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## Uncertain growth, ambiguity aversion and asset prices

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# Working Paper Series

## Uncertain Growth, Ambiguity Aversion and Asset Prices

Hening Liu

Manchester Business School Working Paper No 620

## Manchester Business School

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## **Keywords**

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# Uncertain Growth, Ambiguity Aversion and Asset Prices

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September 2011

## Abstract

I develop a stochastic growth model with production where there is a hidden state governing productivity growth regimes, and the hidden state evolves according to a Markov chain. Economic agents learn about the hidden state and display ambiguity aversion in the spirit of Klibanoff et al. (2005). I calibrate the model to the post-war US data. The main findings are (1) the model can generate a high and volatile equity premium while a low and smooth risk-free rate, (2) agents' fluctuating beliefs induce countercyclical variation in equity premium and in the expected volatility of returns, and moreover volatility clustering and persistence; and (3) Bayesian learning itself is unable to generate a significant and positive risk premium once time variation in investment opportunities is accounted for; in most cases, Bayesian learning lowers the unconditional mean of equity premium.

JEL CLASSIFICATION: C61; D81; G11; G12.

KEYWORDS: Countercyclical, Equity premium, Markov switching, Smooth ambiguity, Stochastic growth

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# 1 Introduction

In this paper, I develop a stochastic growth model with Bayesian learning and ambiguity attitude to account for the dynamic behavior of equity premium and the expected volatility of equity returns observed in the data. In the empirical literature, there exists abundant evidence that the equity premium in US stock market moves countercyclically (see, for example, Fama and French (1988a), Fama and French (1988b), Fama and French (1989) and Poterba and Summers (1988)), and the stock market volatility exhibits countercyclical variation as well (Schwert (1989)). In addition, empirical studies also find that the stock market volatility is persistent and tends to cluster together, featuring the well-documented GARCH behavior of returns (Bollerslev et al. (1992)). These dynamics properties of returns however, are difficult to explain in the standard dynamic stochastic general equilibrium (DSGE) framework. Rouwenhorst (1995) finds that the standard neoclassical growth model with a single source of uncertainty and frictionless markets is incapable of explaining the dynamic behavior of returns over the business cycle. Neither does the standard model generate an empirically plausible equity premium. Recent developments along the line including Jermann (1998), Boldrin et al. (2001), Kaltenbrunner and Lochstoer (2010), Campanale et al. (2010) and Croce (2010), among others, are still silent about the dynamic behavior of equity premium and the expected return volatility.<sup>1</sup> This paper aims to fill in the gap. I propose a nonlinear stochastic growth model that takes into account learning about investment opportunities, the cautious attitude toward model uncertainty (or rephrased as “ambiguity aversion”) and the separation between risk aversion and the attitude toward intertemporal substitution.

Productivity growth is specified as a hidden Markov model (HMM). Economic agents cannot observe the hidden state governing regime switching but can learn about the hidden state based on the historical productivity growth data. I assume there are a “high productivity growth” regime and a

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<sup>1</sup> Guvenen (2009) examines a general equilibrium model with limited stock market participation and different attitudes towards intertemporal substitution. He finds that the model can produce countercyclical variation in equity premium and its volatility.

“low productivity growth” regime. The quantitative estimates of the hidden Markov model suggest that the post-World War II US economy features transitory recessions and persistent booms. Due to the difficulty in distinguishing the two regimes, agents are endowed with incomplete information about the mean growth rate. This specification has been widely used in studying asset prices in endowment economies, for example, by David (1997), Veronesi (1999) and Ju and Miao (2011). The assumed hidden state provides us a tractable way to investigate the impacts of fluctuating beliefs over time. The mechanism through endogenous Bayesian learning induces the time-varying evolution of the market price of risk and equity premia. But with Epstein-Zin recursive utility, which allows for the separation between risk aversion and the attitude toward intertemporal substitution, the model still cannot produce a high enough equity premium to be empirically plausible.

To generate a high equity premium, I also assume that economic agents exhibit ambiguity aversion when they learn about the hidden state. Model uncertainty and the hidden state put agents in a more cautious position to guard against uncertainty in possible probability distributions of productivity growth rates. Agents do so by deviating from Bayesian filtered beliefs and assigning more probability mass to lower continuation value states. This feature is accommodated by the generalized recursive smooth ambiguity model recently proposed by Ju and Miao (2011) and axiomatized by Hayashi and Miao (2011). The model extends the recursive smooth ambiguity model of Klibanoff et al. (2009) by further disentangling risk aversion and the elasticity of intertemporal substitution (EIS) in the spirit of Kreps and Porteus (1978) and Epstein and Zin (1989). In the model, agents are ambiguity averse in the sense that they dislike any mean-preserving spread of the conditional expected utility (or the continuation value, in recursive form) induced by state probabilities. This attitude can be captured by imposing some concave transformation of the certainty equivalent in the recursive formulation of the value function. As noted by Klibanoff et al. (2005), this way of modeling ambiguity has the advantage of relaxing the tight link between ambiguity and

ambiguity aversion.<sup>2</sup>

Here ambiguity is modeled by fluctuating uncertainty about the hidden state while ambiguity aversion is reflected by effectively distorting agents' beliefs towards the lower continuation value states in a pessimistic way. When agents learn about the hidden state based on the historical data, the formed posterior beliefs are naturally embedded into the generalized recursive ambiguity utility model as “second-order probabilities” (Gollier (2011)). These state probabilities are updated in Bayesian fashion. The certainty equivalent is obtained by taking expectation of the conditional expected continuation value over posterior beliefs. Thus, unlike those asset pricing models with pure Bayesian learning (for example Veronesi (1999)), the model with smooth ambiguity has that the conditional distribution given a regime cannot be integrated over the posterior beliefs to yield a predictive distribution.

Suppose the hidden state can be observed, then agents do not need to learn about it, and the model reduces to the one examined by Jahan-Parvar and Liu (2011). In such a case, productivity growth is regime-switching, and the second-order probabilities are not updated over time but instead conform to the transition probabilities estimated from the data. Although their model is successful in explaining salient features of asset-returns data such as a high and volatile equity premium and a low and smooth risk-free rate, it generates counterfactually procyclical equity premia. The model in this paper fundamentally reverses this pattern by introducing Bayesian state beliefs as an additional state variable. In recessions, state beliefs deteriorates given a series of productivity shocks. As a result, ambiguity-averse agents tend to be very uncertain about future investment opportunities. The high state uncertainty, therefore, leads to a high conditional equity premium required to compensate for the uncertainty arose and a high expected return volatility. Thus, the model provides theoretical support to the hypothesis made by Merton (1980) that expected excess returns

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<sup>2</sup> The separation between ambiguity and ambiguity aversion, however, cannot be achieved in the multiple priors utility framework. See for example Chen and Epstein (2002) Leippold et al. (2008) and Epstein and Schneider (2008).

are proportional to the expected stock market volatility. In addition, since Bayesian state beliefs are persistent and move slowly over time, the model implies persistent changes in the volatility of equity returns, characterizing the well-known GARCH-type behavior of asset returns (Bollerslev et al. (1992)). These findings are consistent with the results obtained from those endowment economies studied by Veronesi (1999) and Ju and Miao (2011), where expected consumption or dividend growth rate is assumed to follow a Markov switching process. In a production economy, consumption and dividends are endogenously determined as the outcomes of equilibrium allocations. Thanks to the regime-switching behavior of productivity growth and the mechanism of learning, the model still implies that aggregate risks rise when state beliefs worsen during recessions.

Ambiguity aversion, on the other hand, raises the unconditional mean of equity premium to an empirically plausible range and alters the time series evolution of the market price of risk as well. The former effect is mainly through increased volatility of the marginal rate of substitution, a mechanism similar to Jahan-Parvar and Liu (2011), Ju and Miao (2011) and Hansen and Sargent (2010), among others. The latter effect is via changing the relationship between conditional equity premia and state beliefs. Because ambiguity aversion induces pessimistic evaluation, a productivity shock during recessions may lead to a higher equity premium required by ambiguity-averse agents than ambiguity-neutral agents. This effect could significantly alter the cyclical variation of equity premia.

Another main finding of this paper is that in spite of its key role in replicating equity premia in relation to the business cycle, Bayesian learning is not crucial in raising equity premia as long as time variation of investment opportunities is accounted for. In the presence of regime-switching productivity growth, I find that Bayesian learning, in most cases, even lowers the unconditional mean of equity premium. This observation is robust to various degrees of ambiguity aversion. It is worth noting that the trivial role of learning in matching unconditional moments of asset-returns



data is not a generic feature to most of consumption-based models but due to the production economy environment considered in this paper. Productivity growth and the capital stock are also state variables characterizing equilibrium allocations and asset prices.<sup>3</sup>

The rest of the paper is organized as follows. Section 2 presents the generalized smooth ambiguity model employed to represent preferences and beliefs of agents. Section 3 describes the production economy model and the market equilibrium. Section 4 discusses the calibration exercise and the main findings. Section 5 concludes. A detailed explanation of the numerical algorithm used in the paper is included in the Appendix.

## 2 The Smooth Ambiguity Model

In this section, we describe the framework that we use to model the agent's preferences and beliefs. This framework is embedded in a general equilibrium model with nontrivial production in the next section. The static version of the utility preferences used in this paper is of the smooth ambiguity type, introduced by Klibanoff et al. (2005). Klibanoff et al. (2009) develop the recursive version of this class of preferences in a dynamic setting. In the spirit of Epstein and Zin (1989), Hayashi and Miao (2011) further generalize the model by disentangling risk aversion and intertemporal substitution. In this paper, we assume that the agent's preferences and beliefs are characterized by the generalized recursive smooth ambiguity model of Hayashi and Miao (2011). Interested readers could refer to these papers for more details.

### 2.1 The static and recursive formulation

We start with a formulation ordinally equivalent to the smooth ambiguity model of Klibanoff et al. (2005):

$$v^{-1} \mathbb{E}_{\zeta} v \left( u^{-1} \left( \mathbb{E}_{\pi_{\theta}} u \circ f \right) \right) \equiv v^{-1} \left( \int_{\Theta} v \left( u^{-1} \left( \int_S u(f) d\pi_{\theta} \right) \right) d\zeta(\theta) \right) \quad (1)$$

<sup>3</sup> In the consumption-based model studied by Ju and Miao (2011), Bayesian learning has a profound impact in matching the key financial moments.

where  $f$  is an act that maps states to decisions (for example, a policy function),  $\mathbb{E}$  is an expectation operator,  $u$  is a von Neumann-Morgenstern utility function,  $v$  is an increasing function,  $\zeta$  is a subjective probability measure over a set of parameters denoted by  $\Theta$  in which each element induces a probability measure  $\pi_\theta$  over the state space  $S$ . The decision maker prefers act  $f$  to act  $g$  if and only if

$$v^{-1}\mathbb{E}_\zeta v(u^{-1}(\mathbb{E}_{\pi_\theta} u \circ f)) \geq v^{-1}\mathbb{E}_\zeta v(u^{-1}(\mathbb{E}_{\pi_\theta} u \circ g)).$$

This formulation is ordinally equivalent to the model of Klibanoff et al. (2005) if we define  $\phi \equiv v \circ u^{-1}$ . In the above formulation,  $\pi_\theta$  yields beliefs over outcomes given a certain parameter value, while  $\zeta$  reflects the decision maker's uncertainty as to which probability distribution in the set of probability distributions induced by  $\Theta$  truly governs the state space. As noted by Klibanoff et al. (2005), a key feature of the smooth ambiguity utility model is that it achieves a separation between ambiguity, identified as a characteristic of the decision maker's subjective beliefs, and ambiguity attitude, identified as a characteristic of the decision maker's tastes. In particular, ambiguity is captured by multiplicity of the subjective set of probability measures induced by the set  $\Theta$ . Attitudes toward pure risk and ambiguity are characterized, respectively, by the shape of  $u$  and  $v$ . The decision maker is risk averse if and only if  $u$  is concave, and he is ambiguity averse if and only if  $v$  is a concave transformation of  $u$ . According to Klibanoff et al. (2005), ambiguity aversion is defined to be an aversion to mean preserving spreads in the distribution over expected utility values,  $\mathbb{E}_{\pi_\theta} u \circ f$ , induced by  $\zeta$  and the act  $f$ . This distribution represents the probabilities of different evaluations of the act  $f$  under different probability measures deemed as relevant. Gollier (2011) call this distribution the “second-order” distribution. This class of preferences implies that ambiguity averse decision makers prefer acts whose evaluation is more robust to the possible variation in probabilities than those who display risk aversion only. In addition, this class of preferences implies the *irreducibility of compound distributions*. That is, the model does not impose the compound

reduction between  $\zeta$  and the  $\pi_\theta$ s in the support of  $\zeta$ . In the special case of  $\phi$  being linear, such reduction is feasible, and the decision maker displays ambiguity neutrality which is observationally equivalent to a subjective expected utility decision maker with a subjective prior  $\zeta$ .

Klibanoff et al. (2009) embed the static model (1) in a dynamic setting and develop a recursive formulation of the smooth ambiguity model. In a discrete-time setting, the state space is denoted by  $S$ . The decision-maker's information in period- $t$  is summarized by history  $s^t = \{s_0, s_1, s_2, \dots, s_t\}$  with the root node  $s_0 \in S$  given and  $s_t \in S$ . The decision maker chooses among consumption plans  $C \equiv (C_t)_{t \geq 0}$ , each of which maps a history  $s^t$  to a payoff. That is,  $C_t$  is adapted to  $s^t$  and is a measurable function of  $s^t$ . The decision maker is uncertain as to which probability distribution governs the full state space  $S^\infty$ . This uncertainty is represented by a parameter space  $\Theta$ , a set of candidate models or a state evolving over time according to a Markov chain. The decision maker is allowed to make inference on the set of parameters (if unobservable) based on history  $s^t$ . Suppose  $\pi_\theta(s_{t+1}|s^t)$  denotes the probability distribution that the next observation will be  $s_{t+1}$ , given the parameter  $\theta \in \Theta$  and the history  $s^t$ . We denote by  $\zeta$  the decision maker's prior on the parameter space  $\Theta$ . Klibanoff et al. (2009) develop the following recursive version of the smooth ambiguity model:

$$V_{s^t}(C) = u(C(s^t)) + \beta \phi^{-1} \left[ \int_{\Theta} \phi \left( \int_{S_{t+1}} V_{(s^t, s_{t+1})}(C) d\pi_\theta(s_{t+1}|s^t) \right) d\zeta(\theta|s^t) \right] \quad (2)$$

where  $V_{s^t}(C)$  is a indirect value function,  $\beta \in (0, 1)$  is the subjective discount factor,  $\zeta(\theta|s^t)$  denotes the Bayesian posterior updated given the history  $s^t$ , and  $u$  and  $\phi$  are defined in the same way as in the static model. Collard et al. (2011) study the asset pricing implications of ambiguity using this model. The utility function (2) is always well defined for the specification  $\phi(x) = -\exp(-x/\lambda)$ ,  $\lambda > 0$ . This specification has a straightforward connection with the robust control approach of Hansen (2007) and Hansen and Sargent (2010). However, in such a case the utility function is not homogeneous, which is a desirable property for numerical value function iteration.

## 2.2 The generalized recursive smooth ambiguity model

In this paper, we use the generalized recursive smooth ambiguity model, which further extends the recursive version of the smooth ambiguity model by allowing for the separation between risk aversion and intertemporal substitution. This class of preferences is recently proposed by Ju and Miao (2011) and axiomatized by Hayashi and Miao (2011). Inspired by Kreps and Porteus (1978) and Epstein and Zin (1989), Ju and Miao (2011) propose the following formulation:

$$V_t(C) = W(C_t, \mathcal{R}_t(V_{t+1}(C))), \quad \mathcal{R}_t(V_{t+1}) = v^{-1}(\mathbb{E}_{\zeta_t}[v \circ u^{-1} \mathbb{E}_{\pi_{\theta,t}}[u(V_{t+1})]]) \quad (3)$$

where  $V_t(C)$  is the continuation value at date  $t$ ,  $W$  is a time aggregator that associates period- $t$  continuation value to the payoff generated from period- $t$  consumption plan and some certainty equivalent of period- $t+1$  continuation value,  $\mathcal{R}_t$  is an uncertainty aggregator that maps period- $t+1$  continuation value to its period- $t$  certainty equivalent, and  $u$  and  $v$  have the same interpretation as in the static setting. When  $v \circ u^{-1}$  is linear, that is, the decision maker is ambiguity neutral, we obtain recursive utility of Epstein and Zin (1989). In that case, we can integrate the probability distribution  $\pi_{\theta,t+1}$  over the Bayesian posterior  $\zeta_t$  to obtain a predictive distribution, which is one of the fundamental concepts in Bayesian analysis. When  $v \circ u^{-1}$  is nonlinear, the decision maker displays aversion to uncertainty about which probability distribution governs the state space. Allowing for non-indifference to the timing of the resolution of uncertainty, Ju and Miao (2011) consider a time aggregator in the spirit of Kreps and Porteus (1978) and Epstein and Zin (1989) in the following form:

$$W(c, y) = [(1 - \beta)c^{1-\rho} + \beta y^{1-\rho}]^{\frac{1}{1-\rho}}, \quad \rho > 0, \neq 1 \quad (4)$$

with

$$\begin{aligned} u(x) &= \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \neq 1 \\ v(x) &= \frac{x^{1-\eta}}{1-\eta}, \quad \eta > 0, \neq 1 \end{aligned}$$

where  $\rho$  is the inverse of the EIS parameter  $\psi$ ,  $\gamma$  is the relative risk aversion parameter, and  $\eta$  is the ambiguity aversion parameter.

Applying the aggregator (4) to (3), we obtain

$$\begin{aligned} V_t(C) &= \left[ (1-\beta)C_t^{1-\rho} + \beta \{\mathcal{R}_t(V_{t+1}(C))\}^{1-\rho} \right]^{\frac{1}{1-\rho}} \\ \mathcal{R}_t(V_{t+1}(C)) &= \left( \mathbb{E}_{\zeta_t} \left[ \left( \mathbb{E}_{\pi_{\theta,t}} \left[ V_{t+1}^{1-\gamma}(C) \right] \right)^{\frac{1-\eta}{1-\gamma}} \right] \right)^{\frac{1}{1-\eta}} \end{aligned}$$

It is worth noting that the decision maker is ambiguity averse if and only if  $\eta > \gamma$ . If  $\eta = \gamma$ , the decision maker is ambiguity neutral and his preferences are represented by recursive utility of Epstein and Zin (1989) and Weil (1989).

In the limiting case  $\rho = 1$ , the utility model becomes

$$U_t = (1-\beta) \ln C_t + \frac{\beta}{1-\eta} \ln \left\{ \mathbb{E}_{\zeta_t} \exp \left( \frac{1-\eta}{1-\gamma} \ln (\mathbb{E}_{\pi_{\theta,t}} [\exp ((1-\gamma) U_{t+1})) \right] \right\}$$

where  $U_t = \ln V_t$ . Ju and Miao (2011) note that this specification is isomorphic to the risk sensitive preferences studied by Hansen (2007) and Hansen and Sargent (2010). Specifically, the two risk sensitivity adjustments for the distributions  $\pi_{\theta}(s_{t+1}|s^t)$  and  $\zeta_t$ , which are both in the form of “log-exp”, capture the decision maker’s concern about the misspecification in  $\pi_{\theta}(s_{t+1}|s^t)$  given a parameter  $\theta$  (or alternatively, a hidden state) and in the Bayesian posteriors  $\zeta_t$ , respectively.

Now I adapt notations specific to the production economy to be examined in the next section. The structure of the parameter set and the decision maker’s subjective beliefs is similar to that in the endowment economy studied by Ju and Miao (2011). I assume that uncertainty is represented by a hidden state  $z$  evolving over time as a Markov chain with transition probabilities given. The

Markovian hidden state can switch between a finite number of regimes. Each possible regime corresponds to a probability distribution over the state space. The representative agent cannot observe regimes but can learn about the hidden state based on the current and past signals (productivity growth data). Denote  $\zeta(z | s^t)$  as the Bayesian posterior after the history  $s^t$  is observed. The utility function has the following form:

$$\begin{aligned} V_t(C) &= \left[ (1 - \beta)C_t^{1-\rho} + \beta \{\mathcal{R}_t(V_{t+1}(C))\}^{1-\rho} \right]^{\frac{1}{1-\rho}} \\ \mathcal{R}_t(V_{t+1}(C)) &= \left( \mathbb{E}_{\zeta_t} \left[ \left( \mathbb{E}_{\pi_{z,t}} \left[ V_{t+1}^{1-\gamma}(C) \right] \right)^{\frac{1-\eta}{1-\gamma}} \right] \right)^{\frac{1}{1-\eta}} \end{aligned}$$

where  $\pi_{z,t}$  is the probability distribution of  $\mathcal{S}_{t+1} \subset S^\infty$  given a regime and the history  $s^t$ .

### 3 The Production Economy

This section presents the production economy model that is employed to examine the effects of model uncertainty and ambiguity aversion on asset prices and other macroeconomic fundamentals. For simplicity, I only present the social planner's problem in which labor supply is kept constant. I assume there exist convex capital adjustment costs, following the specification in Campanale et al. (2010).

The representative agent has the following smooth ambiguity utility defined over aggregate consumption

$$V_t(C) = \left[ (1 - \beta)C_t^{1-\rho} + \beta \{\mathcal{R}_t(V_{t+1}(C))\}^{1-\rho} \right]^{\frac{1}{1-\rho}} \quad (5)$$

$$\mathcal{R}_t(V_{t+1}(C)) = \left( \mathbb{E}_{\zeta_t} \left[ \left( \mathbb{E}_{\pi_{z,t}} \left[ V_{t+1}^{1-\gamma}(C) \right] \right)^{\frac{1-\eta}{1-\gamma}} \right] \right)^{\frac{1}{1-\eta}} \quad (6)$$

The consumption good is produced according to a constant return-to-scale Cobb-Douglas production function:

$$Y_t = K_t^\alpha (A_t N_t)^{1-\alpha} \quad (7)$$

where  $Y_t$  is the output,  $K_t$  is the capital stock,  $N_t$  is the amount of labor hours, and  $A_t$  is the level

of aggregate productivity. Labor input is assumed to be exogenous and equal to  $\bar{N}$ . Uncertainty in this economy is driven by stochastic productivity growth. The growth rate  $\Delta a_{t+1} \equiv \log \left( \frac{A_{t+1}}{A_t} \right)$  follows a hidden Markov model with two different regimes

$$\Delta a_t = \mu(s_t) + \sigma \epsilon_t, \quad \epsilon_t \sim N(0, 1) \quad (8)$$

where the hidden state  $s_t$  is unobservable and switch between two different values representing possible states of the economy. I denote by  $s_t = 1$  the high mean growth state (the good regime), and by  $s_t = 2$  the low mean growth state (the bad regime). The transition probability matrix is given by

$$\mathbf{P} = \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix}. \quad (9)$$

I assume that the volatility of the growth rate is constant and attempt to generate a rich set of interesting results through endogenous Bayesian learning.<sup>4</sup>

The agent learns about the hidden state and update his beliefs in Bayes fashion. The Bayesian posterior belief, or one-step-ahead conditional probability, is denoted by  $\zeta_t$  and defined by  $\zeta_t = \Pr(s_{t+1} = 1 \mid \mathcal{I}_t)$ , where ‘1’ represents the high mean growth regime. Given that the prior belief  $\zeta_0$  is known, the posterior beliefs are updated according to Bayes rule:

$$\zeta_{t+1} = \frac{p_{11} f(\Delta a_{t+1} \mid z = 1) \zeta_t + (1 - p_{22}) f(\Delta a_{t+1} \mid z = 2) (1 - \zeta_t)}{f(\Delta a_{t+1} \mid z = 1) \zeta_t + f(\Delta a_{t+1} \mid z = 2) (1 - \zeta_t)}$$

where  $f(s|i) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ - (s - \mu_i)^2 / (2\sigma^2) \right]$  is the density function of the normal distribution with mean  $\mu_i$  and variance  $\sigma^2$ .

The capital stock evolves according to

$$K_{t+1} = (1 - \delta_k) K_t + I_t - G(K_t, K_{t+1}) \quad (10)$$

$$G(K_t, K_{t+1}) = \left| \left( \frac{K_{t+1}}{K_t} - \omega \right) \right|^\iota K_t, \quad \iota > 1, \omega > 0 \quad (11)$$

<sup>4</sup> Croce (2010) explores time variation in the conditional variance of productivity growth and examine the implication for asset prices.

where  $\delta_k$  is the depreciation rate,  $I_t$  denotes investments, and  $G$  is a convex capital adjustment cost function that introduces real market frictions. The functional form is adapted from Campanale et al. (2010).<sup>5</sup> The parameter  $\iota$  determines the size of the marginal capital adjustment costs. A lower  $\iota$  implies higher adjustment costs. The idea of introducing capital adjustment costs dates back to Jermann (1998), who finds that adjustment costs prevent endogenous consumption smoothing from absorbing too much consumption volatility. Without capital adjustment costs, Tobin's  $q$  remains constant at 1.

### *Market equilibrium*

In equilibrium, outputs are equal to the sum of consumption and investment:

$$C_t + I_t = Y_t, \quad (12)$$

In financial markets, the equilibrium condition requires that households hold all outstanding equity shares and all other assets are in zero net supply. By solving the social planner's problem, I can find the equilibrium allocation in competitive markets. Once the social planner's problem is solved, asset prices and returns are then determined as outcomes of equilibrium allocations. The social planner's problem, in recursive form, is fully presented in the Appendix. The agent chooses consumption and capital stock to maximize his welfare. Given that the generalized recursive smooth ambiguity utility model satisfies homogeneity, the optimization problem can be formulated in terms of stationary variables.

Ju and Miao (2011) shows that the pricing kernel for the generalized recursive smooth ambiguity preferences is given by

$$M_{z_{t+1},t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})} \right)^{\rho-\gamma} \left( \frac{\left( \mathbb{E}_{z_{t+1},t} [V_{t+1}^{1-\gamma}] \right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_t(V_{t+1})} \right)^{-(\eta-\gamma)}, \quad (13)$$

where  $z_{t+1} = 1, 2$ , and  $\mathbb{E}_{z_{t+1},t}$  denotes the expectation operator for the distribution of the consump-

<sup>5</sup> In (11),  $\omega$  is a constant that will be given a value in the calibration exercise.



tion process conditioned on the history  $s^t$  and the period- $t+1$  state  $z_{t+1}$ . If the agent is ambiguity neutral ( $\eta = \gamma$ ), the last term in (13) vanishes, and the pricing kernel has a similar form as in Croce (2010) and Kaltenbrunner and Lochstoer (2010). Further, if the agent displays constant relative risk aversion ( $\gamma = \rho$ ), then we obtain the familiar pricing kernel for expected utility. The Euler equation is

$$\mathbb{E}_t [M_{z_{t+1}, t+1} R_{t+1}^e] = 1. \quad (14)$$

where  $\mathbb{E}_t$  is the period- $t$  conditional expectation operator for the posterior distribution of the hidden state. The risk-free rate is the reciprocal of the expectation of the price kernel,  $R_t^f = 1/\mathbb{E}_t [M_{z_{t+1}, t+1}]$ . The return on equity is denoted by  $R_t^e$  (unlevered equity claim) and is given by

$$\begin{aligned} R_{t+1}^e &= \frac{P_{t+1} + D_{t+1}}{P_t} \\ R_{t+1}^e &= \frac{D_{t+1} + [1 + G_{K_{t+2}}(K_{t+1}, K_{t+2})]K_{t+2}}{[1 + G_{K_{t+1}}(K_t, K_{t+1})]K_{t+1}}, \end{aligned} \quad (15)$$

where the subscript of  $G$  stands for the partial derivative, and dividends are equal to  $D_t = \alpha Y_t - I_t$  in equilibrium.

### **The full information case: Jahan-Parvar and Liu (2011)**

Suppose the hidden state is fully observable, that is, in each period, the agent know the current regime of the economy but is still uncertain as to which regime the mean growth rate will switch to in the next period. In this case, the model reduces to the one examined by Jahan-Parvar and Liu (2011). The utility function becomes

$$\begin{aligned} V_{z_t, t}(C) &= \left[ (1 - \beta) C_t^{1-\rho} + \beta \{ \mathcal{R}_{z_t, t}(V_{z_{t+1}, t+1}(C)) \}^{1-\rho} \right]^{\frac{1}{1-\rho}} \\ \mathcal{R}_{z_t, t}(V_{z_{t+1}, t+1}(C)) &= \left( \mathbb{E}_{z_t} \left[ \left( \mathbb{E}_{\pi_{z, t}} \left[ V_{z_{t+1}, t+1}^{1-\gamma}(C) \right] \right)^{\frac{1-\eta}{1-\gamma}} \right] \right)^{\frac{1}{1-\eta}} \end{aligned}$$

where  $V_{z_t, t}(C)$  is the period- $t$  continuation value of consumption plans  $C$  given period- $t$  state, and  $\mathcal{R}_{z_t, t}(V_{z_{t+1}, t+1}(C))$  is the certainty equivalent of future continuation value given period- $t$  state.

The Euler equation is given by the Euler equation

$$\mathbb{E}_{z_t} [M_{z_{t+1}, t+1} R_{t+1}^e] = 1. \quad (16)$$

where  $\mathbb{E}_{z_t}$  is the period- $t$  conditional expectation operator for the current state being  $z_t$ .

## 4 Calibration and Results

In this section, I first summarize stylized facts about historical asset-returns data. Then I calibrate the production economy model to closely replicate unconditional moments of asset returns typically observed in the data. Under the calibrated parameter values, I further examine cyclical variation in the equity premium and the volatility of returns. In the analysis of properties of unconditional and conditional moments of asset returns, I also perform comparative statics analysis by changing the degree of ambiguity aversion.

Due to nonlinearities, the model does not admit an explicit analytical solution. Thus, the model is solved using numerical methods. In particular, the value function iteration is employed to solve the representative agent's consumption-saving problem, and unconditional and conditional moments of financial and macroeconomic fundamentals are calculated using Monte Carlo simulations. A detailed description of the numerical procedure is included in the Appendix.<sup>6</sup>

### 4.1 Data and calibration

I begin by summarizing the key properties of asset-returns data documented in the empirical asset pricing literature. Data on financial variables (equity returns, risk-free interest rates, dividends and price-dividend ratios) and consumption are downloaded from Shiller's website.<sup>7</sup> All nominal variables are deflated using the Consumer Price Index (CPI) data. I use the post-war annual data 1951–2009 for calibration. Data on productivity are drawn from a data set on total factor productivity (TFP) available in Fernald (2009). To keep consistency, the data also span from 1951

<sup>6</sup> The computer code is written in Compaq Visual FORTRAN 6.6 and available upon request from the author.

<sup>7</sup> I thank Robert Shiller for making the data available.

to 2009.

I calibrate the model at the annual frequency and focus on the model's ability to reproduce annual statistics calculated from the data. To estimate parameters in the productivity growth process (8), I apply the expectation maximization (EM) algorithm of Hamilton (1990) to the quarterly productivity growth data for the period 1951–2009 and then annualize the estimates. The parameter estimates are displayed in Table 2. The results suggest that the hidden Markov model clearly identifies two distinct regimes for productivity growth. The good regime is highly persistent with a transition probability of  $p_{11} = 0.94$  and a positive mean growth rate of 0.0178, while the bad regime is relatively transitory with a transition probability of  $p_{22} = 0.58$  and a negative mean growth rate of -0.0234. The existence of economic regimes is also well documented in Hamilton (1989), Rouwenhorst (1995) and Cagetti et al. (2002), among others.

Table 2 also other parameter values that are held invariant throughout the calibration exercise unless otherwise stated. As in the standard literature on DSGE models, I set the capital share ( $\alpha$ ) to 0.35, and the annual depreciation rate of capital to  $\delta = 0.06$ . Labor hours, which is assumed to be constant, is fixed at 0.20. The exponent parameter in the capital adjustment costs function,  $\iota$ , is set at 1.09 to match the standard deviation of consumption growth for the sample period 1951–2009. The annual subjective discount factor  $\beta$  is set at  $\beta = 0.975$ . The relative risk aversion parameter  $\gamma$  is fixed at  $\gamma = 3$ .

## 4.2 Calibration results: unconditional moments

In the calibration exercise, the EIS parameter,  $\psi$ , is set at 1.5, following Bansal and Yaron (2004) and Ju and Miao (2011). As will be seen below, an EIS greater than 1 is crucial to generate countercyclical variation in conditional equity premium and conditional volatility of equity returns. The ambiguity aversion parameter  $\eta$  is set such that the model-implied mean equity premium can closely match with the data. In the light of works by Ju and Miao (2011) and Chen et al. (2011),

the value of  $\eta$  considered in the benchmark calibration is modest ( $\eta = 33$ ).

Table 3 reports the unconditional moments of the key macroeconomic and financial variables generated from the benchmark calibration (labeled “Model I”) and compares the results with the data to see the performance of the model. I also do comparative statics analysis by considering two other alternative values for the ambiguity aversion parameter,  $\eta = 20$  (labeled “Model II”) and  $\eta = 3$  (labeled “Model III”).<sup>8</sup> The latter case corresponds to Epstein-Zin recursive utility with ambiguity neutrality.

The statistics calculated from the data are displayed in Table 3. The macroeconomic statistics include the volatility and the first autocorrelation of consumption growth where consumption growth is defined as the percentage change in real consumption goods. Data on US per capital real consumption (1951–2009) suggest that consumption growth is not volatile in the post-war period, with the standard deviation being equal to 1.8 percent. On the other hand, the mean equity premium remains high at 5.9 percent while the mean risk-free rate is low at 1.98 percent. Moreover, the stock market exhibits much volatility with the standard deviation of excess market returns being equal to 16.50 percent, but the risk-free rate is not volatile at all, and its standard deviation is only 2.46 percent.<sup>9</sup> Overall, the Sharpe ratio for US stock market is about 0.36. These stylized facts are well-documented in the literature (see Campbell (1999) for a survey). Although there have been numerous efforts in explaining these puzzles in the consumption-based framework (see Campbell and Cochrane (1999), Cecchetti et al. (2000), Bansal and Yaron (2004) and Ju and Miao (2011), among others), doing so in a production economy poses further challenges as consumption and dividends are endogenously determined. The habit persistence models developed by Jermann (1998) and Boldrin et al. (2001) can explain the equity premium puzzle but generate excessive volatility in risk-free rates. Kaltenbrunner and Lochstoer (2010) use Epstein-Zin preferences and

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<sup>8</sup> Each set of results is generated from 10,000 simulations, where each simulation containing 100 periods.

<sup>9</sup> Jahan-Parvar and Liu (2011) consider a much lower target level for the volatility of risk-free rates.

can reproduce Sharpe ratio observed in the data, but their model is unable to generate a high mean equity premium even under the counterfactual assumption that the subjective discount factor  $\beta$  is greater than 1. Croce (2010) relies on long-run productivity risk and Epstein-Zin utility but can only generate a 3 percent mean equity premium when the coefficient of relative risk aversion is 30.

The benchmark calibration (Model I) represents a significant improvement over the existing production economy models in terms of asset pricing implications. The benchmark model produces both a low mean risk-free rate and a low level of risk-free rate volatility. The unconditional mean of risk-free rates is 1.92 percent, which is very close to the data. The model, on the other hand, under-predicts the volatility of risk-free rates, with a standard deviation of 0.5 percent generated in the simulation. This level is rather low compared to the data (2.46 percent). Thus, the model avoids producing excessive volatility in the risk-free rate. Both the attitude toward intertemporal substitution and the attitude toward model uncertainty affect moments of the risk-free rate. A high EIS or a high degree of ambiguity aversion decreases the risk-free rate in all states of the economy. The former effect is through the intertemporal substitution effect, and the latter via the endogenous pessimistic evaluation of the future continuation value. Both effects increase savings motive and lower the mean risk-free rate in equilibrium. The low volatility of risk-free rates implied by the model is driven by (1) a high EIS and (2) the strong persistence of Bayesian updated beliefs. Previous results (Jermann (1998), Kaltenbrunner and Lochstoer (2010) and Campanale et al. (2010)) suggest that a rather low EIS tends to imply a very volatile risk-free rate. In addition, because state beliefs are highly persistent in the model, the strong persistence translates into the conditional expectation operator applied to the pricing kernel. Panel A of Figure 2 plots one sample path of simulated state beliefs. It shows that the state probability  $\zeta_t$  spends most of time at high levels. Occasional productivity shocks temporarily drive down the state probability, but it reverts back to high levels immediately. The agent's beliefs are thus characterized by a persistent and slow-moving state

probability. Recall that the risk-free rate is the reciprocal of the conditional expectation of the pricing kernel. As a result, slow movements in Bayesian beliefs lead to low volatility of the risk-free rate.

Turning to the mean equity premium, Table 3 shows that a higher degree of ambiguity aversion results in a higher mean equity premium.<sup>10</sup> In the benchmark calibration, the mean equity premium implied by Model I is 5.88 percent, which is extremely close to the level observed in the data (5.90 percent). The other two models with lower degrees of ambiguity aversion cannot produce a realistic mean equity premium. Under Epstein-Zin recursive utility ( $\gamma = 3, \eta = 3, \psi = 1.5$ ), Model III can only generate a mean equity premium of 1.18 percent. This finding is consistent with Croce (2010) where the mean equity premium is 3 percent when productivity growth consists of a persistent component and a GARCH component, and the relative risk aversion is very high at  $\gamma = 30$  in the calibration. Due to capital adjustment costs, the model is able to generate volatile enough equity returns, with the standard deviation of equity premium being equal to 16.96 percent, which also replicates the observation in the data (16.50 percent). With no adjustment costs, Tobin's  $q$  remains constant, and the variation of equity returns is only driven by changes of dividends. In such a case, the volatility of equity returns could be very low. In the light of these results, it is not surprising to see that among Model I–III, the benchmark calibration produces a Sharpe ratio (0.36) closest to the data (0.35).

The implied high Sharpe ratio is attributed to the increased price of risk under ambiguity aversion. Table 3 shows that under ambiguity neutrality ( $\gamma = \eta = 3$ ), the price of risk is almost negligible ( $\sigma(M)/\mathbb{E}(M) = 0.07$ ). Thus, the pricing kernel (or the marginal rate of substitution) is too smooth to deliver a sizable risk premium. In the benchmark calibration, the volatility of the pricing kernel increases dramatically to yield a high price of risk ( $\sigma(M)/\mathbb{E}(M) = 1.39$ ). The enlarged variation of the pricing kernel is mostly driven by the last multiplicative term in (13),

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<sup>10</sup> Recall that the equity claim is defined as unlevered claim to aggregate dividends.

which vanishes under ambiguity neutrality. To see it more clearly, Figure 1 shows the price of risk plotted against the state probability  $\zeta_t$  for different degrees of ambiguity aversion, where the capital stock is set to its mean in the simulation and productivity growth to its stationary level. Two observations are noteworthy. First, regardless of ambiguity aversion, the conditional price of risk exhibits a hump-shape. Further, the shape looks almost symmetric under ambiguity neutrality. This is a typical finding in asset pricing models with Bayesian learning. See Veronesi (1999) for results in an endowment economy and Cagetti et al. (2002) in a nonlinear production economy. The intuition is that the price of risk is high during periods of high uncertainty. This usually happens in recessions when a series of bad shocks to productivity have occurred and Bayesian filtered state probability drops toward 0.5. Second, ambiguity aversion increases the price of risk for all possible state beliefs. In particular, this effect is more pronounced for  $\zeta$  being close to  $p_{11} = 0.937$  where Bayesian state belief spends most of its time near around. Informally, this is to say that ambiguity aversion leads to “negative skewness” of the price of risk against state beliefs. The asymmetric shape can be intuitively explains in the following. Suppose the filtered probability  $\zeta$  is at its usual level around  $p_{11}$ . Now a few small bad shocks to productivity occur. This will lead to very modest downward revisions to the filtered probability through Bayesian updating. The ambiguity-neutral agent does not deem this scenario as a strong indication of future investment opportunities worsening. Thus, the price of risk rises but not in a great magnitude. However, the ambiguity averse agent, whose attitude toward model uncertainty is manifested via endogenous pessimism, effectively distorts Bayesian beliefs in such a way that more probability weights are assigned to lower continuation value states. As a result, he seriously worries the possibility that future investment opportunities become very bad, the price of risk, therefore, rises substantially.

The quantitative results regarding unconditional moments presented here are similar to those of Jahan-Parvar and Liu (2011), where economic regimes are fully observable and Bayesian updat-

ing is not permitted. Nevertheless, because of different information structure, the model in this paper clearly distinguishes from Jahan-Parvar and Liu (2011). Here, the second-order probabilities entering into the smooth ambiguity utility model are updated taking into account the most recent available information, while in Jahan-Parvar and Liu (2011) these probabilities are not adapted but conform to the transition probabilities. Moreover, the calibration exercise carried out in this paper is not trivial for good reasons. First, the calibration can be used to screen out the impact of learning on the mean equity premium and thus answer the question whether Bayesian updating accounts for any risk premium. Second, the exercise can provide useful guidance on values of preference parameters that will be used to investigate the cyclical variation of equity premium and its conditional volatility. Table 4 summarizes the impact of Bayesian updating on the mean risk-free rate and equity premium. This exercise is executed by implementing the full information model of Jahan-Parvar and Liu (2011) under the parameterization of this paper. It turns out that in all cases learning even dampens the equity premium. Part of this effect comes from the higher mean risk-free rate in the presence of learning. These results suggest that as long as investment opportunities are time varying (for example, regime switching), introducing a hidden state and elements of learning does not contribute to the explanation of the equity premium puzzle. However, the presence of Bayesian learning is crucial to generate countercyclical equity premia.

### **4.3 Cyclical variation**

The empirical literature has extensively documented that both equity premium and the volatility of equity premium move countercyclically (Fama and French (1989), Poterba and Summers (1988) and Schwert (1989)). Further, empirical studies (Merton (1980), French et al. (1987) and Ghysels et al. (2005)) have found evidence of the “volatility feedback” effect, that is, movements in expected returns are driven by changes in the expected future return volatility. In addition, changes in return volatility tend to be persistent, and returns feature volatility clustering and GARCH-type



behavior (see Bollerslev et al. (1992) for a survey). These dynamic regularities of returns are hard to explain in the standard DSGE framework. For example, Rouwenhorst (1995) finds that the standard neoclassical growth model hardly predicts any cyclical variation of equity premium and equity volatility. This is due to the lack of two important ingredients. First, the volatility of equity returns is too low in the absence of market frictions. Second, the standard model lacks additional state variables that reinforce the link between time varying expected returns and productivity shocks. In the present model, convex capital adjustment costs account for the former feature, and highly persistent beliefs capture the latter aspect. I now examine the dynamic properties of expected returns and the expected return volatility under the benchmark calibration ( $\gamma = 3, \eta = 33, \psi = 1.5$ ).

I use a random number generator to generate a long time series (100 years) artificial productivity growth rates from the Markov switching process (8). Then I obtain filtered state beliefs as a result of Bayesian updating, assuming that the prior belief  $\zeta_0$  is in the stationary level. Equilibrium allocations and Euler equation determine conditional equity premium and conditional return volatility in each period. The results are plotted in Figure 2. Panel A reveals that simulated state beliefs are highly persistent and slow-moving. During periods of high uncertainty when  $\zeta$  drops from its stationary level near to 0.5, both conditional equity premium and return volatility rise significantly. It is important to note that even when state beliefs become modestly worse following a few small shocks to productivity, conditional equity premium and return volatility react much more strongly under ambiguity aversion. As noted above, if state beliefs start from being persistent and good, the price of risk for the ambiguity averse agent increases substantially even there happens to be a small decline in his beliefs, giving rise to the negatively skewed shape in Figure 1. Here, it is indeed this effect that drives countercyclical movements in expected returns and the expected return volatility. In this example, the correlation of conditional equity premium and return volatility with simulated productivity growth are, respectively, -0.60 and -0.77. In addition, the expected return volatility

displays persistence and clustering, where high volatility periods tend to cluster around relatively bad state beliefs.

To investigate the performance of the model with application to historical data, I compute equilibrium allocations and conditional moments of returns for the historical annual data on productivity. Figure 3 and 4 plot consumption and dividends endogenously determined in the model and compare the results with the data. Figure 3 shows that in most periods the level of consumption growth implied by the model is less than in the data. In addition, the model correctly captures the cyclical pattern in consumption growth data, and the contemporaneous correlation is high and equal to 0.75. Dividend growth rates generated from the model, however, are not highly correlated with the data (the correlation is 0.29). It is obvious from both figures that the model produces dividend growth much more volatile than consumption growth. This is consistent not only with the data but also with most of consumption-based asset pricing models with leverage ratios (see, for example, Abel (1999), Bansal and Yaron (2004) and Ju and Miao (2011), among others).

Turning to predicted equity premium and return volatility, Figure 5 presents the results for the historical data. Panel A plots total factor productivity growth rates and Hodrick-Prescott filtered consumption growth rates during the period 1951–2009. The predicted conditional equity premium and conditional return volatility for the sample period are plotted in Panel B and C. It reveals that the agent tends to seek more risk premium compensation during periods including 1970s, early 1980s and 1990s, and more recent recession times near 2008. The correlations of predicted equity premium and conditional return volatility with HP-filtered consumption growth are equal to -0.53 and -0.61 respectively. Furthermore, the equity premium predicted by the model has a very high positive correlation with the expected return volatility, which is equal to 0.90. Thus, the model provides theoretical support to the volatility feedback effect documented in the literature and implies a significant risk-return tradeoff.

#### 4.4 Return predictability

The empirical literature has extensively studied predictability of returns by considering various predictive regressions. See, for example, Campbell and Shiller (1988), Fama and French (1988a) and Fama and French (1988b) and Welch and Goyal (2008) for a comprehensive review. Campbell and Shiller (1988) derive the following log-linear relationship for the price-dividend ratio, dividend growth and returns:

$$p_t - d_t = \overline{p - d} + \sum_{j=0}^{\infty} \kappa^j \mathbb{E}_t(\Delta d_{t+1+j}) - \sum_{j=0}^{\infty} \kappa^j \mathbb{E}_t(r_{t+1+j})$$

where  $\overline{p - d}$  is the average price-dividend ratio in the long run, and  $|\kappa| < 1$  is a positive approximation constant. Empirical evidence suggests that the price-dividend ratio forecasts returns, especially at long horizons (Campbell and Shiller (1988), Fama and French (1988a) and Fama and French (1988b)). Moreover, as expected, high prices relative to dividends predict future low returns. The leftmost two columns labeled “Data (1951-2009)” in Table 5 present the results of regressing cumulative log returns on the current log price-dividend ratio for different horizons. Panel A and Panel B contain both the regression slopes and the  $R^2$ ’s, respectively, for equity returns and excess return being the dependent variable. The results confirm previous findings in that both the magnitude of the slope and  $R^2$  are increasing in the horizon.

The results generated from the simulation are presented in Table 5. I consider the benchmark calibration (Model I) as well as the model with Epstein-Zin recursive utility (Model III) for a comparison. To avoid small sample issues, which may bias the regression estimates and the  $R^2$ , I run predictive regressions on each simulated sample of the total 10,000 simulations. Table 5 reports the average values of the two statistics for Model I and Model III. Model III based on Epstein-Zin utility, unfortunately, is not able to generate significant predictability of returns. Results based on the benchmark calibration, on the other hand, suggest predictability of returns, with the  $R^2$  being

about 20% for different horizons. The implied predictability is much stronger than the finding of Ju and Miao (2011) in their consumption-based model with learning and ambiguity aversion. The slope has a negative sign in all cases, which implies that a high price today is associated with low returns in the future. By comparing Panel A and Panel B, it reveals that the predictability is not attributed to that of the risk-free rate because equity premium has even slightly better predictability than equity returns. But the model generates the counterfactual result that the  $R^2$  is declining in the horizon.

## 5 Conclusion

This paper explores how a representative agent's aversion toward model uncertainty is embedded in a production economy and affects asset prices. The agent learns about a hidden state that governs the mean productivity growth regimes, giving rise to model uncertainty and fluctuating beliefs. The agent's averse attitude is manifested through distorting Bayesian updated beliefs in a pessimistic way. This ambiguity aversion alters the time series evolution of the price of risk, making the agent's perception of aggregate risks more sensitive to productivity shocks. Thus, small perturbations of persistent state beliefs in response to productivity shocks could lead to significant changes in the price of risk and the equity premium in equilibrium. The calibration exercise shows that in addition to matching the unconditional moments of the risk-free rate and equity premium observed in the data, the model can generate countercyclical equity premia and expected volatility of equity premia, which implies the volatility feedback effect. Moreover, the model captures the volatility clustering and persistence behavior of returns.

This work can be extended in several ways. In this paper, I assume convex capital adjustment costs to produce volatile equity returns. Alternative specifications of adjustment costs such as the one proposed by Croce (2010) may also be investigated. Additionally, although this paper focuses on the one-sector model, models with multiple sectors could be developed to produce richer

implications of macroeconomic quantities including consumption, investment, output and human wealth.

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## 6 Appendix

### 6.1 The Social Planner's Problem

This section presents the social planner's consumption-saving problem. Labor supply is assumed to be inelastic. Define the following stationary variables:

$$\{c_t, i_t, y_t, k_t, v_t\} = \left\{ \frac{C_t}{A_{t-1}}, \frac{I_t}{A_{t-1}}, \frac{Y_t}{A_{t-1}}, \frac{K_t}{A_{t-1}}, \frac{U_t}{A_{t-1}} \right\}$$

The social planner's problem can be written as

$$v(k_t, \Delta a_t, \zeta_t) = \max_{c_t, k_{t+1}} \left\{ (1 - \beta) c_t^{1-\rho} + \beta e^{(1-\rho)\Delta a_t} \left( \mathbb{E}_{\zeta_t} \left[ \left( \mathbb{E}_{z_{t+1}, t} \left[ v_{t+1}^{1-\gamma}(k_{t+1}, \Delta a_{t+1}, \zeta_{t+1}) \right] \right)^{\frac{1-\eta}{1-\gamma}} \right] \right)^{\frac{1-\rho}{1-\eta}} \right\}^{\frac{1}{1-\rho}} \quad (17)$$

subject to the following constraints:

$$c_t + i_t = y_t \equiv e^{(1-\alpha)\Delta a_t} k_t^\alpha \bar{n}^{1-\alpha} \quad (18)$$

$$e^{\Delta a_t} k_{t+1} = (1 - \delta_k) k_t + i_t - \left| \frac{e^{\Delta a_t} k_{t+1}}{k_t} - \omega \right|^\iota k_t \quad (19)$$

$$\Delta a_t = \mu(s_t) + \sigma \epsilon_t, \quad \epsilon_t \sim N(0, 1) \quad (20)$$

$$c_t \geq 0, k_{t+1} \geq 0 \quad (21)$$

Ju and Miao (2011) show that the pricing kernel is given by

$$M_{s_{t+1}, t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{U_{t+1}}{\mathcal{R}_t(U_{t+1})} \right)^{\rho-\gamma} \left( \frac{\left( \mathbb{E}_{z_{t+1}, t} \left[ U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_t(U_{t+1})} \right)^{-(\eta-\gamma)}, \quad z_{t+1} = 1, 2$$

which is equivalent to

$$M_{s_{t+1}, t+1} = \beta e^{-\rho \Delta a t} \left( \frac{c_{t+1}}{c_t} \right)^{-\rho} \left( \frac{v_{t+1}}{\mathcal{R}_t(v_{t+1})} \right)^{\rho - \gamma} \left( \frac{\left( \mathbb{E}_{z_{t+1}, t} \left[ v_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_t(v_{t+1})} \right)^{-(\eta - \gamma)}, z_{t+1} = 1, 2$$

The conditional expectation operator  $\mathbb{E}_{z_{t+1}, t}$  is applied because the variable  $v_{t+1}^{1-\gamma}$  depends on the history of state variables up to time  $t$  and also on the period- $t + 1$  state  $z_{t+1}$ .

## 6.2 Value Function Iteration Algorithm

I first solve the full information model of Jahan-Parvar and Liu (2011) numerically, using value function iteration. Then the numerical solution to the value function in the full information model is used as an initial guess to solve the model with Bayesian updating in this paper. For completeness, I present the algorithm for the full information model and extends it to incorporate learning.

### *Full Information Model*

1. Compute the steady-state in the deterministic economy, assuming that the productivity growth rate is constant and equal to  $\Delta a_{ss}$  where  $\Delta a_{ss}$  is the stationary growth rate and given by

$$\begin{aligned} p_{ss} &= \frac{1 - p_{22}}{2 - p_{22} - p_{11}} \\ \Delta a_{ss} &= P_{ss} \mu_1 + (1 - P_{ss}) \mu_2 \end{aligned}$$

2. Discretization of the state space: I use (1)  $N_{\Delta a}$  equidistant points for  $\Delta a$  on the interval  $[\underline{\Delta a}, \overline{\Delta a}] = [\Delta a_{ss} - \underline{\lambda}_a \sigma, \Delta a_{ss} + \overline{\lambda}_a \sigma]$  where the constants  $\underline{\lambda}_a$  and  $\overline{\lambda}_a$  are set such that the interval of the grid is wide enough to embrace Gaussian quadrature nodes; (2)  $N_k$  equidistant points for  $k$  on the interval  $[\underline{k}, \overline{k}] = [0.1k_{ss}, 1.9k_{ss}]$  where  $k_{ss}$  is the value of capital at the deterministic steady state.
3. In the VFI algorithm, the value functions  $v(\Delta a, k, 1)$  and  $v(\Delta a, k, 2)$  are arrays of  $N_a \times N_k$ , where 1 and 2 denote productivity regimes.

4. To compute  $\mathbb{E}_{z,t} [v^{1-\gamma}(k_{t+1}, \Delta a_{t+1}, z_{t+1})]$ , note that

$$\begin{aligned} \mathbb{E}_z [v^{1-\gamma}(\Delta a', k', z')] &= p_{11} \mathbb{E}_{z'} [v^{1-\gamma}(\Delta a', k', z') \mid z' = 1] \\ &\quad + (1 - p_{11}) \mathbb{E}_{z'} [v^{1-\gamma}(\Delta a', k', z') \mid z' = 2] \end{aligned}$$

when  $z = 1$

$$\begin{aligned} \mathbb{E}_z [v^{1-\gamma}(\Delta a', k', z')] &= (1 - p_{22}) \mathbb{E}_{z'} [v^{1-\gamma}(\Delta a', k', z') \mid z' = 1] \\ &\quad + p_{22} \mathbb{E}_{z'} [v^{1-\gamma}(\Delta a', k', z') \mid z' = 2] \end{aligned}$$

when  $z = 2$

The conditional expectation  $\mathbb{E}_{z'} [v^{1-\gamma}(\Delta a', k', z') \mid z' = 1 \text{ or } 2]$  can be approximated using the Gauss-Hermite quadrature method. For example,

$$\mathbb{E} [v^{1-\gamma}(\Delta a', k', s') \mid s' = 1] = \int_{\underline{\Delta a}}^{\overline{\Delta a}} v^{1-\gamma}(\Delta a', k', s') \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\Delta a - \mu(1))^2}{2\sigma^2}} d\Delta a'$$

where  $k'$  is on the  $N_k$  grid. However, since quadrature nodes generally imply values of  $\Delta a'$  not exactly on the  $N_a$  grid, I need to interpolate  $v^{1-\gamma}(\Delta a', k', s')$  (the initial guess and each iteration only give the value of  $v(\Delta a, k, s)$  on the grid  $N_a \times N_k$ ). To do interpolation, I first construct Chebyshev polynomials in  $\Delta a$  and then for every  $k'$  I regress  $\log(v(:, k', 1))$  onto the Chebyshev polynomials and obtain Chebyshev regression coefficients. These coefficients allow me to interpolate  $\log(v(\Delta a', k', 1))$  for possible values of  $\Delta a'$ .

5. Interpolate expectation with respect to capital. In solving the consumption-saving problem, we need to choose the policy function  $k'$  to maximize the value function. The usual numerical optimization methods require the choice of  $k'$  that may not be on the  $N_k$  grid. For this reason, we need to evaluate the right-hand side of the recursion in (17) for every admissible value of

the policy function  $k'$ . In particular, we need to approximate the following two expectations

$$\mathbb{E}_{s'} [v^{1-\gamma} (\Delta a', k', s') \mid s' = 1]$$

$$\mathbb{E}_{s'} [v^{1-\gamma} (\Delta a', k', s') \mid s' = 2]$$

To achieve this task, I first create Chebyshev polynomials in  $k'$  on the  $N_k$  grid and then regress the two expectations onto the created Chebyshev polynomials. The regression coefficients can be used to interpolate the two expectations for every admissible  $k'$ .

6. Maximization and iteration. Given the states  $(\Delta a, k, s)$ , the algorithm first searches for  $k'$  on the  $N_k$  grid that maximizes the value function. To refine the optimal policy function, a numerical optimization procedure is employed to search for the optimal  $k'$ . The objective function is updated once an iteration is completed. The stopping rule is that the new value function and the old value function has a standard sup-norm  $\frac{|v' - v|}{|v|} < 1.e - 8$ .

### *The Model with Bayesian learning and ambiguity aversion*

When the representative agent cannot observe the hidden state, his belief about the economy is summarized by the state probability  $\zeta_t$ . The algorithm for the full information model is extended to the current setup by introducing the belief updating step. The algorithm begins in a similar way as Step 1, 2 and 3 in the full information case, with the exception that a discretized grid for  $\zeta$  is also created between 0 and 1.

The recursion requires the computation of the following expectation

$$\mathbb{E}_{\zeta_t} \left[ \left( \mathbb{E}_{z_{t+1}, t} \left[ v_{t+1}^{1-\gamma} (k_{t+1}, \Delta a_{t+1}, \zeta_{t+1}) \right] \right)^{\frac{1-\eta}{1-\gamma}} \right]$$

which can be rewritten as

$$\begin{aligned}
& \mathbb{E}_{\zeta_t} \left[ \left( \mathbb{E}_{z_{t+1}, t} \left[ v^{1-\gamma} (k_{t+1}, \Delta a_{t+1}, \zeta_{t+1}) \right] \right)^{\frac{1-\eta}{1-\gamma}} \right] \\
&= \zeta_t \left( \mathbb{E}_{z_{t+1}, t} \left[ v^{1-\gamma} (k_{t+1}, \Delta a_{t+1}, \zeta_{t+1}) | s_{t+1} = 1 \right] \right)^{\frac{1-\eta}{1-\gamma}} \\
&\quad + (1 - \zeta_t) \left( \mathbb{E}_{z_{t+1}, t} \left[ v^{1-\gamma} (k_{t+1}, \Delta a_{t+1}, \zeta_{t+1}) | z_{t+1} = 2 \right] \right)^{\frac{1-\eta}{1-\gamma}}
\end{aligned}$$

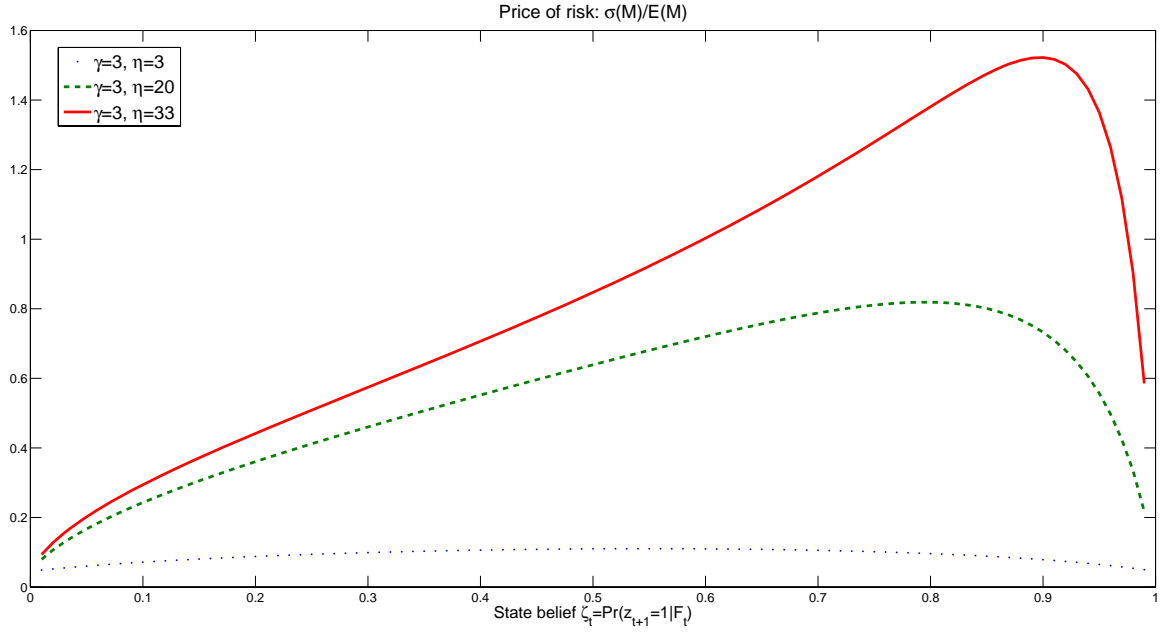
Note that the expectations on the right-hand side are invariant with respect to  $\Delta a$  (because  $\Delta a$  is integrated out). These two expectations can be approximated using Gauss-Hermite quadrature. In particular, the expectation conditioned on next period's regime being regime 1 is approximated in the following

$$\begin{aligned}
& \mathbb{E}_{z_{t+1}, t} \left[ v^{1-\gamma} (\zeta_{t+1}, \Delta a_{t+1}, k_{t+1}) | z_{t+1} = 1 \right] \\
&= \int_{\underline{\Delta a}}^{\overline{\Delta a}} v^{1-\gamma} (\zeta_{t+1}, \Delta a_{t+1}, k_{t+1}) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\Delta a_{t+1} - \mu(1))^2}{2\sigma^2}} d\Delta a_{t+1}
\end{aligned}$$

where  $\zeta_{t+1}$  is updated from  $\zeta_t$  by applying Bayes rule for each quadrature node  $\Delta a_{t+1}$ . To this end, I need to interpolate  $v(\zeta_{t+1}, \Delta a_{t+1}, k_{t+1})$  on possible values of  $\zeta_{t+1}$  and  $\Delta a_{t+1}$ . To do this task, I create a tensor product of Chebyshev polynomials in  $\zeta$  and  $\Delta a$  for their grid values. Then for every  $k_{t+1}$  on the grid I regress  $\log(v(\cdot, \cdot, k_{t+1}))$  onto the tensor product in  $\zeta$  and  $\Delta a$ , and obtain the Chebyshev regression coefficients. These coefficients are used to interpolate  $\log(v(\zeta_{t+1}, \Delta a_{t+1}, k_{t+1}))$  for any possible value of  $(\zeta_{t+1}, \Delta a_{t+1})$ . The interpolation of the expectations conditioned on regimes with respect to  $k$  proceeds in the same way as in the full information case.

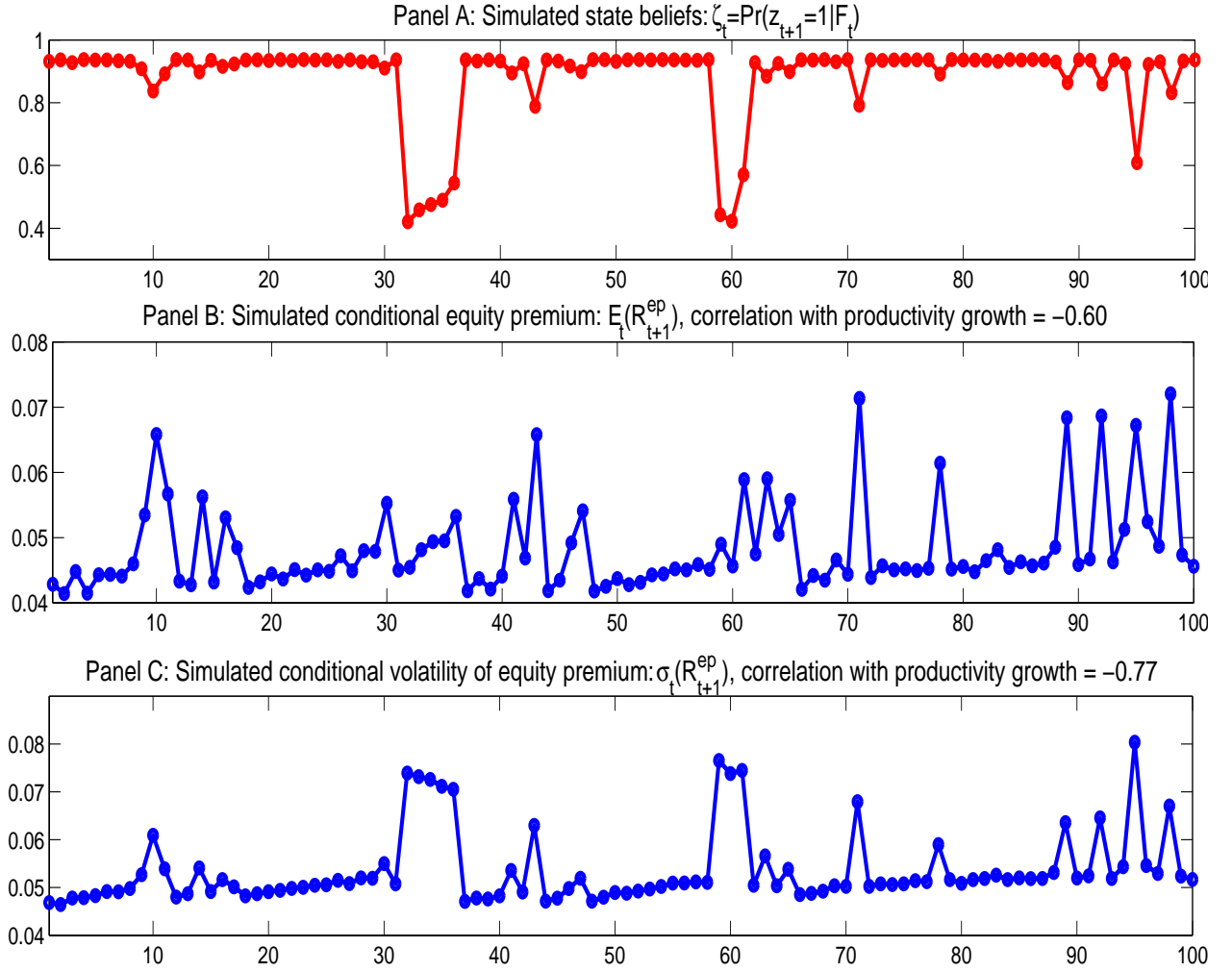
The VFI algorithm with three state variables takes a long time to achieve convergence. To improve convergence, I employ a multigrid scheme, following ????. I first solve the model on a small number of grids for  $k$ . Once the algorithm achieves convergence, it is extended to a larger number of grids for  $k$  by first interpolating the value from the previous step onto the current finer grid and then using the interpolated values as starting values for the new round of the algorithm. This procedure continues until the grid of  $k$  is fine enough and the algorithm finally converges.

Figure 1: Price of risk for different degrees of ambiguity aversion



This figure plots the price of risk, defined as  $\sigma_t(M)/\mathbb{E}_t(M)$ , against different state beliefs (denoted as  $\zeta_t$ ) ranging from 0 to 1. The pricing kernel,  $M$ , is defined in (13). In plotting the series, the capital stock is set to its mean in Monte Carlo simulations, and productivity growth is set to its steady-state level.

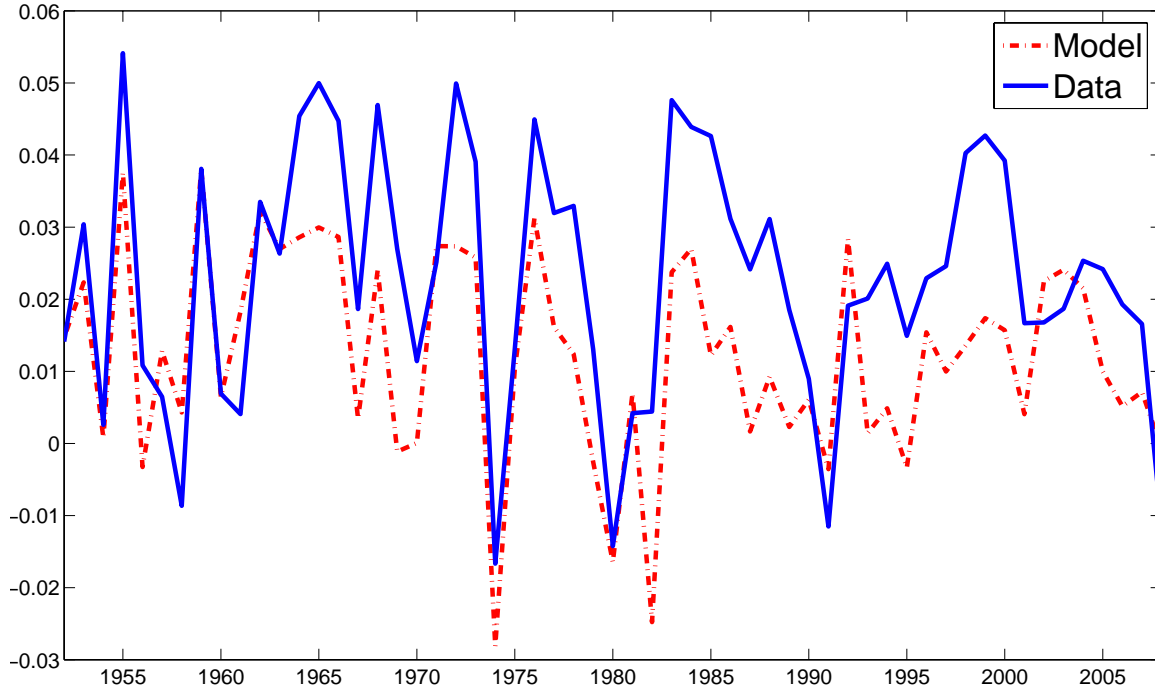
Figure 2: Simulated beliefs, equity premium and return volatility



This figure plots simulated state beliefs (Panel A), conditional mean equity premium (Panel B) and conditional expected volatility of equity premium (Panel C) for the benchmark calibration ( $\gamma = 3, \eta = 33, \psi = 1.5$ ). To simulate beliefs, the prior belief  $\zeta_0$  is set at the steady state level  $(1 - p_{22})/(2 - p_{22} - p_{11})$ , and artificial productivity growth rates are simulated from the Markov-switching process (8). Posterior beliefs are updated according to Bayes rule. Conditional mean equity premium and conditional expected return volatility are calculated as functions of the state variables. Conditional moments are approximated using Gauss-Hermite quadrature method.

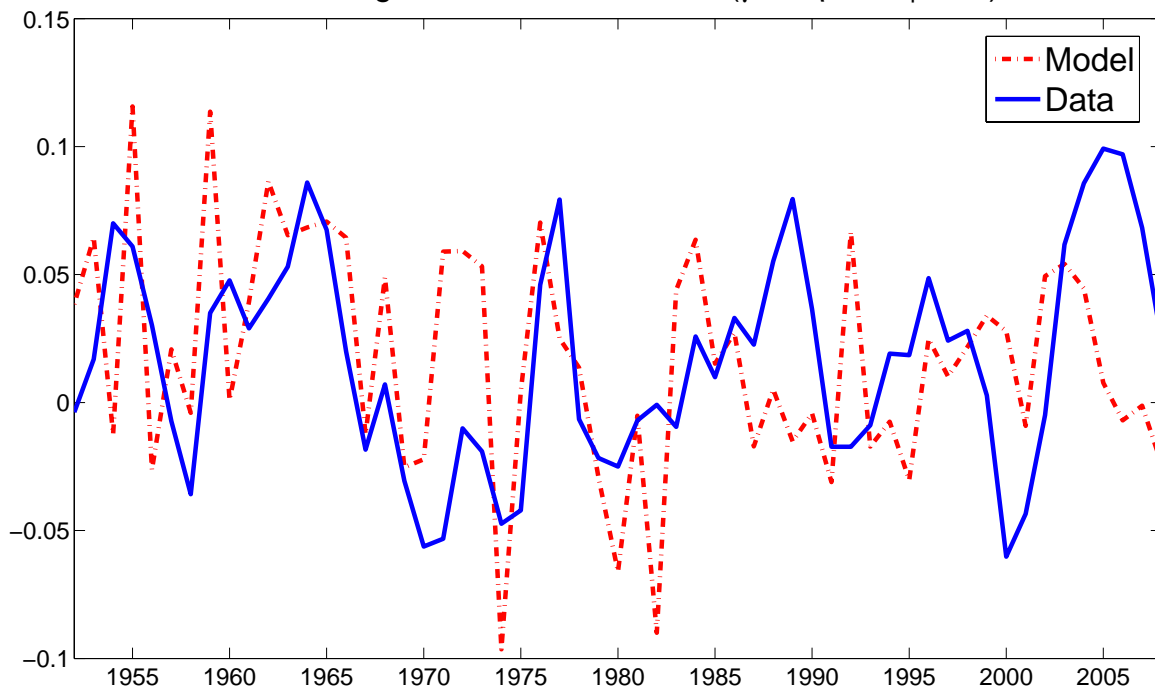


Figure 3: Consumption growth: data and model  
Consumption growth: data and model ( $\gamma=3$ ,  $\eta=33$ ,  $\psi=1.5$ )



This figure plots the historical data on real consumption growth rates and consumption growth rates predicted by the benchmark model (Model I in Table 3) for the period 1951–2009. The correlation between real consumption growth data and predicted consumption growth is 0.75.

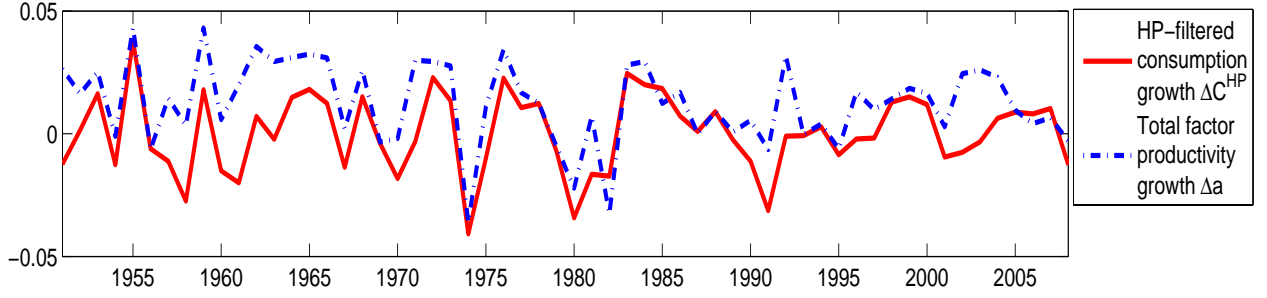
Figure 4: Dividend growth: data and model  
 Dividend growth: data and model ( $\gamma=3$ ,  $\eta=33$ ,  $\psi=1.5$ )



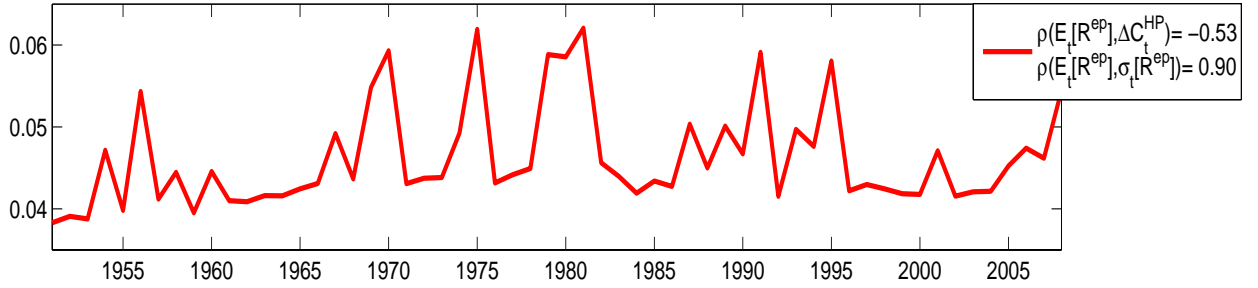
This figure plots the historical data on real dividend growth rates and dividend growth rates predicted by the benchmark model (Model I in Table 3) for the period 1951–2009. The correlation between real dividend growth data and predicted dividend growth is 0.29.

Figure 5: Predicted equity premium and return volatility

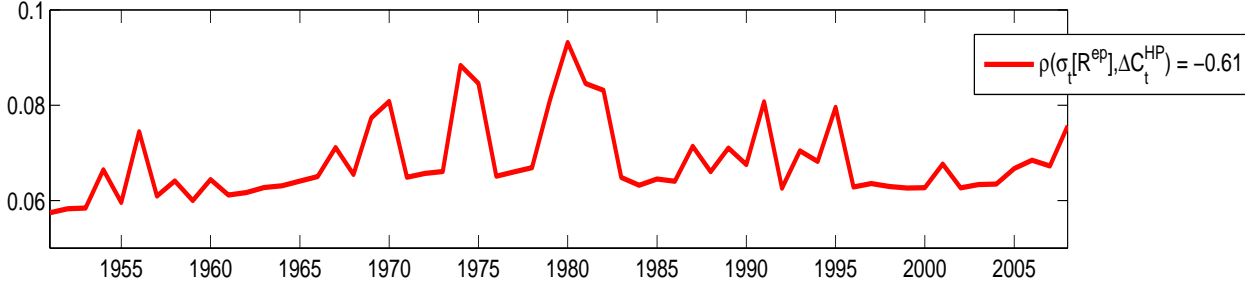
Panel A: Historical consumption growth and productivity growth



Panel B: Predicted conditional equity premium



Panel C: Predicted conditional volatility of equity premium



Panel A plots Hodrick-Prescott filtered consumption growth rates and total factor productivity growth rates for the period 1951–2009. Panel B and Panel C display, respectively, conditional mean equity premium ( $\mathbb{E}_t(R_{t+1}^e - R_t^f)$ ) and conditional expected return volatility ( $\sigma_t(R_{t+1}^e - R_t^f)$ ) predicted by the benchmark model (Model I in Table 3). The prior belief  $\zeta_0$  is set at the steady state level. Posterior beliefs are updated according to Bayes rule given the historical productivity data. Conditional mean equity premium and conditional expected return volatility are calculated as functions of the state variables. Conditional moments are approximated using Gauss-Hermite quadrature method. The correlations of  $\mathbb{E}_t(R_{t+1}^e - R_t^f)$  and  $\sigma_t(R_{t+1}^e - R_t^f)$  with the historical HP-filtered consumption growth are equal to, respectively, -0.53 and -0.61. The correlation between  $\mathbb{E}_t(R_{t+1}^e - R_t^f)$  and  $\sigma_t(R_{t+1}^e - R_t^f)$  is 0.90.

Table 1: **Maximum likelihood estimates of the productivity process**

Parameter	Description	Estimate
$\mu_1$	Mean growth rate (regime 1)	0.0178
$\mu_2$	Mean growth rate (regime 2)	-0.0234
$\sigma$	Standard deviation	0.016755
$p_{11}$	Transition probability (regime 1 to regime 1)	0.937
$p_{22}$	Transition probability (regime 2 to regime 2)	0.582

This table reports the maximum likelihood estimates of the parameters in the Markov switching model for productivity. The estimates are obtained using Hamilton's expectation maximization (EM) algorithm. The data for estimation are quarterly total factor productivity growth rates for 1951–2009.

Table 2: **Parameter values of model parameters**

Parameter	Description	Value
$\delta$	Depreciation rate of capital	0.06
$\alpha$	Capital share	0.35
$\bar{N}$	Exogenous labor input	0.20
$\iota$	Exponent of adjustment costs function	1.09
$\gamma$	Coefficient of risk aversion	3
$\beta$	Time discount parameter	0.975

Table 3: **Unconditional moments and comparative statics**

		Model I*	Model II	Model III
		$\gamma = 3$	$\gamma = 3$	$\gamma = 3$
	U.S. data	$\eta = 33$	$\eta = 20$	$\eta = 3$
Statistic	1951–2009	$\psi = 1.5$	$\psi = 1.5$	$\psi = 1.5$
Panel A: Moments of consumption growth				
$\sigma_{\Delta C}$	1.80	1.77	1.78	1.82
$Corr(\Delta C_{t+1}, \Delta C_t)$	0.22	0.22	0.22	0.22
Panel B: Financial moments				
$\mathbb{E}(R^f)$	1.98	1.92	2.66	3.30
$\sigma(R^f)$	2.46	0.50	0.59	0.42
$\mathbb{E}(R^{ep})$	5.90	5.88	3.83	1.18
$\sigma(R^{ep})$	16.50	16.96	16.63	15.15
Sharpe ratio	0.36	0.35	0.23	0.08
$\sigma(M)/\mathbb{E}(M)$	n.a.	1.39	0.65	0.07

This table reports the key moments for the calibrated model with different degrees of ambiguity aversion. Model I is the benchmark calibration. The ambiguity aversion parameter  $\eta$  is set to 20 and 3, respectively, in Model II and III. The relative risk aversion parameter  $\gamma$  is set to 3 across all the models. Other parameters are given in Table 1 and Table 2. The equity claim is an unlevered claim on aggregate dividends. The calibration is in the annual frequency. The statistics for Model I–III are calculated based on 10,000 Monte Carlo simulations, after equilibrium allocations are solved using value function iteration. The macroeconomic moments reported include: (1) the standard deviation of consumption growth  $\sigma_{\Delta C}$  and (2) the 1st autocorrelation in consumption growth. The financial moments reported include: (1) the mean risk-free rate  $\mathbb{E}(R^f)$ , (2) the standard deviation of the risk-free rate  $\sigma(R^f)$ , (3) the mean equity premium  $\mathbb{E}(R^{ep})$ , defined by  $\mathbb{E}(R_{t+1}^e - R_t^f)$ , (4) the standard deviation of equity premium  $\sigma(R^{ep})$ , defined by  $\sigma(R_{t+1}^e - R_t^f)$ , (5) Sharpe ratio, defined by  $\mathbb{E}(R^{ep})/\sigma(R^{ep})$ , and (6) the price of risk, defined by  $\sigma(M)/\mathbb{E}(M)$ . The statistics  $\sigma_{\Delta C}$ ,  $\mathbb{E}(R^f)$ ,  $\sigma(R^f)$ ,  $\mathbb{E}(R^{ep})$  and  $\sigma(R^{ep})$  are in percentage.

Table 4: **Equity premium: the effect of Bayesian learning**

	$\gamma = 3$	$\gamma = 3$	$\gamma = 3$
	$\eta = 33$	$\eta = 20$	$\eta = 3$
	$\psi = 1.5$	$\psi = 1.5$	$\psi = 1.5$
Panel A: Unconditional mean of risk-free rate $\mathbb{E}(R^f)$			
Bayesian learning	1.92	2.66	3.30
Full information	1.57	2.40	3.30
Panel B: Unconditional mean of equity premium $\mathbb{E}(R^{ep})$			
Bayesian learning	5.88	3.83	1.18
Full information	6.48	4.94	1.44

Panel A and Panel B report, respectively, the mean risk-free rate and the mean equity premium calculated from the model in this paper and that in the full information model of Jahan-Parvar and Liu (2011). The results are for those sets of the preference parameters in Table 3.

Table 5: **Long-horizon predictability: calibration results**

		Data (1951–2009)		Model I $\gamma = 3, \eta = 33$ $\psi = 1.5$		Model III $\gamma = 3, \eta = 3$ $\psi = 1.5$	
Horizon(s)		Slope	$R^2$	Slope	$R^2$	Slope	$R^2$
Panel A: $r_{t,t+s}^e = a + b(p_t - d_t) + e_{t+s}$							
1 year		-0.106	0.070	-0.472	0.216	-0.295	0.120
2 years		-0.196	0.118	-0.493	0.205	-0.294	0.101
3 years		-0.282	0.177	-0.510	0.197	-0.290	0.087
5 years		-0.415	0.218	-0.538	0.184	-0.276	0.068
7 years		-0.610	0.316	-0.560	0.174	-0.261	0.056
Panel B: $r_{t,t+s}^e - r_{t,t+s}^f = a + b(p_t - d_t) + e_{t+s}$							
1 year		-0.098	0.059	-0.486	0.220	-0.306	0.120
2 years		-0.172	0.099	-0.520	0.213	-0.316	0.103
3 years		-0.241	0.146	-0.549	0.208	-0.322	0.090
5 years		-0.333	0.172	-0.598	0.200	-0.330	0.073
7 years		-0.484	0.261	-0.638	0.192	-0.335	0.062

This table reports predictive regression results for the benchmark calibration (Model I) and Model II (Epstein-Zin recursive utility). The table presents the slope coefficients and the  $R^2$ s estimated from regressing equity returns and excess returns (both in log terms) onto the end-of-year log price-dividend ratio. Results are obtained by taking the average of the OLS estimates in 10,000 simulations. The horizon of returns includes 1 year, 2, 3, 5 and 7 years. The empirical estimates for the historical data (1951–2009) are also presented.