2) - 2\kappa\theta/\sigma^2} (1 - \eta\omega)^{-2\eta\lambda/(2\eta\kappa - \sigma^2)}$$

which is product of two Characteristic functions of Gamma distribution. So $v_{1,t}$ is the sum of two Gamma distributed random variables.

When $\eta < \frac{\sigma^2}{2\kappa}$, there is no such simple solution, so we expand the Characteristic function and take the first order approximation of η , we have,

$$\Phi(\omega) = (1 - \frac{\sigma^2}{2\kappa}\omega)^{-2(\lambda\eta + \kappa\theta)/\sigma^2}$$

which is just the Characteristic function of a Gamma distribution.

For the second state variable, it is just the degenerate case of $v_{1,t}$ as $\lambda \rightarrow 0$, so the Characteristic function is,

$$\Phi(\omega) = (1 - \frac{\sigma^2}{2\kappa}\omega)^{-2\kappa\theta/\sigma^2}$$

which is the characteristic function of Gamma distribution $\Gamma(\frac{2\kappa\theta}{\sigma^2}, \frac{\sigma^2}{2\kappa})$ for the CIR process. Here, it is clear to see that when $2\kappa\theta/\sigma^2 < 1$, the probability of $v_{2,t}$ of being zero is positive.⁵

⁵Strictly speaking, it is the density that is positive, not the probability. There is still a debate whether one should stick with such so called Feller condition when using CIR process.

Appendix A.2 Stationary distribution of self exciting jump intensity

The intensity process for self exciting jump is as follows,

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \int_{\mathbb{R} \times \mathbb{R}^+} 1_{J_x < 0} J_\lambda \mu(dJ_x, dJ_\lambda, dt)$$

The forward equation is,

$$\frac{\partial f(\lambda, t)}{\partial t} = -\frac{\partial}{\partial \lambda} \kappa(\theta - \lambda) f(\lambda, t) - \lambda f(\lambda, t) + P(J_x < 0) \int (\lambda - J_\lambda) f(\lambda - J_\lambda, t) \nu(dJ_\lambda)$$

The stationary distribution is the solution of

$$0 = -\frac{\partial}{\partial \lambda} \kappa(\theta - \lambda) f(\lambda) - \lambda f(\lambda) + P(J_x < 0) \int (\lambda - J_\lambda) f(\lambda - J_\lambda) \nu(dJ_\lambda)$$

Applying Fourier transform, we have,

$$0 = \omega [\kappa \theta F(\omega) - \kappa F'(\omega)] - P(J_x < 0) \left[F'(\omega) - F'(\omega) \int e^{J_\lambda \omega} \nu(dJ_\lambda) \right]$$

The ODE to be solved is,

$$\left(\frac{P(J_x < 0) \eta \omega}{1 - \eta \omega} - \kappa \omega \right) F'(\omega) = -\omega \kappa \theta F(\omega)$$

The solution is,

$$F(\omega) = e^{\theta\omega} \left(1 - \frac{\eta\kappa}{\kappa - \eta P(J_x < 0)} \omega \right)^{-\theta P(J_x < 0)/\kappa}$$

which corresponds to a Gamma distribution, $\Gamma(\frac{\theta P(J_x < 0)}{\kappa}, \frac{\eta\kappa}{\kappa - \eta P(J_x < 0)})$ shifted with θ . Unlike normal Gamma distribution, which is supported in $(0, \infty)$, the shifted Gamma distribution here is supported in (θ, ∞) .