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Working Paper

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Manchester Business School Working Paper, No. 637

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Manchester Business School, The University of Manchester

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Working Paper Series

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http://www.mbs.ac.uk/cgi/apps/research/working-papers/

ISSN 0954-7401

Multi-Level Monte Carlo Simulations with Importance Sampling

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June 2, 2013

Abstract

We present an application of importance sampling in a Monte Carlo simulation for multi-asset options and in a Multi-Level Monte Carlo simulation. We demonstrate that applying importance sampling only on the first level of the Multi-Level Monte Carlo significantly improves its effective performance. We extend the Likelihood Ratio Method Based on Characteristic Function to estimate the Greeks of multi-asset options and in a Multi-Level Monte Carlo in a computationally efficient manner. Moreover, we combine it with the importance sampling to reduce the variance of the Greeks. Finally, we study the impact of the skew on the effective performance of importance sampling.

Keywords: Importance sampling; Simulation; Stochastic volatility

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1 Introduction

In practice, the valuation of contingent claims is typically a multi-dimensional problem that involves Monte Carlo simulation. The rate of convergence of a Monte Carlo simulation is $n^{-1/2}$, where n is the number of sample paths. Hence, improving the accuracy by a factor of 10 requires 100 times as many sample paths. For this reason variance reduction techniques become essential. Importance sampling reduces the variance by changing the drift of the simulated sample paths. Variance reduction achieved through importance sampling very much depends on the change of drift. Much of research effort focuses on how to change the drift to fully exploit the variance reduction potential of importance sampling. Multi-Level Monte Carlo is a Monte Carlo simulation performed on different levels. The main advantage of Multi-Level Monte Carlo is that it has lower computational complexity due to reduced variance compared to the basic Monte Carlo. Moreover, the variance of Multi-Level Monte Carlo can be further reduced by combining it with other variance reduction technique such as importance sampling.

In this paper, we focus on importance sampling for multi-asset options and importance sampling in a Multi-Level Monte Carlo simulation. Our contribution is as follows. First, we present an application of importance sampling with a stochastic change of drift to multi-asset options. Next, we provide an efficient importance sampling scheme in a Multi-Level Monte Carlo simulation. Finally, we extend the Likelihood Ratio Method Based on Characteristic Function to estimate the Greeks of multi-asset options and in a Multi-Level Monte Carlo and we combine it with the importance sampling to reduce the variance of the Greeks.

There is relatively little work on variance reduction for multi-asset options. Barraquand (1995) introduces quadratic resampling and combines it with the importance sampling. Avramidis (2002) proposes an algorithm that selects the importance sampling density as a mixture of multivariate Normal densities. Neddermeyer (2011) develops non-parametric importance sampling in conjunction with quasi-random numbers. Su and Fu (1999), Bouhari (2004) as well as Caprotti (2008) use importance sampling with a deterministic change of drift. In Su and Fu (1999) the change of drift is depends on stochastic optimization. In Bouhari (2004) the change of drift relies on the Robbins-Monro algorithms whereas in Caprotti (2008) it is based on the least squares minimization. Pellizzari (1998) suggests the use of control variate based on unconditional and conditional expectations of asset prices as a variance reduction technique for multi-asset options in the Black-Scholes model.

The rest of the paper is organized as follows. In Section 2 we present an application of importance sampling with a stochastic change of drift to multi-asset options in the Heston stochastic volatility model and the Bates stochastic volatility model with jumps. We consider basket, best-of, worst-of, spread, absolute and composite options. We also extend the Likelihood Ratio Method Based on the Characteristic Function to multi-asset options. In Section 3 we discuss the application of importance sampling in a Multi-Level Monte Carlo using the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model. We demonstrate that applying importance sampling only on the first level can significantly improve the effective performance of the Multi-Level Monte Carlo. We use the Likelihood Ratio Method Based on the Characteristic Function to estimate the Greeks in a Multi-Level Monte Carlo and combine it with the importance sampling to reduce the variance of the Greeks. We also study the impact of the skew on the effective performance of importance sampling. What we mean by the skew is the correlation between asset returns and their volatility. We study this feature because it is an important stylized fact of financial time series. Finally, Section 4 concludes the paper.

2 Importance Sampling

Here, we present an application of importance sampling for multi-asset option in the Heston model. We use a stochastic change of drift for a stochastic volatility model that was derived by Fouque and Tullie in [7].

In the Heston model the stock price dynamics under the risk-neutral measure $\mathbb{Q}$ is

$$dS_{i,t} = r_i S_{i,t} dt + \sqrt{v_{i,t}} S_{i,t} dW_{i,t}^{S_i} \tag{1}$$

whereas the variance dynamics is given by

$$dv_{i,t} = \kappa_i \left(\theta_i - v_{i,t}\right) dt + \xi_i \sqrt{v_{i,t}} dW_{i,t}^{v_i} \tag{2}$$

where κ_i is the i-th mean-reversion rate, θ_i is the i-th long-term variance and ξ_i is the i-th volatility of the volatility. The correlation matrix is

$$C = \begin{bmatrix} C_1 & C_2 \\ C_2^\top & C_3 \end{bmatrix} \tag{3}$$

where

$$C_1 = \begin{bmatrix} \rho_{1,1} & \cdots & \rho_{1,n} \\ \vdots & \ddots & \vdots \\ \rho_{n,1} & \cdots & \rho_{n,n} \end{bmatrix} \tag{4}$$

is the correlation corresponding to the asset price processes

$$C_2 = \begin{bmatrix} \rho_{1,n+1} & \cdots & \rho_{1,2n} \\ \vdots & \ddots & \vdots \\ \rho_{n,n+1} & \cdots & \rho_{n,2n} \end{bmatrix} \tag{5}$$

is the correlation corresponding to the asset price processes and the variance processes and

$$C_3 = \begin{bmatrix} \rho_{n+1,n+1} & \cdots & \rho_{n+1,2n} \\ \vdots & \ddots & \vdots \\ \rho_{2n,n+1} & \cdots & \rho_{2n,2n} \end{bmatrix} \tag{6}$$

is the correlation corresponding to the variance processes.

The difference between the Heston model and the Bates model is that in the Bates model the stock price dynamics under the risk-neutral measure $\mathbb{Q}$ becomes

$$dS_{i,t} = S_{i,t}\left(r_i - \lambda_i \bar{k}_i\right) dt + S_{i,t}\sqrt{v_{i,t}} dW_{i,t}^{S_i} + S_{i,t} dZ_{i,t} \tag{7}$$

where $Z_{i,t}$ is a compound Poisson process with intensity λ_i and log-normal distribution of jump sizes such that if k_i is its jump size then $\ln(1+k_i) \sim \mathcal{N}\left(\ln\left(1+\bar{k}_i\right) - \frac{1}{2}\delta_i^2, \delta_i^2\right)$.

To apply this result to multi-asset options we will express the Heston model in matrix notation. Let

$$dX_t = b(X_t)\, dt + a(X_t)\, d\eta_t \tag{8}$$

where $C = \Sigma\Sigma^\top$, η_t is a $2n$-dimensional correlated $\mathbb{Q}$-Brownian motion and

$$dX_t = \begin{pmatrix} S_{1,t} \\ \vdots \\ S_{n,t} \\ v_{1,t} \\ \vdots \\ v_{n,t} \end{pmatrix}, \ \eta_t = \begin{pmatrix} W_{1,t}^{S_1} \\ \vdots \\ W_{n,t}^{S_n} \\ W_{1,t}^{v_1} \\ \vdots \\ W_{n,t}^{v_n} \end{pmatrix}, \ b(x) = \begin{pmatrix} r_1 s_1 \\ \vdots \\ r_n s_n \\ \kappa_1(\theta_1 - v_1) \\ \vdots \\ \kappa_n(\theta_n - v_n) \end{pmatrix}$$

$$a(x) = \begin{pmatrix} \sqrt{v_1}s_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \sqrt{v_n}s_n & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \xi_1\sqrt{v_1} & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \xi_n\sqrt{v_n} \end{pmatrix}$$

Let $h(t, X_t)$ be η-adapted process. We introduce the martingale

$$H_t = \exp\left(\int_0^T \Sigma^{-1}h(s, X_s)\cdot\Sigma^{-1}d\eta_t + \frac{1}{2}\int_0^T \Sigma^{-1}h(s, X_s)\cdot\Sigma^{-1}h(s, X_s)ds\right) \tag{9}$$

and define a new probability measure $\tilde{\mathbb{Q}}$ by the density

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = (H_T)^{-1} \tag{10}$$

By the Girsanov theorem for correlated Brownian motions, the process

$$\tilde{\eta}_t = \eta_t + \int_0^t h(s, X_s)\, d\eta_s \tag{11}$$

is a 2-dimensional correlated $\tilde{\mathbb{Q}}$-Brownian motion. Thus under a new probability measure $\tilde{\mathbb{Q}}$ the model dynamics becomes

$$dX_t = (b(X_t) - a(X_t)\, h(t, X_t))\, dt + a(X_t)\, d\tilde{\eta}_t \tag{12}$$

The optimal choice of h for which the variance of the Monte Carlo estimator under $\tilde{\mathbb{Q}}$ is minimized is

$$h(t, X_t) = -\frac{1}{P(t, X_t)}a(t, X_t)^\top \nabla P(t, X_t) \tag{13}$$

This result is also valid for the Bates model with the difference that

$$a(x) = \begin{pmatrix} \left(\sqrt{v_1} + \frac{dZ_{1,t}}{dW_{1,t}^{S_1}}\right)s_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \left(\sqrt{v_n} + \frac{dZ_{n,t}}{dW_{n,t}^{S_n}}\right)s_n & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \xi_1\sqrt{v_1} & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \xi_n\sqrt{v_n} \end{pmatrix}$$

We refer to [13] for details.

For both models we approximate $P(t, X_t)$ using its fast mean-reversion expansion as in [7] which is given by the price of a European option in the Black-Scholes model with volatility

$$\bar{\sigma} = \sqrt{\sum_{i=1}^{N} \sigma_i^2 - 2 \sum_{1 \le i < j \le N}^{N} \sigma_i \sigma_j \rho_{i,j}} \tag{14}$$

and initial stock price $\bar{S} = \sum_{i=1}^{N} S_i$. We use the fast mean-reversion expansion because it can be used in conjunction with a look-up table introduced in [13] which dramatically reduces the computational time.

2.1 Basket Option

The payout of a basket option depends on the performance of a basket of underlying assets, each with its own corresponding weight. The weights w_i must satisfy the constraints $0 \le w_i \le 1$ for all $i = 1, \cdots, n$ and $\sum_{i=1}^{n} w_i = 1$. The payout of the basket call with maturity T is given by

$$\max\left(w_1 S_1(T), \cdots, w_n S_n(T) - K, 0\right) \tag{15}$$

The main advantage of a basket option is that it offers great flexibility in the construction of the underlying basket and it is usually cheaper than buying vanilla options on each of the underlying assets.

As an example we will consider basket call on three underlying assets. We assume that the time to maturity is 1 year. Other parameters are

i	w	S	r	v_0	ξ	κ	θ
1	50%	70	0.05	0.04	0.4	3	0.09
2	30%	35	0.05	0.09	0.3	0.5	0.25
3	20%	40	0.05	0.25	0.2	5	0.04

Table 1: Model parameters

and the correlation matrix is

$$\begin{bmatrix} 1 & 0.4 & 0.2 & -0.6 & -0.28 & -0.1 \\ 0.4 & 1 & 0.5 & -0.24 & -0.7 & -0.25 \\ 0.2 & 0.5 & 1 & 0.0282 & -0.35 & -0.5 \\ -0.6 & -0.24 & 0.0282 & 1 & 0.168 & 0.0294 \\ -0.28 & -0.7 & -0.35 & 0.168 & 1 & 0.175 \\ -0.1 & -0.25 & -0.5 & 0.0294 & 0.175 & 1 \end{bmatrix} \tag{16}$$

We note that the Feller condition is satisfied for each underlying asset.

In a Monte Carlo simulation we use 10 000 sample paths and a time increment of 0.001. In Table 2 we report the results for basic Monte Carlo and importance sampling for the Heston model and the Bates model. For the Bates model, we assume that the jump intensity is 1 per year, standard deviation of the jumps is 2% and the mean jump size is -5%.

$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
Heston									
price	MC	23.0003	18.0939	13.4929	9.4321	6.1308	3.6583	2.0060	0.9916
	IS	23.0432	18.1301	13.5388	9.4302	6.0855	3.6288	1.9740	0.9759
variance	MC	139.1319	131.7882	116.7452	93.7079	66.5853	41.7007	22.8492	11.1104
	IS	23.9918	22.8241	27.9146	15.9959	11.1225	10.7502	5.6949	1.8306
Bates									
price	MC	22.9964	18.0972	13.5079	9.4722	6.1935	3.7289	2.0761	1.0521
	IS	22.9257	18.0389	13.4853	9.4724	6.2015	3.7453	2.0880	1.0719
variance	MC	143.5208	135.8888	120.4728	96.8471	69.1762	43.8538	24.5189	12.2881
	IS	32.9773	31.0928	26.9779	21.0307	14.3609	8.3187	4.1190	1.7322

Table 2: Price and variance of price for basket call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance 5 times compared to the basic Monte Carlo.

2.2 Best-of Option

Best-of option depends on the performance of the best performing asset in a basket. The payout of the best-of call with maturity T is given by

$$\max\left(\max\left(S_1(T), \cdots, S_n(T)\right) - K, 0\right) \tag{17}$$

Best-of call has a higher upside potential compared to a call option on the same a basket of underlying assets.

As an example we will consider best-of three underlying assets call option. We assume that the time to maturity is 1 year. Other parameters are

i	S	r	v_0	ξ	κ	θ
1	30	0.05	0.04	0.4	3	0.09
2	35	0.05	0.09	0.3	0.5	0.25
3	40	0.05	0.25	0.2	5	0.04

Table 3: Model parameters

and the correlation matrix is given by (16). We also note that the Feller condition is satisfied for each underlying asset.

In a Monte Carlo simulation we use 10 000 sample paths and a time increment of 0.001. In Table 4 we report the results for basic Monte Carlo and importance sampling for the Heston model and the Bates model. For the Bates model, we assume that the jump intensity is 1 per year, standard deviation of the jumps is 2% and the mean jump size is -5%.

$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
Heston									
price	MC	23.3939	20.0806	16.8117	13.6660	10.7524	8.1722	5.9940	4.2450
	IS	23.3965	20.0837	16.8275	13.6950	10.7276	8.1459	5.9882	4.2398
variance	MC	114.9330	114.2491	112.0612	106.5591	96.5682	82.5957	66.5238	50.5769
	IS	58.0854	49.6236	40.2004	31.9864	24.6266	19.9598	15.4075	11.2659
Bates									
price	MC	23.5387	20.2257	16.9608	13.8165	10.8969	8.3088	6.1219	4.3675
	IS	23.4815	20.1805	16.9499	13.8360	10.8988	8.3327	6.1667	4.4003
variance	MC	118.2053	117.5027	115.1506	109.5496	99.5909	85.5577	69.3061	53.0277
	IS	57.7069	49.5691	40.9534	33.3357	26.6619	21.6881	16.7958	12.4169

Table 4: Price and variance of price for best-of call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance 3 times compared to the basic Monte Carlo.

2.3 Worst-of Option

Worst-of option depends on the performance of the worst performing asset in a basket. The payout of the worst-of call with maturity T is given by

$$\max\left(\min\left(S_1(T), \cdots, S_n(T)\right) - K, 0\right) \tag{18}$$

Worst-of call has a lower upside potential compared to a call option on the same a basket of underlying assets.

As an example we will consider worst-of three underlying assets call option. We use the same parameters as in the previous section. In a Monte Carlo simulation we use 10 000 sample paths and a time increment of 0.001. In Table 5 we report the results for basic Monte Carlo and importance sampling for the Heston model and the Bates model. For the Bates model, we assume that the jump intensity is 1 per year, standard deviation of the jumps is 2% and the mean jump size is -5%.

$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
Heston									
price	MC	7.5960	5.0870	3.1251	1.7429	0.8760	0.3895	0.1544	0.0590
	IS	7.6359	5.1117	3.1181	1.7279	0.8628	0.3849	0.1524	0.0533
variance	MC	42.3979	32.2608	21.3585	12.1857	5.9854	2.5606	0.9838	0.3423
	IS	19.3028	13.8514	8.7862	4.7116	2.1009	0.7885	0.2509	0.0701
Bates									
price	MC	7.4994	5.0152	3.0794	1.7277	0.8753	0.3964	0.1649	0.0615
	IS	7.5106	5.0174	3.0583	1.6931	0.8489	0.3835	0.1561	0.0580
variance	MC	42.7988	32.4943	21.5437	12.3510	6.1418	2.6990	1.0523	0.3754
	IS	20.1260	14.3138	8.9745	4.7930	2.1609	0.8268	0.2723	0.0798

Table 5: Price and variance of price for worst-of call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance 3 times compared to the basic Monte Carlo.

2.4 Spread option

Spread option depends on the difference between two underlying assets. Seller of such an option is long correlation which differentiates it from the majority of multi-asset options that leave the seller short correlation. The payout of the spread call with maturity T is given by

$$\max\left(S_1(T) - S_2(T) - K, 0\right) \tag{19}$$

As an example we will consider spread call. We assume that the time to maturity is 1 year. Other parameters are

i	S	r	v_0	ξ	κ	θ
1	30	0.05	0.04	0.4	3	0.09
2	5	0.05	0.09	0.3	0.5	0.25

Table 6: Model parameters

and the correlation matrix is

$$\begin{bmatrix} 1 & 0.4 & -0.6 & -0.28 \\ 0.4 & 1 & -0.24 & -0.7 \\ -0.6 & -0.24 & 1 & 0.168 \\ -0.28 & -0.7 & 0.168 & 1 \end{bmatrix} \tag{20}$$

We note that the Feller condition is satisfied for each underlying asset.

In a Monte Carlo simulation we use 10 000 sample paths and a time increment of 0.001. In Table 7 we report the results for basic Monte Carlo and importance sampling for the Heston model

and the Bates model. For the Bates model, we assume that the jump intensity is 1 per year, standard deviation of the jumps is 2% and the mean jump size is -5%.

$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
Heston									
price	MC	10.9116	8.7664	6.8078	5.0821	3.6278	2.4616	1.5897	0.9777
	IS	10.8962	8.7438	6.7701	5.0316	3.5795	2.4298	1.5730	0.9676
variance	MC	51.5956	47.0730	40.6259	32.9627	25.0319	17.7671	11.7748	7.3314
	IS	3.4245	3.7729	2.9216	1.7977	1.3673	1.0649	0.7246	0.3987
Bates									
price	MC	10.9352	8.8015	6.8525	5.1433	3.7056	2.5509	1.6764	1.0552
	IS	10.9055	8.7716	6.8252	5.1218	3.6918	2.5521	1.6944	1.0786
variance	MC	54.2643	49.5032	42.8706	34.9473	26.7347	19.1973	12.9460	8.2356
	IS	5.4703	4.7225	4.1768	3.4911	2.5868	1.9515	1.4214	0.9444

Table 7: Price and variance of price for spread call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance 13 times compared to the basic Monte Carlo.

2.5 Absolute option

Absolute option is an option written on the absolute value of the difference between the two underlying assets at maturity. The payout of the spread call with maturity T is given by

$$\max\left(\max\left(S_1(T), S_2(T)\right) - \min\left(S_1(T), S_2(T)\right) - K, 0\right) \tag{21}$$

As an example we will consider absolute call on two underlying assets. We assume that the time to maturity is 1 year. Other parameters are

i	S	r	v_0	ξ	κ	θ
1	30	0.05	0.04	0.4	3	0.09
2	35	0.05	0.09	0.3	0.5	0.25

Table 8: Model parameters

and the correlation matrix is given by (20). We also note that the Feller condition is satisfied for each underlying asset.

In a Monte Carlo simulation we use 10 000 sample paths and a time increment of 0.001. In Table 9 we report the results for basic Monte Carlo and importance sampling for the Heston model and the Bates model. For the Bates model, we assume that the jump intensity is 1 per year, standard deviation of the jumps is 2% and the mean jump size is -5%.

$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
Heston									
price	MC	0.8110	0.4772	0.2775	0.1561	0.0860	0.0470	0.0260	0.0137
	IS	0.8157	0.4738	0.2703	0.1517	0.0837	0.0457	0.0241	0.0124
variance	MC	8.4177	4.9509	2.8182	1.5635	0.8538	0.4603	0.2410	0.1229
	IS	4.3064	1.9271	0.9153	0.4107	0.1753	0.0765	0.0256	0.0071
Bates									
price	MC	0.8988	0.5351	0.3091	0.1733	0.0947	0.0509	0.0271	0.0143
	IS	0.8731	0.5180	0.3015	0.1726	0.0973	0.0530	0.0289	0.0158
variance	MC	9.2814	5.4544	3.0895	1.6995	0.9159	0.4850	0.2541	0.1307
	IS	3.9397	1.8787	0.8458	0.3651	0.1514	0.0554	0.0207	0.0082

Table 9: Price and variance of price for absolute call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance 7 times compared to the basic Monte Carlo.

2.6 Composite

Composite option is an option on a foreign underlying asset with a strike denominated in the domestic currency. The holder of a composite option faces foreign exchange risk, but benefits from having fixed the strike in the domestic currency. The payout of the composite call with maturity T is given by

$$\max\left(S(T)FX(T) - K, 0\right) \tag{22}$$

where FX is the foreign exchange rate.

As an example we will consider composite call. We assume that the time to maturity is 1 year. Other parameters are

i	S	r	v_0	ξ	κ	θ
1	30	0.05	0.04	0.4	3	0.09
2	2	0.05	0.09	0.3	0.5	0.25

Table 10: Model parameters

and the correlation matrix is given by (20). We also note that the Feller condition is satisfied for each underlying asset.

In a Monte Carlo simulation we use 10 000 sample paths and a time increment of 0.001. In Table 11 we report the results for basic Monte Carlo and importance sampling for the Heston model and the Bates model. For the Bates model, we assume that the jump intensity is 1 per year, standard deviation of the jumps is 2% and the mean jump size is -5%.

$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
Heston									
price	MC	14.7229	11.8809	9.2202	6.8346	4.8012	3.1750	1.9658	1.1369
	IS	14.7025	11.8549	9.1903	6.8037	4.7791	3.1696	1.9776	1.1546
variance	MC	132.9938	113.6243	93.6035	73.2430	53.7984	36.7750	23.3859	13.9662
	IS	65.9776	54.7920	44.3035	34.0895	24.4774	16.1298	9.6617	5.2622
Bates									
price	MC	14.7446	11.9122	9.2596	6.8880	4.8743	3.2598	2.0555	1.2193
	IS	14.7270	11.8902	9.2423	6.8705	4.8617	3.2663	2.0775	1.2507
variance	MC	138.4657	118.5183	98.0459	77.1913	57.2165	39.7261	25.8310	15.8694
	IS	70.2021	58.6839	47.6199	36.9331	26.9349	18.1927	11.4240	6.8354

Table 11: Price and variance of price for composite call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance 2 times compared to the basic Monte Carlo.

2.7 Greeks

Price of a multi-asset option can be expressed as expectation under the risk-neural measure $\mathbb{Q}$

$$P = \mathbb{E}\left[e^{-r(T-t)}\phi\left(S_1(T), \cdots, S_n(T)\right)\right] \tag{23}$$

where $\phi\left(S_1(T), \cdots, S_n(T)\right)$ is the payout function. Using the definition of expectation, this can be written as

$$P = \int_0^\infty \cdots \int_0^\infty e^{-r(T-t)}\phi\left(S_1(T), \cdots, S_n(T)\right) f(x_1, \cdots, x_n) dx_1 \cdots dx_n \tag{24}$$

where $f(x_1, \cdots, x_n)$ is the joint density.

Let us consider the first derivative of the option price with respect to $S_1(0)$

$$\begin{aligned} \Delta &= \frac{\partial}{\partial S_1(0)} \int_0^\infty \cdots \int_0^\infty e^{-r(T-t)}\phi\left(S_1(T), \cdots, S_n(T)\right) f(x_1, \cdots, x_n) dx_1 \cdots dx_n \\ &= \int_0^\infty \cdots \int_0^\infty e^{-r(T-t)}\phi\left(S_1(T), \cdots, S_n(T)\right) \frac{\frac{\partial}{\partial S_1(0)} f(x_1, \cdots, x_n)}{f(x_1, \cdots, x_n)} f(x_1, \cdots, x_n) dx_1 \cdots dx_n \end{aligned}$$

where $\frac{\frac{\partial}{\partial S_1(0)} f(x_1, \cdots, x_n)}{f(x_1, \cdots, x_n)}$ is the likelihood ratio. By Sklar's Theorem there exists a copula C such that

$$F\left(x_1, \cdots, x_n\right) = C\left(F_1\left(x_1\right), \cdots, F_n\left(x_n\right)\right) = C\left(u_1, \cdots, u_n\right) \tag{25}$$

In [13] we showed that the CDF and PDF for both the Heston model and the Bates model can

be obtained as

$$\begin{aligned} F_1(x_1) &= Pr(S_1(T) \leq x_1) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Re\left[\frac{\exp(-i\omega \ln(x_1))\,\phi_T(\omega)}{i\omega}\right] d\omega \\ f_1(x_1) &\approx \frac{F_1(x_1+\Delta x) - F_1(x_1)}{\Delta x} \end{aligned}$$

where ϕ is the characteristic function. n^{th} order differentiation of (25) gives an expression for the joint density.

$$f(x_1, \cdots, x_n) = \prod_{i=1}^{n} f_i(x_i)\, c(u_1, \cdots, u_n) \tag{26}$$

where $c(u_1, \cdots, u_n) = \frac{\partial^n C(u_1, \cdots, u_n)}{\partial u_1 \cdots \partial u_n}$. Other Greeks can be computed in a similar manner. In order to estimate the Greeks with respect to correlation between assets we will use an analytical copula as an approximation of copula in (25).

As an example let us again consider basket call. As an approximation of the joint PDF for the Heston model we will use t copula with 62 degrees of freedom and correlation matrix

$$\begin{bmatrix} 1 & 0.4 & 0.2 \\ 0.4 & 1 & 0.5 \\ 0.2 & 0.5 & 1 \end{bmatrix} \tag{27}$$

In Figure 1 we display the joint PDF for the first and second underlying asset in Table 1. The joint PDF in the upper panel was obtained using the Monte Carlo simulation. The joint PDF in the lower panel was obtained using t copula with 62 degrees of freedom and marginal PDFs of the first and second underlying asset in Table 1. Correlation matrix and the number of degrees of freedom were estimated using maximum likelihood.

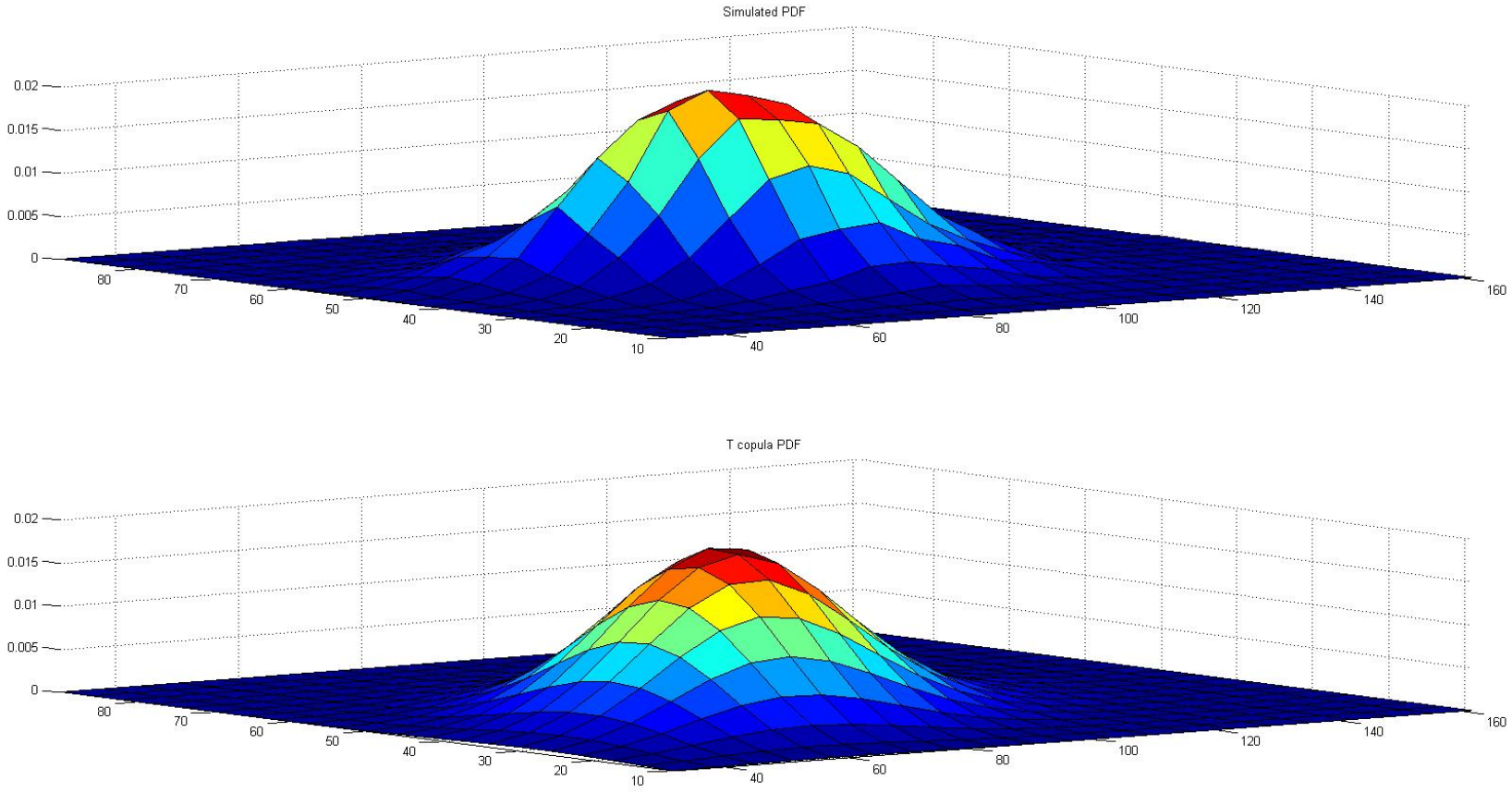

Figure 1: Joint PDF for the first and second underlying asset in Table 1 for the Heston model.

In Tables 12 and 13 we report delta and gamma of each underlying asset computed by the Likelihood Ratio Method Based on the Characteristic Function for the Heston model.

i	$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
1	delta	MC	0.4973	0.4850	0.4572	0.4070	0.3346	0.2500	0.1678	0.1005
		IS	0.4978	0.4832	0.4528	0.4005	0.3290	0.2499	0.1702	0.1045
	variance	MC	3.9044	2.9080	2.0993	1.4687	0.9938	0.6445	0.3937	0.2223
		IS	2.9194	1.9176	1.1636	0.6470	0.3326	0.2544	0.1250	0.0439
2	delta	MC	0.2968	0.2873	0.2671	0.2329	0.1870	0.1366	0.0906	0.0524
		IS	0.2961	0.2865	0.2673	0.2365	0.1910	0.1352	0.0874	0.0501
	variance	MC	10.9800	7.8665	5.3811	3.4908	2.1292	1.2105	0.6341	0.3048
		IS	9.8116	6.5965	4.1291	2.3213	1.1894	0.6826	0.2735	0.0868
3	delta	MC	0.1991	0.1886	0.1728	0.1506	0.1219	0.0897	0.0583	0.0340
		IS	0.2025	0.1932	0.1789	0.1593	0.1315	0.0995	0.0689	0.0430
	variance	MC	7.7937	5.4444	3.6204	2.2803	1.3568	0.7634	0.4083	0.2102
		IS	6.2314	4.0029	2.3606	1.2534	0.6082	0.2816	0.1127	0.0388

Table 12: Delta and variance of delta for basket call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance of delta 2 times.

i	$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
1	gamma	MC	0.0001	0.0009	0.0025	0.0048	0.0071	0.0084	0.0083	0.0067
		IS	0.0020	0.0026	0.0039	0.0060	0.0079	0.0089	0.0086	0.0071
	variance	MC	0.0479	0.0378	0.0291	0.0217	0.0157	0.0108	0.0070	0.0042
		IS	0.0372	0.0262	0.0174	0.0107	0.0061	0.0033	0.0018	0.0009
2	gamma	MC	0.0142	0.0115	0.0093	0.0077	0.0065	0.0054	0.0042	0.0027
		IS	0.0197	0.0169	0.0144	0.0120	0.0100	0.0079	0.0055	0.0036
	variance	MC	0.4378	0.3224	0.2279	0.1536	0.0977	0.0581	0.0322	0.0164
		IS	0.4136	0.2865	0.1852	0.1098	0.0587	0.0298	0.0127	0.0046
3	gamma	MC	0.0104	0.0082	0.0063	0.0047	0.0036	0.0029	0.0024	0.0020
		IS	0.0127	0.0103	0.0081	0.0062	0.0047	0.0034	0.0025	0.0017
	variance	MC	0.1819	0.1294	0.0883	0.0576	0.0360	0.0218	0.0130	0.0078
		IS	0.1433	0.0928	0.0554	0.0301	0.0150	0.0079	0.0032	0.0011

Table 13: Gamma and variance of gamma for basket call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance of gamma 2 times.

As an approximation of the joint PDF for the Bates model we will use t copula with 30 degrees of freedom and the correlation matrix given by (27).

In Figure 2 we display the joint PDF for the first and second underlying asset in Table 1. The joint PDF in the upper panel was obtained using the Monte Carlo simulation. The joint PDF in the lower panel was obtained using t copula with 30 degrees of freedom and marginal PDFs of the first and second underlying asset in Table 1. Correlation matrix and the number of degrees of freedom were estimated using maximum likelihood.

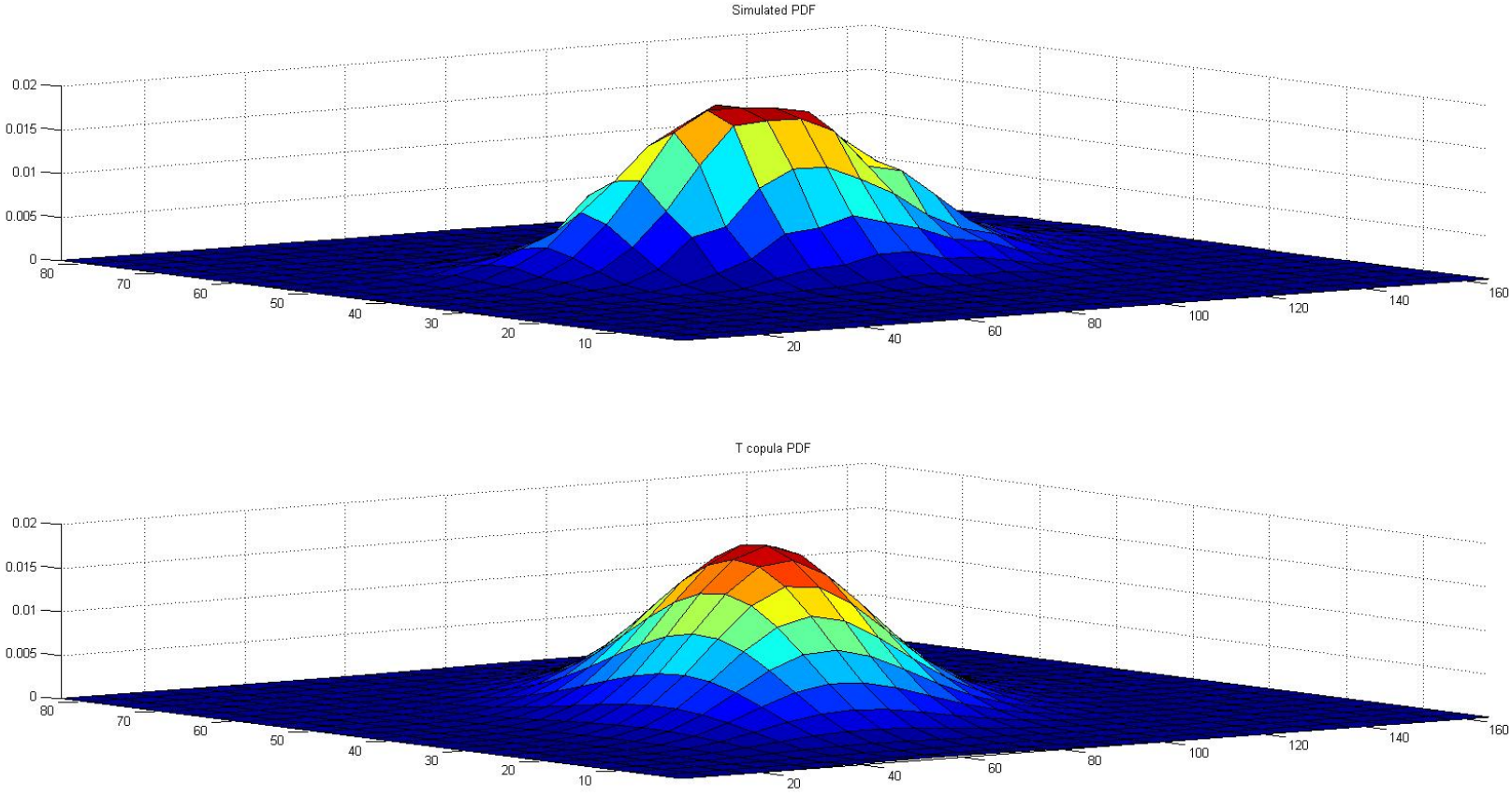

Figure 2: Joint PDF for the first and second underlying asset in Table 1 for the Bates model.

In Tables 14 and 15 we report delta and gamma of each underlying asset computed by the Likelihood Ratio Method Based on the Characteristic Function for the Bates model.

i	$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
1	delta	MC	0.4966	0.4845	0.4567	0.4066	0.3347	0.2512	0.1706	0.1044
		IS	0.4912	0.4769	0.4489	0.3976	0.3292	0.2488	0.1722	0.1092
	variance	MC	3.8657	2.8944	2.1049	1.4874	1.0193	0.6713	0.4182	0.2421
		IS	2.8849	1.9074	1.1758	0.6710	0.3565	0.1826	0.0906	0.0463
2	delta	MC	0.3056	0.2933	0.2708	0.2342	0.1862	0.1364	0.0906	0.0529
		IS	0.2895	0.2812	0.2598	0.2284	0.1803	0.1296	0.0833	0.0477
	variance	MC	10.8908	7.8325	5.3818	3.5098	2.1576	1.2400	0.6602	0.3245
		IS	9.6904	6.5213	4.1040	2.3413	1.2146	0.5691	0.2377	0.0889
3	delta	MC	0.1803	0.1722	0.1588	0.1390	0.1136	0.0831	0.0543	0.0317
		IS	0.1766	0.1701	0.1611	0.1442	0.1206	0.0939	0.0655	0.0404
	variance	MC	7.5413	5.2603	3.4916	2.1943	1.3011	0.7289	0.3872	0.1981
		IS	6.0178	3.8632	2.2887	1.2276	0.6000	0.2648	0.1058	0.0389

Table 14: Delta and variance of delta for basket call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance of delta 2 times.

i	$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
1	gamma	MC	0.0023	0.0028	0.0040	0.0060	0.0080	0.0091	0.0087	0.0071
		IS	0.0037	0.0040	0.0055	0.0071	0.0089	0.0100	0.0094	0.0078
	variance	MC	0.0477	0.0379	0.0295	0.0223	0.0163	0.0114	0.0076	0.0047
		IS	0.0342	0.0242	0.0168	0.0105	0.0063	0.0036	0.0020	0.0011
2	gamma	MC	0.0210	0.0174	0.0142	0.0115	0.0093	0.0072	0.0053	0.0033
		IS	0.0267	0.0227	0.0199	0.0163	0.0132	0.0098	0.0072	0.0048
	variance	MC	0.4307	0.3207	0.2294	0.1565	0.1009	0.0608	0.0340	0.0175
		IS	0.3641	0.2523	0.1679	0.1049	0.0561	0.0279	0.0123	0.0051
3	gamma	MC	0.0123	0.0096	0.0071	0.0050	0.0035	0.0025	0.0019	0.0014
		IS	0.0117	0.0094	0.0075	0.0057	0.0041	0.0032	0.0021	0.0013
	variance	MC	0.1674	0.1184	0.0801	0.0515	0.0315	0.0185	0.0105	0.0059
		IS	0.1210	0.0781	0.0472	0.0258	0.0129	0.0061	0.0026	0.0009

Table 15: Gamma and variance of gamma for basket call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance of gamma 2 times.

3 Multi-Level Monte Carlo with Importance Sampling

Multi-Level Monte Carlo is a Monte Carlo simulation with different number of time steps $h_l = 2^{-l}T$ on each level $l = 0, 1, \cdots, L$. Let P denote the payout and P_l denote its approximation on level l. The Multi-Level Monte Carlo estimator is given by

$$\widehat{Y} = \mathbb{E}[P_L] = \mathbb{E}[P_0] + \sum_{l=1}^{L} \mathbb{E}[P_l - P_{l-1}] \tag{28}$$

Let $\hat{Y}_0$ denote an estimator for $\mathbb{E}[P_0]$ using N_0 sample paths and let $\hat{Y}_l$ denote an estimator for

$\mathbb{E}[P_l - P_{l-1}]$ using N_l sample paths. $\hat{Y}_l$ is calculated as a mean of N_l independent samples

$$\hat{Y}_l = \frac{1}{N_l} \sum_{i=1}^{N_l} \left(P_l^i - P_{l-1}^i \right) \tag{29}$$

It is important to note that P_l and P_{l-1} are obtained using the same Brownian path. This is done by first constructing N_l Brownian increments to evaluate P_l^i and then summing them in pairs, yielding $\frac{N_l}{2}$ Brownian increments, to evaluate P_{l-1}^i.

The variance of the Multi-Level Monte Carlo estimator is

$$Var\left[\hat{Y}\right] = \sum_{l=0}^{L} Var\left[\hat{Y}_l\right] = \sum_{l=0}^{L} \frac{1}{N_l} V_l \tag{30}$$

where $V_l = Var[P_l - P_{l-1}]$.

The mean square error is given by

$$MSE = \mathbb{E}\left[\left(\hat{Y} - \mathbb{E}[P]\right)^2\right] = Var\left[\hat{Y}\right] + \left(\mathbb{E}\left[\hat{Y}\right] - \mathbb{E}[P]\right)^2 \tag{31}$$

where $\left(\mathbb{E}\left[\hat{Y}\right] - \mathbb{E}[P_L]\right)$ is the discretization error. By Theorem 3.1 in [8] $\left(\mathbb{E}\left[\hat{Y}\right] - \mathbb{E}[P_L]\right)^2 \leq \frac{\epsilon^2}{2}$ and $Var\left[\hat{Y}\right] \leq \frac{\epsilon^2}{2}$ so the upper bound on the mean square error is ϵ^2.

We wish to minimize the variance of Y for a given computational cost $C = \sum_{l=0}^{L} \frac{N_l}{h_l}$. It can be shown that the optimal number of sample paths for level l is

$$N_l = \left\lceil 2\epsilon^{-2} \sqrt{V_l h_l} \left(\sum_{l=0}^{L} \sqrt{\frac{V_l}{h_l}} \right) \right\rceil \tag{32}$$

Overall, Monte Carlo for a given ϵ has computational complexity proportional to ϵ^{-3} whereas that of the Multi-Level Monte Carlo is proportional to $\epsilon^{-2} (\log \epsilon)^2$ due to reduced variance.

To illustrate the performance of Multi-Level Monte Carlo with the importance sampling we will use the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model. The dynamics of these models under the risk neutral measure $\mathbb{Q}$ is given by

$$dS_t = r_t S_t dt + \sqrt{v_t} S_t dW_t^S \tag{33}$$

$$dv_t = \kappa(\bar{v} - v_t)\, dt + \gamma \sqrt{v_t} dW_t^v \tag{34}$$

$$dr_t = \lambda(\theta - r_t)\, dt + \eta r_t^p dW_t^r \tag{35}$$

where $\left\langle dW_t^S dW_t^v \right\rangle = \rho_{S,v} dt$, $\left\langle dW_t^S dW_t^r \right\rangle = \rho_{S,r} dt$ and $\left\langle dW_t^r dW_t^v \right\rangle = 0$. If $p = 0$ we have the Heston-Hull-White model and if it is 0.5 we have the Heston-Cox-Ingersoll-Ross model.

We compare the performance of the Multi-Level Monte Carlo with the importance sampling and without importance sampling for European call option. We use the following parameters: $\kappa = 2$, $\gamma = 0.06$, $v_0 = \bar{v} = 0.04$, $\lambda = 0.05$, $r_0 = \theta = 0.07$, $\eta = 0.01$, $S_0 = 100$, $\rho_{S,v} = -0.3$, $\rho_{S,r} = 0.2$, $T = 1$. We set $\epsilon = 0.01$ and $L = 8$. We consider five strikes: 60, 80, 100, 120 and 140. In Table 16 we report European option prices and relative errors with respect to the semi-analytical solution for different strikes under the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model.

Strike	60	80	100	120	140
Heston-Hull-White					
Semi closed-form (price)	44.0682	26.0077	11.5943	3.7583	0.9221
MLMC (price)	44.0918	26.0258	11.6100	3.7636	0.9219
MLMC (relative error)	0.05%	0.07%	0.13%	0.14%	0.02%
MLMC+IS (price)	44.0879	26.0256	11.6085	3.7648	0.9229
MLMC+IS (relative error)	0.04%	0.07%	0.12%	0.17%	0.08%
Heston-Cox-Ingersoll-Ross					
Semi closed-form (price)	44.0686	25.9996	11.5668	3.7296	0.9071
MLMC (price)	44.0724	26.0067	11.5696	3.7320	0.9037
MLMC (relative error)	0.01%	0.03%	0.02%	0.06%	0.37%
MLMC+IS (price)	44.0769	26.0045	11.5693	3.7326	0.9041
MLMC+IS (relative error)	0.02%	0.02%	0.02%	0.08%	0.33%

Table 16: Option price and relative error with respect to the semi-analytical solution.

Given the nature of the Multi-Level Monte Carlo, we observe that it is possible to use importance sampling on all levels or some levels. We will refer to the former case as full importance sampling. We note that the first level, where $l = 0$, is the coarsest, because the time step at this level is T. The point is that variance at level l decreases as l increases because both P_{l-1} and P_l accurately approximate P as they are obtained using the same Brownian path. Therefore, as an alternative to the full importance sampling we will consider importance sampling on the first level only. In Figure 3 we illustrate the effective performance defined as the ratio of variance reduction from importance sampling to speed for different strikes. Speed is defined as the ratio of computational time of the Multi-Level Monte Carlo with the importance sampling to computational time of the Multi-Level Monte Carlo without importance sampling. We compare Multi-Level Monte Carlo with full importance sampling, Multi-Level Monte Carlo with the importance sampling on the first level only and Multi-Level Monte Carlo without importance sampling. The results indicate that the Multi-Level Monte Carlo with the importance sampling is more efficient than Multi-Level Monte Carlo without importance sampling and Multi-Level Monte Carlo with the importance sampling on the first level only is much more efficient than both Multi-Level Monte Carlo without importance sampling and Multi-Level Monte Carlo with full importance sampling. The performance improvement compared to the Multi-Level Monte Carlo full importance sampling comes from two

sources. The first one is variance reduction from importance sampling. The second one is reduced computational time. This is due to the fact that the number of sample paths at level l which is given by (32) depends on the variance at level l. Since importance sampling reduces the variance at the first level, the required number of sample paths at this level is less compared to the Multi-Level Monte Carlo without importance sampling.

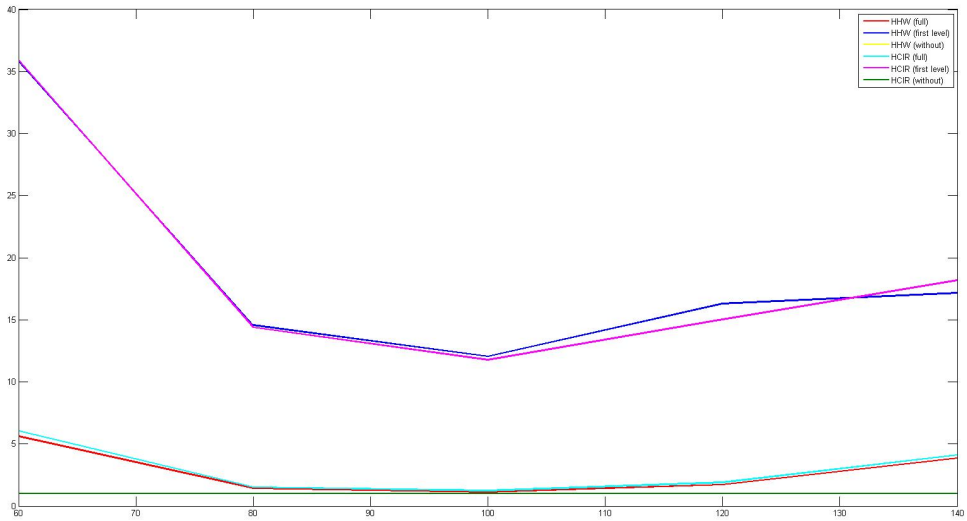

Figure 3: Effective performance for different strikes.

It is also possible to use the Likelihood Ratio Method Based on Characteristic Function to estimate the Greeks in a Multi-Level Monte Carlo. Let us consider the first derivative of the Multi-Level Monte Carlo estimator (28) with respect to S_0.

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial S_0}\mathbb{E}\left[P_L\right] \\
&= \frac{\partial}{\partial S_0}\mathbb{E}\left[P_0\right] + \sum_{l=1}^{L}\frac{\partial}{\partial S_0}\mathbb{E}\left[P_l - P_{l-1}\right] \\
&= \int_0^\infty e^{-r(T-t)} P_0 \frac{\frac{\partial}{\partial S_0} f(x)}{f(x)} f(x)dx + \sum_{l=1}^{L}\int_0^\infty e^{-r(T-t)}\left(P_l - P_{l-1}\right)\frac{\frac{\partial}{\partial S_0} f(x)}{f(x)} f(x)dx
\end{aligned}$$

where $\frac{\frac{\partial}{\partial S_0} f(x)}{f(x)}$ is the likelihood ratio which can be obtained from the characteristic function as in [13].

In Tables 17 and 18 we report delta and gamma computed by the Likelihood Ratio Method Based on the Characteristic Function for European option under the Heston-Hull-White model and the Heston-Cox-Ingersoll-Ross model.

Strike	60	80	100	120	140
Heston-Hull-White					
MLMC (delta)	0.9595	0.9187	0.6691	0.3227	0.1068
MLMC (variance)	8.5492	4.7461	2.5166	1.1986	0.4324
MLMC+IS (delta)	0.9621	0.9172	0.6690	0.3235	0.1058
MLMC+IS (variance)	4.3355	1.1883	0.2358	0.0869	0.0379
Heston-Cox-Ingersoll-Ross					
MLMC (delta)	0.9570	0.9179	0.6698	0.3231	0.1062
MLMC (variance)	8.6794	4.8099	2.5586	1.2152	0.4403
MLMC+IS (delta)	0.9577	0.9195	0.6686	0.3224	0.1057
MLMC+IS (variance)	4.4050	1.2070	0.2374	0.0861	0.0350

Table 17: Delta and variance of delta.

On average importance sampling reduces the variance of delta 8 times.

Strike	60	80	100	120	140
Heston-Hull-White					
MLMC (gamma)	0.0001	0.0042	0.0165	0.0175	0.0092
MLMC (variance)	0.0569	0.0370	0.0219	0.0119	0.0055
MLMC+IS (gamma)	0.0001	0.0042	0.0164	0.0174	0.0092
MLMC+IS (variance)	0.0264	0.0108	0.0035	0.0011	0.0005
Heston-Cox-Ingersoll-Ross					
MLMC (gamma)	0.0001	0.0041	0.0165	0.0176	0.0092
MLMC (variance)	0.0591	0.0383	0.0227	0.0123	0.0057
MLMC+IS (gamma)	0.0002	0.0042	0.0164	0.0175	0.0091
MLMC+IS (variance)	0.0271	0.0112	0.0036	0.0011	0.0005

Table 18: Gamma and variance of gamma.

On average importance sampling reduces the variance of gamma 7 times.

Another interesting question is whether the skew has an impact on the effective performance of importance sampling. To answer this question we consider two scenarios. In the first scenario $\rho_{S,v} = -0.3$, and in the second scenario $\rho_{S,v} = -0.7$. In Figure 4 we illustrate the effective performance in each scenario for different strikes. The results indicate that the skew has no significant impact on the effective performance of importance sampling.

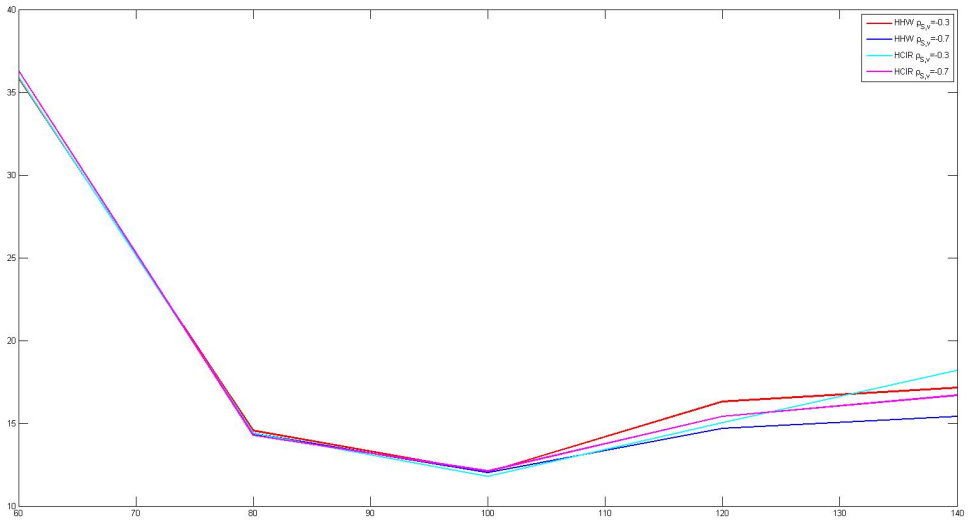

Figure 4: Effective performance for different strikes.

Finally, we will use the Multi-Level Monte Carlo to price basket call on three underlying assets. We will use the same parameters as in Section 2.1 and $\lambda = 0.05$, $r_0 = \theta = 0.05$, $\eta = 0.01$, $\rho_{S,v} = -0.3$, $\rho_{S,r} = 0.2$, $\rho_{r,v} = 0$. We set $\epsilon = 0.05$ and $L = 8$. We note that a combination of Multi-Level Monte Carlo and hybrid stochastic volatility model such as Heston-Hull-White or Heston-Cox-Ingersoll-Ross is particularly suitable for pricing variable annuities which are in principle long-dated basket put options. In Table 19 we report the results for basket call on three underlying assets and in Figure 5 we illustrate the effective performance.

$\frac{K}{S_0}$		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
Heston-Hull-White									
price	MLMC	23.0163	18.1578	13.5499	9.4680	6.1793	3.6732	2.0032	1.0110
	IS	23.0566	18.1401	13.5361	9.5040	6.1853	3.6966	2.0461	1.0250
variance	MLMC	167.2807	163.5697	151.1191	128.0302	98.3946	68.8361	45.2237	28.9471
	IS	82.6510	83.0224	79.2609	68.8662	53.9843	39.5120	27.5075	18.7768
Heston-Cox-Ingersoll-Ross									
price	MLMC	23.0473	18.1467	13.5498	9.4906	6.1786	3.7034	2.0303	1.0340
	IS	23.0272	18.1399	13.5098	9.4610	6.1423	3.6764	2.0344	0.9912
variance	MLMC	167.5398	163.5396	151.2621	128.0371	98.0752	69.0209	45.4317	28.6720
	IS	82.6687	82.9464	79.3410	68.9031	53.9411	39.3100	27.4990	18.6551

Table 19: Price and variance of price for basket call as a function of $\frac{K}{S_0}$.

On average importance sampling reduces the variance 2 times compared to the Multi-Level Monte Carlo without importance sampling.

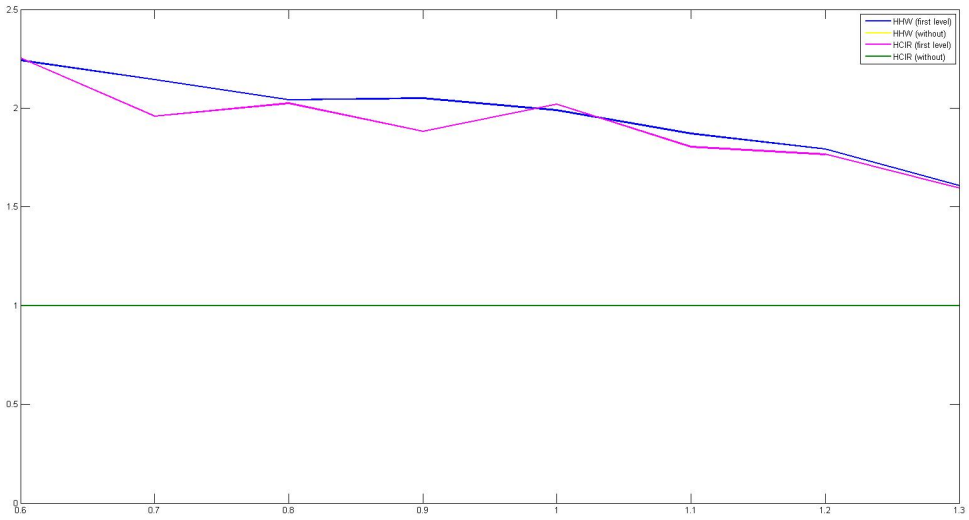

Figure 5: Effective performance as a function of $\frac{K}{S_0}$.

4 Conclusion

We have presented an application of importance sampling with stochastic change of drift to multi-asset options. We have illustrated the use of importance sampling with basket, best-of, worst-of, spread, absolute and composite options as examples. Based on our results, importance sampling reduces variance of multi-asset options by a factor of 3-13 on average across the strikes.

We have provided an extension of the Likelihood Ratio Method Based on the Characteristic Function to multi-asset options and combined it with the importance sampling to reduce the variance of the Greeks. Based on our results, importance sampling reduces variance of Greeks of multi-asset options by a factor of 2 on average across the strikes.

We applied importance sampling in a Multi-Level Monte Carlo and demonstrated that applying importance sampling on the first level significantly improves its effective performance. For European option the Multi-Level Monte Carlo with full importance sampling the effective performance is on average almost 3 times better than that of the Multi-Level Monte Carlo without importance sampling. The Multi-Level Monte Carlo with the importance sampling on the first level only the effective performance is on average almost 19 times better than that of the Multi-Level Monte Carlo without importance sampling. For basket option the Multi-Level Monte Carlo with the importance sampling on the first level only the effective performance is on average almost 2 times better than that of the Multi-Level Monte Carlo without importance sampling. We have also used the Likelihood Ratio Method Based on the Characteristic Function to estimate the Greeks in a

Multi-Level Monte Carlo and combined it with the importance sampling to reduce the variance of the Greeks. Based on our results, importance sampling reduces variance of Greeks of by a factor of 6-8 on average across the strikes.

Finally, we have studied the impact of the skew on the effective performance of importance sampling. Our results suggest that the skew has no significant impact on the effective performance of importance sampling.

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