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Thomas, Jonathan P.

Working Paper Minimax punishments in discounted two-person supergames

Diskussionsbeiträge - Serie II, No. 67

Provided in Cooperation with: Department of Economics, University of Konstanz

Suggested Citation: Thomas, Jonathan P. (1988) : Minimax punishments in discounted two-person supergames, Diskussionsbeiträge - Serie II, No. 67, Universität Konstanz, Sonderforschungsbereich 178 - Internationalisierung der Wirtschaft, Konstanz

This Version is available at: https://hdl.handle.net/10419/101764

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Sonderforschungsbereich 178 "Internationalisierung der Wirtschaft"

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Universität Konstanz

Diskussionsbeiträge

Juristische Fakultät Fakultät für Wirtschaftswissenschaften und Statistik

Jonathan Thomas

Minimax Punishments in Discounted Two-Person Supergames

Postfach 5560 D-7750 Konstanz Serie II — Nr. 67 Juni 1988 MINIMAX PUNISHMENTS IN DISCOUNTED

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TWO-PERSON SUPERGAMES

Jonathan, Thomas *

Serie II - Nr. 67

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A = 3304188

Juni 1988

* Visiting from Department of Economics, University of Warwick, Coventry, U. K. In the study of perfect equilibria of repeated-games it is known from the work of Abreu [2], that the question of whether a particular outcome path can be sustained in equilibrium can be answered by loking at whether "optimal punishments" are sufficiently severe to discourage deviations from the path. The optimal punishment for a particular player corresponds simply to an equilibrium which gives him his lowest pay-off among all equilibria. Sometimes optimal punishments can deliver minimax utility levels, and are therefore the most severe possible. In Abreu [1] in a particular symmetric oligopolistic game, it is shown that minimax utility levels are attainable in equilibrium for values of the discount factor near unity. In Lambson [7], in a Bertrand pricing model, minimax punishments are attainable for all values of the discount factor.

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In Section III a complete analysis is given of the possibility of using minimax punishments in equilibrium in two-person discounted supergames. In the work of Fudenberg and Maskin [5] it is shown that, subject only to individual rationality, minimax payoffs can be approached as the discount factor tends to unity. Here, in contrast, necessary and sufficient conditions on the stage-game are given, under which minimax payoffs can actually be attained for all discount factors above a critical value. Provided minimaxing is individually rational (Condition A below) then minimax payoffs can be received in equilibrium by both players unless the following (condition C below) holds: one of the players is (always) indifferent about how he responds to being minimaxed in the stage-game, and moreover all of these responses always give the minimaxer a payoff no greater than his own minimax payoff and strictly less if the response

I.

itself is also a minimax strategy of the player being minimaxed. This condition is necessary and sufficient for the non-existence of equilibria in which both players receive their minimax payoffs. Since the condition is so strict, the pessimistic conclusion that discounting does not help much in reducing the plethora of equilibria is reinforced.

This pessimism will be further reinforced in Section IV by a consideration of 2×2 symmetric games. For a given distribution over payoffs, the proportion of such games in which minimaxing is possible in equilibrium, given a value for the discount factor, will be derived.

II. PRELIMINARIES

Briefly G is a one-shot simultaneous game with 2 players. Player i has a <u>pure strategy set</u> A_i and chooses an action $a_i \in A_i$. <u>An action</u> pair $a \equiv (a_1, a_2)$ belongs to $A \equiv A_1 \times A_2$; π_i is the ith player's payoff function, $\pi_i : A \rightarrow R$. G^{∞} is the <u>supergame</u> obtained by repeating G infinitely often. The <u>discount factor</u> is δ where $0 < \delta < 1$, and player i's payoff for the repeated game is $(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_i(a^t)$, where a^t is the action pair played at t.

Assume that each A_i is endowed with a topology, and give all product spaces referred to below the product topology. Assume

further:

A.1 Each A_i is sequentially compact.

A.2
$$\pi_i$$
 is continuous for each i.

We will employ a dynamic programming approach to construct the equilibria. The same approach has been used by Abreu, Pearce and Stacchetti [3, 4] and Fudenberg and Maskin [6]. We define a mapping T which takes bounded sets into bounded sets.

Definition. Let X be a bounded subset of \mathbb{R}^{N} . Define $\underline{v}_{i} = \inf_{v \in X} \{v_{i}\}$ and $\underline{v} = (\underline{v}_{1}, \underline{v}_{2})$: Then $v \in T(X)$ if and only if there exists a $v \in X$ and an action profile $\hat{a} \in A$ such that

$$v = (1 - \delta) \pi(a) + \delta v$$
 (1)

and

$$(1 - \delta) \pi_{i} (a_{i}, a_{-i}) + \delta \underline{v}_{i} \leq (1 - \delta) \pi_{i}(a) + \delta \overline{v}_{i}$$
(2)
for all $a_{i} \in A_{i}$.

With any action profile $a \in A$, we associate the <u>deviation</u> <u>utility</u> d(a) whose ith element

$$d_{i}(a) \equiv -(1-\delta) \max \{\pi_{i}(a_{i}, a_{-i}) - \pi_{i}(a)\}$$
$$a_{i} \in A$$

That is, $-d_1(a)$ is the best gain i can make by deviating unilaterally from a. Note that $d(a) \leq 0$. The maximum is attained by Al and A2. Then the mapping T may be thought about as follows. For any $\hat{v} \in X$, and $a \in A$, add d(a) to $\delta \hat{v}$ to form $\delta \hat{v} + d(a)$. If $\delta \hat{v} + d(a) \geq \delta \underline{v}$ then we say that the addition of $(1 - \delta) \pi(a)$ to δv is "admissable" and $(\delta \hat{v} + (1 - \delta) \pi(a)) \in T(X)$. Then T(X) is just the set of all such admissable additions.

The notation used in the definition will be used throughout. We say that a bounded set X <u>self-generates</u> if $X \subseteq T(X)$. Let U be the set of Perfect Equilibrium payoffs. U is completely characterised in terms of T, and this is the content of Lemmas 1 and 2 and the corollary. The proofs may be found in Thomas [8].^{1/}

Lemma 1: If $X \subseteq \mathbb{R}^N$ is a bounded set which self-generates $(X \subseteq T(X))$ then $X \subseteq U$. Moreover $\tilde{X} \subseteq T(\bar{X}) \subseteq U$, where \bar{X} is the closure of X.

Lemma 2: U = T(U).

Corollary: U is closed, and is the largest bounded subset of $\ensuremath{\mathbb{R}}^N$ which self-regenerates.

III. OPTIMAL PUNISHMENTS FOR TWO-PERSON GAMES WHEN THE DISCOUNT RATE IS LOW

There is a clear limit to the punishment which can be imposed on player i in the one-shot game, which we shall refer to as

his minimax level, v_i^M:

$$v_i^M = \min \max \pi_i(a_i, a_i).$$

Let F be the convex hull of the set of feasible payoffs and let P be that part of F which lies no lower than the minimax levels:

$$P = F \bigcap \{ \pi: \pi_i \ge v_i^M \text{ for all } i \}$$

Thus U can never be greater than P, and the various folk theorems demonstrate how U approximates P when there is no discounting or for low discount rates. Of interest here is the theorem of Fudenberg and Maskin (5): for any $\pi \epsilon P$ with $\pi_i > v_i^M$, there is a $\overline{\delta}$, $0 < \overline{\delta} < 1$, such that if $\overline{\delta} < \delta < 1$ then $\pi \epsilon U$. The method of proof used (for the two-player case) is to construct (simple) punishments consisting of a number of periods of mutual minimaxing and then reverting to the proposed equilibrium. The argument only works for payoffs strictly greater than minimax levels.

There is one case where it is clear that the minimax level can never be attained from one of the players in equilibrium. This is where for one i there is no $\pi \epsilon P$ with $\pi_i = v_i^M$. Notice that payoffs have to be quite asymmetric for this to occur. The following rules out such games.

Condition A: There exists $\tilde{\pi}^i \in P$ such that $\tilde{\pi}^i_i = v^M_i$ and $\tilde{\pi}^i_{-i} > v^M_{-i}$, for i = 1, 2.

This says that for each player there is some payoff in P which delivers the minimax payoff except that we are excluding the knife-edge case where (v_i^M, v_{-i}^M) is the only such point for one of the players.

From now on it will simplify matters to normalise payoffs in G so that $v_i^M = 0$ for all i. This does not alter the structure of the game in any way. After normalisation $(1 - \delta) \pi(a) \ge d(a)$ for all a ε A. Let $A_i^M = \arg \min_{a_i} \max_{a_{-i}} \pi_{-i}(a_i, a_{-i})$, that is the set of i's actions which minimax his opponent. When Condition C below holds, one of the players can never be minimaxed; Condition B, which is weaker, is required for the proof.

Condition B: For either i = 1 or 2, for all $a_i^M \epsilon A_i^M$ and all $a_{-i} \epsilon A_{-i}$, $\pi_{-i}(a_i^M, a_{-i}) = 0$.

Proposition 1: Assume Condition A is satisfied. Then there exists a $\overline{\delta}$, $0 < \overline{\delta} < 1$, such that if $\overline{\delta} < \delta < 1$ then $\underline{v}_i = 0$ for i = 1 and 2 (that is, minimax payoffs are equilibrium payoffs) if and only if Condition C does not hold.

Proof. (i) We start by showing that $\underline{v}_i = 0$, i = 1 and 2, for δ near 1, under the stronger assumption that Condition B does not hold. This implies that this is some \hat{a}_i with $\pi_i(a_{-i}^M, \hat{a}_i) < 0$, i = 1, 2. For the moment concentrate on player 2. By the definition of condition A there is a $\tilde{\pi}^2 \in P$ such that $\tilde{\pi}_2^2 = 0$ and $\tilde{\pi}_1^2 > 0$, and

suppose that $\tilde{\pi}^2$ is in fact the largest such vector (P is closed). Either $\tilde{\pi}^2$ is a convex combination of two payoff vectors of the game G, say π' and π'' where $\pi'_2 < 0$ and $\pi''_2 > 0$, or $\tilde{\pi}^2$ itself is a payoff vector of G and by the existence of $\tilde{\pi}^1$ (defined symmetrically) there is another payoff vector of G, π'' , with $\pi''_2 > 0$. In the latter case denote $\tilde{\pi}^2$ by π' to allow a uniform treatment. Let d' and d' be the respective deviation vectors associated with π' and π'' . Consider a chord between $\pi(a_1^M, a_2)$ and $\tilde{\pi}^2$. Since $\tilde{\pi}_1^2 > 0$ it is possible by continuity to find a set of chords also emanating from $\pi(a_1^M, a_2)$ but ending higher up the chord between $\pi'_{K\pi''}$ than $\tilde{\pi}^2$ but still intersecting the $\pi_2 = 0$ axis where $\pi_1 > 0$. Formally define $\pi^X \equiv \lambda \pi'' + (1-\lambda)\pi'$ to be the end-point, and the chord itself by

$$C(\lambda) = \{ \pi: \pi = \mu \pi^{\lambda} + (1-\mu)\pi(a_1^M, a_2^{\lambda}); 0 \le \mu \le 1 \}$$

So there is a set of these chords $\{C(\lambda): \lambda' \leq \lambda \leq \lambda''\}$, where $0 < \lambda' < \lambda'' < 1$, such that $(\pi_1, 0) \in C(\lambda)$ implies $\pi_1 > 0$, and moreover the end-points π^{λ} lie in the interior of the positive orthant $(\pi'' \text{ itself need not})$. This is illustrated in Figure 1.

[FIGURE 1 ABOUT HERE]

Notice that the existence of this set could not be asserted if $\pi_2(a_1^M, \hat{a}_2) = 0$. Let E_2 be the set of endpoints: $E_2 = {\pi^{\lambda} : \lambda \leq \lambda \leq \lambda''}$. Fix a particular value of $\lambda(\lambda \leq \lambda \leq \lambda'')$, and add to E_2 a sequence of points in the non-negative part of $C(\lambda)$:

$$\begin{split} \mathbf{X}_{2}(\lambda) &= \mathbf{E}_{2} \bigcup \{ \mathbf{\pi}^{r} : \mathbf{\pi}^{r} = \delta^{r} \mathbf{\pi}^{\lambda} + (1 - \delta^{r}) \mathbf{\pi}(\mathbf{a}_{1}^{M}, \mathbf{a}_{2}^{\lambda}), \ r = 0, \ 1, \ 2 \dots, \ \mathsf{R}; \\ \text{where } \mathsf{R} \text{ satisfies } \mathbf{\pi}_{2}^{R} \geq 0, \ \mathbf{\pi}_{2}^{R+1} < 0 \}. \end{split}$$

We shall show that for δ near unity such a set, together with a similarly defined set $X_1(\lambda)$ for player one, is self-generating whilst including minimax payoffs. We shall be particularly interested in the intersection of $C(\lambda)$ with the $\pi_2 = 0$ axis: identify this point by the weight $\mu(\lambda)$, $0 < \mu(\lambda) < 1$. So $\mu(\lambda) \pi_2^{\lambda} + (1-\mu(\lambda)) \pi_2(a_1^{M}, a_2) = 0.$ It is an elementary fact that μ is strictly increasing in λ , hence $\mu(\lambda') > \mu(\lambda')$ and $\{\mu(\lambda): \lambda \leq \lambda \leq \lambda''\} = [\mu(\lambda'), \mu(\lambda'')]. \text{ Suppose that } \delta \geq \mu(\lambda')/\mu(\lambda'').$ Then it is possible to find an integer r > 0 such that $\delta^r \epsilon [\mu(\lambda'),$ $\mu(\lambda'')$ (since otherwise there must be some integer r such that $\delta^r > \mu(\lambda'')$ and $\delta^{r+1} < \mu(\lambda')$ which implies $\delta < \mu(\lambda')/\mu(\lambda'')$. Thus for this r we can find λ such that $\delta^{r} = \mu(\lambda)$, and denote this by $\lambda_2(\delta)$. The point of all this is that $X_2(\lambda_2(\delta))$ now includes a payoff which gives player two zero. To show that $X^{*}(\delta) = X_{2}(\lambda_{2}(\delta)) \bigcup X_{1}(\lambda_{1}(\delta))$ self-generates for δ high enough, let $\delta \geq 1/(1 + \lambda'' - \lambda')$. Ignore constraint (2) for the moment. Take any point π^{λ} in E₂. Then there is some point $\delta \pi^{\lambda}$ in δE_2 such that either $\pi^{\lambda} = (1 - \delta) \pi' + \delta \pi^{\lambda}$ or $\pi^{\lambda} = (1 - \delta) \pi' + \delta \pi^{\lambda}$. To see this first let $\hat{\lambda}$ satisfy $\pi^{\lambda} - (1 - \delta) \pi' = \delta \pi^{\lambda}$, which implies that $\dot{\lambda} = \lambda/\delta$, so $\dot{\lambda} > \lambda'$ (since $\lambda \ge \lambda'$). If also $\dot{\lambda} \le \lambda''$ then $\pi^{\dot{\lambda}} \in E_2$ and we will have found a $\pi^{\hat{\lambda}}$ that will do. So assume that this is not the case, i.e. $\hat{\lambda} > \lambda''$, that is $\lambda/\delta > \lambda''$. Now let $\hat{\lambda}$ satisfy $\pi^{\lambda} - (1 - \delta) \pi'' = \delta \pi^{\lambda}$, which implies $\hat{\lambda} = \lambda/\delta - (1 - \delta)/\delta$, or $\bigwedge_{\lambda} = \lambda - (1 - \delta)(1 - \lambda)/\delta \text{ so } \bigwedge_{\lambda} < \lambda'' \text{ (since } \lambda \leq \lambda''). \text{ Also}$

$$\lambda = \lambda/\delta - (1 - \delta)/\delta > \lambda'' - (1 - \delta)/\delta \ge \lambda'$$

where the first inequality follows from $\lambda/\delta > \lambda''$ and the second from $\delta \ge 1/(1 + \lambda'' - \lambda')$. Hence $\pi^{\lambda} \in E_2$ as required, and so $E_2 \subseteq T(E_2)$ provided (2) is ignored, and likewise for E_1 . Next consider the points π^r in the definition of $X_2(\lambda)$. We have for r > 0, $\pi^r = \delta \pi^{r-1} + (1 - \delta) \pi(a_1^M, a_2^A)$, and $\pi^o \in E_2$. Hence $X_2(\lambda) \subset T(X_2(\lambda))$. Finally we have to worry about (2). For $\delta \geq \max [\mu(\lambda')/\mu(\lambda''), 1/(1 + \lambda'' - \lambda')], X_2(\lambda_2(\delta))$ contains a payoff π^{r} with $\pi_{2}^{r} = 0$. Likewise for δ above some point $X_{1}(\lambda_{1}(\delta))$ contains a π^{S} with $\pi_{l}^{S} = 0$. Consequently for sufficiently high δ the set $X^*(\delta) \equiv X_2(\lambda_2(\delta)) \bigcup X_1(\lambda_1(\delta))$ touches both axes. Provided the deviation vector associated with each addition discussed above does not extend past either axis then (2) is satisfied since $\underline{v} = (0, 0)$. Consider first \mathbf{q}' and d", associated with additions respectively of π' and π'' to E_2 . Since E_2 lies in the interior of the positive orthant we only have to choose δ sufficiently high for (2) to hold. Likewise for δ high enough $d_1(a_1^M, a_2^A)$ causes no problem for the construction of the π 's since they are all bounded away from the axis $\pi_1 = 0$. Notice also that $d_2(a_1^M, a_2) = (1 - \delta) \pi_2(a_1^M, a_2)$ so all π^r 's are formed by admissable additions (even, crucially, where $\pi_{0}^{l^{\circ}} = 0$: $\pi^{r} = \delta \pi^{r-1} + (1 - \delta) \pi (a_{1}^{M}, a_{2}^{A})$. There must therefore be some $\overline{\delta}$ such that $X^{\star}(\delta) \subseteq T(X^{\star}(\delta))$ for $\delta > \overline{\delta}$, and $X^{\star}(\delta)$ contains minimax payoffs for both players. Appealing to Lemma One comletes the proof of (i).

(ii) We consider what happens when B holds. Suppose w.l.o.g. that it is player 2 who is indifferent about his response, that is

$$\pi_2(a_1^M, a_2) = 0$$
 for all $a_2 \in A_2$, all $a_1^M \in A_1^M$.

(a) If C does not hold, then either $\pi_1(a_1^M, a_2^M) = 0$ for some $a_2^M \varepsilon A_2^M$, in which case (a_1^M, a_2^M) is a Nash-equilibrium of G and both players can be minimaxed in G^{∞} for any δ , or there is a response a_2 such that $\pi_1(a_1^M, a_2) > 0$. For δ high enough $\pi(a_1^M, a_2) \bigcup X_1(\lambda_1(\delta))$ self-generates: the argument for $X_1(\lambda_1(\delta))$ is the same as before; to minimax player 2 we only have to consider $\pi(a_2^M, a_2) = \delta \pi(a_1^M, a_2) + (1 - \delta) \pi(a_1^M, a_2)$, and $d_2(a_1^M, a_2) = 0$ while $\delta \pi_{\underline{f}}(a_1^M, a_2) + d_1(a_1^M, a_2) \ge 0$ for δ high enough so (1) and (2) are both satisfied.

(b) If C holds, $\pi_1(a_1^M, a_2) \leq 0$ for all $a_2 \in A_2$. Suppose that in fact player two can be minimaxed in G° for some δ ; thus there is a $\pi \in U$ such that $\pi_2 = 0$. By Lemma 2 $\pi = \delta \pi + (1 - \delta) \pi(\tilde{a})$ for some $\tilde{a} \equiv A$, and $\delta \pi + d(\tilde{a}) \geq 0$. It must be the case that $\tilde{a}_1 = a_1^M$ (which need not be unique), since

$$d(\tilde{a}) \stackrel{>}{=} - \delta \pi = -\pi + (1 - \delta) \pi(\tilde{a});$$

hence $d_2(\tilde{a}) \ge (1 - \delta) \pi_2(\tilde{a})$, which means that player two can only guarantee himself a non-positive payoff. By the assumption that $\pi_2(a_1^M, a_2) = 0$, we have $\pi_2(\tilde{a}) = 0$ so $\pi_2 = 0$. Also by the assumption that $\pi_1(a_1^M, a_2) \le 0$, $\pi_1 < \delta \pi_1$. Since $\pi \in U$ and $\pi_2 = 0$ we can repeat the argument using π instead of π . If $\pi_1 > 0$ then eventually successive repetitions of the argument will yield a point which lies outside P (each time π_1 rises by a factor of at least $1/\delta$) and there is a contradiction. If on the other hand it had been

the case that $\pi_1 = 0$ then either $\pi_1 > 0$, in which case the above argument still applied (start from π), or $\pi_1 = 0$, but this would imply that \tilde{a} was a Nash equilibrium with $\pi(\tilde{a}) = 0$, contrary to assumption.

Q.E.D.

It is straightforward to derive strategies which support any point in a set which regenerates as an equilibrium by using the admissable actions implicit in the application of T. For example to minimax player two in the generic case when B does not hold, (a_1^M, a_2^n) is followed for R periods (see part (i) of the proof). Thereafter the actions with payoffs π' and π'' , say a' and a'' (respectively), are used exclusively in such a way as to keep future utility in E_2 . Specifically after R periods the future utility of adhering to the proposed strategy is $\pi^{\lambda} \in E_{2}$. As was shown in the proof, either $\pi^{R+1} = (\pi^{\lambda} - (l - \delta) \pi')/\delta \epsilon E_2$ in which case follow a' in period R + 1 or $\pi^{R+1} = (\pi^{\lambda} - (1 - \delta) \pi')/\delta \epsilon E_2$ in which case follow a". Whichever is chosen, the same process is followed for π^{R+1} allowing an action for period R + 2 to be determined and so on. If player two ever deviates from this, go back to the beginning again and if player one deviates start the similar strategy which minimaxes him.

Example 1 is a case where player two can never be minimaxed and the set of equilibria is illustrated in Figure 2 for $\delta = .95$, for which $\underline{v}_2 = .2195$. The larger arrow represents d(a) and whenever, starting from a point in δU , this does not extend below $\delta \underline{v}$, the addition of $(1 - \delta) \pi(a)$ (the finer arrow) is admissable and the sum belongs to T(U).

Example 1.

Player 2

Player l	1,1	0,-5	
	-5,0	-4,0	

[[]FIGURE 2 ABOUT HERE]

Of course whenever condition A is satisfied the Folk Theorem asserts that the minimax payoff can nevertheless be approached for high δ .

There is one last, unlikely, case to be dealt with. As mentioned earlier, even if Condition A is failed we cannot rule out the possibility of being able to minimax both players in the extreme case where (0, 0) ϵ P. First assume that $P \bigwedge R_{++}^2 \neq \phi$, that is P contains an interior point of the positive orthant. Then the necessary and sufficient condition for the possibility of asymptotic minimaxing of both players is that there exists a $\mu > 0$ such that $-\mu\pi(a_1^M, a_2^M) \in P$. (Recall that $\pi(a_1^M, a_2^M)$ is the payoff when both players attempt to minimax each other). If a_1^M is not unique then the condition only has to hold for some (a_1^M, a_2^M) . To establish necessity assume that $(0, 0) \in U$ for some δ (if Condition A fails, as assumed, this is necessary for both players to be minimaxed). Then by Lemma 2 there is some $\pi \in U$ and $a \in A$ such that $(0, 0) = (1 - \delta) \pi(a) + \delta \pi$, and $d(a) + \delta \pi > 0$. So $d(a) \ge (1 - \delta) \pi(a)$, but since $d(a) \le (1 - \delta) \pi(a)$ (by the $\underline{v}_i = 0$ normalisation) we have $d(a) = (1 - \delta) \pi(a)$. This says that each player can at most get zero by deviating, hence both are being simultaneously minimaxed; $a = (a_1^M, a_2^M)$. So $-(1 - \delta) \pi(a_1^M, a_2^M) = \delta \pi \varepsilon \delta U \subseteq \delta P$. Hence $\mu = (1 - \delta)/\delta$ is as

required. To prove sufficiency is more difficult. The condition implies the existence of a ε A, $\mu > 0$ such that $-\mu\pi(a_1^M, a_2^M) = \pi(a)$. This is demonstrated as follows. Τf $\pi(a_1^M, a_2^M) = (0, 0)$ there is no problem; in fact (a_1^M, a_2^M) is a Nash-equilibrium. Otherwise consider the largest μ such that $-\mu\pi(a_1^M, a_2^M) \in P$ (it clearly exists). It must be a convex combination of $\pi(\tilde{a})$ and $\pi(\tilde{a})$ say. If $\pi_1(a_1^M, a_2^M) \neq 0$, i = 1 and 2, then if neither $\pi(\tilde{a})$ nor $\pi(\tilde{a})$ are multiples of $\pi(a_1{}^M,\,a_2{}^M)$ Condition A must be satisfied contrary to assumption, so one must be a multiple of $\pi(a_1^M, a_2^M)$; and a, μ are as required. If $\pi_i(a_1^M, a_2^M) = 0$ for one i only then a convex combination of the assumed interior point and $\pi(a_1^M, a_2^M)$ will ensure Condition A holds, again contrary to assumption. The rest of the proof is as in Proposition 1, except now $\pi\,(a_1{}^M,\,a_2{}^M)\,$ plays the role of both $\pi(a_1^M, a_2^N)$ and $\pi(a')$, while $\pi(a)$ substitutes for $\pi(a')$. Because $-\mu\pi(a_1^M, a_2^M) = \pi(a)$ all the chords of type $C(\lambda)$ pass through (0, 0), and so it is this point which is shown to belong to U. A game for which the conditions fails is given in Example 2.

Example 2.

Player 2

-2,-2

1,1

-3,0

0,2

-1,1

-2,-1

Player]
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Lastly, and most obscurely, if $P \bigwedge R_{++}^2 = \phi$ then the necessary and sufficient condition for minimaxing both players is that either $\pi(a_1^M, a_2^M) = 0$ (so there is a Nash equilibrium at (0, 0)) or there is some $\epsilon A, \mu > 0$, such that $\pi(a) = \mu \pi(a_1^M, a_2^M)$ and $a_i = a_i^M$ for one i. This condition is in fact only a strengthening of the previous one to cope with the fact that P is confined, at most, to one of the axes, and the argument is very similar and is omitted. This completes the analysis.

Provided there is some equilibrim existing in the limit (true whenever $P \bigwedge R_{++}^2 \neq \phi$) then it should be clear from the above arguments that <u>one</u> of the players can always be minimaxed in equilibrium for δ high enough.

While is is usually possible to achieve minimax payoffs as we have seen, it does not follow for such games that U = P for δ near one. Problems may still appear at corners, as illustrated by Example 3, Figure 3 for $\delta = .9$. The problem diminishes as δ approaches one, but does not go away.

Example 3.

Player 2

Player l	2,-1	0,1
	0,-3	-1,0

[[]FIGURE 3 ABOUT HERE]

IV. 2X2 SYMMETRIC GAMES

The general results given above concern only low discount rates, and it may be that in many games minimax punishments can also be imposed at relatively high discount rates. As an example,

symmetric games with two actions will be considered, and assuming the payoffs are uniformly distributed over the unit interval, the probability that minimax punishments can be imposed will be calculated as a function of the discount factor.



	A	В
A	a,a	b,đ
В	d,b	c,c

and a, b, c, d are all assumed to be independently and uniformly distributed over [0,1]. With probability 0.5, max (a,d) \leq max (b,c), in which case strategy A is used to minimax the other player, and we concentrate on this case (the other case being symmetric). Then if a \geq d (conditional probability of 0.5), a is the minimax utility level and (A,A) is a Nash equilibrium, so minimaxing is possible in the stage-game. If a < d then the minimax utility is d, and moreover if b \geq c (conditional probability of 0.25) then (B,A) and (A,B) are also Nash equilibria. So altogether minimax is a Nash equilibrium of the stage-game in 75% of cases.

We are left with the case a < d, b < c and max $(a,d) \leq max$ (b,c), for which minimaxing is not an equilibrium of the stage game. The last inequality can be replaced by $d \leq c$. For simplicity it is assumed that publicly correlated strategies are permitted. Normalise, as before, by deducting the minimax utility d from each payoff: $a^* = a - d$, $b^* = b - d$, $c^* = c - d$. So we are considering the case $a^* < 0$, $c^* \geq 0$ and $b^* < c$.*

Allowing publicly correlated strategies means that the new set-valued mapping, say $T^{C}(X)$, is simply the convex hull of the previously defined mapping T(X). With this new mapping, Lemmas 1 and 2 continue to hold. We shall consider minimaxing player 2. If this is possible, then from Lemma 2 there exists a $\hat{v} \in U$, $\hat{a} \in A$, such that $0 = (1 - \delta) \pi_{2}(\hat{a}) + \delta \hat{v}_{2}$ and $\delta \hat{v} + d(\hat{a}) \geq (0, 0)$. $\hat{a} = (B,B)$ can be ruled out, since this would imply $\hat{v}_{2} < 0$, which is inconsistent with $\hat{v} \in U$. $\hat{a} = (A,A)$ is a possibility, but if (A,A) is admissable at all, it must be from (c^{*}, c^{*}) since this lies farthest from the origin. The condition for this is

$$\delta c^* + (1 - \delta) a^* > 0$$
 (3)

But when (3) holds, the line segment betwen the origin and (c^*, c^*) is self-generating: $(c^*, c^*) = (1 - \delta) \pi(B,B) + \delta(c^*, c^*)$, where this addition is admissable since $d(B,B)^{\bullet} = (0,0)$; and $(0,0) = (1 - \delta)$ $\pi(A,A) + \delta(v,v)$ where $v = -(1 - \delta) a^*/\delta$, and under (3), $0 < v \le c^*$ so this addition is also admissable. So (c^*, c^*) and (0,0) belong to T, hence T^C, the convex hull, is the original segment. Thus (3) is necessary and sufficient for (A,A) to be used in achieving minimax punishments.

The other possibility is $\hat{a} = (A,B)$. Since $v_1 \leq b$, this can only be used when $\delta b^* + (1 - \delta)(b^* - c^*) \geq 0$, that is when

$$b^* \ge (1 - \delta) c^* . \tag{4}$$

When this holds, however, the set $\{(0,c^*), (c^*, 0)\}$ is clearly self-generating. So minimaxing is possible if and only if either of (3) and (4) holds.

Then the unconditional probability, for a given value of δ , of the case a < d, b < c and d < c occurring and minimaxing being possible is

$$\begin{aligned} 1 & c & c & max(\delta c + (1 - \delta) a, c - (c - b)/\delta) \\ \int \int \int a & 1 & dd & da & db & dc \\ & = \frac{\delta}{24} (4 - 2\delta + \delta^2) . \end{aligned}$$

Hence, overall, minimaxing is possible with a probability of $3/4 + (\delta/12)(4 - 2\delta + \delta^2)$. For example, even when the discount rate is as high as 10% (δ = .909), minimaxing is still possible in 97.79% of all games.

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FOOTNOTES

1/ Available on request from the author. An independent characterisation of U is also given: let F be the convex hull of all feasible payoffs; then $U = \bigcap_{r=0}^{\infty} T^{r}(F)$.

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FIGURE 2

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FIGURE 3