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Managing portfolio risk using multivariate extreme value methods

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Abstract

This paper provides a strategy for portfolio risk management by inferring extreme movements in financial markets. The core of the provided strategy is a statistical model for the joint tail distribution that attempts to capture accurately the data generating process through an extremal modelling for the univariate margins and the multivariate dependence structure. It takes into account the asymmetric behavior of extreme negative and positive returns, the heterogeneous temporal and cross-sectional lead-lag extremal dependencies among the portfolio constituents. The strategy facilitates scenario generation for future returns, estimation of portfolio profit-and-loss distribution and calculation of risk measures, and hence, enabling us to answer several questions of economic interest. We illustrate the usefulness of our proposal by an application to stock market returns for the G5 economies.

JEL classification: C14; C15; C32

Keywords: ARMA-GARCH filtering; Asymptotic dependence; Asymptotic independence; Copula; Multivariate extreme values

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1. Introduction

Financial returns are in general serially uncorrelated although they are not independent. They exhibit pronounced volatility clustering which manifests itself through the significant autocorrelations of the squared and absolute returns. The unconditional distributions of financial returns are typically leptokurtic and heavy-tailed (see e.g., Cont, 2001; Mikosch, 2001). Financial returns across assets and markets tend to move together during stress periods, a phenomenon that has attracted a lot of attention in the empirical research (see e.g., Longin & Solnik, 2001; Hartmann et al., 2004). The latter two features naturally introduce extreme value theory (EVT) as a modelling tool for financial returns.

Thus to provide a strategy that is well-suited for portfolio risk management, one needs to represent as realistically as possible the data generating process of the corresponding return vector \mathbf{R}_t by accounting for the empirical properties of financial returns and taking into account that extreme events (represented by potential losses or substantial gains) typically constitute the primary interest. To this end, we adopt a strategy that integrates three main aspects: return filtering, marginal modelling and dependence modelling. While the econometric and statistical tools are well-developed to handling the first two aspects, dealing with the third aspect (representing the key aspect of our strategy) is far from being a trivial exercise due to the complex dependence structure and high dimensionality of financial applications. In what follows, we briefly outline each aspect.

The temporal dependence and volatility clustering of financial returns preclude the direct application of statistical methods that require independent and identically distributed (IID) observations and make filtering financial returns a necessary first modelling step. We use the ARMA-GARCH processes to obtain the filtered return vector \mathbf{X}_t on which the subsequent analysis is based upon (see e.g., Weiss, 1984; Li et al., 2002).

Since we are specifically interested in extreme (negative or positive) returns, we need an extreme value model so that inferences can be made independently of the distribution of the entire data and hence avoiding any bias that might be induced from the use of non-extreme

values. For the marginal distribution, we make use of the generalized Pareto distribution (GPD) with separate models for the lower and upper tails aiming at capturing the asymmetric behavior of extreme negative and positive returns often found in financial data (see Chapter 6 in Embrechts et al., 1997). Next, in order to disentangle the marginal and dependence aspects, we marginally standardize (using the probability integral transform) the filtered return vector \mathbf{X}_t to have a common distribution and denote the resulting vector by \mathbf{Y}_t . We aim to model the (extremal) dependence structure of \mathbf{Y}_t .

The tail dependence coefficients (TDCs) - introduced by Sibuya (1960) - provide natural measures to quantify the amount of dependence in the tail regions. They play a significant role in the finance literature to assess the tendency of large negative (for the lower TDC) and large positive (for the upper TDC) returns on two financial assets to occur simultaneously.¹

For two continuous random variables Y_1 and Y_2 with a common marginal distribution F_Y , we define the lower and upper tail dependence coefficients respectively as

$$\lambda^- = \lim_{q \rightarrow 0} \Pr[Y_2 < F_Y^{-1}(q) | Y_1 < F_Y^{-1}(q)] \quad \text{and} \quad \lambda^+ = \lim_{q \rightarrow 1} \Pr[Y_2 > F_Y^{-1}(q) | Y_1 > F_Y^{-1}(q)], \quad (1)$$

provided the limits exist.² Here $F_Y^{-1}(q) = \text{VaR}_q$ is the well-known quantile risk measure Value-at-Risk. If $\lambda^\dagger \neq 0$ (respectively $\lambda^\dagger = 0$) then Y_1 and Y_2 are said to be asymptotically dependent (respectively asymptotically independent) in the lower tail region (for $\dagger = -$) or in the upper tail region (for $\dagger = +$).

In many multivariate statistical modelling approaches, asymptotic dependence is traditionally assumed once independence of the variables is rejected. However, it is important to note that the resulting multivariate model will overestimate the probability of joint occurrence of extreme events if the variables of interest are asymptotically independent (see

¹The concept of TDC has been extended for lagged dependencies within a time series in Ledford & Tawn (2003) and between two time series in Zhang & Huang (2006).

²In Section 3.5 we introduce a finite-level multivariate version of the lower and upper TDCs.

Ledford & Tawn, 1996; Coles et al., 1999; Poon et al., 2004). Thus it is important to adopt a model that is able to accommodate both asymptotic dependence and asymptotic independence.

The adopted model must also account for the asymmetric behavior in the extremal dependence between financial return series which often takes the form of a stronger dependence during market downturns than during market upturns (see e.g., Longin & Solnik, 2001; Martens & Poon, 2001). Moreover, the model must capture the different extremal dependencies among the portfolio constituent assets. Thus we require that the temporal extremal dependence within each individual series as well as the cross-sectional contemporaneous and lagged extremal dependencies over several consecutive days (representing the horizon of interest) to be taken into account. Therefore, for any two assets i and j in the investment dataset, the model characterizes the extremal dependence between $Y_{i,t}$ and $Y_{j,t+\ell}$ where ℓ is a time lag. Furthermore, the model should facilitate extrapolation beyond the observations.

We are not aware of a statistical model fulfilling all the above demands. As a baseline to develop such a statistical model, we use the conditional multivariate extreme value approach of Heffernan & Tawn (2004) since it offers the flexibility of modelling both types of extremal dependence (asymptotic dependence and asymptotic independence) that might arise and its statistical inference is straightforward.

Our approach is a generalization of McNeil & Frey (2000) who propose an extreme value approach for estimating the tail-related risk measures. While their approach is basically univariate, ours is truly multivariate. Our work is also closely related to that of Zhang & Huang (2006) who propose a model that combines a three-state Markov chain controlling the changes in return signs with two separate M4 processes (see Smith & Weissman, 1996 for details) representing the dynamics of negative and positive returns. Despite the flexibility of their proposal, its application is restricted to the situation when the variables of interest are asymptotically dependent. Nyström & Skoglund (2002) present a multivariate approach

that differs from ours as the full dependence structure is modelled by the copula function.³ Although copulas are widely used tools, there is generally no clear approach for specifying the appropriate copula in practice. Furthermore, dependence in the main body of the data may differ from that of extreme events and most of the copulas used in practice are not consistent with multivariate EVT (see e.g., Bouye, 2002). For example, the normal copula is a model of tail independence and considered as a benchmark model in risk management. However, the normality assumption for the dependence structure of financial data is often rejected due to the simultaneous drawdown across assets observed in stress periods (see e.g., Mashal & Zeevi, 2002; Breyman et al., 2003). The Student's t-copula accounts for this phenomenon by allowing tail dependence between the variables of interest. A comparison between these two copulas with our strategy will be given in Section 4.

To illustrate our strategy empirically, we apply it for a stock portfolio diversified among the G5 economies. Specifically, we utilize daily returns measured at closing times on five international stock market indices, namely, S&P 500, FTSE 100, CAC 40, DAX 30 and NIKKEI 225. Selected pairwise scatter plots of the return series are displayed in Figure 1, showing different degrees and types of extremal dependence. While the French-Germany stock market pair exhibits persistent extremal dependence in the sense that simultaneous large negative/positive returns are realized, this behavior is much weaker for the US-Japan stock market pair.

³A d -dimensional copula is a function $\mathcal{C} : [0, 1]^d \rightarrow [0, 1]$ with standard uniform margins. The existence of such a function is due to the fundamental theorem of Sklar (1959). Let \mathbf{X} be a d -dimensional random vector with joint distribution $F_{\mathbf{X}}$ and continuous marginal distributions F_{X_i} for $i = 1, \dots, d$. There exists a unique d -dimensional copula \mathcal{C} such that for every $\mathbf{x} \in \mathbb{R}^d$ we have

$$F_{\mathbf{X}}(x_1, \dots, x_d) = \mathcal{C}(F_{X_1}(x_1), \dots, F_{X_d}(x_d)).$$

Conversely, for every $\mathbf{u} \in [0, 1]^d$ we have

$$\mathcal{C}(u_1, \dots, u_d) = F_{\mathbf{X}}(F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)).$$

Thus the copula \mathcal{C} represents the dependence structure of \mathbf{X} . See Nelsen (1999) for details and Embrechts et al. (2003) regarding financial applications.

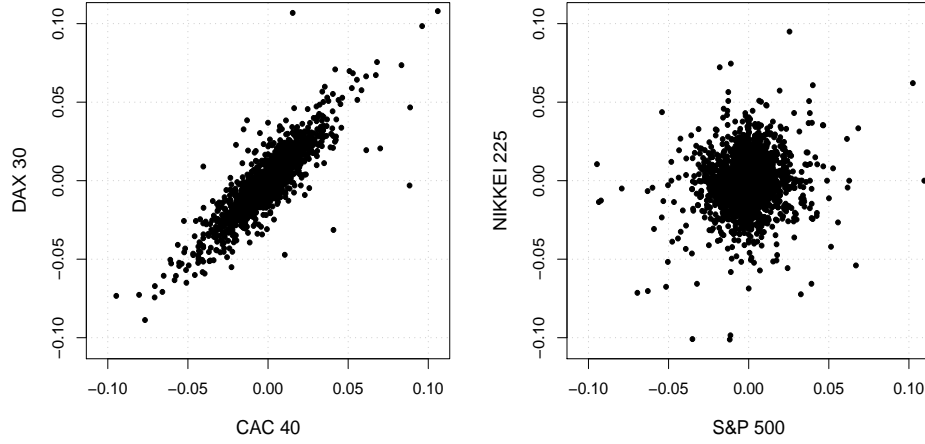


Figure 1: Selected scatter plots for daily returns of the G5 stock market indices spanning the period from January 04, 2000 to December 31, 2009.

The remainder of the paper is organized as follows. Section 2 describes the strategy adopted in the study. The subsequent section deals with the empirical analysis. More specifically, Section 3.1 introduces the data to be analyzed and provides descriptive statistics. Section 3.2 addresses the filtering step of the investigated return series, while the marginal modelling step is addressed in Section 3.3. The (extremal) dependence modelling is the subject of Section 3.4. Section 3.5 ends the study with a numerical example concerning risk assessment for an international portfolio as a demonstration for the practical relevance and the economic significance of our approach. Section 4 compares the proposed strategy with some typical strategies in risk management. Section 5 summarizes the study findings and provides several possible extensions to the current work.

2. Strategy

Consider a sample of n observations of returns on d financial assets, say $\mathbf{R}_t = (R_{1,t}, \dots, R_{d,t})$ for $t = 1, \dots, n$. The following subsections describe the three aspects of our strategy for representing the data generating process of \mathbf{R}_t .

2.1. Return filtering

To produce approximately IID return observations, we filter each return series under consideration by the ARMA-GARCH processes. For the analyzed data, the AIC criterion suggests using the ARMA(1,1)-GARCH(2,1) model. Specifically, we model the dynamics of a generic random variable R_t representing stock market returns as follows

$$\begin{aligned} R_t &= \mu_t + \sigma_t \cdot X_t, \\ \mu_t &= \mu + \psi \cdot R_{t-1} + \theta \cdot \varepsilon_{t-1}, \\ \sigma_t^2 &= \omega + \alpha_1 \cdot \varepsilon_{t-1}^2 + \alpha_2 \cdot \varepsilon_{t-2}^2 + \beta \cdot \sigma_{t-1}^2, \\ \varepsilon_t &= R_t - \mu_t, \\ X_t &\sim \mathcal{N}(0, 1). \end{aligned}$$

For stationarity and invertibility of the conditional mean μ_t , we impose the constraints $|\psi| < 1$ and $|\theta| < 1$. Moreover, $\omega > 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\beta \geq 0$ and $\alpha_1 + \alpha_2 + \beta < 1$ are assumed to ensure positivity and stationarity of the conditional variance σ_t^2 . Model parameters are estimated jointly via quasi-maximum likelihood method to obtain a sequence of estimates $\{(\hat{\mu}_t, \hat{\sigma}_t), t = 1, \dots, n\}$. The corresponding filtered returns are extracted as

$$X_t = \frac{R_t - \hat{\mu}_t}{\hat{\sigma}_t},$$

with X_t are assumed to be IID over t .

2.2. Marginal modelling

2.2.1. Structure

Let X be a generic random variable with the same distribution as X_t and denote its distribution by F_X . The intention is to approximate the unknown marginal distribution F_X , while bearing in mind that the tails of the distribution constitute the primary interest. It is natural then to adopt a semi-parametric approach in which only the tails of the distribution

are modelled parametrically. More precisely, we use the generalized Pareto distribution for both the lower and upper tails of F_X and consider the empirical distribution in the central part of F_X . This representation of the marginal model was proposed by McNeil & Frey (2000) and independently by Danielsson & de Vries (1997) but specific to heavy-tailed data.

Let u_X^- be an appropriate low threshold that indicates the beginning of the lower tail of F_X such that the excesses $\{(u_X^- - X_t) : X_t < u_X^-\}$ are approximately distributed according to the GPD with scale and shape parameters $\varphi^- > 0$ and $\xi^- \in \mathbb{R}$, respectively. Similarly, let u_X^+ be a high threshold that indicates the beginning of the upper tail of F_X such that the excesses $\{(X_t - u_X^+) : X_t > u_X^+\}$ are approximately distributed according to the GPD with scale and shape parameters $\varphi^+ > 0$ and $\xi^+ \in \mathbb{R}$, respectively. Therefore, the adopted marginal model is given by

$$\hat{F}_X(x) = \begin{cases} \tilde{F}_X(x) & \text{if } x \in [u_X^-, u_X^+], \\ \tilde{F}_X(u_X^-) [1 + \xi^-(u_X^- - x)/\varphi^-]_+^{-1/\xi^-} & \text{if } x < u_X^-, \\ 1 - \tilde{F}_X(u_X^+) [1 + \xi^+(x - u_X^+)/\varphi^+]_+^{-1/\xi^+} & \text{if } x > u_X^+, \end{cases} \quad (2)$$

where $\tilde{F}_X(x) = \sum_{t=1}^n \mathbb{I}(X_t \leq x) / n$ is the empirical distribution function of the variable X and $\tilde{\bar{F}}_X(x) = 1 - \tilde{F}_X(x)$ denotes the corresponding survivor function.

2.2.2. Estimation

There exists a wide variety of statistical methods to estimate the GPD parameters, though the most widely used method is the maximum likelihood. As shown by Smith (1985), provided that $\xi > -0.5$, the familiar regularity conditions hold so that the maximum likelihood estimators possess the desirable asymptotic properties of consistency, normality and efficiency. Since the case $\xi \leq -0.5$ rarely occurs in statistical applications, the method of maximum likelihood provides an attractive choice for estimating the GPD parameters.

The marginal threshold u_X^\dagger for $\dagger \in \{-, +\}$ is chosen at the lowest possible level for

efficient estimation of the GPD parameters such that statistical inferences are insensitive to any increase of threshold level. Several diagnostic methods are available to aid marginal threshold selection (see e.g., Davison & Smith, 1990; Embrechts et al., 1997; Coles, 2001; Choulakian & Stephens, 2001). Here we primarily follow the latter diagnostic method by using the Anderson-Darling statistic.

2.2.3. Standardization

Finally we transform the marginal distribution of the original vector \mathbf{X} to another vector \mathbf{Y} having a common marginal distribution F_Y . Following Hilal et al. (2011), we transform our variables to have approximately standard Laplace margins. Thus we define

$$Y = F_Y^{-1} [\hat{F}_X(X)] = \begin{cases} \log [2\hat{F}_X(X)] & \hat{F}_X(X) \leq 0.5, \\ -\log [2(1 - \hat{F}_X(X))] & \hat{F}_X(X) > 0.5. \end{cases} \quad (3)$$

Now define the following random variable:

$$\dagger Y_t = \begin{cases} (-1) \times Y_t & \text{if } \dagger = -, \\ (+1) \times Y_t & \text{if } \dagger = +. \end{cases}$$

Note that both variables $-Y_t$ and $+Y_t$ have the same marginal distribution due to the symmetry property of Laplace distribution. It is also important to note that the dependence structure (copula) of the original vector \mathbf{X} is preserved under the above transformation due to the invariance property of the copula function under strictly increasing transformations of the margins. The focus in the remainder of this section is on characterizing and modelling the extremal dependence structure of the transformed vector \mathbf{Y} .

2.3. Dependence modelling

2.3.1. Formulation

Let $\{\mathbf{Y}_t = (Y_{1,t}, \dots, Y_{d,t}) : t = 1, \dots, n\}$ be the sample of transformed (on Laplace scale) filtered returns on the d financial assets under study, of which (\mathbf{Y}_t) constitutes a strictly stationary process. Then, for some horizon H , we would like to develop a model that allows us to capture the extremal dependence structure of the system $(\mathbf{Y}_{t-H+1}, \dots, \mathbf{Y}_t, \dots, \mathbf{Y}_{t+H-1})$. However, it might be reasonable to conjecture that extreme return fluctuations are short-lived events in the sense that their impact on future returns will not last for a long time (see Chapter 4 in McNeil et al., 2005). To substantiate this conjecture, we set $H = 2$, that is, the developed model intends to capture the dependence structure of the reduced system $(\mathbf{Y}_{t-1}, \mathbf{Y}_t, \mathbf{Y}_{t+1})$ whenever one of its variables experiencing an extreme (large negative or large positive) observation.

Our aim could be stated more formally as follows. Let $\ell \in \{-1, 0, 1\}$ be a time lag. We want to capture the dependence structure of the variables $Y_{i,t}$ and $Y_{j,t+\ell}$ when the former is extreme (large in modulus). Note that only the time lag ℓ matters rather than the actual time t due to the stationarity of the process (\mathbf{Y}_t) . Moreover, we want to allow for asymmetric behavior in this dependence structure by differentiating between the cases when the conditioning variable $Y_{i,t}$ is negative and when it is positive.

We would like to formulate a statistical model that identifies the behavior of quantiles of the distribution of $Y_{j,t+\ell} | \dagger Y_{i,t} = y$ for large y . We denote such a *conditional extremal dependence model* by $\mathbf{M}_{j|i}^{(\ell)\dagger}$, while the collection of all these models comprises what we refer to as the *dependence model*.

To develop the model $\mathbf{M}_{j|i}^{(\ell)\dagger}$, we exploit the conditional approach of Heffernan & Tawn (2004) for Laplace transformed variables. For tail region \dagger and component i denote by Δ_i^\dagger all possible combinations $(j, \ell) \in \{\{1, \dots, d\} \times \{-1, 0, 1\}\} \setminus \{(i, 0)\}$. We then make the modelling assumption that there exist normalizing constants $\alpha_{j|i}^{(\ell)\dagger} \in [-1, 1]$ and $\beta_{j|i}^{(\ell)\dagger} \in$

$(-\infty, 1)$ such that for all fixed $\mathbf{z} = (z_{j|i}^{(\ell)\dagger}) \in \mathbb{R}^{3d-1}$ and $y > 0$ the following condition holds:

$$\lim_{u \rightarrow \infty} \Pr \left[\dagger Y_{i,t} - u > y, \frac{Y_{j,t+\ell} - \alpha_{j|i}^{(\ell)\dagger} \dagger Y_{i,t}}{\dagger Y_{i,t}^{\beta_{j|i}^{(\ell)\dagger}}} \leq z_{j|i}^{(\ell)\dagger} : (j, \ell) \in \Delta_i^\dagger \middle| \dagger Y_{i,t} > u \right] = \exp(-y) G_{j|i}^\dagger(\mathbf{z}), \quad (4)$$

where the joint distribution $G_{j|i}^\dagger$ has non-degenerate marginal distributions $G_{j|i}^{(\ell)\dagger}$ satisfying the property $\lim_{z \rightarrow \infty} G_{j|i}^{(\ell)\dagger}(z) = 1$.⁴ A direct consequence of this condition, given that $\dagger Y_{i,t} > u$ as $u \rightarrow \infty$, is the independence property of the variables $\dagger Y_{i,t} - u$ and

$$Z_{j|i,t}^{(\ell)\dagger} = \frac{Y_{j,t+\ell} - \alpha_{j|i}^{(\ell)\dagger} \dagger Y_{i,t}}{\dagger Y_{i,t}^{\beta_{j|i}^{(\ell)\dagger}}},$$

for each $(j, \ell) \in \Delta_i^\dagger$ with limiting distributions being standard exponential for the former and $G_{j|i}^{(\ell)\dagger}$ for the latter.

Thus our statistical model $\mathbf{M}_{j|i}^{(\ell)\dagger}$ is motivated by the assumption that condition (4) holds exactly and that the associated independence property is satisfied over $u \geq u_{j|i}^{(\ell)\dagger}$. Denote by $\dagger Y_{i(t)}$ for $t = 1, \dots, n_{j|i}^{(\ell)\dagger}$ the observations of the variable $\dagger Y_{i,t}$ that exceed the threshold $u_{j|i}^{(\ell)\dagger}$ and denote by $Y_{j(t+\ell)}$ and $Z_{j|i(t)}^{(\ell)\dagger}$ the corresponding observations of the variables $Y_{j,t+\ell}$ and $Z_{j|i,t}^{(\ell)\dagger}$ respectively. The model $\mathbf{M}_{j|i}^{(\ell)\dagger}$ is then assumed to be well-represented by the following semi-parametric regression model:.

$$\begin{aligned} Y_{j(t+\ell)} &= \alpha_{j|i}^{(\ell)\dagger} \dagger Y_{i(t)} + \dagger Y_{i(t)}^{\beta_{j|i}^{(\ell)\dagger}} Z_{j|i(t)}^{(\ell)\dagger}, \\ Z_{j|i(t)}^{(\ell)\dagger} &\stackrel{\text{iid}}{\sim} G_{j|i}^{(\ell)\dagger} \quad \text{and} \quad \dagger Y_{i(t)} \text{ independent of } Z_{j|i(t)}^{(\ell)\dagger}. \end{aligned} \quad (5)$$

Finally, the dependence structure of the variable $Y_{j,t+\ell}$ and extreme $\dagger Y_{i,t}$ that is im-

⁴The formulation of the limiting condition (4) is slightly different from the one given in Heffernan & Tawn (2004).

plied by the model $\mathbf{M}_{j|i}^{(\ell)\dagger}$ can be categorized into either asymptotic dependence class or asymptotic independence class including the three sub-classes of the latter by the parameters $(\alpha, \beta)_{j|i}^{(\ell)\dagger} = (\alpha_{j|i}^{(\ell)\dagger}, \beta_{j|i}^{(\ell)\dagger})$. More precisely, the variables $(\dagger Y_{i,t}, Y_{j,t+\ell})$ are asymptotically dependent when $(\alpha, \beta)_{j|i}^{(\ell)\dagger} = (1, 0)$, otherwise the variables are asymptotically independent and exhibit according to the following categories:

1. Positive extremal dependence if $0 < \alpha_{j|i}^{(\ell)\dagger} < 1$ or $\left\{ \alpha_{j|i}^{(\ell)\dagger} = 0, \beta_{j|i}^{(\ell)\dagger} > 0 \right\}$.
2. Negative extremal dependence if $-1 \leq \alpha_{j|i}^{(\ell)\dagger} < 0$.
3. Near extremal independence if $\alpha_{j|i}^{(\ell)\dagger} = 0$ and $\beta_{j|i}^{(\ell)\dagger} \leq 0$.

These three categories respectively imply that the probability of both variables $\dagger Y_{i,t}$ and $Y_{j,t+\ell}$ being extreme is greater than, less than and equal to what would be expected under the case of independence.

2.3.2. Inference

To facilitate statistical inference for the dependence model formulated in the preceding subsection, we construct the samples $\mathbf{Y}^{(-1)}$, $\mathbf{Y}^{(0)}$ and $\mathbf{Y}^{(+1)}$ which represent respectively yesterday's, today's and tomorrow's return observations on the d financial assets under consideration. More specifically, for each time lag ℓ , we define a d -variate sample as

$$\mathbf{Y}^{(\ell)} = \{(Y_{1,t}, \dots, Y_{d,t}) : t = \ell + 2, \dots, \ell + n - 1\}.$$

Then the samples $(\dagger \mathbf{Y}^{(0)}, \mathbf{Y}^{(\ell)})$ are used to fit the conditional extremal dependence model $\mathbf{M}_{j|i}^{(\ell)\dagger}$ given by (5). Model parameters $(\alpha, \beta)_{j|i}^{(\ell)\dagger}$ are estimated by the quasi-maximum likelihood method assuming that model residuals $Z_{j|i(t)}^{(\ell)\dagger}$ are normally distributed with finite mean $\mu_{j|i}^{(\ell)\dagger}$ and standard deviation $\sigma_{j|i}^{(\ell)\dagger}$, that is, by maximizing the following log-likelihood:

$$\log L = - \sum_{t=1}^{n_{j|i}^{(\ell)\dagger}} \left[\log \left\{ \sigma_{j|i}^{(\ell)\dagger} \dagger Y_{i(t)}^{\beta_{j|i}^{(\ell)\dagger}} \right\} + \frac{1}{2} \left\{ \frac{Y_{j(t+\ell)} - \alpha_{j|i}^{(\ell)\dagger} \dagger Y_{i(t)} - \mu_{j|i}^{(\ell)\dagger} \dagger Y_{i(t)}^{\beta_{j|i}^{(\ell)\dagger}}}{\sigma_{j|i}^{(\ell)\dagger} \dagger Y_{i(t)}^{\beta_{j|i}^{(\ell)\dagger}}} \right\}^2 \right],$$

where notation follows the earlier conventions.

The residual distribution $G_{j|i}^{(\ell)\dagger}$ is then estimated empirically from the sample

$$\left\{ \frac{Y_{j(t+\ell)} - \hat{\alpha}_{j|i}^{(\ell)\dagger} \dagger Y_{i(t)}}{\dagger Y_{i(t)}^{\hat{\beta}_{j|i}^{(\ell)\dagger}}} : t = 1, \dots, n_{j|i}^{(\ell)\dagger} \right\}.$$

Note that for all indices j and lags ℓ that were considered, the sample $\dagger \mathbf{Y}^{(0)}$ was used to define the conditioning variable $\dagger Y_{i,t}$. Accordingly, we have $n_{j|i}^{(\ell)\dagger} = n_{j'|i}^{(\ell')\dagger}$ for any indices $j, j' \in \{1, \dots, d\}$ and time lags $\ell, \ell' \in \{-1, 0, 1\}$ with the obvious constraint that $i \neq j$ (respectively $i \neq j'$) when $\ell = 0$ (respectively $\ell' = 0$).

2.3.3. Simulation

To be able to simulate from the proposed dependence model, we define three $2d$ -dimensional vectors corresponding to d financial assets that are tracked over two consecutive trading days as follows:

$$\mathbf{Y} = (\mathbf{Y}_t, \mathbf{Y}_{t+1}), \quad \mathbf{L} = (1, \dots, d, 1, \dots, d), \quad \mathbf{T} = (0, \dots, 0, 1, \dots, 1).$$

Then consider a common dependence threshold $u_{\mathcal{Y}}$ defined as $u_{\mathcal{Y}} = \max \left\{ u_{j|i}^{(\ell)-}, u_{j|i}^{(\ell)+} \right\}$ where the maximum is taken over $\ell \in \{-1, 0, 1\}$ and $i, j \in \{1, \dots, d\}$ with $i \neq j$ when $\ell = 0$. Therefore, decompose the space \mathbb{R}^{2d} into the following regions:

$$\begin{aligned} \mathbb{R}_I^- &= \left\{ \mathbf{Y} \in \mathbb{R}^{2d} : \max_{J \in \{1, \dots, 2d\}} |\mathcal{Y}_J| = -\mathcal{Y}_I > u_{\mathcal{Y}} \right\} \quad \text{for } I = 1, \dots, 2d, \\ \mathbb{R}_I^+ &= \left\{ \mathbf{Y} \in \mathbb{R}^{2d} : \max_{J \in \{1, \dots, 2d\}} |\mathcal{Y}_J| = +\mathcal{Y}_I > u_{\mathcal{Y}} \right\} \quad \text{for } I = 1, \dots, 2d, \\ \mathbb{R}_0 &= \left\{ \mathbf{Y} \in \mathbb{R}^{2d} : \max_{J \in \{1, \dots, 2d\}} |\mathcal{Y}_J| \leq u_{\mathcal{Y}} \right\}, \end{aligned}$$

with the associated probabilities

$$\begin{aligned} P_I^- &= \Pr(\mathbf{y} \in \mathbb{R}_I^-) = \Pr[\mathbf{y} \in \mathbb{R}_I^- \mid -\mathcal{Y}_I > u_{\mathcal{Y}}] \Pr(-\mathcal{Y}_I > u_{\mathcal{Y}}), \\ P_I^+ &= \Pr(\mathbf{y} \in \mathbb{R}_I^+) = \Pr[\mathbf{y} \in \mathbb{R}_I^+ \mid +\mathcal{Y}_I > u_{\mathcal{Y}}] \Pr(+\mathcal{Y}_I > u_{\mathcal{Y}}), \\ P_0 &= \Pr(\mathbf{y} \in \mathbb{R}_0) = 1 - \sum_I \{P_I^- + P_I^+\}. \end{aligned}$$

Since the variable $\dagger\mathcal{Y}_I$ is marginally distributed according to the standard Laplace distribution, the marginal probability $\Pr(\dagger\mathcal{Y}_I > u_{\mathcal{Y}}) = \exp(-u_{\mathcal{Y}})/2$. On the other hand, the conditional probability $\Pr[\mathbf{y} \in \mathbb{R}_I^\dagger \mid \dagger\mathcal{Y}_I > u_{\mathcal{Y}}]$ can be replaced with its Monte Carlo approximation so that

$$P_I^\dagger \approx \left\{ \frac{1}{K} \sum_{k=1}^K \mathbb{I} \left\{ \tilde{\mathbf{y}}_{|I}^\dagger \in \mathbb{R}_I^\dagger \right\} \right\},$$

where the K vectors $\tilde{\mathbf{y}}_{|I}^\dagger = (\dagger\tilde{\mathcal{Y}}_I, \tilde{\mathbf{y}}_{-I}^\dagger)$ ⁵ are generated according to the following algorithm:

Algorithm 1

1. Sample a $(4d + 1)$ -dimensional vector of natural numbers $(k_0, k_I^-, k_I^+)_{I=1, \dots, 2d}$, such that $K = k_0 + \sum_I \{k_I^- + k_I^+\}$, from a multinomial distribution with probability vector $(P_0, P_I^-, P_I^+)_{I=1, \dots, 2d}$.
2. For each $I = 1, \dots, 2d$, iterate the steps of Algorithm 2 until k_I^\dagger simulated points $\tilde{\mathbf{y}}_{|I}^\dagger$ are accepted.
3. Sample with replacement k_0 points from the observed data which fall within the empirical region \mathbb{R}_0 .
4. Randomize the order of the resulting sequence of simulated points.

Algorithm 2

1. Set $\dagger\tilde{\mathcal{Y}}_I = u_{\mathcal{Y}} + E$ where $E \sim \text{Exponential}(1)$.

⁵The subscript k of the generated vectors $\tilde{\mathbf{y}}_{|I}^\dagger$ is suppressed for notational simplicity.

2. For each index pair (I, J) where $J \in \{1, \dots, 2d\} \setminus \{I\}$, find the corresponding index pair $(i, j) = (\mathcal{L}[I], \mathcal{L}[J])$ and time lag $\ell = \mathcal{T}[J] - \mathcal{T}[I]$.⁶ Then consider using the following estimates for the subsequent steps:

$$\left(\hat{\mathcal{A}}_{J|I}^\dagger, \hat{\mathcal{B}}_{J|I}^\dagger\right) = \left(\hat{\alpha}_{j|i}^{(\ell)\dagger}, \hat{\beta}_{j|i}^{(\ell)\dagger}\right) \quad \text{and} \quad \hat{\mathcal{G}}_{J|I}^\dagger = \hat{G}_{j|i}^{(\ell)\dagger}.$$

The latter is the empirical distribution associated with the residual sample

$$\left\{ \mathcal{Z}_{J|I(t)}^\dagger = Z_{j|i(t)}^{(\ell)\dagger} : t = 1, \dots, n_{j|i}^{(\ell)\dagger} \right\}.$$

3. Denote by $\hat{\mathcal{A}}_I^\dagger$ and $\hat{\mathcal{B}}_I^\dagger$ the $(2d - 1)$ -dimensional vectors whose components are respectively given by $\hat{\mathcal{A}}_{J|I}^\dagger$ and $\hat{\mathcal{B}}_{J|I}^\dagger$ for $J \in \{1, \dots, 2d\} \setminus \{I\}$. Similarly, denote by $\hat{\mathcal{G}}_I^\dagger$ the $(2d - 1)$ -dimensional distribution whose marginals are given by $\hat{\mathcal{G}}_{J|I}^\dagger$.
4. Simulate $\tilde{\mathcal{Z}}_I^\dagger \sim \hat{\mathcal{G}}_I^\dagger$ independently of $\dagger\tilde{\mathcal{Y}}_I$.
5. Obtain $\tilde{\mathcal{Y}}_{-I}^\dagger = \hat{\mathcal{A}}_I^\dagger \left\{ \dagger\tilde{\mathcal{Y}}_I \right\} + \left\{ \dagger\tilde{\mathcal{Y}}_I \right\}^{\mathcal{B}_I^\dagger} \tilde{\mathcal{Z}}_I^\dagger$.
6. Reject the simulated point if $\tilde{\mathcal{Y}}_I^\dagger = \left(\dagger\tilde{\mathcal{Y}}_I, \tilde{\mathcal{Y}}_{-I}^\dagger \right) \notin \mathbb{R}_I^\dagger$.

3. Empirical analysis

3.1. Investment dataset

Our dataset consists of stock market indices for the G5 economies: United States (US), United Kingdom (UK), France (FR), Germany (GR) and Japan (JP). Index values are in terms of local currency and measured on a daily frequency at closing times (without adjustment for splits or dividends) collected from Datastream. The basic unit of analysis is the daily logarithmic return (referred to as *return* thereafter) over the period from January 04, 2000 to December 31, 2009. Holidays and unscheduled market closures are treated as missing values only if they are common to all investigated stock markets, otherwise they

⁶This is due to the stationarity of the process (\mathbf{Y}_t) .

are represented as zero returns. By doing so, we end up with a dataset comprising $n = 2600$ daily return observations per series, as shown in Figure 2. Visual inspection of the series plots reveals that our data clearly possess non-stationarity in the variability.

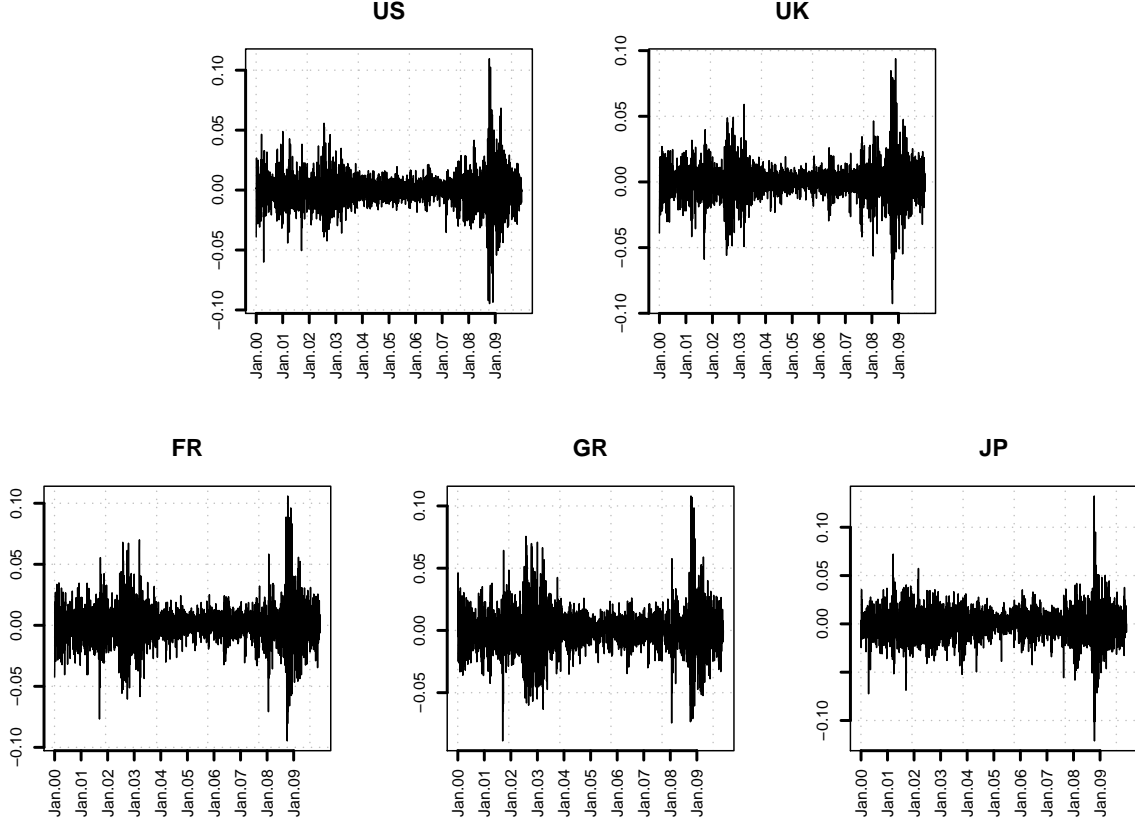


Figure 2: Time series plot for the returns associated with the stock market indices: S&P 500 (US), FTSE 100 (UK), CAC 40 (FR), DAX 30 (GR) and NIKKEI 225 (JP). The data span the period from January 04, 2000 to December 31, 2009.

Summary statistics of the data are reported in Table 1 which are generally consistent with previous studies. Average return is neither economically large nor statistically different from zero (at 5% significance level) for each index. The variance estimates are comparable among the five return series with the US and UK series having the lowest estimates. Over the studied period, both the minimum and maximum return are realized for the Japanese series. Finally, all series exhibit negative skewness and pronounced excess kurtosis relative to the normal distribution and they show insignificant autocorrelations in their levels at almost

all lags but positive and overwhelmingly significant autocorrelations in their absolute values (cf. Figure 3).

Statistic	Multiplier	US	UK	FR	GR	JP
Mean	$\times 10^4$	+2.175	+1.536	+1.312	+2.285	−2.510
Variance	$\times 10^4$	1.333	1.286	1.937	2.142	2.364
Kurtosis		9.541	6.830	4.978	5.261	5.708
Skewness	$\times 10$	−2.024	−1.135	−0.375	−1.338	−0.212
Minimum	$\times 100$	−9.470	−9.265	−9.472	−9.871	−12.111
Maximum	$\times 100$	10.957	9.384	10.595	10.797	13.235

Table 1: Summary statistics of the return series on the stock market indices: S&P 500 (US), FTSE 100 (UK), CAC 40 (FR), DAX 30 (GR) and NIKKEI 225 (JP). The data span the period from January 04, 2000 to December 31, 2009.

3.2. Return filtering results

As detailed in Section 2.1, our approach for filtering returns is market-specific in the sense that the respective volatility process is expressed as a function of own past information only through the use of the ARMA(1,1)-GARCH(2,1) process. It is evident from Figure 4 that standardizing return series by their respective estimated volatility processes successfully filtered out the autocorrelation functions that were persistent in the absolute values of the original return series. In view of this, we conclude that the filtered returns can be treated as IID.

3.3. Marginal modelling results

To make statistical inferences about the marginal tail behavior of the filtered return data, we implement the method described in Section 2.2. The Anderson-Darling statistic suggests that one can choose more or less the same threshold exceedance probability for the five filtered return series. In particular, the GPD models fitted to the smallest/largest 78 observations (equivalent to 3% of the filtered return data) were found to provide a reasonable approximation for the tails of the marginal distributions. The selected marginal thresholds

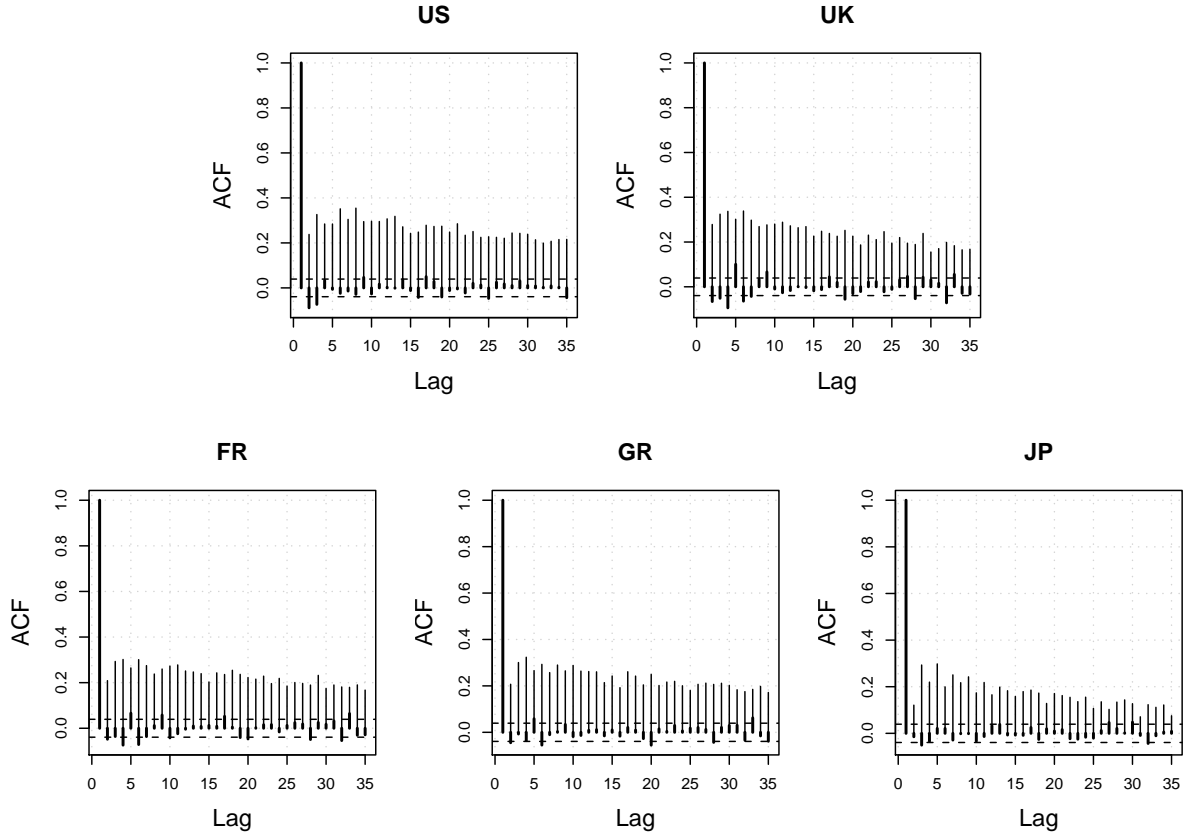


Figure 3: The sample autocorrelation functions of the return series (thick bars) as well as the absolute return series (thin bars) associated with the stock market indices: S&P 500 (US), FTSE 100 (UK), CAC 40 (FR), DAX 30 (GR) and NIKKEI 225 (JP). The data span the period from January 04, 2000 to December 31, 2009. The bands of the 95% confidence intervals for the autocorrelations of IID Gaussian noise are shown as dashed lines for reference.

and the associated parameter estimates of the GPD models are given in Table 2. Selected quantile plots are shown in Figure 5 to reinforce our judgment about the fitted marginal models.

For each series of filtered returns, the lower tail threshold has slightly larger magnitude than the corresponding upper tail threshold indicating asymmetric marginal distribution. The point estimates suggest that the marginal distributions of the US, French and German series exhibit heavy lower tails and short upper tails with a clear similarity in the marginal tail behavior of the French and German series. This is in contrast with the UK series. The marginal distribution of the Japanese series, on the other hand, is characterized with

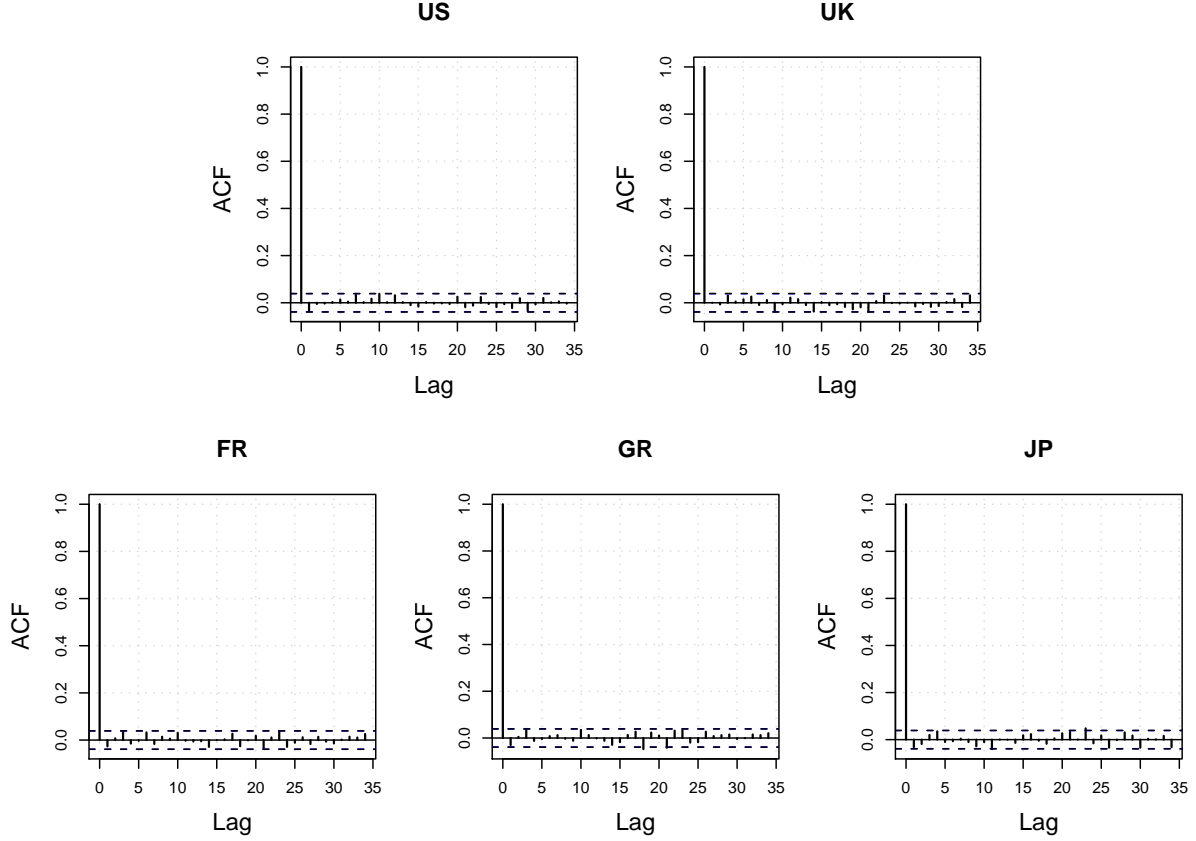


Figure 4: The sample autocorrelation functions of the absolute values of the filtered return series associated with the stock market indices: S&P 500 (US), FTSE 100 (UK), CAC 40 (FR), DAX 30 (GR) and NIKKEI 225 (JP). The data span the period from January 04, 2000 to December 31, 2009. The bands of the 95% confidence intervals for the autocorrelations of IID Gaussian noise (given by $\pm 2/\sqrt{n}$ where n is the sample size) are shown as dashed lines for reference.

both tails being heavy. However, these results are not conclusive since all shape parameter estimates are not significantly different from zero (at 5% level).

3.4. Dependence modelling results

3.4.1. The model fit

An important assumption for the validity of the the conditional extremal dependence model $\mathbf{M}_{j|i}^{(\ell)\dagger}$ is the independence of its residuals from the value of the conditioning variable used to define them. Thus the dependence threshold $u_{j|i}^{(\ell)\dagger}$ can be chosen on the basis of this feature besides assessing the stability of model parameter estimates over higher thresholds. Ideally, we would like to go through these diagnostics for all the considered 140 models.

Panel A. Lower Tail

i	US	UK	FR	GR	JP
$u_{X_i}^-$	-2.093	-2.139	-2.007	-1.991	-2.022
φ_i^-	0.414 (0.069)	0.496 (0.089)	0.431 (0.078)	0.397 (0.070)	0.469 (0.086)
ξ_i^-	0.228 (0.126)	-0.010 (0.138)	0.213 (0.142)	0.218 (0.137)	0.209 (0.146)
AD	0.656	0.356	0.121	0.203	0.280
CV	0.713 {0.867}	0.814 {0.997}	0.713 {0.867}	0.713 {0.867}	0.713 {0.867}

Panel B. Upper Tail

i	US	UK	FR	GR	JP
$u_{X_i}^+$	1.810	1.734	1.728	1.801	1.784
φ_i^+	0.470 (0.076)	0.305 (0.051)	0.445 (0.065)	0.372 (0.067)	0.339 (0.056)
ξ_i^+	-0.185 (0.116)	0.068 (0.122)	-0.066 (0.094)	-0.077 (0.141)	0.074 (0.122)
AD	0.302	0.251	0.274	0.480	0.172
CV	0.852 {1.047}	0.781 {0.955}	0.814 {0.997}	0.814 {0.997}	0.781 {0.955}

Table 2: The marginal modelling results for the filtered returns on the G5 stock market indices over the period from January 04, 2000 to December 31, 2009. The lower/upper tail marginal thresholds all have empirical exceedance probability equals 3%. The maximum likelihood (ML) estimates of the GPD parameters are obtained under cross-sectional independence of the investigated series and the associated standard errors (between parentheses) are calculated based on the asymptotic normality of the ML estimators. The last two lines in each panel report the Anderson-Darling AD statistics and the corresponding (approximated) critical values CV at 10% level and {5%} level.

Since this is practically infeasible, we examine selected models in order to decide on the appropriate threshold non-exceedance probability $F_Y(u_{ji}^{(\ell)\dagger})$. The maximum value of these probabilities will then be used to determine a common threshold u_Y above which all the conditional extremal dependence models are fitted. In our implementation, the dependence threshold is taken $F_Y(u_Y) = 0.9$.

Although some insights into the characteristics of the extremal dependence structure of the data can be gained by looking at the model parameter estimates (not reported), it is important to stress that the interpretation of the results is not straightforward and conclusions regarding the strength of the extremal dependencies among stock markets should not be solely based on the relevant estimates of $(\alpha, \beta)_{ji}^{(\ell)\dagger}$ because this is likely to miss the importance of the distribution of model residuals $Z_{ji}^{(\ell)\dagger}$. To overcome this difficulty, in Section 3.4.2 we introduce a convenient way to quantify the amount of association between

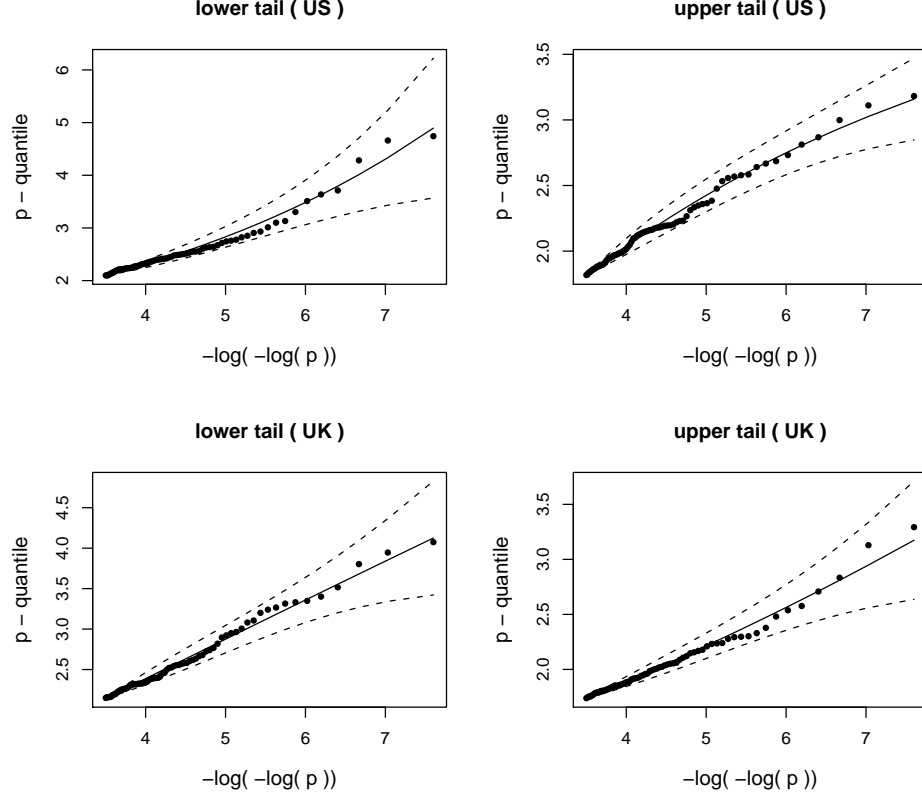


Figure 5: Selected quantile plots for the marginal tails. The model-based quantiles estimated over a range of threshold probabilities $p \geq 0.97$ are shown by the solid line, while the empirical counterparts are shown as points. The dashed lines in each plot represent the 95% confidence intervals calculated based on the asymptotic normality of the maximum likelihood estimators.

the variables $Y_{i,t}$ and $Y_{j,t+\ell}$ when the former is attaining a large negative/positive value (relative to the threshold u_Y) under the assumption that the conditional extremal dependence model $\mathbf{M}_{j|i}^{(\ell)\dagger}$ provides a good description to this association.

3.4.2. The model summary

For an extreme level $y > u_Y$, we summarize the conditional distribution of $Y_{j,t+\ell} | \dagger Y_{i,t} = y$ by its median \tilde{y} under the fitted model $\mathbf{M}_{j|i}^{(\ell)\dagger}$ which is given by

$$\tilde{y} = \hat{\alpha}_{j|i}^{(\ell)\dagger} y + y^{\hat{\beta}_{j|i}^{(\ell)\dagger}} \text{median} \left\{ Z_{j|i}^{(\ell)\dagger} \right\}.$$

We then define the following summary measure

$$T_{j|i}^{(\ell)\dagger}(y) = \begin{cases} \Pr\{\dagger Y_{i,t} > y\} / \Pr\{Y_{j,t+\ell} \leq \tilde{y}\} & \text{if } \dagger = -, \\ \Pr\{\dagger Y_{i,t} > y\} / \Pr\{Y_{j,t+\ell} > \tilde{y}\} & \text{if } \dagger = +. \end{cases}$$

The above formulation of the summary measure implies that $T_{j|i}^{(\ell)\dagger}(y)$ takes values between 0 and 1. Thus a value of $T_{j|i}^{(\ell)\dagger}$ close to zero (close to one) can be interpreted as follows. Return in stock market j is nearly independent of (almost identical to) the extreme return in stock market i when the former is realized ℓ -step from the latter.

We have evaluated $T_{j|i}^{(\ell)\dagger}(y)$ for $F_Y(y) \in \{0.99, 0.999, 0.9999\}$ with $i \neq j$. In what follows, we discuss the results for zero-lag and one-lag ahead relationships only (i.e. $\ell \in \{0, 1\}$) because of their obvious interpretation from a forecasting point of view. Overall, the measure $T_{j|i}^{(\ell)\dagger}(y)$ has small to moderate values that are steadily decreasing with the increase of severity of the conditioning event y , indicating that asymptotic dependence is not an appropriate description for the extremal dependence structure of the G5 stock markets. Since results were found to be qualitatively similar over y , we decided to present those associated with $F_Y(y) = 0.99$ only. We have plotted the corresponding values of $T_{j|i}^{(\ell)\dagger}(y)$ in Figures 6 and 7 along with their bootstrap-based confidence intervals.

The measure $T_{j|i}^{(\ell)\dagger}(y)$ corresponding to the contemporaneous extremal dependencies among the European stock markets takes values within the range (0.3, 0.6) in both tail regions, pointing to a relatively strong degree of extremal dependence particularly when the French and German series are concerned. The results regarding the lagged extremal dependencies are mixed. For example, the Japanese stock market is fairly dependent on extreme one-day lagged (negative or positive) returns originated in the other stock markets. On the other hand, the US stock market appears to be only weakly dependent on the other stock markets when conditioning upon their extreme one-day lagged returns. A similar finding is obtained for the three European stock markets when conditioning on the Japanese extreme one-day lagged returns.

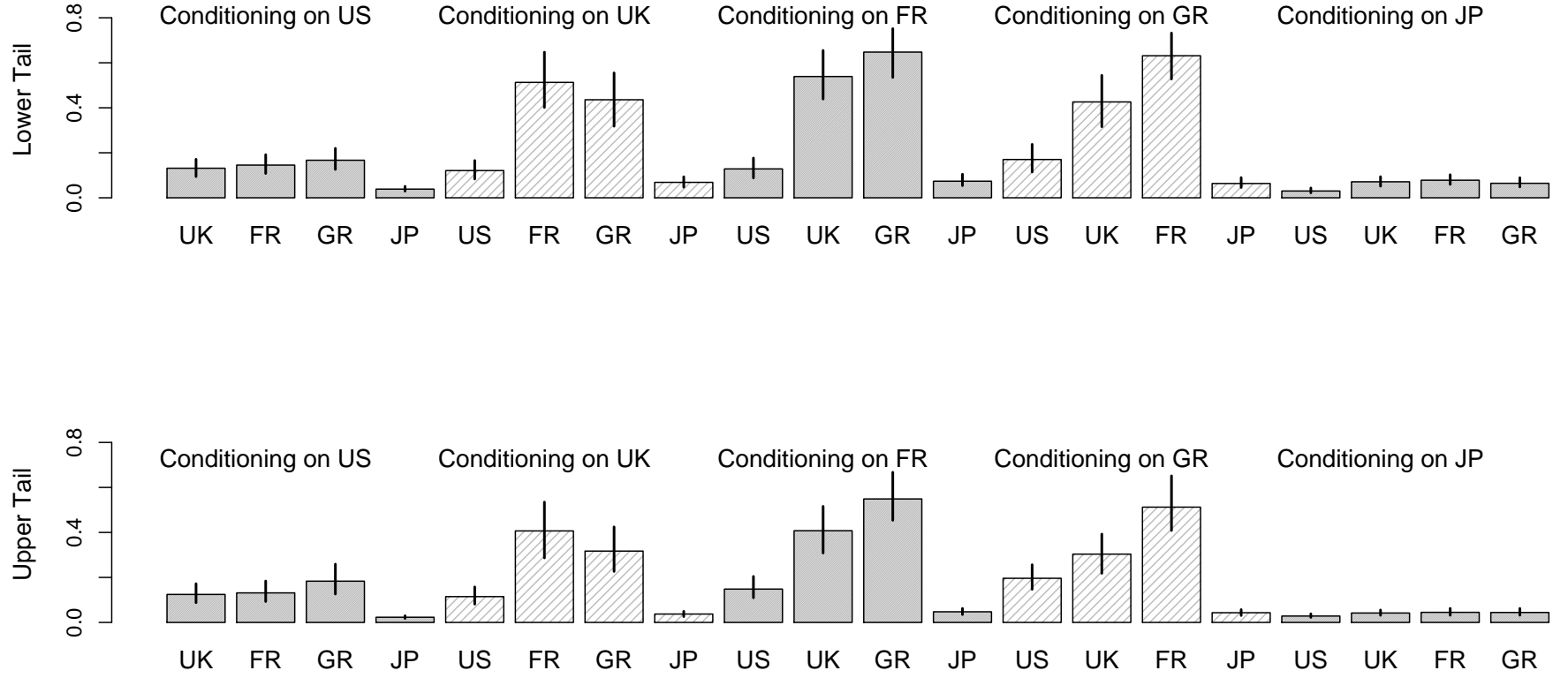


Figure 6: Summarizing the contemporaneous ($\ell = 0$) extremal dependencies among the G5 stock markets by the measure $T_{j|i}^{(\ell)\dagger}(y)$ evaluated at $F_Y(y) = 0.99$ with $\dagger = -$ for the upper panel and $\dagger = +$ for the lower panel. The vertical lines along the top of each bar represent the 95% bootstrap confidence intervals calculated on the basis of 1000 replicates.

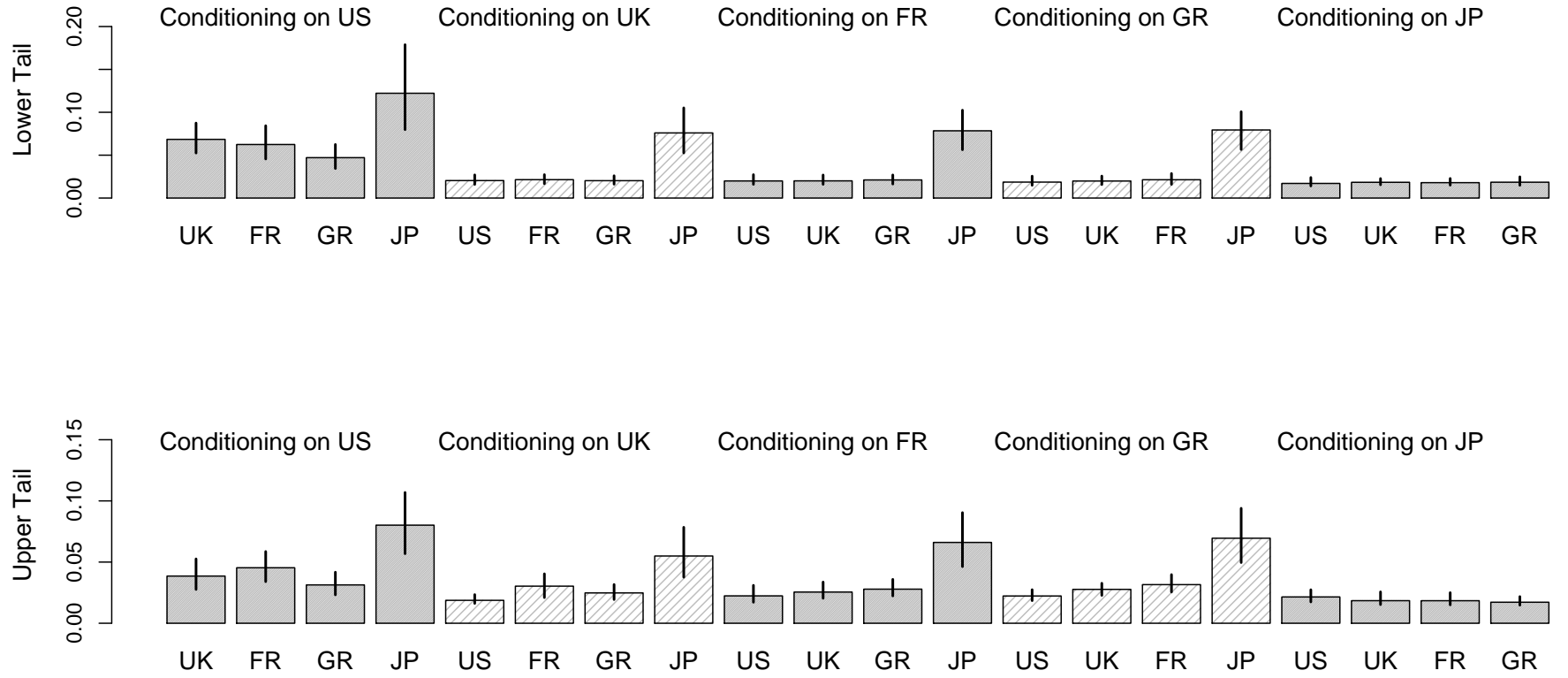


Figure 7: Summarizing the lagged ($\ell = 1$) extremal dependencies among the G5 stock markets by the measure $T_{j|i}^{(\ell)\dagger}(y)$ evaluated at $F_Y(y) = 0.99$ with $\dagger = -$ for the upper panel and $\dagger = +$ for the lower panel. The vertical lines along the top of each bar represent the 95% bootstrap confidence intervals calculated on the basis of 1000 replicates.

3.5. Portfolio risk assessment

We introduce a multivariate version of the tail dependence coefficients defined in equations (1) to quantify the amount of dependence between portfolio constituent assets in the tail regions. It should be noted, however, that the tail dependence coefficient is a limit property that cannot be achieved in practice. For this reason, we focus our discussion on a finite approximation. For two disjoint index sets $I, J \subset \{1, \dots, d\}$, consider the following measures:

$$\begin{aligned}\lambda_{J(\ell_J)|I(\ell_I)}^-(p|q) &= \Pr \left\{ \bigcap_{j \in J} \{Y_{j,t+\ell_j} < \text{VaR}_p\} \middle| \bigcap_{i \in I} \{Y_{i,t+\ell_i} < \text{VaR}_q\} \right\}, \\ \lambda_{J(\ell_J)|I(\ell_I)}^+(p|q) &= \Pr \left\{ \bigcap_{j \in J} \{Y_{j,t+\ell_j} > \text{VaR}_p\} \middle| \bigcap_{i \in I} \{Y_{i,t+\ell_i} > \text{VaR}_q\} \right\},\end{aligned}$$

where $\ell_i, \ell_j \in \{-1, 0, 1\}$ are time lags and $p, q \in (0, 1)$ are probability levels. Given that losses (gains) of each asset $i \in I$ exceed their respective VaR on certain days, the first (second) measure gives the probability that all the other relevant assets indexed by $j \in J$ will realize losses (gains) worse (better) than their respective VaR on some days which are specified by the time lags used in the definition of the measure of interest. Moreover, when both index sets reduce to single elements i, j such that the associated probability levels p, q are equal, then $\lambda_{j(\ell_j)|i(\ell_i)}^\dagger(p|q)$ can be considered as a probability measure of dependence with values less than, equal to and greater than q indicating that the associated variables are negatively dependent, independent and positively dependent at the specified probability level $p = q$ and in the tail region \dagger , respectively.

When empirical estimates of these measures are reliable, then a comparison of these measures with their model-based counterparts can serve as a diagnostic tool to verify the ability of the proposed dependence model to quantify the amount of dependence among the variables of interest in the lower and upper tail regions. We have done this for our data. More specifically, for all the possible combinations $(i, j, \dagger) \in \{\text{US, UK, FR, GR, JP}\}^2 \times \{-, +\}$, we estimated the measures $\lambda_{j(0)|i(0)}^\dagger(p|q)$ and $\lambda_{j(1)|i(0)}^\dagger(p|q)$. Two typical values for the conditioning probability level q have been considered so as to capture extreme market

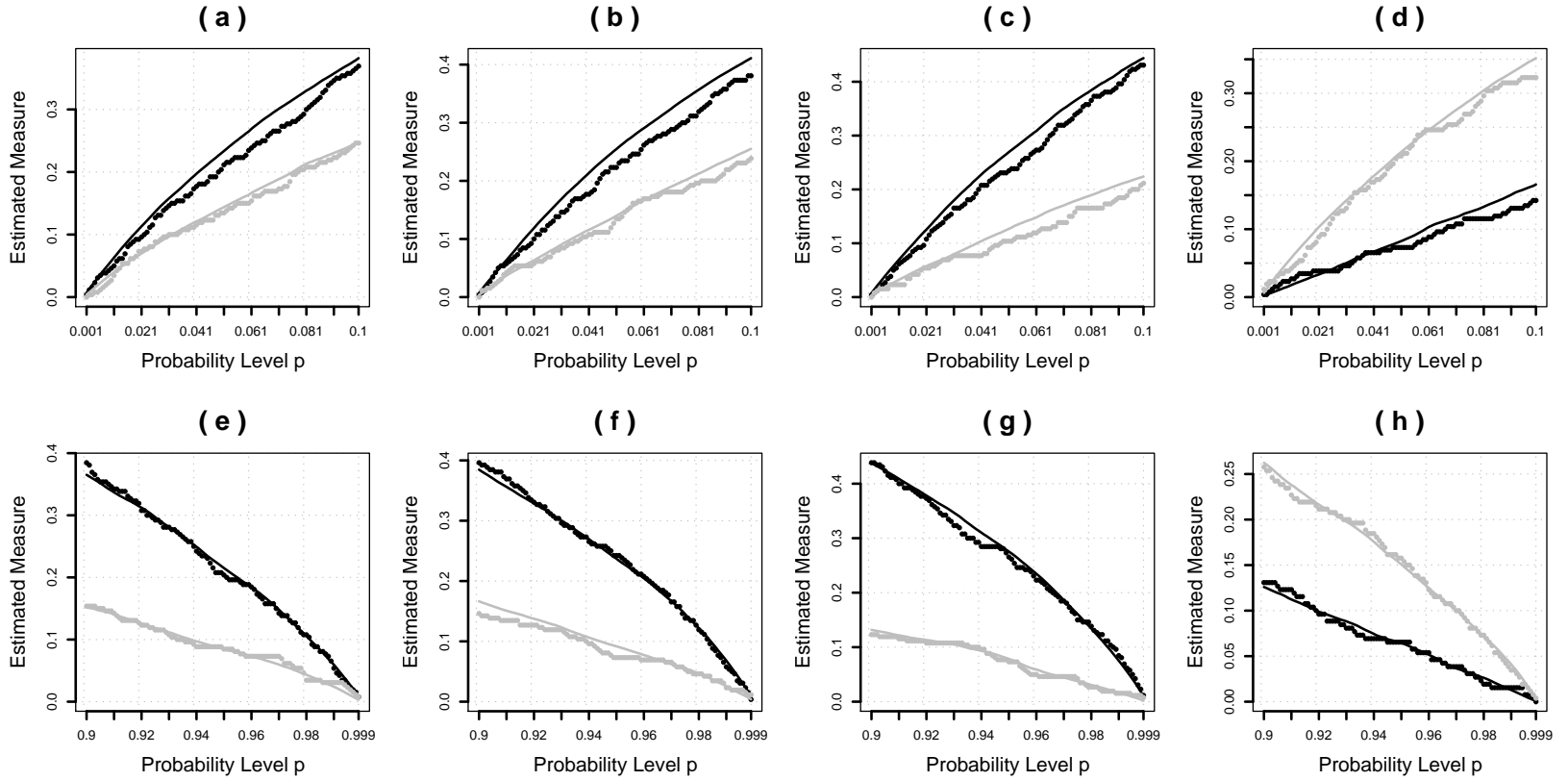
fluctuations, namely $q = 0.1$ when looking at the lower tail relationships (i.e. when $\dagger = -$) and $q = 0.9$ when looking at the upper tail relationships (i.e. when $\dagger = +$). On the other hand, we have set $p \leq q$ in the former case and $p \geq q$ in the latter.

Selected plots are displayed in Figure 8 which show that the model-based estimates closely match their empirical counterparts, demonstrating that the model captures the extremal dependence structure of the data. It is also interesting to note that when $p = q$, the corresponding estimates of the measure $\lambda_{j^{(0)}|i^{(0)}}^\dagger(p|q) > 10\%$. This signifies the international stock markets with positive contemporaneous extremal dependencies. More precisely, the probability that two stock markets experiencing large negative/positive returns on a particular day is greater than what would be expected in the case of independence. The same is true for the lagged extremal dependencies for which $\lambda_{j^{(1)}|i^{(0)}}^\dagger(p|q) > 10\%$ when conditioning on large US returns ($i = \text{US}$) or when focusing on the Japanese stock market returns ($j = \text{JP}$).

We have also looked at assessing risk of a portfolio diversified among the G3 (the US, UK and Japanese) stock markets. These markets open and close almost sequentially. This special time structure of the three markets should be accounted for when assessing how likely are extreme events generated from any two of these markets to exert extreme events in the remaining market. For example, valuable information for UK investors comes from the Japanese stock market on the same day as well as that coming from the US stock market on the previous day. Thus the following measures are of interest:

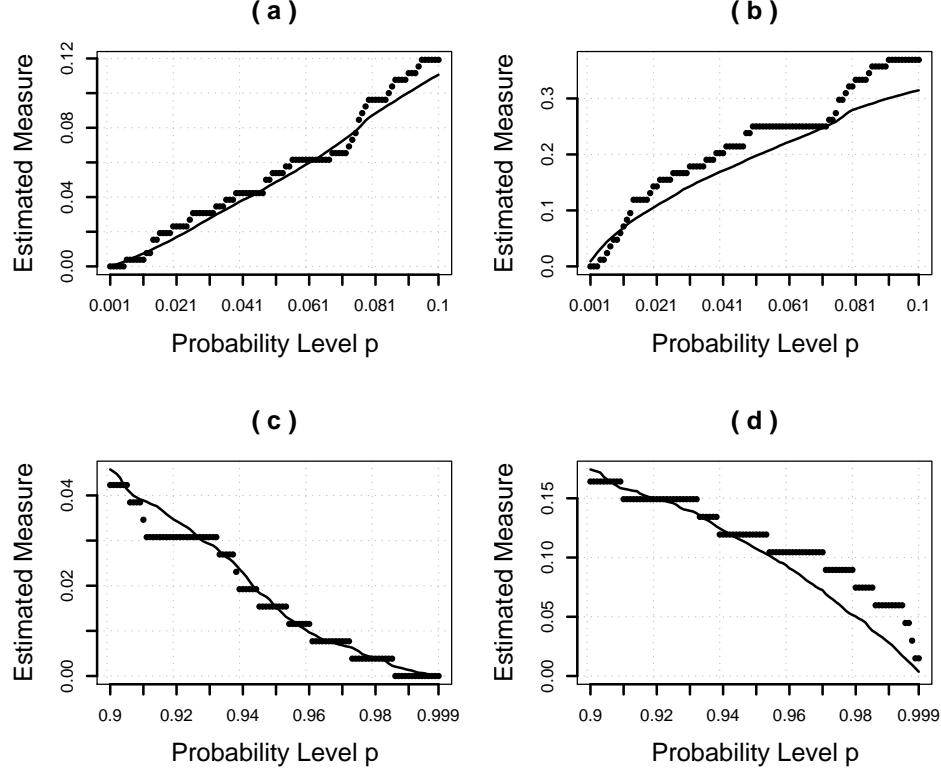
$$\begin{aligned}\lambda_{\text{UK}^{(0)}|(\text{JP}^{(0)}, \text{US}^{(-1)})}^-(p|q) &= \Pr \{Y_{\text{UK},t} < \text{VaR}_p \mid (Y_{\text{JP},t} < \text{VaR}_q) \cap (Y_{\text{US},t-1} < \text{VaR}_q)\}, \\ \lambda_{(\text{UK}^{(0)}, \text{JP}^{(0)})|\text{US}^{(-1)}}^\dagger(p|q) &= \Pr \{(Y_{\text{UK},t} < \text{VaR}_p) \cap (Y_{\text{JP},t} < \text{VaR}_q) \mid Y_{\text{US},t-1} < \text{VaR}_q\}.\end{aligned}$$

In our empirical analysis, these measures have been estimated at the same probability levels p and q that were previously considered. The corresponding plots, displayed in Figure 9, are fairly acceptable with a good agreement between the model-based and empirical estimates.



(a) : $\lambda_{UK^{(\ell)}|US^{(0)}}^-(p|0.1)$ (b) : $\lambda_{FR^{(\ell)}|US^{(0)}}^-(p|0.1)$ (c) : $\lambda_{GR^{(\ell)}|US^{(0)}}^-(p|0.1)$ (d) : $\lambda_{JP^{(\ell)}|US^{(0)}}^-(p|0.1)$
 (e) : $\lambda_{UK^{(\ell)}|US^{(0)}}^+(p|0.9)$ (f) : $\lambda_{FR^{(\ell)}|US^{(0)}}^+(p|0.9)$ (g) : $\lambda_{GR^{(\ell)}|US^{(0)}}^+(p|0.9)$ (h) : $\lambda_{JP^{(\ell)}|US^{(0)}}^+(p|0.9)$

Figure 8: The plots display empirical estimates (shown as points) versus model-based estimates (shown as continuous lines) of the measures specified above where the time lag $\ell = 0$ (contemporaneous relationships) for the black colored plots and $\ell = 1$ (one-day lagged relationships) for the gray colored plots.



$$\begin{aligned}
 \text{(a)} : \lambda_{(\text{UK}^{(0)}, \text{JP}^{(0)}) | \text{US}^{(-1)}}^-(p|0.1) & \quad \text{(b)} : \lambda_{\text{UK}^{(0)} | (\text{JP}^{(0)}, \text{US}^{(-1)})}^-(p|0.1) \\
 \text{(c)} : \lambda_{(\text{UK}^{(0)}, \text{JP}^{(0)}) | \text{US}^{(-1)}}^+(p|0.9) & \quad \text{(d)} : \lambda_{\text{UK}^{(0)} | (\text{JP}^{(0)}, \text{US}^{(-1)})}^+(p|0.9)
 \end{aligned}$$

Figure 9: The plots display empirical estimates (shown as points) versus model-based estimates (shown as continuous lines) of the measures specified above.

4. Risk management strategies

The emphasis here is on the economic value of adequately capturing the extremal dependence structure of the data at hand. Three strategies are considered to model the joint distribution of financial asset returns, all identical in the filtering and marginal models but differing in their assumptions regarding the (extremal) dependence structure. The competing dependence models are our extreme-value-based model (referred to as EV-strategy) and two copula-based models which use the normal copula (NC-strategy) or Student's t-copula (TC-strategy).

By working with these two copulas, we impose a constraint on the type of tail dependence for each pair of the variables under study. The extreme-value-based model, on the other hand, incorporates both types of tail dependence and allows for heterogeneity among the different pairs.

To illustrate the differences between the three strategies, we estimate the one-day joint and conditional loss probabilities for the G5 stock market index returns. The day of interest is January 04, 2010 (the first trading day in the year 2010) and will be denoted by t^* . To be more precise, define the following Bernoulli random variables:

$$\mathbb{I}_i(q) := \mathbb{I}\{R_{i,t^*} < \mu_{i,t^*} + \sigma_{i,t^*} \times F_{X_i}(q)\} \sim \text{Bernoulli}(q),$$

where $i \in \{\text{US, UK, FR, GR, JP}\}$ and $q \in \{1\%, \dots, 10\%\}$. Interest lies in estimating the following probabilities:

$$\begin{aligned} \lambda^{\text{joint}}(q) &= \Pr\left\{\sum_i \mathbb{I}_i(q) = 5\right\}, \\ \lambda^{\text{cond}}(q) &= \Pr\left\{\sum_i \mathbb{I}_i(q) = 5 \left| \sum_i \mathbb{I}_i(q) \geq 1\right.\right\}. \end{aligned}$$

To this end, we have implemented the three strategies using return observations from January 04, 2000 through December 31, 2009. After estimating the dependence model upon which a specific risk management strategy is based, a K -sample of transformed filtered returns $\{\tilde{Y}_{i,t^*(k)} : k = 1, \dots, K\}$ is generated by Algorithms 1 and 2. The corresponding sample of original returns is given by

$$\tilde{R}_{i,t^*(k)} = \hat{\mu}_{i,t^*} + \hat{\sigma}_{i,t^*} \times \hat{F}_{X_i}^{-1}\left[F_Y\left(\tilde{Y}_{i,t^*(k)}\right)\right].$$

It is this sample which we use to estimate the probabilities of interest. Specifically,

$$\begin{aligned}\widehat{\lambda}^{\text{joint}}(q) &= \frac{1}{K} \sum_{k=1}^K \left\{ \sum_i \mathbb{I}_i^{(k)}(q) = 5 \right\}, \\ \widehat{\lambda}^{\text{cond}}(q) &= \frac{\sum_{k=1}^K \left\{ \sum_i \mathbb{I}_i^{(k)}(q) = 5 \right\}}{\sum_{k=1}^K \left\{ \sum_i \mathbb{I}_i^{(k)}(q) \geq 1 \right\}},\end{aligned}$$

where

$$\mathbb{I}_i^{(k)}(q) := \mathbb{I} \left\{ R_{i,t^*(k)} < \hat{\mu}_{i,t^*} + \hat{\sigma}_{i,t^*} \times \widehat{F}_{X_i}(q) \right\}.$$

The results are displayed in Figure 10 along with the estimates obtained under exact independence and perfect dependence. If the EV-strategy represents the true data generating process, then Figure 10 shows that the NC-strategy (tail independence strategy) tends to underestimate the loss probabilities whereas the TC-strategy (tail dependence strategy) tends to overestimate them. It is also apparent from the figure (left plot) that the dependence inferred by each strategy is inconsistent with the exact independence and perfect dependence.

5. Discussion

We have proposed a statistical model based on multivariate extreme value theory that is amenable to financial applications. The model incorporates several empirical properties of financial returns with respect to their marginal and dependence tail behavior. It provides flexibility in modelling both asymptotically dependent and asymptotically independent asset returns in a unified framework. The proposed model was subsequently used to analyze the extremal interdependencies across the world's five major economies over 10 years of market activity. The model was shown to be successful in representing the observed extremal dependence structure of the analyzed data in terms of several performance measures.

We have considered one-day ahead forecast in this work, but the generalization is

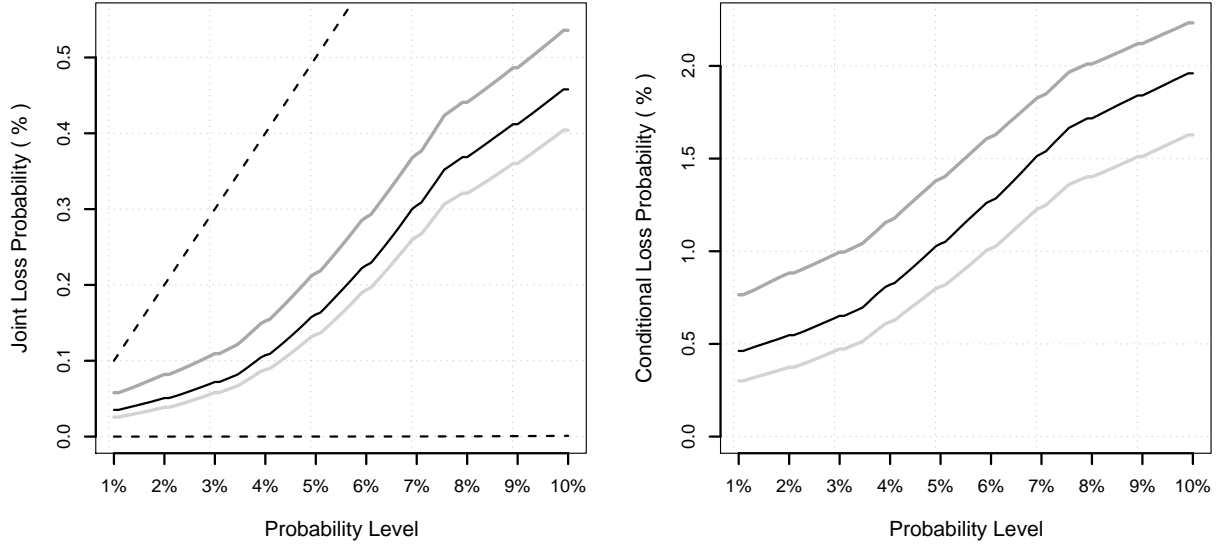


Figure 10: The estimated one-day (January 04, 2010) joint loss probability $\lambda^{\text{joint}}(q)$ (left plot) and conditional loss probability $\lambda^{\text{cond}}(q)$ (right plot) for the G5 stock market index returns evaluated at different probability levels q and under several degrees/types of dependence: lower dashed line \equiv exact independence; upper dashed line \equiv perfect dependence (scaled by 10); light-gray line \equiv tail independence (NC-strategy); black line \equiv modelled dependence (EV-strategy) and dark-gray line \equiv tail dependence (TC-strategy). The parameters pertain to the marginal models as well as to the latter three dependence models were estimated using return observations from January 04, 2000 through December 31, 2009.

straightforward by extending the investment horizon H though the computational burden will increase. Moreover, despite the flexibility of the proposed model, it suffers from different levels of uncertainty surrounding its estimation. This is a typical problem in extreme value analysis. The problem is intensified when one considers a multivariate situation. These uncertainties can be naturally accounted for by Bayesian methods. However, details of such improvement are left for future research.

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