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## Share equations in econometrics: A story of repression, frustration and dead ends

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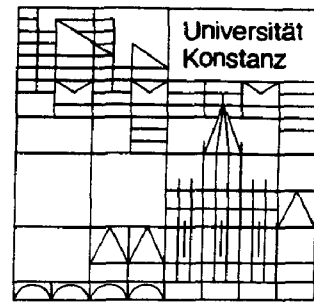
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A Story of Repression, Frustration  
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**SHARE EQUATIONS IN ECONOMETRICS:  
A STORY OF REPRESSION, FRUSTRATION AND DEAD ENDS**

**Gerd Ronning**

Serie II - Nr. 118

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# Share Equations in Econometrics: A Story of Repression, Frustration and Dead Ends

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## Abstract

Share equations play an important role in applied economic research, notably in marketing and demand analysis. Both market shares and budget shares have been used as dependent variables in econometric models which were partly motivated by microeconomic theory. However attempts of econometricians (and other statisticians) to treat share equations adequately led mostly to unsatisfactory approaches: Some researchers although admitting that shares satisfy a sum constraint simply repressed the fact that shares cannot be normally distributed. Some researchers looked in vain for a stochastic specification which at the same time is consistent and allows a flexible covariance structure. Last not least almost nobody has properly taken care of additional problems arising from dynamic share models. The paper discusses these three issues and proposes a possible way out of this dilemma which was first suggested by Aitchison (1982) and has been applied to econometric demand analysis by Considine and Mount (1984). Demand-theoretic implications as well as methods of estimation are discussed. An example using German import data illustrates some of the results.

Keywords: demand systems, market attraction models, linear logit models, Dirichlet distribution.

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# 1 Introduction

Shares, proportions or percentages play an important role in the presentation of data since they provide direct information about the 'share' of the total for some category.

<sup>1</sup> Market shares of firms and budget shares of households are prominent examples in the economic area. Statistical analysis of shares has attracted interest of scientists from various fields of research; geologists, biologists and economists have been most active in this area. They noted that the analysis of shares creates special problems. For example, two shares which add up to one will always be negatively correlated. More generally, for more than two categories there will be always at least one pair of negatively correlated shares. Aitchison (1986 p.53) called this the 'negative bias' and mentions other 'difficulties of compositional data analysis' in his book. As will be seen below, the most natural stochastic specification for shares, the Dirichlet distribution, is even more restrictive in specifying negative correlations throughout. On the other hand this distribution recognizes that shares vary only within the  $[0, 1]$  interval whereas it is custom in econometrics to assume that the shares are normally distributed. Shares have also been used as dependent variables in 'linear share models'. A prominent example is Deaton and Muellbauer's (1980) 'Almost Ideal Demand System', another is the Translog production function under special assumptions.<sup>2</sup> 'Adding up' of shares implies corresponding constraints for parameters and residuals some of which have been always and some of which were almost never observed. Such problems disappear if we switch to *nonlinear* share equations which led some researchers from the marketing area to the conclusion that nonlinear specifications should be preferred.

In this paper two types of shares will be considered: *Shares related to quantities* and *shares related to count data*. This is illustrated by some fictitious data given in table 1: In the upper part we have the amount of investment from domestic and foreign investors given in Deutschmark. The resulting shares of domestic and foreign investors are related to the quantity 'amount of investment'. In the lower part we have the number of domestic and foreign employees in a certain region. The share (or proportion) of domestic employees is related to the number of employees and is therefore again a discrete variable, although constrained to the unit interval, for which binomial or multinomial distribution would be the adequate model whereas for the quantity-related shares the continuous random variables following, for example, the Dirichlet distribution would be the proper specification. My paper is concerned primarily with

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<sup>1</sup>However, the total which indicates the 'size' disappears in this kind of presentation if not reported additionally.

<sup>2</sup>See, for example, Berndt and Christensen (1973).

the latter type of shares although in descriptive analysis both types of shares will be considered.<sup>3</sup>

*Table 1: Two Types of Shares*

A. Investment by domestic and foreign investors		
category	investment (billion DM)	share of investment
domestic	850	0.85
foreign	150	0.15
B. Employment of domestic and foreign workers		
	number of employees	share of employment
domestic	500000	0.833
foreign	100000	0.167

My paper tries to bring together results related to the analysis of shares most of which pronounce the problems inherent in this kind of analysis. The emphasis is on *linear* models although special forms of nonlinear models seem to be the only way out of the dilemma as will be discussed below. There is an alternative available which should be also considered seriously: Very often one could analyse the quantities from which the shares are computed. For example, why not use the expenditures instead of the budget shares in econometric demand analysis? The choice of results reported here is partly dictated by personal experience in this field which I gathered during the last fifteen years.

The paper is organized as follows: Section 2 reports results for the linear share model, section 3 those for the nonlinear case and section 4 gives extensions for dynamic share models. Some concluding remarks are added in section 5.

## 2 Linear Share Models

In this section for expository purpose we first consider the case of only two categories and then move on to more than two categories. We also distinguish between descriptive

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<sup>3</sup>Sometimes shares related to count data are computed from weighted units. For example, the IFO institute uses employment or sales of firms as a weighting factor in determining proportions of firms answering 'up' and 'down', respectively. There is then no longer a clear distinction between the two types of shares.

methods such as fitting by ordinary least squares and the estimation of stochastic models.

## 2.1 Fitting linear share models

Let  $y_t$  be the dependent variable<sup>4</sup> for observation  $t$  and  $x_{tk}$  the corresponding value of the  $k$ -th explanatory variable where  $t = 1, \dots, n$ . Since  $y_t$  is a share, it satisfies for all  $t$

$$0 \leq y_t \leq 1. \tag{2-1}$$

Marketing researchers call this the 'range constraint'. Furthermore we define by  $\mathbf{y}$  the (column) vector of all  $y_t$ 's and by  $\mathbf{X}$  the  $(n \times k)$ -matrix of the explanatory variables with columns  $\mathbf{x}_k$  and rows  $\mathbf{x}(t)'$ . We also use the  $n$ -dimensional vector of ones denoted by  $\mathbf{1}$ . It appears as the first column in  $\mathbf{X}$  if a constant term is specified (see below).

The simplest linear share model is given by

$$y_t = \beta_1 + \beta_2 x_{t2} + u_t, \quad t = 1, \dots, n, \tag{2-2}$$

where  $u_t$  is a random error term with expectation zero. This model is given graphically in figure 1. Note that the model is meaningful only for values of the explanatory variable within a certain range. One could of course argue that the linear relationship approximates the true relation within this range. Another feature of this picture almost never mentioned is that the conditional variance of  $y$  given  $x$  in general will not be constant, that is heteroskedasticity is inherent in linear and (some - see below) nonlinear share models.

\*\*\*\*\*figure 1 about here\*\*\*\*\*

More compactly the model can also be written as

$$\begin{aligned} \mathbf{y} &= \beta_1 \mathbf{1} + \beta_2 \mathbf{x}_2 + \mathbf{u} \\ &= \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \end{aligned} \tag{2-3}$$

where  $\boldsymbol{\beta} = (\beta_1, \beta_2)'$ ,  $\mathbf{u}$  is the  $n$ -dimensional vector of error terms and the  $(n \times 2)$  matrix  $\mathbf{X}$  contains the vectors  $\mathbf{1}$  and  $\mathbf{x}_2$ .

Let  $\mathbf{b}$  be the least squares solution of  $\boldsymbol{\beta}$  which for full column rank of  $\mathbf{X}$  is given by

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \tag{2-4}$$

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<sup>4</sup>Following common usage in econometrics  $y_t$  denotes both the random variable and its realization except in section 3.3 (Dirichlet share model).

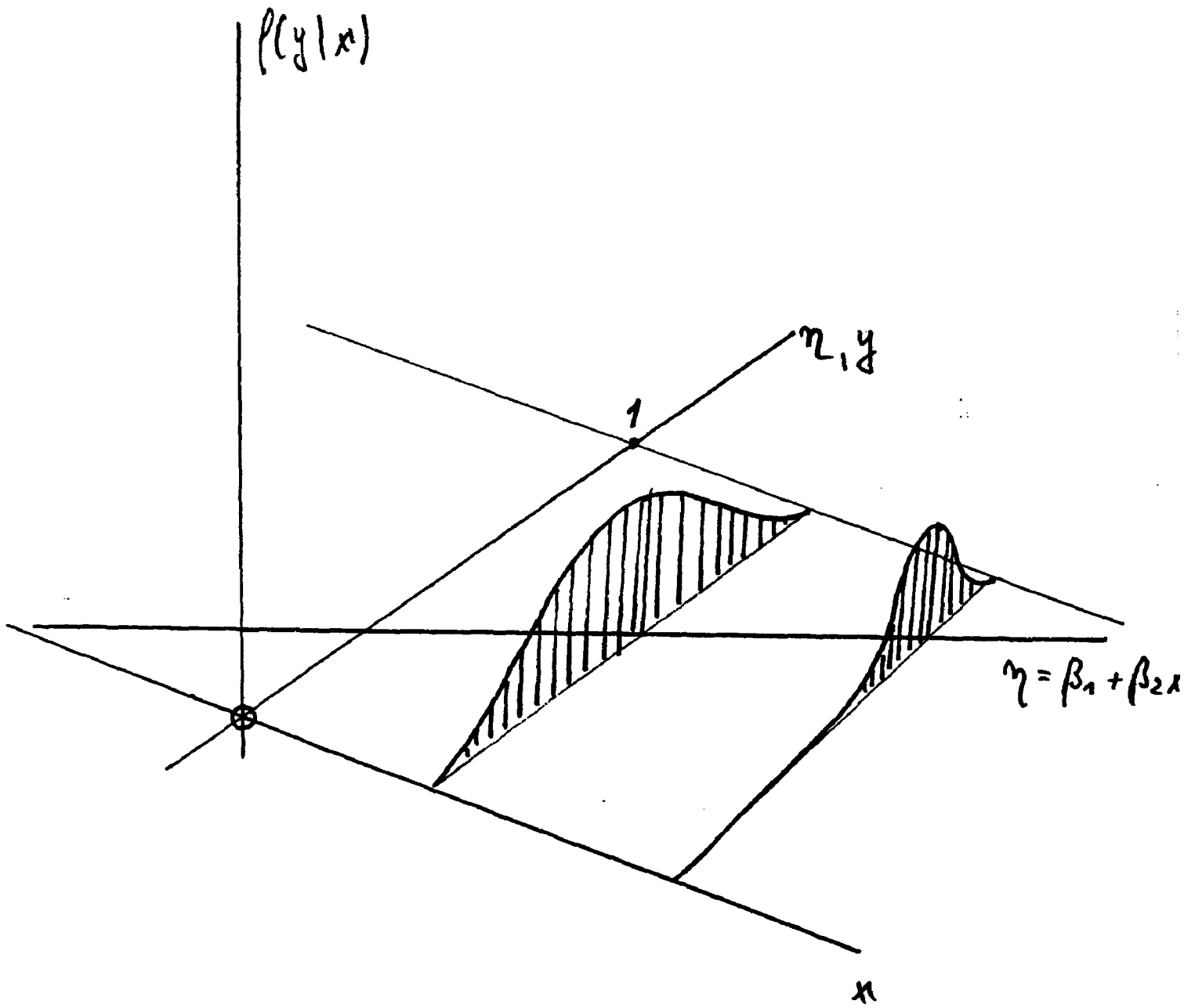


Figure 1



although (an infinite set of) solutions also exist for linear dependent columns of the regressor matrix. The estimate of the 'systematic part'

$$\eta \equiv X\beta \tag{2-5}$$

however is always unique and is given by

$$\hat{\eta} = XX^+y \tag{2-6}$$

where  $X^+$  is the Moore-Penrose inverse of  $X$ . It is well-known that the matrix  $XX^+$  is symmetric idempotent. Now observe that  $\hat{\eta}$  would be nonnegative for any share vector  $y$  if the matrix  $XX^+$  would be nonnegative. We therefore ask: Under what conditions is this matrix nonnegative? <sup>5</sup> For the case of a constant term, that is a vector of ones added to the regressor matrix, the answer is surprisingly simple: There must not be more than two distinct elements in the vector  $x_2$ , that is for the explanatory variable at most two distinct values are observed. Note that nothing is said about the values of the dependent variable. For  $r = 2$  distinct values in  $n = 5$  observations figure 2 shows the result from least squares fitting.

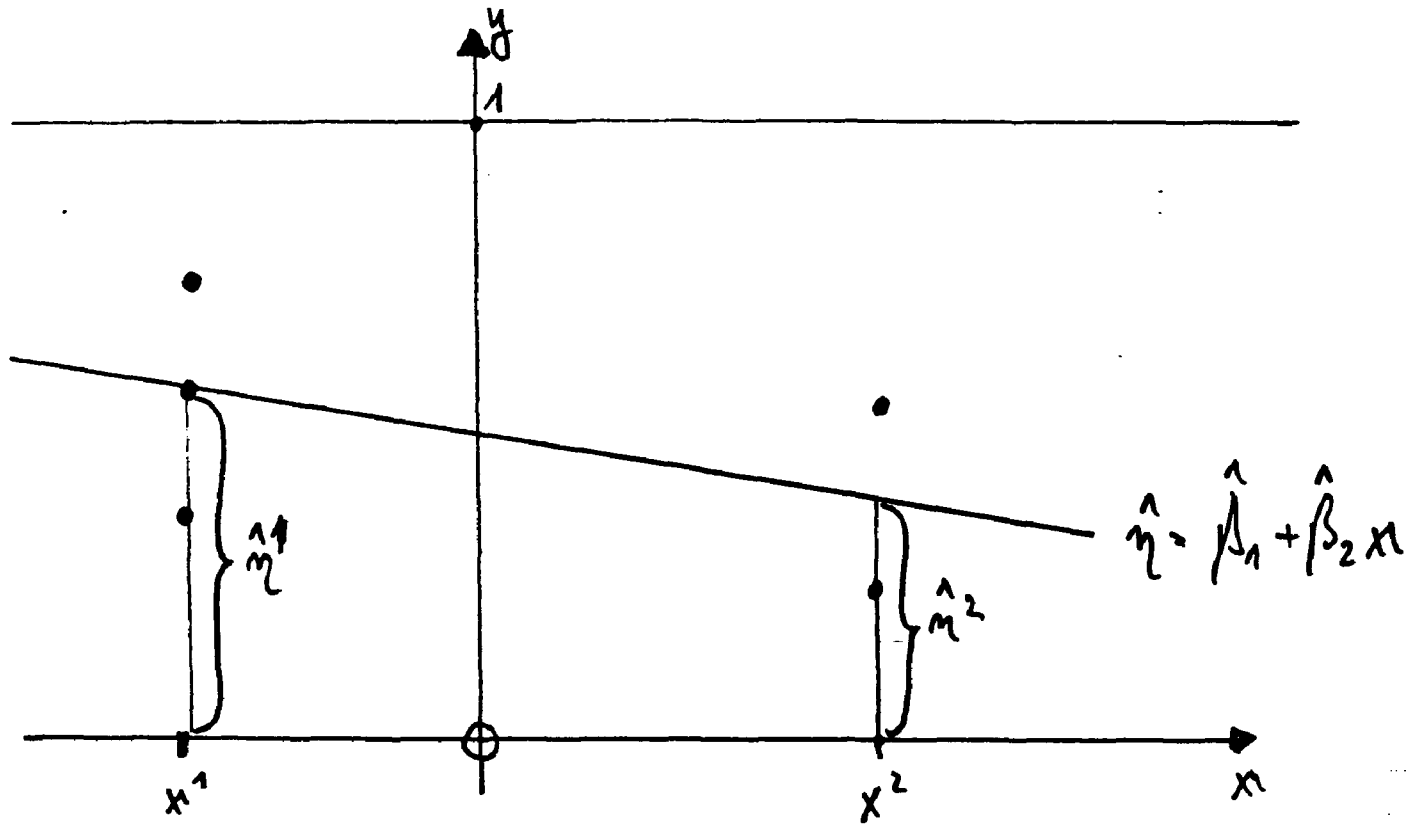
\*\*\*\*\*figure 2 about here\*\*\*\*\*

More generally, for the case of  $K$  regressors (including the vector  $\iota$ ) and  $\text{rank}(X) = m \leq K$  and  $n \geq K$  the condition of nonnegativity can be checked as follows: Pick the  $r$  distinct rows from  $X$  and form the  $(r \times K)$  matrix  $B$  from these rows.  $XX^+$  is nonnegative if and only if the rank of  $X$  equals the number  $r$  of distinct rows in  $B$ . Since the rank of  $X$  cannot be greater than  $K$ , nonnegativity will never be achieved if the number of distinct rows is greater than  $K$ , the number of regressors. If however  $r \leq K$ , then always nonnegativity of  $XX^+$  is achieved as will become evident from the following example with  $n = 4$  and  $K = 3$ : Matrix  $X_1$  has four distinct rows and is of full column rank whereas matrix  $X_2$  has only two distinct rows and has rank 2.

Example 1

$$\begin{array}{l}
 \mathbf{X}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & 4 & -2 \end{bmatrix} \quad \mathbf{X}_1\mathbf{X}_1^+ = \begin{bmatrix} 0.7012 & 0.3841 & 0.1280 & -0.2134 \\ 0.3841 & 0.5060 & -0.1646 & 0.2743 \\ 0.1280 & -0.1646 & 0.9451 & 0.0914 \\ -0.2134 & 0.2743 & 0.0914 & 0.8475 \end{bmatrix} \\
 \\
 \mathbf{X}_2 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 4 & -2 \\ 1 & 4 & -2 \end{bmatrix} \quad \mathbf{X}_2\mathbf{X}_2^+ = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}
 \end{array}$$

<sup>5</sup>The following results are taken from Ronning (1985).



$x^1$  and  $x^2$  are the two distinct values of  $x$   
 $\hat{\eta}^1$  gives estimated shares for all observations  
 with  $x = x^1$ .  
 $\hat{\eta}^2$  correspondingly for  $x = x^2$

Figure 2

The second matrix shows the typical shape of nonnegative symmetric idempotent matrices: The matrix is block-diagonal and each block is a symmetric idempotent matrix of rank one. <sup>6</sup>.

One could also ask what conditions the *observed* vector  $\mathbf{y}$  should satisfy such that for any regressor matrix  $\mathbf{X}$  the vector  $\hat{\boldsymbol{\eta}}$  is nonnegative. Ronning and Ringwald (1979) derived the following inequalities as a sufficient condition:

$$\begin{aligned} \frac{y_{max}}{y_{min}} &\leq 1 + \frac{4}{\sqrt{n-1}} \\ \frac{1-y_{min}}{1-y_{max}} &\leq 1 + \frac{4}{\sqrt{n-1}} \end{aligned} \quad (2-7)$$

Note that the second inequality considers the shares of the second category which are given from the adding-up constraint (2-1). See the following example.

### Example 2

The following table <sup>7</sup> shows the proportion of women which would accept an increase of the meat price. Both working and non-working persons were interviewed.

Table 2: Acceptance of Price Increase

No.	proportion of women		price increase (DM)	estimated share	
	working	non-working		working	non-working
1	0.81	0.84	0.20	0.664	0.625
2	0.53	0.48	0.30	0.561	0.522
3	0.39	0.34	0.40	0.457	0.419
4	0.31	0.23	0.50	0.354	0.315
5	0.13	0.10	0.60	0.251	0.212
6	0.10	0.05	0.70	0.148	0.109
7	0.08	0.04	0.80	0.044	0.006
8	0.07	0.03	0.90	-0.059	-0.098

$y_{max} = 0.84$  ,  $y_{min} = 0.03$  ,  $\frac{y_{max}}{y_{min}} = 28.00$   
 $1 - y_{min} = 0.97$  ,  $1 - y_{max} = 0.16$  ,  $\frac{1-y_{min}}{1-y_{max}} = 6.06$

We have  $n = 16$  observed shares (8 for working and 8 for non-working women) of those accepting the price increase. The linear share model with constant term, price

<sup>6</sup>See Berman and Plemmons (1979) p. 66.

<sup>7</sup>The data are taken from Strecker (1973). 465 working and 938 non-working women were asked.

increase and a dummy for nonworking women as explanatory variables was estimated by ordinary least squares. The last two columns show the implied estimated shares some of which are negative. Since

$$1 + \frac{4}{\sqrt{n-1}} = 2.03$$

both ratios in (2-7) violate the condition and nonnegativity of estimated shares is not guaranteed.<sup>8</sup>

Results for both  $\mathbf{XX}^+$  and  $\mathbf{y}$  were obtained in the aim to avoid constrained least squares as, for example, suggested by Lee et al (1977). This leads to a quadratic program under inequality constraints and has been applied to the estimation of 'popularity functions' by Ronning and Ringwald (1977).<sup>9</sup> This approach is not very attractive since properties of estimators are not well known. In the paper just mentioned also the nonlinear logistic model was used to which we return in the next section.

## 2.2 Heteroskedasticity

We already remarked in passing that heteroskedasticity is inherent in linear and (some - see below) nonlinear share models. Therefore weighted least squares (WLS) is more efficient, that is the 'systematic part'  $\boldsymbol{\eta} \equiv \mathbf{X}\boldsymbol{\beta}$  should be estimated by

$$\tilde{\boldsymbol{\eta}} = \mathbf{X}(\mathbf{X}'\mathbf{V}^{-2}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-2}\mathbf{y} = \mathbf{V}\mathbf{W}\mathbf{W}^+\mathbf{V}^{-1}\mathbf{y} \quad (2-8)$$

where  $\mathbf{V}^2$  is the diagonal covariance matrix of the error terms and  $\mathbf{W} = \mathbf{V}^{-1}\mathbf{X}$ . Since  $\mathbf{V}\mathbf{W}\mathbf{W}^+$  is nonnegative whenever  $\mathbf{XX}^+$  is nonnegative<sup>10</sup> we can again use the number of distinct rows of the regressor matrix as the criterion for a nonnegative estimate of the mean share vector  $\boldsymbol{\eta}$ .

Amemiya (1977) has pointed out that for the linear probability model the ordinary least squares estimate and the weighted least squares estimate of the parameter vector  $\boldsymbol{\beta}$  are identical if and only if the number of distinct rows equals the number of regressors. Clearly this is a special case of the condition for nonnegativity of  $\mathbf{XX}^+$ . Ronning (1985) has elaborated the relation between nonnegativity of  $\mathbf{XX}^+$  and the equality of OLS and WLS regarding the systematic part, that is

$$\tilde{\boldsymbol{\eta}} = \hat{\boldsymbol{\eta}} \quad (2-9)$$

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<sup>8</sup>Since our inequality provides only a sufficient condition violation does not exclude the possibility of a nonnegative  $\tilde{\boldsymbol{\eta}}$

<sup>9</sup>Unrestricted estimation procedures for the same set of data were used by Ronning and Schneider (1976).

<sup>10</sup>See Ronning 1985 p. 43.

where he exploits the special structure of variances in share models: For example, both in the linear probability model and the Beta share model (to be discussed below) the variance of the error term  $u_t$  depends on the 'level' of the systematic part  $\eta_t$  where  $\eta_t$  is the  $t$ -th component of the vector  $\boldsymbol{\eta}$ :

$$\text{var}(u_t) = c_t \eta_t (1 - \eta_t) \quad (2 - 10)$$

where  $c_t \equiv 1$  for the linear probability model.

Ronning (1985 theorem 2) obtains the following result which we here state for the case of a constant term added<sup>11</sup>:

Assume that  $\text{var}(u_t) \neq \text{var}(u_s)$  iff  $\mathbf{x}(t) \neq \mathbf{x}(s)$  for  $t \neq s$ , that is the two variances are only identical if the corresponding rows of  $\mathbf{X}$  are identical. Then the following three conditions are equivalent.

- (a)  $\mathbf{X}\mathbf{X}^+$  is a nonnegative matrix.
- (b) OLS estimator and WLS estimator of  $\boldsymbol{\eta}$  coincide, that is equation (2-9) holds.
- (c) The condition  $[\mathbf{V}(\mathbf{I} - \mathbf{X}\mathbf{X}^+)]^+ = (\mathbf{I} - \mathbf{X}\mathbf{X}^+)\mathbf{V}^{-1}$  is satisfied.

Two remarks are in order: First, condition (c) is well-known. See, for example Albert (1972). The proof exploits the blockdiagonal structure of  $\mathbf{X}\mathbf{X}^+$ . Second, the variance specification (2-10) is excluded by assumption.

Heteroskedasticity in share equations has also been considered by Chavas and Seger-son (1987 section 4) who claim to give conditions of *homoskedastic* shares from demand-theoretic restrictions. One should again look at figure 1 and then ask what the relevance of these conditions could be. So we conclude that these authors have offered demand theoretic structure for the heteroskedasticity which is inherent in share equations. At the same time they derived conditions for homoskedasticity which are inconsistent with share equations under the 'range constraint'.<sup>12</sup>

<sup>11</sup>This excludes the possibility of 'proportional rows'.

<sup>12</sup>The same remarks would apply to the analysis of *expenditures* under a budget constraint since again the dependent variables (expenditures) can only vary within a certain range and have to obey the budget constraint. So we would expect heteroskedasticity in such demand systems, too. Chipman and Tian (1989) show that for the Linear Expenditure System price induced heteroskedasticity is implied if a stochastic utility function is maximized.

## 2.3 More than two categories and adding-up

So far we have considered only two categories of the dependent variable. We already used the fact that the share of the second category is given by  $1 - y_t$  since the two shares must add up to 1. We now denote the first share by  $y_{1t}$  and the second share by  $y_{2t}$  which satisfy the constraint

$$y_{1t} + y_{2t} = 1 \quad \text{for all } t. \quad (2-11)$$

We then write the linear share model (2-2) as follows:

$$\begin{aligned} y_{1t} &= \beta_{11} + \beta_{12}x_{t2} + u_{1t} \\ y_{2t} &= \beta_{21} + \beta_{22}x_{t2} + u_{2t} \end{aligned} \quad (2-12)$$

for  $t = 1, \dots, n$ . Summing the two equations gives

$$1 = (\beta_{11} + \beta_{21}) + (\beta_{12} + \beta_{22})x_{t2} + (u_{1t} + u_{2t}) \quad (2-13)$$

It is an straightforward exercise <sup>13</sup> to show that the coefficients of the model have to satisfy

$$\begin{aligned} \beta_{11} + \beta_{21} &= 1 \\ \beta_{12} + \beta_{22} &= 0 \end{aligned} \quad (2-14)$$

and that for the error terms

$$u_{1t} + u_{2t} = 0 \quad (2-15)$$

must hold for all  $t$ , that is the two error terms are linear dependent random variables so that the covariance matrix will be singular. These are the well-known '*adding-up conditions*' which are easily generalized to the case of  $r$  categories and  $K$  explanatory variables:

$$\begin{aligned} \sum_{j=1}^r \beta_{j1} &= 1, \\ \sum_{j=1}^r \beta_{jk} &= 0, \quad k = 2, \dots, K \\ \sum_{j=1}^r u_{jt} &= 0 \end{aligned} \quad (2-16)$$

These conditions need however some qualifying remarks:

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<sup>13</sup>Since this equation is valid for arbitrary  $x_{t2}$  we may choose the values 0 and 1 taking at the same time the expectation on both sides. This gives (2-14). (2-15) then follows from (2-14).

- (i) Both sets of conditions are not independent.  
(ii) These conditions hold only if for both equations the same explanatory variables are specified.

The second remark is much more important. It has been discussed primarily in the marketing literature whereas in econometric demand analysis this point played a minor role. Why? In demand systems like the 'Almost Ideal Demand System' prices of all goods and income enter into all  $r$  equations. Therefore all equations face the same set of explanatory variables. Contrary to this marketing people very often specify firm-specific variables like amount of advertisement or change of own selling price in market share systems. For the simplest version with two categories (two firms) and only one explanatory variable (firm specific advertisement  $z_{t2j}, j = 1, 2$ ) we now get instead of (2-12) the following system:

$$\begin{aligned} y_{1t} &= \gamma_{11} + \gamma_{12}z_{t21} + u_{1t} \\ y_{2t} &= \gamma_{21} + \gamma_{22}z_{t22} + u_{2t} \end{aligned} \quad (2-17)$$

and summing up this time gives

$$1 = (\gamma_{11} + \gamma_{21}) + (\gamma_{12}z_{t21} + \gamma_{22}z_{t22}) + (u_{1t} + u_{2t}) \quad (2-18)$$

The above adding-up constraints are no longer valid. Even worse, this system is not consistent if not something is assumed with respect to the two explanatory variables.<sup>14</sup> Let us make the rather strange assumption that total advertisement of the two firms is equal to  $\tau$  for all  $t$ :

$$z_{t21} + z_{t22} = \tau, \quad t = 1, \dots, n. \quad (2-19)$$

Then the following adding-up constraints have to be satisfied:

$$\begin{aligned} \gamma_{11} + \gamma_{21} &= 1 - \overline{\gamma_2}\tau \\ \gamma_{12} &= \gamma_{22} = \overline{\gamma_2} \end{aligned} \quad (2-20)$$

This means that the two firm-specific explanatory variables must have the same coefficient with regard to advertisement, a rather annoying result.

Marketing people who were aware of the strange assumption about time-invariant total advertisement used the following trick: Instead of the financial amount they used the *shares* of advertisement which automatically add up to one for all  $t$ .<sup>15</sup>

<sup>14</sup>Of course, one could restrict both coefficients to zero. See the discussion in section 4.1.

<sup>15</sup>See, for example, Ghosh et al (1984) p. 204. Those authors also used 'price shares' which have no easy interpretation.

This result can be stated more generally<sup>16</sup> as follows: Assume that a share system for  $r$  categories with  $K > r$  explanatory variables is given. We have  $K - r$  category-independent variables (including the constant term) denoted by  $x_{tk}, k = 1, \dots, K - r$ , and one<sup>17</sup> set of  $r$  category-specific variables denoted by  $z_{tj}, j = 1, \dots, r$  which might be the firm-specific shares of advertisement:

$$y_{jt} = \sum_{k=1}^{K-r} \beta_{jk} x_{tk} + \sum_{i=1}^r \gamma_{ji} z_{ti} + u_{jt}, \quad j = 1, \dots, r \quad (2-21)$$

In this system the market share  $y_{jt}$  of firm  $j$  is explained by advertisement shares of *all*  $r$  firms together with some other variables like the oil price or total sales of industry. The category-specific  $z$ -variables satisfy the constraint

$$\sum_j z_{tj} = \tau \quad \text{for all } t. \quad (2-22)$$

In case of shares we would have  $\tau = 1$ . Then the following constraints must be satisfied by the coefficients:

$$\begin{aligned} \sum_{j=1}^r \gamma_{ji} &= \kappa, & i &= 1, \dots, r, \\ \sum_{j=1}^r \beta_{jk} &= 0, & k &= 2, \dots, K - r, \\ \sum_{j=1}^r \beta_{j1} &= 1 - \kappa\tau \end{aligned} \quad (2-23)$$

So both the  $\beta$ 's and the  $\gamma$ 's are constrained over the  $r$  different share equations.

Please note that the system (2-21) *without* restrictions (2-22) creates no special problems as long as *all* category-specific variables appear in a symmetric manner. We mentioned already above the example of econometric demand analysis where usually prices of all goods are specified. The trouble starts (and restrictions (2-22) become important) only when some of the  $\gamma$ 's in (2-21) are set equal to zero. A particular interesting case arises if only firm's own advertisement share is specified as explanatory variable, that is we restrict all  $\gamma_{ji}$  to zero for  $j \neq i$ . Then the share equations (2-21) can be written as follows:

$$y_{jt} = \sum_{k=1}^{K-r} \beta_{jk} x_{tk} + \bar{\gamma} z_{tj} + u_{jt}, \quad j = 1, \dots, r \quad (2-24)$$

with additional constraints on the  $\beta$ 's:

$$\sum_{j=1}^r \beta_{jk} = 0, \quad k = 2, \dots, K - r,$$

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<sup>16</sup>For the following see Ronning (1986).

<sup>17</sup>A more general result can be formulated for various sets.



$$\sum_{j=1}^{\tau} \beta_{j1} = 1 - \bar{\gamma}\tau. \quad (2-25)$$

which for the two-category case has already been discussed above.

There has been an intensive discussion in the marketing literature especially about the restriction with regard to category-specific variables. Naert and Weverbergh (1981 p. 147) write that '...(this) restriction will often be considered as unacceptable by marketing managers.' They therefore estimate the model without this restriction. How are the results to be interpreted then? In my opinion the results are simply *uninterpretable* since they result from a non-identified model.

McGuire and Weiss (1976) have pointed out that the particularly annoying restriction of identical  $\gamma$ 's in all equations can be avoided if in each equation at least two variables from the set of  $z$ -variables is specified. Since such an asymmetric specification usually does not make sense this suggestion implies specification of the whole set of  $z$ -variables such as in (2-21). We noted already above that then only the usual adding-up constraints would be valid if not restriction (2-22) has to be taken into account..

One further remark concerns the error terms in the share models just considered. Of course, they are also category-specific variables and therefore should have the same coefficient in all equations which is true since all coefficients equal 1. We come back to this point in section 4 where we consider autoregressive residuals.

Another remark concerns the OLS estimation of share equations with more than two categories. If all explanatory variables are category-independent then the system of shares can be written - in analogy to (2-6) for the two-category case - as follows:

$$\mathbf{y}_j = \mathbf{X}\boldsymbol{\beta}_j + \mathbf{u}_j, \quad j = 1, \dots, \tau, \quad (2-26)$$

and the OLS estimator of the systematic part  $\boldsymbol{\eta}_j = \mathbf{X}\boldsymbol{\beta}_j$  is given by

$$\hat{\boldsymbol{\eta}}_j = \mathbf{X}\mathbf{X}^+ \mathbf{y}_j. \quad (2-27)$$

It is well-known that the estimated shares sum up to a vector of ones<sup>18</sup>, that is

$$\sum_j \hat{\boldsymbol{\eta}}_j = \boldsymbol{\iota} \quad (2-28)$$

However this is no longer true for systems in which category-dependent variables appear as could be illustrated by an example. Moreover OLS estimation would not recognize the adding-up restrictions for the  $\gamma$ 's. Therefore restricted least squares satisfying restrictions over equations should be used !

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<sup>18</sup>We use the fact that  $\mathbf{X}\mathbf{X}^+ \boldsymbol{\iota} = \boldsymbol{\iota}$  and  $\sum_j \mathbf{y}_j = \boldsymbol{\iota}$ .

## 2.4 Linear Dirichlet share models

A number of statisticians has tried to find an adequate stochastic specification for shares. A 'natural' formulation is offered by the Dirichlet distribution with its special case of the Beta distribution in case of two categories. Woodland (1979) has proposed this model for demand analysis and Ronning (1989b) has analysed its properties. Since this approach has a nonlinear analogue which seems to be more suitable for the formulation of share models we postpone the discussion to section 3.3.

## 3 Nonlinear Share Models

### 3.1 Normalizing linear share models

Since shares are normalized variables which are derived from *nonnegative* quantities such as expenditures or sales the models which try to explain them should be constructed in a corresponding manner. One could think of a normalized version of the linear model with sales or expenditures as dependent variables. This gives

$$\begin{aligned} y_{1t} &= \frac{\beta_{11} + \beta_{12}x_{t2}}{\beta_{11} + \beta_{12}x_{t2} + \beta_{21} + \beta_{22}x_{t2}} + u_{1t} \\ y_{2t} &= \frac{\beta_{21} + \beta_{22}x_{t2}}{\beta_{11} + \beta_{12}x_{t2} + \beta_{21} + \beta_{22}x_{t2}} + u_{2t} \end{aligned} \quad (3-1)$$

or

$$\begin{aligned} y_{1t} &= \frac{\beta_{11} + \beta_{12}x_{t2} + u_{1t}}{\beta_{11} + \beta_{12}x_{t2} + u_{1t} + \beta_{21} + \beta_{22}x_{t2} + u_{2t}} \\ y_{2t} &= \frac{\beta_{21} + \beta_{22}x_{t2} + u_{2t}}{\beta_{11} + \beta_{12}x_{t2} + u_{1t} + \beta_{21} + \beta_{22}x_{t2} + u_{2t}} \end{aligned} \quad (3-2)$$

Although the two sets of equations look very similar they are fundamentally different: In (3-1) we have normalized the *deterministic* variables  $\eta_{jt}$  and then added an error term whereas in (3-2) the *stochastic* variables were normalized. The first type has been used extensively in econometric demand analysis under the heading 'translog demand systems' first suggested by Christensen, Jorgenson and Lau (1975). The second type, or rather a multiplicative version of it<sup>19</sup> has been favored by market researchers. We note the following characteristics:

(i) Both models have to observe parameter restrictions over the equations.

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<sup>19</sup>See Naert and Weverbergh (1981 table 1). They call it 'market attraction model'.

- (ii) Both models satisfy the sum constraint automatically.
- (iii) Both models satisfy the 'range constraint' only if additional assumption with regard to error terms and/or values of explanatory variables are made.
- (iv) Coefficients in both models are identified only up to a scale factor.

### 3.2 A logistic specification

For the multiplicative version version of (3-2), the so-called market-attraction models, the third remark is no longer true if only nonnegative explanatory variables are considered and the error terms are assumed to be lognormally distributed. These two specifications can be forced into the model if we consider the so-called 'linear logit share equations' suggested by Considine and Mount (1984)<sup>20</sup> for the analysis of input demand systems:

$$\begin{aligned}
 y_{1t} &= \frac{e^{\beta_{11} + \beta_{12}x_{t2} + u_{1t}}}{e^{\beta_{11} + \beta_{12}x_{t2} + u_{1t}} + e^{\beta_{21} + \beta_{22}x_{t2} + u_{2t}}} \\
 y_{2t} &= \frac{e^{\beta_{21} + \beta_{22}x_{t2} + u_{2t}}}{e^{\beta_{11} + \beta_{12}x_{t2} + u_{1t}} + e^{\beta_{21} + \beta_{22}x_{t2} + u_{2t}}}
 \end{aligned}
 \tag{3-3}$$

This model clearly satisfies both the 'range constraint' and the 'summing-up constraint' for arbitrary values of the explanatory variables and for any distribution of the error term (including the normal). The same suggestion for treating shares was made quite independently by Aitchison (1982, 1986) who assumed normality for the error terms<sup>21</sup> and called the implied distribution of shares the 'logistic normal distribution'. Again this model has parameters which are identified only up to a scale factor. This is also evident if we use the so-called logit-transformation and write the model in the equivalent form:

$$\log \left( \frac{y_{1t}}{y_{2t}} \right) = (\beta_{11} - \beta_{21}) + (\beta_{12} - \beta_{22})x_{t2} + (u_{1t} - u_{2t})
 \tag{3-4}$$

which is *linear in parameters* as already indicated by the name of this model. Under the assumption of independently and identically (normally) distributed errors the equation (3-4) can be estimated by ordinary least squares<sup>22</sup> using the 'log-ratios' as dependent variables. Therefore this approach seems to be very attractive for empirical work although it is rather heuristic when seen from an, say, economic view: Both the Translog

<sup>20</sup>The authors use the term 'linear logit model'. See also Bewley (1986, pp.31-33 and chap. 7.6).

<sup>21</sup>Aitchison considers only models without explanatory variables but his idea is immediately generalized to the model considered here.

<sup>22</sup>See Aitchison (1986 p. 143).

model and the 'Almost ideal Demand System' were derived from behavioral assumptions which lead to linear share equations such as (2-12) or nonlinear share equations such as (3-1) (or (3-2)). We therefore present in the next subsection a approach which tries to specify a direct stochastic analogue of deterministic demand systems using the Dirichlet distribution.

A last remark with regard to the linear logit model in (3-3) or (3-4) considers the *use of category-specific variables*. Assume that instead of explanatory variable  $x_{t2}$  we would have used  $z_{t1}$  in the numerator of the first share equation and  $z_{t1}$  in the numerator of the second share equation leading to the following two-equation system:

$$\begin{aligned} y_{1t} &= \frac{e^{\gamma_{11} + \gamma_{12} z_{t1} + u_{1t}}}{e^{\gamma_{11} + \gamma_{12} z_{t1} + u_{1t}} + e^{\gamma_{21} + \gamma_{22} z_{t2} + u_{2t}}} \\ y_{2t} &= \frac{e^{\gamma_{21} + \gamma_{22} z_{t2} + u_{2t}}}{e^{\gamma_{11} + \gamma_{12} z_{t1} + u_{1t}} + e^{\gamma_{21} + \gamma_{22} z_{t2} + u_{2t}}} \end{aligned} \quad (3-5)$$

If we would write down the corresponding linear model of log-ratios it would become evident that the two coefficients related to the category-dependent variables must be equal, that is

$$\gamma_{12} = \gamma_{22} \quad (3-6)$$

since otherwise the parameters would not be identified.<sup>23</sup>

### 3.3 Dirichlet share equations

We assume that for each observation point  $t$  the random shares  $Y_{it}$  for the  $r$  categories are generated by a Dirichlet distribution with parameters  $\theta_{it}$ . For two categories ( $r = 2$ ) this specification specializes to the better-known Beta distribution. Figure 3 shows the typical shape of this distribution. Note that the density is bell-shaped only if both parameters are greater than 1.

\*\*\*\*\*figure 3 about here\*\*\*\*\*

For each  $t$  the joint density function of the shares  $Y_{it}$  is given by

$$f(y_{1t}, y_{2t}, \dots, y_{rt}) = \Gamma \left( \sum_{j=1}^r \theta_{jt} \right) \prod_{i=1}^r \frac{y_{it}^{\theta_{it}-1}}{\Gamma(\theta_{it})} \quad (3-7)$$

---

<sup>23</sup>This result appears also in discrete choice analysis in connection with 'category-dependent characteristics'. See Maddala (1983) p. 42. For this reason Considine and Mount (1984 p. 440) specify a coefficient of lagged demand which is constant for all factors of production.

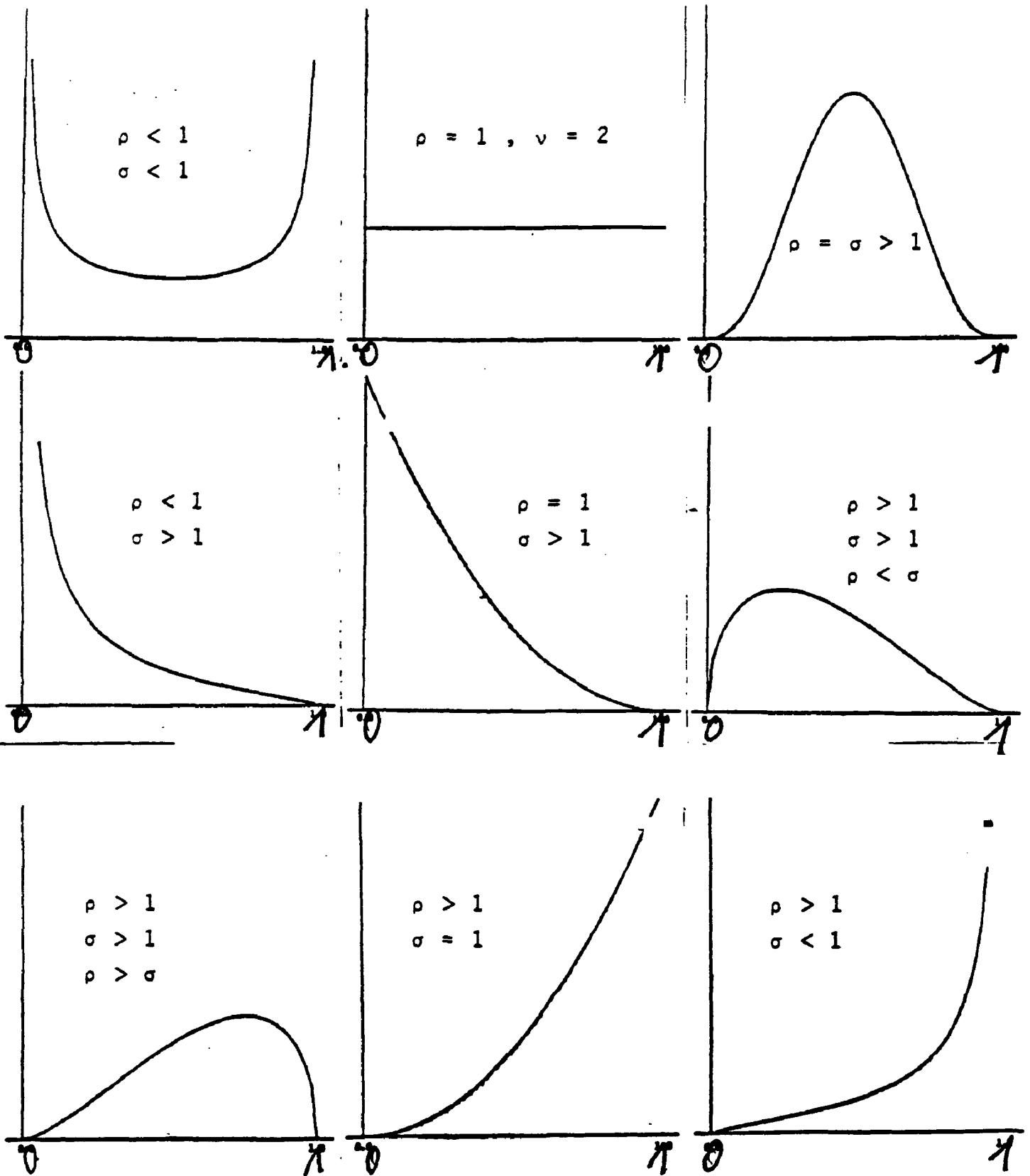


Figure 3 - Beta Distributions

with  $\theta_{it} > 0$  and  $y_{it} \geq 0$  for all  $i$ ,  $\sum_{j=1}^r y_{jt} = 1$  and  $\Gamma$  denotes the gamma function. We now let the parameters  $\theta_{it}$  depend linearly on the explanatory variable  $x_t$ :

$$\theta_{it} = \alpha_i + \beta_i x_t \quad (3-8)$$

where again it is assumed that  $\theta_{it} > 0$ . Then the expected value of the stochastic share  $Y_{it}$  is given by

$$E(Y_{it}) = \frac{\alpha_i + \beta_i x_t}{\sum_{j=1}^r \alpha_j + \beta_j x_t} \quad (3-9)$$

(see, for example, Johnson and Kotz 1972 p.233), that is, the expected value of  $Y_{it}$  has the form of the deterministic part of the *nonlinear* share equations in (3-1).

Following Woodland (1979) one could also derive a *linear* specification from

$$\theta_{it} = c(\alpha_i + \beta_i x_t) \quad , \quad c > 0 \quad (3-10)$$

together with the parameter restrictions ('adding-up constraints')

$$\sum_j \alpha_j = 1 \quad \text{and} \quad \sum_j \beta_j = 0 \quad (3-11)$$

We have under (3-10) and (3-11)<sup>24</sup>

$$E(Y_{it}) = \alpha_i + \beta_i x_t \quad . \quad (3-12)$$

In (3-8) and (3-10) we implicitly assume that  $0 < \alpha_i + \beta_i x_t < 1$  holds.

The unknown  $\alpha$ 's and  $\beta$ 's of the nonlinear share model<sup>25</sup> can be estimated by the maximum likelihood procedure. In Ronning (1989b) it is shown that the log-likelihood function of this model is globally concave<sup>26</sup> which makes computation of the maximum easy although numerical problems may arise for a large number of categories. See the empirical example in section 3.4. Proof of concavity is relatively easy if use is made of the inequality

$$\psi_1(x) < \delta \psi_1(\delta x) \quad , \quad x > 0, \quad 0 < \delta < 1 \quad (3-13)$$

(see Ronning 1986a) where  $\psi_1$  is the trigamma function which is defined as the second derivative of the log of the gamma function, that is  $\psi_1(x) \equiv \frac{d^2 \log \Gamma(x)}{(dx)^2}$ .<sup>27</sup>

<sup>24</sup>Note that the use of (3-10) (instead of (3-8)) implies the flexible scale factor  $1/(1+c)$  (instead of  $1/2$ ) in the covariance matrix of the random variables  $Y_{it}$  for a certain  $t$ . The covariance matrix of the Dirichlet distribution is given, for example, in Johnson and Kotz (1972 p.233).

<sup>25</sup>In the following we disregard results for the linear share model since the nonlinear specification seem to fit better into the Dirichlet distribution. Results for the linear case are given in Ronning (1989b).

<sup>26</sup>See also Ronning (1989a) for some numerical aspects.

<sup>27</sup>See Abramowitz and Stegun (1965, chapter 6).

The fact, that the stochastic specification of error terms in share equations is more than merely 'adding an additive error term', is nicely illustrated for the nonlinear specification: For the deterministic share equations of translog type parameters  $\alpha_i$  and  $\beta_i$  are *not* identified, that is the parameters are only unique up to a scale factor. However in the nonlinear Dirichlet share equations model these parameters *are* identified (as shown by the definiteness of the Hessian matrix - see Ronning (1989b).). This is due to the fact that not only the expected values, but also the covariance matrix of the shares for this model are determined by these parameters.

A main disadvantage of the Dirichlet distribution is that all correlations between shares are negative. More seriously, these nonzero correlations do not necessarily have any economic meaning: If expenditures with regard to  $r$  different goods were *independently* gamma distributed with identical scale parameter and second parameter  $\theta_i$ , then the budget shares were Dirichlet distributed with parameters  $\theta_i$ . See Johnson and Kotz (1972 p. 231-233). This reveals two inconsistencies of the Dirichlet distribution approach: (i) Expenditures for some goods may very well be correlated with each other according to substitutional or complementary relations of the corresponding goods. (ii) The specification of gamma distributed expenditures does not take into account the budget constraint as a (finite) upper bound of the random variable. Needless to say that these points should not be taken as arguments in favour of the purely heuristic specification of *normally* distributed shares. However, the empirical example in the following subsection shows that the fit of the Dirichlet model is quite good and outperforms (for this example) the linear logit specification.

### 3.4 An empirical example

In our empirical example we use yearly German consumer expenditure data for the years 1966 to 1986 ( $n = 21$  observations) for  $r = 8$  expenditure categories. Prices for these groups together with total expenditure (as an indicator of 'income') and a linear trend variable were specified as explanatory variables. Under this specification both the (nonlinear) Dirichlet share equations model and the linear logit model were estimated. No parameter constraints were used in both models and no care was taken of possible autocorrelation. The Dirichlet model was estimated by maximum likelihood<sup>28</sup> and the linear logit model (in its linear form) by least squares. Note that both the Dirichlet model and the linear logit specification use  $r * (r + 2) = 80$  parameters. However the

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<sup>28</sup>Our program is written in GAUSS. It checks at each iteration the positivity of the Dirichlet parameters since otherwise the Gamma function is no longer defined. See Ronning (1989a, 1989b). Convergence was achieved after 17 iteration starting from arbitrary (small positive) values.

latter loses  $r + 2$  parameters through normalization.

The upper part of table 3 shows the empirical correlations of the observed shares from the 8 categories. The two other parts of this table show the correlation matrices computed from the estimated shares of the two approaches. Furthermore we report the 'distance'  $d$  between estimated and observed shares which was computed from

$$d = \sum_t \sum_j |y_{jt} - \tilde{\eta}_{jt}| \quad (3 - 14)$$

where  $\tilde{\eta}$  is either the ML estimate for the Dirichlet model or the OLS estimate for the linear logit model with  $\eta$ . Without going into the details of our estimation results (which we consider as preliminary) we would like to mention the following points:

- (i) The Dirichlet model shows both positive and negative correlations.
- (ii) The Dirichlet model shows a better fit than the linear logit specification.

Note that we report for the Dirichlet specification the usual estimate of the correlation matrix which assumes a constant correlation for all observation points  $t$ . The same estimation procedure is used also for the two other parts of the table. On the contrary Woodland reports correlation of shares which result from the estimates when inserted into the formula for the theoretical covariance matrix. Of course, then all covariances have to be negative! However our results demonstrate that this approach is unfair towards the Dirichlet model when compared with other models! We therefore conclude that our Dirichlet model does not exclude positive correlations for *estimated shares*!



Table 3: Estimation Results

Correlation matrix of observed shares							
1.000	0.920	-0.860	-0.842	0.431	-0.949	0.695	-0.897
0.920	1.000	-0.843	-0.882	0.485	-0.881	0.567	-0.803
-0.860	-0.843	1.000	0.762	-0.566	0.703	-0.417	0.706
-0.842	-0.882	0.762	1.000	-0.549	0.776	-0.441	0.663
0.431	0.485	-0.566	-0.549	1.000	-0.445	0.238	-0.345
-0.949	-0.881	0.703	0.776	-0.445	1.000	-0.768	0.857
0.695	0.567	-0.417	-0.441	0.238	-0.768	1.000	-0.801
-0.897	-0.803	0.706	0.663	-0.345	0.857	-0.801	1.000
Correlation matrix of estimated shares							
LINEAR LOGIT							
1.000	0.956	-0.863	-0.854	0.543	-0.960	0.742	-0.915
0.956	1.000	-0.871	-0.897	0.577	-0.910	0.620	-0.832
-0.863	-0.871	1.000	0.766	-0.659	0.710	-0.426	0.724
-0.854	-0.897	0.766	1.000	-0.659	0.799	-0.459	0.670
0.543	0.577	-0.659	-0.659	1.000	-0.492	0.232	-0.443
-0.960	-0.910	0.710	0.799	-0.492	1.000	-0.817	0.894
0.742	0.620	-0.426	-0.459	0.232	-0.817	1.000	-0.874
-0.915	-0.832	0.724	0.670	-0.443	0.894	-0.874	1.000
Correlation matrix of estimated shares							
DIRICHLET							
1.000	0.955	-0.866	-0.863	0.540	-0.962	0.726	-0.917
0.955	1.000	-0.868	-0.902	0.582	-0.915	0.607	-0.831
-0.866	-0.868	1.000	0.768	-0.660	0.721	-0.415	0.727
-0.863	-0.902	0.768	1.000	-0.663	0.815	-0.477	0.681
0.540	0.582	-0.660	-0.663	1.000	-0.496	0.193	-0.453
-0.962	-0.915	0.721	0.815	-0.496	1.000	-0.792	0.905
0.726	0.607	-0.415	-0.477	0.193	-0.792	1.000	-0.864
-0.917	-0.831	0.727	0.681	-0.453	0.905	-0.864	1.000
d(linear logit) = 0.334 , d(Dirichlet) = 0.315							

## 4 Dynamic Share Models

### 4.1 Autoregressive linear share models and distributed lags

Although the adding-up restrictions (2-16) were recognized already quite early<sup>29</sup>, additional constraints for dynamic share equations were recognized only during the mid-sixties. Berndt and Savin (1975) showed that it is not possible to specify a category-specific autoregressive residual of the form

$$u_{jt} = \rho_j u_{j,t-1} + \varepsilon_{jt}, \quad j = 1, \dots, r \quad (4-1)$$

but that all share equations must show the same autoregressive coefficient:

$$\rho_j = \rho \quad \text{for all } j. \quad (4-2)$$

This, of course, is again the result already used in section 2 for category-specific explanatory variables. See (2-25) or the restrictions in (2-24) for the more general case of all  $r$  residuals specified in each equation. The same result applies to the nonlinear share system (3-1) which has an *additive* error term (but not to (3-2) or (3-3)).

Lagged dependent variables have been used quite extensively in econometric work to capture phenomena such as learning and partial adjustment both in econometric demand systems and in market share systems. Most researchers specified lagged shares as additional explanatory variables<sup>30</sup> although some studies have preferred lagged quantities which they consider as the more adequate measure of adjustment.<sup>31</sup>

The two following *linear* share systems show the two specifications for just two categories and one explanatory variable ( $x_t$ ) besides the lagged variable: For lagged shares we have

$$\begin{aligned} y_{1t} &= \beta_{11} + \beta_{12}x_t + \gamma_1 y_{1,t-1} + u_{1t} \\ y_{2t} &= \beta_{21} + \beta_{22}x_t + \gamma_2 y_{2,t-1} + u_{2t} \end{aligned} \quad (4-3)$$

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<sup>29</sup>See chapter 1 of Deaton and Muellbauer (1980b) and the bibliographical notes on page 24 of that book.

<sup>30</sup>This seems to be the case for all studies in the marketing area. See, for example, Naert and Weverbergh (1981) and Ghosh et al (1984). For studies in econometric demand analysis using lagged shares see Klevmarken (1979) and Anderson and Blundell (1982).

<sup>31</sup>See, for example, Blanciforti and Green (1983) and Considine and Mount (1984).

and for lagged quantities  $q_{j,t-1}$  we get

$$\begin{aligned} y_{1t} &= \beta_{11} + \beta_{12}x_{t2} + \gamma_1 q_{1,t-1} + u_{1t} \\ y_{2t} &= \beta_{21} + \beta_{22}x_{t2} + \gamma_2 q_{2,t-1} + u_{2t} \end{aligned} \quad (4-4)$$

For the first system it is clear from (2-25) that the two  $\gamma$ 's must be equal. For the second system however which was used, for example, by Blanciforti and Green (1983) we run into trouble: Since there is no meaningful restriction on the *physical quantities* of goods<sup>32</sup> we cannot use (2-25). On the other hand, without a constraint on the variable  $q$  the only admissible parameter value for the  $\gamma$ 's is zero which eliminates the lagged variables completely from both equations.<sup>33</sup> We here again have a good example of how strange the rules in share equations are.

Since lagged dependent variables often are motivated by a reformulation of the Koyck distributed lag model we should mention here another peculiarity of the share model.<sup>34</sup> Let us assume that a set of shares is explained by the usual distributed lag specification:

$$\begin{aligned} y_{it} &= \alpha_i + \sum_{k=0}^{\infty} \beta_{ik} x_{t-k} + u_{it} \\ \beta_{ik} &= \gamma(1 - \lambda_i)\lambda_i^k, \quad 0 \leq \lambda_i < 1 \end{aligned} \quad (4-5)$$

where  $\gamma > 0$  is some normalizing constant. The 'adding-up constraint'  $\sum_j \beta_{jk} = 0$  then implies that

$$\lambda_i = 0, \quad , i = 1, \dots, r. \quad (4-6)$$

must hold, that is the Koyck distributed lag model is incompatible with share equations !

## 4.2 An autoregressive share process with stochastic coefficients

To my best knowledge no attempt has been made in econometrics to consider the properties of stochastic share processes. For example, nothing is known about stationarity for the autoregressive models considered above.<sup>35</sup> Note that the usual rules do not apply since the shares should stay within the unit interval and will have a varying variance over time. An interesting alternative for the two-category case has been

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<sup>32</sup>As explained in section 2.3 the restriction should also be constant for all  $t$ .

<sup>33</sup>Blanciforti and Green (1983) estimate the model without constraint.

<sup>34</sup>The following is taken from Ronning (1988).

<sup>35</sup>See Ronning (1986b) for some speculative remarks.

suggested by McKenzie (1985) who considers an first order autoregressive process of a beta-distributed random share with stochastic coefficients. He uses the following properties of the Beta distribution:

$$\begin{aligned} 1 - Be(\alpha, \beta) &= Be(\beta, \alpha) \\ Be(\alpha, \beta) \cdot Be(\alpha + \beta, \gamma) &= Be(\alpha, \beta + \gamma) \end{aligned} \quad (4-7)$$

Here  $Be(p, q)$  denotes the beta distribution with parameters  $p$  and  $q$  and '=' denotes equality of distributions. Assume now that  $y_{t-1} \sim Be(\alpha, \beta)$  and that two other random variables,  $u_t$  and  $w_t$ , satisfy

$$u_t \sim Be(\beta, \alpha - p) \text{ and } w_t \sim Be(p, \alpha - p), \quad 0 < p < \alpha. \quad (4 - 8)$$

Both random variables are distributed independently of each other and of  $y_{t-1}$ . Then from the two properties mentioned above it can be shown that  $y_t \sim Be(\alpha, \beta)$ , that is the process is stationary, if the following linear process is specified:

$$y_t = 1 - u_t(1 - w_t y_{t-1}) \quad (4 - 9)$$

Its autocorrelation function is given by  $\rho_y(k) = \rho^k$  where

$$\rho = \frac{p\beta}{\alpha(\alpha + \beta - p)} \quad (4 - 10)$$

so that autocorrelation increases with  $p$  and takes on all *positive* values between 0 and 1 if  $p$  varies within the admissible interval. Note that we can write (4-9) as follows:

$$\begin{aligned} y_t &= A_t + B_t y_{t-1} \\ A_t &= u_t w_t \\ B_t &= 1 - u_t \end{aligned} \quad (4-11)$$

In a similar manner a process for *negative* autocorrelation can be defined.

## 5 Concluding Remarks

The paper presents in a rather provocative manner problems invoked from the use of shares as dependent variables. All kinds of restrictions emerge which are hard to interpret from a econometrician's view. The question then arises: Why do we use share equations at all? The argument that shares are used frequently in descriptive analysis

and should therefore also be used in theoretical (stochastic) models does not convince very much. Certainly one could normalize the estimated quantities instead of starting from a model for normalized variables. If these arguments are not strong enough then one should at least look for a 'robust' specification of the share model. Aitchison's (1982,1986) suggestion of logistic normal distributions which lead to well-known logit regression should then be considered as an attractive alternative as demonstrated by Considine and Mount (1984). However, our empirical example shows that also the Dirichlet specification which has been attacked because of its rather restrictive covariance structure can be seen as a possibility although the computational effort is greater.

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