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and Regression Dependence

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Hedging of exchange rate risk and regression dependence

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Abstract

The paper presents a model of a risk-averse exporting firm under exchange rate risk. We focus on the economic implications of basis risk. It is shown that the regression dependence assumptions between spot and futures exchange rates are essential in analyzing optimal hedging and export decisions. When the spot exchange rate and the futures exchange rate are imperfectly correlated we show that the firm adopts an over hedge when the exchange rate risk exposure is convex and an under hedge when the risk exposure is concave given the unbiasedness of the currency futures market.

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Hedging of exchange rate risk
and regression dependence

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The paper presents a model of a risk-averse exporting firm under exchange rate risk. We focus on the economic implications of basis risk. It is shown that the regression dependence assumptions between spot and futures exchange rates are essential in analyzing optimal hedging and export decisions. When the spot exchange rate and the futures exchange rate are imperfectly correlated we show that the firm adopts an over hedge when the exchange rate risk exposure is convex and an under hedge when the risk exposure is concave given the unbiasedness of the currency futures market.

JEL classification: F21, F31

Keywords: exchange rate risk, hedging, regression dependence

1. Introduction

Exchange rate risk management has received increasing attention nowadays when more and more companies of all sizes and of all industries source and sell abroad. Corporations take it very seriously as response to the continuing incidence of exchange rate fluctuations of major currencies (see, e.g., Meese, 1990) and to its high visibility driven by accounting rules and regulations. Indeed, as documented in a survey by Rawls and Smithson (1990), it is ranked by financial managers as one of their primary objectives.

1The Financial Accounting Standards Board requires transaction and translation gains and losses to be reported in income statements. The Securities and Exchange Commission, on the other hand, requires statements explaining the impact of exchange rate movements on reported profits to be included in annual reports.
Hedging and nonlinear risk exposure

Given the real-world prominence of exchange rate risk management, there have been a great many papers concerning the production and hedging decisions of a risk-averse exporting firm facing exchange rate uncertainty. Two notable results have emerged in the literature. First, the 'separation property' states that the firm's production decision is affected neither by the firm's attitude towards risk nor by the incidence of exchange rate volatility if the firm has access to a foreign currency futures market. Second, the 'full hedging property' states that the firm should eliminate all exchange rate risk by holding a full hedge position if the foreign currency futures market is unbiased.

The purpose of this paper is to re-examine the optimality of the separation and full hedging properties when basis risk exists, i.e., when spot exchange rate and futures exchange rate are imperfectly correlated, possibly in a nonlinear manner. As summarized by Sercu and Uppal (1995, p. 136), the two major sources of basis risk are that (i) expiration dates of futures contracts do not match the maturity of inflows/outflows in foreign currency, and (ii) underlying assets in futures markets do not match the currency one wishes to hedge. Hedging in such an environment is de facto imperfect. It is referred to as a delta hedge when maturities are mismatched and as a cross hedge when currencies are mismatched.

Our model features a competitive firm which exports all of its production to a foreign country. The firm is risk averse and has access to a foreign currency futures market to hedge against its exchange rate risk exposure. Basis risk is modelled by assuming that the spot exchange rate, \( \varepsilon \), is regressive on the futures price, \( \hat{f} \). Specifically, \( \varepsilon = G(\hat{f}) + \tilde{\varepsilon} \), where \( \tilde{\varepsilon} \) is a zero-mean random variable independent of \( \hat{f} \), and \( G(\cdot) \) is an increasing function.

In this setting, we show that the separation property no longer holds. Indeed, the firm produces less than that with perfect hedging as a response

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2See, e.g., Benninga, Eldor, and Zilcha (1985), Kawai and Zilcha (1986), and Broil and Zilcha (1992), to name just a few.

3This property was first established by Danthine (1978) and Holthausen (1979) under price uncertainty. It is a rather surprising result in light of the seminal works of Baron (1970) and Sandmo (1971) which show that the production decision is affected by risk factors and preferences in the absence of hedging opportunities.

4The fixed size nature of futures contracts may also give rise to basis risk.
to the introduction of basis risk. The full hedging property holds if, and only if, the firm's exchange rate risk exposure is linear (i.e., \( G \) is linear).

Given rather reasonable assumptions on the firm's preference and linear exchange rate risk exposure, we show that the firm's optimal hedge position is always smaller than that with perfect hedging when the foreign currency futures market exhibits either unbiasedness or contango. The full hedging property breaks down when the firm's exchange rate risk exposure is nonlinear. Succinctly, we show that the firm adopts an over hedge when the exposure is convex (i.e., \( G'' > 0 \)) and an under hedge when the exposure is concave (i.e., \( G'' < 0 \)) given the unbiased foreign currency futures market.

The organization of the paper is as follows. The next section 2 describes the model of a risk-averse competitive exporting firm facing basis risk and nonlinear exchange rate risk exposure. Section 3 presents some benchmark results in the absence of basis risk. Section 4 examines the production and hedging decisions of the firm in face of basis risk and linear exchange rate risk exposure. Section 5 extends the analysis to the case under nonlinear exchange rate risk exposure. The last section offers some conclusions.

2. Imperfect hedging and regression specification

Consider a competitive firm which has a planning horizon of one period, starting at date 0 and ending at date \( T \). The firm produces in its home country a single good at a cost of \( C(Q) \), where \( Q \) is the quantity produced, and \( C \) is a twice continuously differentiable function with \( C(0) > 0 \), \( C'' > 0 \), and \( C'' > 0 \). The entire output of the firm is sold to a foreign country at a price denominated in the foreign currency. The selling price per unit

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5 The collapse of the separation property is consistent with the findings in a number of recent papers which address the impact of imperfect hedging on production. See, e.g., Broll, Wahl, and Zilcha (1995), Broll and Wahl (1996), and Adam-Müller (1997).

6 Benninga, Eldor, and Zilcha (1983, 1984) show that the validity of the full hedging property depends only on the independence assumption. Lence (1995) further shows that what is really needed is the much weaker assumption of conditional independence.

7 The reasonable assumptions on the firm's preference are the decreasing absolute risk aversion of Arrow (1965) and Pratt (1964) and the standard risk aversion of Kimball (1993).
is exogenously determined and is, without loss of generality, normalized to unity.

The spot exchange rate at date $T$, $\bar{e}$, specifies the amount of the domestic currency for which one can exchange one unit of the foreign currency at date $T$. Since the firm does not know \textit{ex ante} the \textit{ex post} realization of $\bar{e}$, it faces an exchange rate risk exposure of $Q$. The firm, however, has access to a currency futures market which trades infinitely divisible futures contracts for the foreign currency. Each futures contract obligates the holder to deliver one unit of the foreign currency at date $T_1$ at the pre-specified futures price, $f_0$. Transactions in the currency futures market are costless because there are no commission fees, margin requirements, or capital outlays.

Denote $\hat{f}$ as the spot exchange rate at date $T_1$. If the expiration date of futures contracts does not match the maturity of the foreign currency cash flow to which the firm is exposed (i.e., $T_1 \neq T$), basis risk existing as $\bar{e}$ need not equal $\hat{f}$. Given this maturity mismatch, hedging with futures contracts is \textit{de facto} imperfect and is referred to as a delta hedge.\footnote{The qualitative results of the paper remain unchanged if we consider a cross hedge \textit{vis-à-vis} a delta hedge when futures contracts are available for a "related" currency, rather than for the foreign currency.}

In a similar spirit as that of Benninga, Eldor, and Zilcha (1983, 1984), we model the basis risk by assuming that $\bar{e}$ is regressee on $\hat{f}$, possibly in a nonlinear manner:\footnote{Wong (1997) adopts a similar regression dependence structure to explain the real determinants of bank interest margins under credit and interest rate risks.}

$$\bar{e} = G(\hat{f}) + \tilde{\epsilon}, \quad (1)$$

where $G$ is an increasing and deterministic function, and $\tilde{\epsilon}$ is a zero-mean random variable independent of $\hat{f}$. According to Ingersoll (1987, p. 15), $\hat{f}$ and $\tilde{\epsilon}$ are independent if, and only if, $\text{Cov}[M(\hat{f}), N(\tilde{\epsilon})] = 0$ for any pair of functions $M$ and $N$.

The terminal (i.e., date $T$) profit of the firm, $\Pi$, is given by

$$\Pi = [\bar{e}Q - C(Q)] + H(f_0 - \hat{f}), \quad (2)$$

where $H$ is the number of futures contracts sold (purchased if negative) by the firm. The first term in the right-hand side of equation (2) is the firm's...
profit from operations, while the second term is the gain or loss due to the firm's position in the currency futures market.

Note that the firm's terminal profit is linear in \( \bar{f} \) if, and only if, \( G \) is a linear function of \( \bar{f} \). We refer to this case as the case in which the firm faces linear exchange rate risk exposure. Otherwise, the firm faces nonlinear exchange rate risk exposure in general, and convex (concave) exchange rate risk exposure when \( G'' > (\leq) 0 \) in particular.

The firm is risk averse and possesses a von Neumann-Morgenstern utility function, \( U \), which is thrice continuously differentiable with \( U' > 0 \) and \( U'' < 0 \). Before any uncertainty is resolved, the firm chooses an output level, \( Q \), and a hedge position, \( H \), so as to maximize its expected utility of terminal profit:

\[
\max_{Q, H} E[U(\bar{\Pi})],
\]

where \( \bar{\Pi} \) is defined in equation (2).

### 3. Benchmark results with perfect hedging

In this section, we consider the benchmark in which hedging is perfect. It is the case when, e.g., futures contracts mature at date \( T \) so that the maturity mismatch disappears. By means of the convergence property, the regression specification, (1), has to be replaced by \( \bar{e} = \bar{f} \). Thus, in this benchmark case, the first-order conditions for program (3) are given by\(^{10}\)

\[
E\{U'(\bar{\Pi}^0)[\bar{f} - C'(Q)]\} = 0,
\]

\[
E[U'(\bar{\Pi}^0)(f_0 - \bar{f})] = 0,
\]

where \( \bar{\Pi}^0 = \bar{f}Q - C(Q) + H(f_0 - \bar{f}) \). Let \((Q^0, H^0)\) be the solution to the above system of equations. The second-order conditions are easily shown to hold given the concavity of \( U \) and the convexity of \( C \).

Evaluating equations (4) and (5) at \((Q^0, H^0)\) and substituting the latter into the former yields

\[
C'(Q^0) = f_0.
\]

\(^{10}\)Mild regularity assumptions that \( C'(0) = 0 \) and \( C'(\infty) = \infty \) ensure that \( 0 < Q^0 < \infty \).
Equation (6) characterizes the celebrated ‘separation property’ which states that the firm’s optimal output is independent of the probability distribution of the random exchange rate and the attitude towards risk of the firm when a currency futures market is introduced.

To see the intuition underlying the separation property, rewrite the firm’s terminal profit in the absence of basis risk as

\[ \Pi^0 = [f_0Q - C(Q)] + (H - Q)(f_0 - \bar{f}). \]  

(7)

The first term in the right-hand side of equation (7) is the firm’s operating profit wherein the selling price denominated in the domestic currency is locked in at the pre-specified futures price, \( f_0 \), via hedging with futures contracts. The second term is the gain or loss due to the firm’s position in the currency futures market being deviated from a full hedge position (i.e., \( H = Q \)).

It is evident from equation (7) that the firm can completely neutralize the effect of altering its output by \( \Delta Q \) on its exposure of exchange rate risk by simply adjusting its hedge position with exactly the same amount. In other words, the degree of exchange rate risk exposure chosen by the firm should be totally unrelated to the firm’s production decision, thereby invoking the separation property.

For any given output level, \( Q \), equation (5) uniquely determines a hedge position, \( H(Q) \). Substituting \( H(Q) \) into equation (5) and using the property of the covariance operator yields\(^\text{11}\)

\[ E[U'(\Pi^0)][f_0 - E(\bar{f})] = \text{Cov}[U'(\Pi^0), \bar{f}], \]

(8)

where \( \Pi^0 = f_0Q - C(Q) + H(Q)(f_0 - \bar{f}) \). Note that the realization of \( \Pi^0 \) is increasing with, invariant to, or decreasing with the realization of \( \hat{e} \) depending on whether \( H(Q) \) is less than, equal to, or greater than \( Q \). If the currency futures market is unbiased so that the pre-specified futures price, \( f_0 \), equals the expected spot exchange rate, \( E(\bar{f}) \), equation (8) implies a full coverage of the exchange rate risk exposure, i.e., \( H(Q) = Q \). This is the famous ‘full hedging property.’

The intuition is that the unbiased currency futures market essentially provides the firm ‘insurance’ at actuarial terms, rendering full hedging by

\(^{11}\text{For any two random variables, } \tilde{x} \text{ and } \tilde{y}, \text{ Cov}(\tilde{x}, \tilde{y}) = E(\tilde{x}\tilde{y}) - E(\tilde{x})E(\tilde{y}).\)
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the firm optimal. If the currency futures market exhibits contango so that \( f_0 > E(\hat{f}) \), the firm will speculate by holding a more than full hedge position, i.e., \( H(Q) > Q \), hoping to gain from a lower spot exchange rate at expiration. Finally, if the currency futures market exhibits normal backwardation so that \( f_0 < E(\hat{f}) \), the firm will speculate by partially hedging its risk exposure, i.e., \( H(Q) < Q \), expecting to gain from a higher spot exchange rate at expiration.

To summarize, we have the following proposition.

**Proposition 1.** If hedging in the currency futures market involves no basis risk, then the exporting firm’s optimal output, \( Q^0 \), as characterized in equation (6), depends neither on the probability distribution of the random exchange rate nor on the attitude towards risk of the firm. Furthermore, the firm’s optimal hedge position, \( H^0 = H(Q^0) \), as characterized in equation (5), is an over hedge, a full hedge, or an under hedge, depending on whether the currency futures market exhibits contango, unbiasedness, or normal backwardation, respectively.

Proposition 1 simply restates the well-known results derived by Danthine (1978), Holthausen (1979), Katz and Paroush (1979), and others in the hedging literature. These results will serve as some benchmark for later reference.

Before closing this section, let us examine the property of \( H(Q) \) which will prove to be useful in the next section. Substituting \( H(Q) \) into equation (5) and totally differentiating the resulting equation with respect to \( Q \) yields

\[
H'(Q) = -\frac{E[U''(\hat{Q})](f_0 - \hat{f})[\hat{f} - C'(Q)]}{E[U''(\hat{Q})(f_0 - \hat{f})^2]} \]

where \( \hat{Q} = \hat{f}Q - C(Q) + H(Q)(f_0 - \hat{f}) \). Inspection of equation (9) gives rise to the following proposition.

**Proposition 2.** The marginal hedge position is a full hedge, i.e., \( H'(Q) = 1 \), either when \( Q = Q^0 \) or when the currency futures market is unbiased.

**Proof.** Substituting equation (6) into equation (9) yields \( H'(Q^0) = 1 \). If the currency futures market is unbiased, \( H(Q) = Q \) so that \( \hat{Q} \) is no longer a function of \( \hat{e} \). Hence, the second term in the right-hand side of equation (9) vanishes. This completes the proof. \( \square \)
Proposition 2 tells us that, at the optimum, the marginal hedge position is a full hedge. This is true even when the currency futures market is biased. To understand the intuition, inspection of equation (7) reveals that one unit increase in output changes the firm’s operating profit by $f_0 - C'(Q)$, holding the exchange rate risk exposure of the firm constant. At the optimum, this incremental change of operating profit vanishes by equation (6). Thus, there is no reason for the firm to alter its already optimal exposure of exchange rate risk. In other words, the marginal hedge position at the optimum must be a full hedge, irrespective of the characteristics of the currency futures market.

When the currency futures market is biased, the behavior of $H(Q)$ in general depends on the firm’s attitude towards risk. The firm’s utility function is said to exhibit decreasing absolute risk aversion (DARA) if, and only if, the Arrow-Pratt measure of absolute risk aversion, $-U''(\Pi)/U'(\Pi)$, is a decreasing function of $\Pi$.

DARA is regarded as the most reasonable hypothesis concerning risk-taking behavior. It implies that as wealth increases, the firm is apt to take on more risk. Equipped with this reasonable attitude towards risk of the firm, we establish the following proposition.

**Proposition 3.** If the firm’s utility function exhibits decreasing absolute risk aversion (DARA), then, for all $Q < Q^0$, the marginal hedge position is an over hedge, i.e., $H'(Q) > 1$, or an under hedge, i.e., $H'(Q) < 1$, depending on whether the currency futures market exhibits contango or normal backwardation, respectively.

**Proof.** Consider the case where the currency futures market exhibits contango. Then, we know that $H(Q) > Q$. Define $\Pi^0(f)$ as the realization of $\Pi^0$ at $\bar{f} = f$. It follows from $H(Q) > Q$ that $\Pi^0(f)$ decreases with $f$. Thus, given that $U$ exhibits DARA, we have

$$-\frac{U''[\Pi^0(f)]}{U'[\Pi^0(f)]} > (<) - \frac{U''[\Pi^0(f_0)]}{U'[\Pi^0(f_0)]} \quad \text{for} \quad f > (<) f_0.$$

Multiplying $-U''[\Pi^0(f)](f_0 - f)$ to both sides of the above inequality and taking expectations with respect to the distribution function of $\bar{f}$ yields

$$E[U''(\Pi^0)(f_0 - \bar{f})] > \frac{U''[\Pi^0(f_0)]}{U'[\Pi^0(f_0)]}E[U''(\Pi^0)(f_0 - \bar{f})] = 0,$$
where the equality follows from the fact that \( H(Q) \) solves equation (5). Since \( Q < Q^0 \) implies that \( f_0 > C'(Q) \) from equation (6) and the convexity of \( C \), it follows from equation (9) that \( H'(Q) > 1 \). The case in which the currency futures market exhibits normal backwardation can be proved analogously.

The intuition of Proposition 3 is as follows. The variance of the firm's terminal profit is given by

\[
\text{Var}(\Pi_0) = [H(Q) - Q]^2 \text{Var}(\hat{f}).
\] (10)

For \( Q < Q^0 \), equation (6) and the convexity of \( C \) imply that \( f_0 > C'(Q) \). Inspection of equation (7) reveals that one unit increase in output increases the firm's operating profit by \( f_0 - C'(Q) \), holding the exchange rate risk exposure constant. Given DARA, the firm is willing to take on more risk. Differentiating equation (10) with respect to \( Q \) yields

\[
\frac{d\text{Var}(\Pi_0)}{dQ} = 2[H(Q) - Q][H'(Q) - 1]\text{Var}(\hat{f}).
\]

If the currency futures market exhibits contango, we have \( H(Q) > Q \). In order to increase its exchange rate risk exposure in response to the increase in its operating profit, the firm will set \( H'(Q) > 1 \). Similar interpretation applies to the case where the currency futures market exhibits normal backwardation.

4. Linear exchange rate risk exposure

In this section, we resume our original assumption that there is basis risk. Following Benninga, Eldor, and Zilcha (1983, 1984), the regression specification, (1), is taken to be linear. That is,

\[
\tilde{e} = \alpha + \beta \hat{f} + \tilde{e},
\]

where \( \alpha \geq 0 \) and \( \beta > 0 \) are constants. To simplify the notation, we set \( \alpha = 0 \) and \( \beta = 1 \) because no additional insights are gained with arbitrarily chosen \( \alpha \) and \( \beta \).

The first-order conditions for program (3) in this case are given by

\[
EU_Q \equiv E[U'(\Pi)[\hat{f} + \tilde{e} - C'(Q)]] = 0,
\] (11)

\[
EU_H \equiv E[U'(\Pi)(f_0 - \hat{f})] = 0,
\] (12)
where $\tilde{I} = (\tilde{f} + \tilde{e})Q - C(Q) + H(f_0 - \tilde{f})$. Let $EU_{QK}^*$ and $EU_{HK}^*$ be the partial derivatives of $EU_Q$ and $EU_H$ with respect to $K = Q$ and $H$, evaluating at $(Q^*, H^*)$, the solution to the above system of equations. The second-order conditions are

$$EU_{QQ}^* < 0, \ EU_{HH}^* < 0, \text{and} \ EU_{QH}^* EU_{HQ}^* > 0. \quad (13)$$

The first two conditions are easily shown to hold given the concavity of $U$ and the convexity of $C$. The last condition is assumed to hold.\(^{12}\)

### 4.1. Production decision

We focus first on the effect of imperfect hedging and linear exchange rate risk exposure on the firm's production decision. That is, we want to compare $Q^*$ with $Q^0$.

Evaluating equations (11) and (12) at $(Q^*, H^*)$ and substituting the latter into the former yields

$$\tilde{I}^* = (\tilde{f} + \tilde{e})Q^* - C(Q^*) + H^*(f_0 - \tilde{f}).$$

Using the property of the covariance operator, the above equation can be written as

$$\text{E}[U'(\tilde{I}^*)][f_0 - C'(Q^*)] = -\text{Cov}[U'(\tilde{I}^*), \tilde{e}]. \quad (15)$$

Given the fact that $\tilde{I}^*$ increases with $\tilde{e}$ and $U'' < 0$, the covariance term is negative. Thus, we have

$$C'(Q^*) < f_0. \quad (16)$$

The following proposition is an immediate consequence of equations (6) and (16) and the convexity of $C$.

**Proposition 4.** If hedging with futures contracts is imperfect and exchange rate risk exposure is linear, the optimal output of the firm is less than that with perfect hedging, i.e., $Q^* < Q^0$.

\(^{12}\)When the currency futures market is unbiased, it will be shown easily by the results in section 5 that this condition holds.
Inspection of equation (15) reveals that $Q^*$ depends on both the probability distribution of the random exchange rate and the attitude towards risk of the firm. In other words, the separation property fails to hold with imperfect hedging. This result is consistent with the findings by Broil, Wahl, and Zilcha (1995), Broil and Wahl (1996), and Adam-Müller (1997).

To elucidate the intuition underlying Proposition 4, rewrite the firm's terminal profit in this case as

$$
\Pi = f_0 Q - C(Q) + (H - Q)(f_0 - \hat{f}) + \hat{\varepsilon}Q. \tag{17}
$$

Comparing this with equation (7), the additional term, $\hat{\varepsilon}Q$, due to the presence of basis risk makes the firm's terminal profit riskier. This observation is more transparent if we look at the variance of the firm's terminal profit:

$$
\text{Var}(\Pi) = (H - Q)^2 \text{Var}(\hat{f}) + Q^2 \text{Var}(\hat{\varepsilon}). \tag{18}
$$

The first term in the right-hand side of equation (18) can be reduced through trading in the currency futures market. In stark contrast, the second term can be reduced only by lowering the firm's output and hedging with futures contracts plays no role. To wit, $\hat{f}$ is a hedgeable risk whereas $\hat{\varepsilon}$ is an unhedgeable risk. The firm, being risk averse, has to voluntarily decrease its output in order to minimize the unhedgeable risk as compared to the case with perfect hedging.

4.2. Hedging decision

Now, we examine the effect of basis risk and linear exchange rate risk exposure on the firm's hedging decision. That is, we want to compare $H^*$ with $H^0$. To facilitate the comparison, an intermediate step is warranted. Note that for any given output level, $Q$, equation (12) uniquely determines a hedge position, $\hat{H}(Q)$. Our intermediate step is to compare $\hat{H}(Q)$ with $H(Q)$. From this and the results in the previous sections, we are then able to compare $H^*$ with $H^0$.

Following Kihlstrom, Romer, and Williams (1981) and Nachman (1982), for any given output level, $Q$, define the derived utility function, $V$, as

$$
V(\Pi) = \mathbb{E}[U(\Pi + \hat{\varepsilon}Q)], \tag{19}
$$
where the expectation is taken with respect to the distribution function of $\tilde{\epsilon}$. From equation (19), we can write equation (12) as

$$E[V'(\tilde{\Pi}^0)(f_0 - \tilde{f})] = 0,$$

(20)

where $\tilde{\Pi}^0 = \tilde{f}Q - C(Q) + \tilde{H}(Q)(f_0 - \tilde{f})$. Note that $\tilde{\epsilon}$ does not appear in equation (20) and $V$ is a thrice continuously differentiable function with $V' > 0$ and $V'' < 0$ as evident from equation (19).

Inspection of equations (5) and (20) reveals that comparing $\tilde{H}(Q)$ with $H(Q)$ is equivalent to comparing the derived utility functions, $V$, with the underlying utility function, $U$. Furthermore, from Proposition 1, we have $\tilde{H}(Q)$ is greater than, equal to, or less than $Q$ depending on whether the foreign currency futures market exhibits contango, unbiasedness, or normal backwardation, respectively.

Although $U$ and $V$ should be closely related, the Arrow-Pratt theory of risk aversion is too weak to provide any intuitive linkage between these two utility functions. To resolve this problem, we adopt a stronger, yet canonical, notion of risk aversion advocated by Kimball (1990, 1993), known as standard risk aversion. Kimball defines 'prudence' as $U'' > 0$ and the precautionary premium, $\Psi$, as the quantity satisfying (in our context)\(^{13}\)

$$U'[\Pi - \Psi(\Pi)] = E[U'(\Pi + \tilde{\epsilon}Q)].$$

(21)

He shows that $\Psi$ is proportional to the index, $-U''(\Pi)/U''(\Pi)$, which is denoted as the degree of absolute prudence. $U$ is said to exhibit standard risk aversion if, and only if, it exhibits both DARA and decreasing absolute prudence (DAP).

Equipped with standard risk aversion, the following lemma shows that $V$ and $U$ are linked in an intuitive way in that the former is more risk averse than the latter.

**Lemma.** If the firm's utility function exhibits standard risk aversion, then the derived utility function, $V$, as defined in equation (19), is more risk averse than the underlying utility function, $U$. That is,

$$\frac{V''(\Pi)}{V'(\Pi)} > \frac{U''(\Pi)}{U'(\Pi)}, \text{ for all } \Pi.$$

\(^{13}\)In essence, $\Psi$ is analogous to Pratt's risk premium, but for $U'$ rather than $U$. 
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Proof. Differentiating equation (19) with respect to $\Pi$ yields

$$V'(\Pi) = E[U'(\Pi + \xi Q)] = U'[\Pi - \Psi(\Pi)],$$

(22)

where the second equality follows from equation (21). Differentiating equation (22) with respect to $\Pi$ yields

$$V''(\Pi) = E[U''(\Pi + \xi Q)] = U''[\Pi - \Psi(\Pi)][1 - \Psi'(\Pi)].$$

(23)

Using equations (22) and (23), we obtain

$$-\frac{V''(\Pi)}{V'(\Pi)} = -\frac{U''[\Pi - \Psi(\Pi)]}{U'[\Pi - \Psi(\Pi)]}[1 - \Psi'(\Pi)] > -\frac{U''[\Pi - \Psi(\Pi)]}{U'[\Pi - \Psi(\Pi)]} > -\frac{U''(\Pi)}{U'(\Pi)},$$

where the first inequality follows from DAP (i.e., $\Psi' < 0$), and the second inequality follows from prudence (i.e., $\Psi > 0$) and DARA.

The proof of the above lemma is adopted from that of Eeckhoudt and Kimball (1992). Based on this lemma, the following proposition is immediately invoked.

**Proposition 5.** Suppose that hedging with futures contracts is imperfect, exchange rate risk exposure is linear, and the firm's utility function exhibits standard risk aversion. When the currency futures market is unbiased, the firm's optimal hedge position is always a full hedge. When the currency futures market is biased, for any given output level, $Q$, the firm's optimal hedge position is less than or greater than that with perfect hedging depending on whether the currency futures market exhibits contango or normal backwardation, respectively.

Proof. Define $Z(f) = V'[\Pi^0(f)]/U'[\Pi^0(f)]$, where $\Pi^0(f) = fQ - C(Q) + H(Q)(f_0 - f)$. Differentiating $Z$ with respect to $f$ yields

$$Z'(f) = Z(f)[Q - H(Q)]\left\{\frac{V''[\Pi^0(f)]}{V'[\Pi^0(f)]} - \frac{U''[\Pi^0(f)]}{U'[\Pi^0(f)]}\right\}.$$

By the above lemma, the expression inside the curly brackets is negative given standard risk aversion. Consider first the case where the currency futures market exhibits contango. Then, $H(Q) > Q$ so that $Z' > 0$. Thus, we have

$$Z(f) > (\leq) Z(f_0) \quad \text{for} \quad f > (\leq) f_0.$$

\(^{14}\)See also Wong (1996) for extending this result to the case where the independence assumption is relaxed.
Multiplying $U'(\Pi^0(f))(f_0 - f)$ to both sides of the above inequality and taking expectations with respect to the distribution function of $\tilde{f}$ yields

$$E[V'(\Pi^0)(f_0 - \tilde{f})] < E[U'(\Pi^0)(f_0 - \tilde{f})] = 0,$$

where the equality follows from the fact that $H(Q)$ solves equation (5). Thus, using equation (20), we have $\hat{H}(Q) < H(Q)$. The other two cases can be proved in a similar manner.

Proposition 5 says that the presence of basis risk and linear exchange rate risk exposure induces the firm with standard risk aversion to choose a hedge position closer to a full hedge, irrespective of the characteristics of the currency futures market.\(^{15}\) Briys, Crouhy, and Schlesinger (1993) have derived similar results, albeit based on a rather different proof.

The intuition of Proposition 5 is as follows. Since the firm is prudent (i.e., $U'''' > 0$), its marginal utility function is convex, thereby making it more sensitive to low realizations of terminal profit. As such, the firm has an incentive to avoid these low realizations by subsidizing them with high realizations of terminal profit. To this end, it is evident from equation (17) that the firm will choose a hedge position, $H$, closer to the output level, $Q$, so as to minimize the extreme realizations of terminal profit.

When the currency futures market exhibits contango or unbiasedness, we know from Proposition 5 that $H^* = \hat{H}(Q^*) \leq H(Q^*)$. Since $Q^* < Q^0$ from Proposition 4 and $H'(Q) \geq 1$ for $Q < Q^0$ from Proposition 3, we have $H(Q^*) < H(Q^0) = H^0$ and thus $H^* < H^0$. On the other hand, when the currency futures market exhibits normal backwardation, Proposition 5 implies that $H^* = \hat{H}(Q^*) > H(Q^*)$. However, we only know from Proposition 3 that $H'(Q) < 1$ for $Q < Q^0$. Indeed, $H(Q^0)$ can be larger than or smaller than $H(Q^*)$, depending on whether $H'(Q)$ is positive or negative, respectively. As a result, the comparison of $H^*$ and $H^0$ is ambiguous in the normal backwardation case.

To summarize, we have the following proposition.

\(^{15}\)Benninga, Eldor, and Zilcha (1983) show that a firm's optimal hedge position being a full hedge given unbiased futures markets does not depend on the firm's preference other than risk aversion. Indeed, Lence (1995) demonstrates that this utility-free hedge position holds if, and only if, the hedgeable risk in question is conditionally independent of the unhedgeable risk, a much weaker assumption than the independence assumption.
Proposition 6. Suppose that the firm’s utility function exhibits standard risk aversion. The presence of basis risk and linear exchange rate risk exposure reduces the firm’s optimal hedge position as compared to that with perfect hedging when the currency futures market exhibits either unbiasedness or contango. When the currency futures market exhibits normal backwardation, this effect is ambiguous.

Proposition 6 says that the firm with standard risk aversion always engages less futures contracting in face of basis risk and linear exchange rate risk exposure, especially when the currency futures market does not exhibit normal backwardation. This is in stark contrast to the findings of Briys, Crouhy, and Schlesinger (1993), which are essentially those reported in Proposition 5. Unlike Briys, Crouhy, and Schlesinger (1993) who assume the output level to be fixed, we endogenize the firm’s production decision. The presence of basis risk and linear exchange rate risk exposure destroys the separation property and reduces the firm’s optimal output, resulting in less motive for hedging with futures contracts.

5. Nonlinear exchange rate risk exposure

In the previous section, we have restricted our attention to the case where the regression specification, (1), is linear. Now, we relax this assumption by allowing for nonlinearity in a quadratic manner:

\[ \tilde{e} = \alpha + \beta \tilde{f} + \gamma \tilde{f}^2 + \tilde{\epsilon}, \]

where \( \alpha \geq 0, \beta > 0, \) and \( \gamma \neq 0 \) are constants. As in the previous section, we set \( \alpha = 0 \) and \( \beta = 1 \) to simplify the notation. The nonlinear component, \( \gamma \tilde{f}^2 \), is convex or concave depending on whether \( \gamma \) is positive or negative, respectively. We refer to the case when \( \gamma > 0 \) as the case in which the firm faces convex exchange rate risk exposure. Likewise, the firm is said to face concave exchange rate risk exposure when \( \gamma < 0 \).

The first-order conditions for program (3) in this case are given by

\[
\begin{align*}
\mathbb{E}\{U'(\tilde{\Pi})(\tilde{f} + \gamma \tilde{f}^2 + \tilde{\epsilon} - C'(Q))\} &= 0, \\
\mathbb{E}\{U'(\tilde{\Pi})(f_0 - \tilde{f})\} &= 0,
\end{align*}
\]

(24) (25)
Hedging and nonlinear risk exposure

where $\tilde{I} = (\bar{f} + \gamma \bar{f}^2 + \tilde{\epsilon})Q - C(Q) + H(f_0 - \tilde{f})$. We are interested in studying how the nonlinear component, $\gamma \bar{f}^2$, albeit arbitrarily small, would affect the optimal hedging and production decisions of the firm. Note that when $\gamma = 0$, $(Q^*, H^*)$ is the solution for the above system of equations.

Totally differentiating equations (24) and (25) with respect to $\gamma$, evaluating the resulting equations at $\gamma = 0$ and $(Q^*, H^*)$, and applying Cramer’s rule yields

$$\frac{dQ^*}{d\gamma} = \frac{EU_{HQ}^* EU_{QH}^* - EU_{QH}^* EU_{HQ}^*}{EU_{QQ}^* EU_{HH}^* - EU_{QH}^* EU_{HQ}^*},$$

(26)

$$\frac{dH^*}{d\gamma} = \frac{EU_{QH}^* EU_{QH}^* - EU_{HQ}^* EU_{HQ}^*}{EU_{QQ}^* EU_{HH}^* - EU_{QH}^* EU_{HQ}^*},$$

(27)

where $\tilde{I}^* = (\bar{f} + \tilde{\epsilon})Q^* - C(Q^*) + H^*(f_0 - \tilde{f})$,

$$EU_{QQ}^* = E\{U''(\tilde{I}^*)[\bar{f} + \tilde{\epsilon} - C'(Q^*)]^2\} - E[U'(\tilde{I}^*)]C''(Q^*),$$

(28)

$$EU_{HH}^* = E[U''(\tilde{I}^*)](f_0 - \tilde{f})^2,$$

(29)

$$EU_{QH}^* = E\{U''(\tilde{I}^*)[\bar{f} + \tilde{\epsilon} - C'(Q^*)](\tilde{f} - f_0)\},$$

(30)

$$EU_{Q\gamma}^* = E[U'(\tilde{I}^*)\bar{f}^2] + E\{U''(\tilde{I}^*)[\bar{f} + \tilde{\epsilon} - C'(Q^*)]\tilde{f}^2\}Q^*,$$

(31)

$$EU_{H\gamma}^* = E[U''(\tilde{I}^*)(f_0 - \tilde{f})\bar{f}^2]Q^*. $$

(32)

It follows from the concavity of $U$ and the convexity of $C$ that $EU_{QQ}^* < 0$ and $EU_{HH}^* < 0$. By the second-order conditions, (13), the denominators in the right-hand side of equations (26) and (27) are positive.

5.1. Production decision

The effect of introducing nonlinear exchange rate risk exposure on the firm’s optimal output is governed by the sign of the numerator in the right-hand side of equation (26). This turns out to be an extremely difficult exercise without imposing some restrictions on the characterizations of the foreign currency market and the attitude towards risk of the firm. To this end, hereafter we follow the hedging literature by confining ourselves to the case where the currency futures market is unbiased and the firm’s utility function exhibits DARA.
Given that the currency futures market is unbiased, we know from Proposition 5 that $H^* = Q^*$. In fact, as shown by Benninga, Eldor, and Zilcha (1983), the optimality of the full hedge position given the unbiased currency futures market does not depend on the firm’s preference other than risk aversion. Thus, we do not have to assume the firm to have standard risk aversion in this case.

The firm’s terminal profit is $\hat{\Pi}^* = f_0Q^* - C(Q^*) + \bar{\epsilon}Q^*$ which is not a function of $\bar{\epsilon}$. From equations (29) and (30), we have

$$EU_{QH} = E\{U''(\hat{\Pi}^*)[f_0 + \bar{\epsilon} - C'(Q^*)](f_0 - \hat{\epsilon})\} - EU_{HH}$$

$$= -Cov\{U''(\hat{\Pi}^*)[f_0 + \bar{\epsilon} - C'(Q^*)], \hat{\epsilon}\} - EU_{HH}$$

$$= -EU_{HH}^*,$$  \hspace{1cm} (33)

where the second equality follows from unbiasedness and the third equality follows from independence. From equations (31) and (32), we have

$$EU_{Q\gamma} = E\{U''(\hat{\Pi}^*)\hat{\epsilon}^2\} - EU_{H\gamma} + E\{U''(\hat{\Pi}^*)[f_0 + \bar{\epsilon} - C'(Q^*)]\hat{\epsilon}^2\}Q^*$$

$$= E[U'(\hat{\Pi}^*)]E(\hat{\epsilon}^2) + Cov[U'(\hat{\Pi}^*), \hat{\epsilon}^2] - EU_{H\gamma}$$

$$+ E\{U''(\hat{\Pi}^*)[f_0 + \bar{\epsilon} - C'(Q^*)]\}E(\hat{\epsilon}^2)Q^*$$

$$+ Cov\{U''(\hat{\Pi}^*)[f_0 + \bar{\epsilon} - C'(Q^*)], \hat{\epsilon}^2\}Q^*$$

$$= \{E[U'(\hat{\Pi}^*)] + E\{U''(\hat{\Pi}^*)[f_0 + \bar{\epsilon} - C'(Q^*)]\}Q^*\}E(\hat{\epsilon}^2)$$

$$- EU_{H\gamma}^*,$$  \hspace{1cm} (34)

where the third equality follows from independence. Using equations (33) and (34), the numerator in the right-hand side of equation (26) becomes

$$- \{E[U'(\hat{\Pi}^*)] + E\{U''(\hat{\Pi}^*)[f_0 + \bar{\epsilon} - C'(Q^*)]\}Q^*\}E(\hat{\epsilon}^2)EU_{HH}^*.$$  \hspace{1cm} (35)

Given DARA, we show in the following proposition that the second term inside the curly brackets of equation (35) is positive so that $dQ^*/d\gamma > 0$.

**Proposition 7.** If the firm’s utility function exhibits DARA and the currency futures market is unbiased, the presence of convex (concave) exchange rate risk exposure increases (decreases) the firm’s optimal output as compared to that under linear exchange rate risk exposure.
Proof. Define $\Pi^*(\epsilon)$ as the realization of $\tilde{\Pi}^*$ at $\tilde{\epsilon} = \epsilon$. Since $\Pi^*(\epsilon)$ increases with $\epsilon$ and $U$ exhibits DARA, we have

$$-\frac{U''[\Pi^*(\epsilon)]}{U'[\Pi^*(\epsilon)]} < (>) \quad \text{for} \quad \epsilon > (<) \epsilon^* \equiv C'(Q^*) - f_0.$$

Multiplying $-U'[\Pi^*(\epsilon)][f_0 + \epsilon - C'(Q^*)]$ to both sides of the above inequality and taking expectations with respect to the distribution function of $\tilde{\epsilon}$ yields

$$E\{U''(\tilde{\Pi}^*)[f_0 + \tilde{\epsilon} - C'(Q^*)]\} > \frac{U''[\Pi^*(\epsilon^*)]}{U'[\Pi^*(\epsilon^*)]}E\{U'(\tilde{\Pi}^*)[f_0 + \tilde{\epsilon} - C'(Q^*)]\} = 0,$$

where the equality follows from equation (14). The desired results then follow from equations (26) and (35). □

The intuition of Proposition 7 is as follows. Note that the firm's terminal profit under nonlinear exchange rate risk exposure can be written as

$$\tilde{\Pi} = f_0Q - C(Q) + (H - Q)(f_0 - \hat{f}) + \gamma \hat{f}^2 Q + \tilde{\epsilon} Q. \quad (36)$$

The variance of the firm's terminal profit is given by

$$\text{Var}(\tilde{\Pi}) = (H - Q)^2 \text{Var}(\hat{f}) + \gamma^2 Q^2 \text{Var}(\hat{f}^2) - 2\gamma Q(H - Q) \text{Cov}(\hat{f}, \hat{f}^2) + Q^2 \text{Var}(\tilde{\epsilon}). \quad (37)$$

Suppose that the firm does not change its production and hedging decisions when convex exchange rate risk exposure is introduced, i.e., the solution remains at $(Q^*, H^*)$ and $\gamma > 0$. Given that the currency futures market is unbiased, we have $H^* = Q^*$. Inspection of equations (36) and (37) reveals that an infinitesimal increase in $\gamma$ from zero increases the firm's terminal profit by $\hat{f}^2 Q^*$, while leaving the variance unchanged. Given DARA, the firm is willing to take on additional risk. As a result, it produces more in face of convex exchange rate risk exposure. Similar intuition applies to the case under concave exchange rate risk exposure (i.e. an infinitesimal decrease in $\gamma$ from zero).

5.2. Hedging decision

The effect of introducing nonlinear exchange rate risk exposure on the firm's optimal hedging decision is governed by the sign of the numerator in the
right-hand side of equation (27). From equations (28) and (29), we have

\[ EU^*_Q = E\{U''(\tilde{\Pi}^*)(f_0 + \bar{\varepsilon} - C'(Q^*))\} + EU^*_{HH} + 2E\{U''(\tilde{\Pi}^*)(f_0 + \bar{\varepsilon} - C'(Q^*))(|\tilde{f} - f_0|)\} - E[U'(\tilde{\Pi}^*)]C''(Q^*), \]

where the second equality follows from unbiasedness and the third equality follows from independence. From equation (32), we have

\[ EU^*_{HH} = E[U''(\tilde{\Pi}^*)(f_0 - \tilde{f})(f_0^2 - f_0^2)]Q^* + E[U''(\tilde{\Pi}^*)(f_0 - \tilde{f})]f_0^2 Q^* - E[U''(\tilde{\Pi}^*)(f_0 - \tilde{f})]f_0^2 Q^* - E[U'(\tilde{\Pi}^*)]f_0^2 Q^*, \]

where the second equality follows from unbiasedness and the third equality follows from independence. Since \( U'' < 0 \), we have \( EU^*_G > 0 \). Using equations (33), (34), and (38), the numerator in the right-hand side of equation (27) becomes

\[ \{E[U'(\tilde{\Pi}^*)]C''(Q^*) - E[U''(\tilde{\Pi}^*)(f_0 + \bar{\varepsilon} - C'(Q^*))\}EU^*_{HH}, \]

The first term in equation (40) is positive because \( U'' < 0, C'' > 0, \) and \( EU^*_{HH} > 0 \). The second term is exactly equal to equation (35) which has already been shown to be positive given DARA. Thus, we have the following proposition.

**Proposition 8.** If the firm’s utility function exhibits DARA and the currency futures market is unbiased, the presence of convex (concave) exchange rate risk exposure increases (decreases) the firm’s hedge position as compared to that under linear exchange rate risk exposure.

We have shown in Proposition 7 that the firm has an incentive to increase its output in face of convex exchange rate risk exposure. To reduce the associated higher risk level due to its aggressive production decision, the firm
adjusts its hedge position upward by selling more futures contracts. In fact, substracting equation (27) from equation (26) and using equations (35) and (40) yields

\[ \frac{dH^*}{d\gamma} - \frac{dQ^*}{d\gamma} = \frac{\{E[U'(\tilde{\eta}^*)]C''(Q^*) - E[U''(\tilde{\eta}^*)][f_0 + \xi - C'(Q^*)]^2\}EU_{HH}}{EU_{QQ}EU_{HH} - EU_{QH}EU_{HQ}}, \]

which is positive. That is, the firm's optimal hedge position is an over hedge in the presence of convex exchange rate risk exposure. The following proposition summarizes this result.

**Proposition 9.** If the firm's utility function exhibits DARA and the currency futures market is unbiased, the presence of convex (concave) exchange rate risk exposure induces the firm to opt for an over (under) hedge.

The intuition of Proposition 9 is as follows. Differentiating equation (37) with respect to \( \gamma \) and evaluating the resulting equation at \( \gamma = 0 \) yields

\[ \frac{d\text{Var}[(\tilde{\eta})]}{d\gamma} = -2Q(H - Q)Cov(\tilde{f}, \tilde{f}^2). \]

Since \( Cov(\tilde{f}, \tilde{f}^2) > 0 \), the firm can always further reduce its risk level by choosing \( H > Q \). Thus, the firm has an incentive to hold a more than full hedge position in face of convex exchange rate risk exposure.

### 6. Conclusion

Corporations take exchange rate risk management very seriously. Indeed it is ranked by financial managers as one of their most important objectives. Given the importance of exchange rate risk management, in recent years we have witnessed a growing literature about optimal production and hedging by international firms which are facing exchange rate uncertainty.

Two notable results have emerged in the literature: the separation and the full hedging theorem. First, the separation theorem states that the firm's production decision is affected neither by the firm's attitude towards risk nor by the incidence of spot exchange rate volatility if the firm has access to a foreign currency futures market. Second, the full hedging theorem states
that the firm should eliminate all exchange rate risk by holding a full hedge position if the currency futures market is unbiased.

The purpose of this paper is to re-examine the separation and full hedging theorem when basis risk exists, i.e., when spot exchange rate and futures exchange rate are imperfectly correlated. The two major sources of basis risk are: (i) expiration dates of futures contracts do not match the maturity of inflows or outflows in foreign currency, and (ii) underlying assets in futures markets do not match the currency one wishes to hedge. Hedging in such an environment is de facto imperfect. It is shown that the regression specification between spot and futures exchange rates is important in analyzing hedging and export decisions. The regression dependence between the spot and futures exchange rate can be linear or nonlinear.

The study features a competitive firm which exports all of its production to a foreign country. The firm is risk averse and has access to a currency futures market to hedge against its exchange rate risk exposure. With basis risk we show that the separation property no longer holds. Therefore the firm produces less as a response to the introduction of imperfect hedging.

The full hedging property holds if, and only if, the firm’s exchange rate risk exposure is linear. Given reasonable assumptions on the firm’s risk preference and linear exchange rate risk exposure, we show that the firm’s optimal hedge position is always smaller than that with perfect hedging when the currency futures market exhibits either unbiasedness or contango. The full hedging property breaks down when the exchange rate risk exposure is nonlinear. We demonstrate that the firm adopts an over hedge when the exposure is convex and an under hedge when the exposure is concave given the unbiased foreign currency futures market.

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