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## Intra-industry trade in the Heckscher-Ohlin-Lerner-Samuelson model

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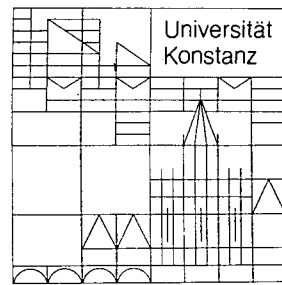
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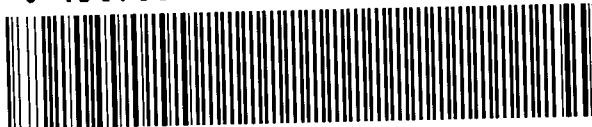
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John S. Chipman

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Heckscher-Ohlin-Lerner-Samuelson  
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HECKSCHER-OHLIN-LERNER-SAMUELSON MODEL

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# Intra-Industry Trade in the Heckscher–Ohlin–Lerner–Samuelson Model

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## Abstract

An  $n$ -commodity,  $n$ -factor,  $K$ -country log-linear model is developed, with identical Cobb–Douglas production coefficients and identical consumer expenditure shares across countries, in which factor endowments are such as to permit positive production of all commodities in all countries, hence equalization of factor rentals among them. Explicit solutions are obtained for the countries' imports, exports, and outputs as well as for equilibrium prices and factor rentals. Allowance is made for interindustrial relationships and trade in intermediate products.

Given any set of production and consumption coefficients and any set of world factor endowments which, when suitably allocated among countries, result in positive production of all commodities in all countries, and given any mode of aggregating the  $n$  commodities into at most  $n - 1$  industrial categories, it is shown that there exists an allocation of the world factor endowments among the  $K$  countries such that each country engages in trade and 100% of each country's trade is intra-industry trade (i.e., the values of imports and exports balance each other in each aggregate category).

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# 1 Introduction

As any international economist knows who has attempted to analyze statistics of international trade flows, at even the finest subdivisions one will find imports to and exports from the same country in virtually every commodity category. This phenomenon has been described by Grubel & Lloyd (1975) as “intra-industry trade”.

Following the lead of these authors, the overwhelming consensus of opinion among international economists appears to have been that the phenomenon of intra-industry trade is incompatible with the Heckscher–Ohlin–Lerner–Samuelson (HOLS) theory, or indeed with the law of comparative advantage itself. A typical point of view is that expressed by Helpman & Krugman (1985, p. 5):

Conventional trade theory explains trade entirely by differences among countries, especially differences in their relative endowments of factors of production. This suggests an inverse relationship between similarity of countries and the volume of trade between them. In practice, however, nearly half the world’s trade consists of trade between industrial countries that are relatively similar in their relative factor endowments.

... actual trade patterns seem to include substantial two-way trade in goods of similar factor intensity. This “intra-industry” trade seems both pointless and hard to explain from the point of view of conventional trade analysis.

One will find many similar statements in the literature.

In fact, the so-called “Heckscher–Ohlin theorem” (cf. Jones 1956)—which states that a country will export the commodity which uses relatively intensively the factor in which it is relatively well endowed—is valid only under very stringent conditions: free and costless trade between two countries in two commodities using two factors of production under conditions of perfectly competitive markets and constant returns to scale, absence of reversal of factor intensities, identical production functions and identical homothetic preferences between countries, and balanced trade. Generally in the literature it is only the absence of perfect competition and constant

returns to scale to which intra-industry trade is attributed, although it is obvious that in the real world most of the other assumptions—in particular that there are only two commodities, factors, and countries—are clearly violated. Furthermore, the Heckscher–Ohlin theorem is concerned only with the *direction* of trade; it says nothing about the *volume* of trade.

It is therefore somewhat surprising that, given the very stringent conditions required for the Heckscher–Ohlin theorem to hold, only two violations of these many conditions have been singled out in the literature as responsible for the observed phenomenon of intra-industry trade. Possibly this is because there has generally been lack of recognition in the profession of how stringent these conditions are, given the scarcity of correct statements to be found in the literature.<sup>1</sup>

In this paper I show that the HOLS model, in its multi-commodity, multi-factor, multi-country version (cf. Samuelson, 1953), is compatible with any degree of intra-industry trade; in particular, it is compatible with a situation in which all countries trade with each other and 100% of their trade is intra-industry trade. I show this for a special case of the HOLS model—that of identical Cobb–Douglas technologies and tastes, in which the number of commodities is equal to the number of factors, and in which trade is balanced. It goes without saying that under less stringent assumptions it would be even easier for the model to be compatible with intra-industry trade. The Cobb–Douglas model—developed in section 2—permits a simple algebraic solution to the world equilibrium problem, avoiding the complications of fixed-point theorems.<sup>2</sup> Thus it provides explicit closed-form expressions for equilibrium trades and prices, making it possible to obtain answers to questions that would otherwise be difficult to obtain. The model of section 2 also allows for interindustrial relationships and trade in intermediate products.

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<sup>1</sup>To my knowledge, the first correct statement and proof of the Heckscher–Ohlin theorem in the literature was that of Riezman (1974), and the next that of Leamer (1980). Most statements to be found in the literature omit at least two crucial conditions. For a discussion see Chipman (1987).

<sup>2</sup>For previous developments of models assuming Cobb–Douglas technology and/or preferences, see Laursen (1952), Radner (1966), Leamer (1980), Chipman (1985, 1986), and Hartigan & Tower (1986). Undoubtedly there have been others.

The main result, contained in section 3, takes as given arbitrary production and consumption coefficients, as well as any arbitrary mode of aggregation of commodities into a smaller number of industrial groups. It assumes that the world factor endowments are such as to make possible, when appropriately allocated among countries, positive production of all  $n$  commodities in all countries and equalization of the  $n$  factor rentals among them. The theorem states that it is possible to find allocations of the world factor endowments among countries (in fact, many such allocations) such that all countries engage in trade and 100% of their trade is intra-industry trade.

## 2 A simple log-linear model of the world economy

Let production functions in all countries be of the identical Cobb–Douglas type

$$(1) \quad q_j^k = f_j(u_{.j}^k, v_{.j}^k) = \mu_j \prod_{i=1}^n (u_{ij}^k)^{\alpha_{ij}} \prod_{i=1}^m (v_{ij}^k)^{\beta_{ij}} \\ (\alpha_{ij} \geq 0, \beta_{ij} \geq 0, \sum_{i=1}^n \alpha_{ij} < 1, \sum_{i=1}^n \alpha_{ij} + \sum_{i=1}^m \beta_{ij} = 1),$$

where  $q_j^k$  is the gross output of commodity  $j$  in country  $k$  and  $u_{ij}^k, v_{ij}^k$  are respectively the amounts of the  $i$ th intermediate and the  $i$ th primary input into the production of commodity  $j$ , and  $u_{.j}^k, v_{.j}^k$  are vectors of these amounts. The minimum-unit-cost function dual to (1) is

$$(2) \quad g_j(p, w^k) = \nu_j \prod_{i=1}^n p_i^{\alpha_{ij}} \prod_{i=1}^m (w_i^k)^{\beta_{ij}} \quad (\nu_j = 1/f_j(\alpha_{.j}, \beta_{.j})),$$

where  $p_i$  and  $w_i^k$  are the  $i$ th commodity price and factor rental in country  $k$  respectively, and  $p, w^k$  denote the corresponding  $n \times 1$  and  $m \times 1$  vectors. Thus, the exponents  $\alpha_{ij}, \beta_{ij}$  may be interpreted both as elasticities of gross outputs with respect to intermediate and primary inputs, and as elasticities of minimum unit costs with respect to prices of intermediate and primary inputs respectively. The cost-minimizing input-output and factor-output coefficients are given by  $a_{ij} = \partial g_j / \partial p_i = p_j \alpha_{ij} / p_i$  and  $b_{ij}^k = \partial g_j / \partial w_i^k = p_j \beta_{ij} / w_i^k$  respectively (cf. Shephard, 1953). Denoting the

$n \times n$  input-output and  $m \times n$  factor-output matrices  $A = [a_{ij}]$  and  $B^k = [b_{ij}^k]$  respectively, as well as the corresponding elasticity matrices  $A = [\alpha_{ij}]$  and  $B = [\beta_{ij}]$  (which are distinguished from  $A$  and  $B$  by being set in roman rather than italic), these relationships may be indicated in matrix notation by

$$(3) \quad A = P^{-1}AP, \quad B^k = (W^k)^{-1}BP$$

where  $P$  and  $W^k$  are diagonal matrices  $P = \text{diag } p$  and  $W^k = \text{diag } w^k$  of commodity prices and factor rentals respectively.

In accordance with the well-known Leontief (1951) theory, the  $n \times 1$  vector of net outputs is defined by

$$(4) \quad y^k = (I - A)q^k, \quad \text{or} \quad Py^k = (I - A)Pq^k,$$

where  $q^k$  is the vector of gross outputs. The resource-allocation constraint (assuming full employment) is given by  $\sum_{j=1}^n v_{ij}^k = \sum_{j=1}^n b_{ij}^k y_j = l_i^k$ , yielding

$$(5) \quad B^k q^k = l^k, \quad \text{or} \quad BPq^k = W^k l^k,$$

where  $l^k$  is the  $m \times 1$  vector of country  $k$ 's factor endowments. Note that in this log-linear model, linear relations between vectors of quantities defined by the input-output and factor-output matrices carry over into corresponding linear relations between vectors of *values* of these quantities defined by the respective matrices of input-output and factor-output elasticities.

It will now be assumed that all commodities are produced in each country, so that prices are equal to minimum unit costs for all  $n$  commodities. Since from (1) it follows that  $I - A$  has a positive dominant diagonal (cf. McKenzie, 1960), its inverse exists and it satisfies the Hawkins-Simon (1949) conditions, hence it satisfies  $(I - A)^{-1} \geq I$  (cf. Chipman, 1954). From (2) we have

$$(6) \quad \log p = [I - A']^{-1} \log \nu + [I - A']^{-1} B' \log w^k.$$

This defines a system of consolidated cost functions  $p = \psi(w^k)$ . Denoting

$$(7) \quad \Gamma = B(I - A)^{-1}, \quad C^k = B^k(I - A)^{-1}, \quad \text{where} \quad C^k = (W^k)^{-1} \Gamma P,$$



clearly  $\Gamma$  has all its elements nonnegative. We verify that it has unit column sums, hence is column-stochastic: since  $[A', B']$  has unit row sums from (1), the matrix  $[I - A', -B']$  has zero row sums hence, letting  $\iota$  denote a column vector of ones, we have

$$(8) \quad \iota' \begin{bmatrix} I \\ -B(I - A)^{-1} \end{bmatrix} = \iota' \begin{bmatrix} I - A \\ -B \end{bmatrix} (I - A)^{-1} = 0$$

and therefore  $\iota' I - \iota' B(I - A)^{-1} = \iota' - \iota' \Gamma = 0$ .

In order for (6) to provide a unique solution  $w^k$  for given  $p$ , it is necessary and sufficient that  $\text{rank}(B) = m$ ,<sup>3</sup> which implies in particular that  $m \leq n$ . This will be assumed in the rest of the paper. From this assumption it follows that factor rentals are equalized throughout the world, and the  $k$  superscript may be dropped from  $w$ ,  $b_{ij}$ ,  $B$ , and  $C$ . From (4) and (5) we obtain the relationship between net outputs and factor endowments:

$$(9) \quad Cy^k = l^k, \text{ or } \Gamma Py^k = Wl^k.$$

If  $n > m$ , the solutions of these equations are not unique, reflecting the fact that country  $k$ 's production-possibility frontier is a ruled surface. In this case there is a basic indeterminacy in trade patterns, and to proceed in our analysis we would have to deal with sets of possible equilibrium trades. In order to obtain definite results, I shall therefore assume henceforth that  $\text{rank}(B)$  (hence  $\text{rank}(\Gamma)$ ) =  $n$ , hence  $m = n$ ,

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<sup>3</sup>Let  $B$  have rank  $r \leq \min(m, n)$ . From Penrose's (1955) theory (see, e.g., Chipman, 1976), (6) has a solution if and only if  $\log p$  is confined to an  $r$ -dimensional affine subspace of  $n$ -dimensional space defined by the projection

$$(I - A') \log p - \log \nu = B' B^{-'} [(I - A') \log p - \log \nu],$$

where  $B^{-}$  is a generalized inverse of  $B$ , i.e., any matrix satisfying  $BB^{-}B = B$ . (If  $r = n$  this entails no restriction; otherwise it reflects the fact that country  $k$ 's production-possibility frontier is a ruled surface, and the budget hyperplane defined by the world prices must be tangent to this ruled surface.) The general solution of (7) is

$$\log w^k = B^{-'} [(I - A') \log p - \log \nu] + (I_m - B^{-'} B') \zeta$$

where  $\zeta$  is an arbitrary  $m \times 1$  vector. This is unique if and only if  $BB^{-} = I_m$ , i.e., if and only if  $\text{rank}(B) = m$ .

which is equivalent to assuming that the production-possibility frontier is strictly concave to the origin (cf. Chipman, 1987). Denoting  $L^k = \text{diag } l^k$ , we may write the solution of the second equation of (9) as

$$(10) \quad Py^k = \Gamma^{-1} W l^k = \Gamma^{-1} L^k w.$$

To complete the model, let us now assume that preferences in each country are generated by the same Mill-Cobb-Douglas utility function

$$(11) \quad U(x^k) = \prod_{j=1}^n (x_j^k)^{\theta_j} \quad (\theta_j > 0, \quad \sum_{j=1}^n \theta_j = 1),$$

where  $x_j^k$  is the consumption of commodity  $j$  in country  $k$ . This implies that in each country, consumers spend a constant proportion  $\theta_j$  of the disposable income  $Y^k$  on commodity  $j$ , i.e.,

$$(12) \quad Px^k = \theta Y^k, \quad \text{where } Y^k = \iota' P y^k + D^k,$$

where  $D^k$  is the deficit in country  $k$ 's balance of payments on current account.

World equilibrium may now be solved for explicitly as follows. Let there be  $K$  countries, and denote the world consumption, net output, and factor-endowment vectors and world income by

$$(13) \quad x = \sum_{k=1}^K x^k, \quad y = \sum_{k=1}^K y^k, \quad l = \sum_{k=1}^K l^k, \quad Y = \sum_{k=1}^K Y^k$$

respectively. World equilibrium is defined by the condition  $x = y$ . Aggregating (10) and (12) over countries and taking account of the fact that  $\sum_{k=1}^K D^k = 0$  we obtain

$$(14) \quad Py = \Gamma^{-1} W l$$

and

$$(15) \quad Px = \theta Y, \quad \text{where } Y = \iota' Px = \iota' Py = \iota' \Gamma^{-1} W l.$$

Denoting  $L = \text{diag } l$ , upon equating (14) and (15) we obtain

$$(16) \quad w = L^{-1} \Gamma \theta Y.$$

The solution (16) is unique up to a proportionality factor. One could set  $w_1 = 1$ , or more symmetrically,  $Y = 1$ . The latter normalization will be used in section 3.

Denoting  $\log \nu^* = [I - A']^{-1} \log \nu$ , the world prices are obtained from (6):

$$(17) \quad \log p = \log \psi(w) = \log \nu^* + \Gamma' \log w = \log \nu^* + \Gamma' \log(L^{-1} \Gamma \theta Y).$$

Country  $k$ 's output and consumption vectors are determined from (10), (12), (14), (15) and (16) by

$$(18) \quad Py^k = \Gamma^{-1} W l^k = \Gamma^{-1} L^k w = \Gamma^{-1} L^k L^{-1} \Gamma \theta Y$$

and

$$(19) \quad Px^k = \theta Y^k = \theta(\iota' P y^k + D^k) = \theta[\iota' \Gamma^{-1} L^k L^{-1} \Gamma \theta Y + D^k],$$

hence its net imports  $z^k = x^k - y^k$  are determined from

$$(20) \quad Pz^k = P(x^k - y^k) = \theta D^k - (I - \theta \iota') \Gamma^{-1} L^k L^{-1} \Gamma \theta Y.$$

Note that these vectors of values of production, consumption, and net imports (relative to world income,  $Y$ ) depend only on the technology, preferences, and endowments, and may be obtained without solving (17) for the equilibrium world prices. As seen above one solves for the prices by solving (16) for the factor rentals and substituting them into the consolidated cost function (6).

### 3 The main theorem

Assuming that  $n$  traded commodities are produced in each of  $K$  countries with the aid of  $n$  factors of production, let

$$(21) \quad t^k = Pz^k$$

denote the vector of country  $k$ 's *trades*, defined as the values of its net imports (by (20)). Country  $k$ 's gross imports and gross exports of commodity  $j$  may be defined by

$$(22) \quad z_j^k(+)=\max(z_j^k,0), \quad z_j^k(-)=-\min(z_j^k,0)$$

respectively, yielding the corresponding vectors

$$(23) \quad z^k(+) = (z_1^k(+), \dots, z_n^k(+))', \quad z^k(-) = (z_1^k(-), \dots, z_n^k(-))'.$$

In value terms we have the vectors of gross imports and exports

$$(24) \quad t^k(+) = Pz^k(+), \quad t^k(-) = Pz^k(-),$$

which have the property that net imports and total trades (in absolute value) are given by

$$(25) \quad t^k = t^k(+) - t^k(-) \quad \text{and} \quad |t^k| = t^k(+) + t^k(-)$$

respectively.

International trade statistics provide data not on the net and absolute trades (25), but on aggregates of these. Let  $G$  denote an  $\bar{n} \times n$  *grouping matrix*, defined as a matrix with zeros and ones with exactly one unit element in each column (cf. Chipman 1975a, 1975b, 1976), where  $\bar{n} < n$ . Trade statistics provide data on

$$(26) \quad \bar{t}^k(+) = Gt^k(+), \quad \bar{t}^k(-) = Gt^k(-),$$

i.e., on total gross imports and gross exports of all commodities in each category, and therefore also of net imports

$$(27) \quad \bar{t}^k(+) - \bar{t}^k(-) = G[t^k(+) - t^k(-)] = Gt^k$$

and of total trade

$$(28) \quad \bar{t}^k(+) + \bar{t}^k(-) = G[t^k(+) + t^k(-)] = G |t^k|$$

in each aggregated category.<sup>4</sup>

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<sup>4</sup>The process of grouping imports and exports according to (22) is of course itself a form of aggregation. We may suppose the grouping matrix  $G$  to be decomposed into a product  $G = \bar{G}G^*$  of  $\bar{n} \times 2\bar{n}$  and  $2\bar{n} \times n$  grouping matrices  $\bar{G}$  and  $G^*$ , where  $G^*$  is partitioned into two  $\bar{n} \times n$  import and export grouping submatrices  $G_0$  and  $G_1$ . These grouping submatrices distinguish and group together the import (resp. export) subindustries within standard industries, so that

$$G_0 t^k = G t^k(+) \quad \text{and} \quad G_1 t^k = -G t^k(-).$$

By their definitions, the disaggregated vectors of imports and exports  $t^k(+)$  and  $t^k(-)$  are orthogonal to one another; there cannot be both imports and exports in the same disaggregated category. In general this is not true of the aggregated vectors  $\bar{t}^k(+)$  and  $\bar{t}^k(-)$ . If it were true of the aggregate vectors, for all aggregated categories, then we would have  $|Gt^k| = G|t^k|$ , and we would say that there is no intra-industry trade. At the other extreme, if import values exactly balance export values in each aggregated category, then  $Gt^k = 0$ . The *Grubel-Lloyd index of intra-industry trade* (cf. Grubel & Lloyd, 1975, p. 22) is defined (for  $t^k \neq 0$ ) by

$$(29) \quad Q_{GL} = 1 - \frac{\iota' |Gt^k|}{\iota' G |t^k|} = 1 - \frac{\iota' |Gt^k|}{\iota' |t^k|},$$

(the last equality following from the fact that  $\iota' G = \iota'$ , by the definition of  $G$ ). The Grubel-Lloyd index takes on the values 0 and 1 in the two extreme cases just considered.

The second of these extreme cases is the major object of interest in this section. Is it possible, and if so under what conditions, for 100% of trade to be intra-industry trade?

From the form of (29), it is clear that  $Q_{GL} = 1$  if and only if  $Gt^k = 0$ . The Grubel-Lloyd index is an appropriate measure only if trade is balanced,<sup>5</sup> but this assumption will be adhered to here. Accordingly, from (21) and (20) we see that  $Q_{GL} = 1$  if and only if

$$(30) \quad Gt^k = GPz^k = -G(I - \theta\iota')\Gamma^{-1}L^kL^{-1}\Gamma\theta Y = 0.$$

This formula may be simplified in the following way. Define the vector

$$(31) \quad \lambda^k = \Lambda^k \iota, \quad \text{where} \quad \Lambda^k = L^k L^{-1}.$$

The matrix  $\bar{G}$  aggregates these import and export subindustries together into standard industries.

A problem facing the econometrician is that while there are data on trades  $G_i t^k$  as well as on import and export price indices of the form  $G_i^{\dagger} p = (G_i D_i G_i')^{-1} G_i D_i p$  (where  $D_i$  is a diagonal matrix of weights), there are no corresponding data on  $G_i x^k$ ,  $G_i y^k$ ,  $G_i q^k$  ( $i = 0, 1$ ).

<sup>5</sup>A more general measure which reduces to the Grubel-Lloyd index when trade is balanced was provided by Aquino (1978). See also Chipman (1987).

Recalling that  $L^k$  and  $L$  are diagonal matrices (see (10) and (16)), the product of the diagonal matrix  $\Lambda^k$  and the vector  $\Gamma\theta$  may be written as the product of the diagonal matrix  $\text{diag}(\Gamma\theta)$  and the vector  $\lambda^k$ , i.e.,

$$(32) \quad \Lambda^k \Gamma \theta = \text{diag}(\Gamma \theta) \lambda^k,$$

hence (30) yields the condition

$$(33) \quad G(I - \theta \iota') \Gamma^{-1} \text{diag}(\Gamma \theta) \lambda^k = 0.$$

For given  $\Gamma$  and  $\theta$ , (33) expresses a condition on the vector  $\lambda^k$  of country  $k$ 's endowments relative to world endowments.

Three more conditions must be satisfied by  $\lambda^k$ . First, it must be consistent with positive gross outputs. From (4), (3), and (18), and using the normalization  $Y = 1$ , we then have the condition

$$(34) \quad Pq^k = (I - A)^{-1} \Gamma^{-1} \text{diag}(\Gamma \theta) \lambda^k > 0 \quad (k = 1, 2, \dots, K).$$

A second condition is required in order to rule out trivialities, namely that each country be engaged in some trade, i.e.,  $t^k \neq 0$ , or  $\iota' |t^k| > 0$ ; this condition is also needed for the Grubel-Lloyd index to be well defined. From (21) and (20), and given our assumption of balanced trade and the normalization  $Y = 1$ , this condition reduces to

$$(35) \quad t^k = Pz^k = -(I - \theta \iota') \Gamma^{-1} \text{diag}(\Gamma \theta) \lambda^k \neq 0 \quad (k = 1, 2, \dots, K).$$

Finally we of course require that the countries' relative factor endowments be positive and sum to unity, i.e.,

$$(36) \quad \lambda^k > 0 \quad (k = 1, 2, \dots, K), \quad \sum_{k=1}^K \lambda^k = \iota.$$

Note from (1) and (5) that the inequalities (36) are redundant, being implied by (34).<sup>6</sup>

To summarize, our problem is to show that there exists a set of  $\lambda^k$  ( $k = 1, 2, \dots, K$ ) satisfying (33), (34), (35), and (36).

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<sup>6</sup>In fact, zero outputs will be reached when a factor-endowment ray hits the edge of a diversification cone, long before it hits one of the axes.

**Theorem 1** *There exists a solution  $\lambda^k$  ( $k = 1, 2, \dots, K$ ) to (33), (34), (35), and (36); i.e., given any technology matrix  $\Gamma$  and any vector  $\theta$  of consumer expenditure shares, as well as any vector  $l$  of world factor endowments permitting positive production of all commodities in all countries, and given any way of grouping commodities into a smaller number of industries (i.e., given any  $\bar{n} \times n$  grouping matrix  $G$  with  $\bar{n} < n$ ), there exists an allocation of the world factor endowments among countries such that each country engages in trade and 100% of each country's trade is intra-industry trade.*

*Proof.* Defining

$$(37) \quad M = (I - \theta\iota')\Gamma^{-1}\text{diag}(\Gamma\theta),$$

we start by showing that there exists a vector  $\rho$  such that

$$(38) \quad M\rho \neq 0 \quad \text{and} \quad GM\rho = 0.$$

Observe that

$$(39) \quad M\iota = (I - \theta\iota')\Gamma^{-1}\Gamma\theta = 0.$$

Since  $I - \theta\iota'$  is idempotent of rank  $n - 1$ ,  $M$  has rank  $n - 1$  and nullity 1; therefore from (39) it follows that the null space of  $M$  is exactly the space

$$(40) \quad \mathcal{N}(M) = \{\rho \in E^n : \rho = \omega\iota \text{ for some } \omega, -\infty < \omega < \infty\}.$$

Hence  $\rho$  satisfies the first condition of (38) if and only if it is not proportional to  $\iota$ , i.e., if and only if its components are not all equal to one another.

Now  $G$  is  $\bar{n} \times n$  of rank  $\bar{n}$ . Since it satisfies  $\iota'G = \iota'$ , it follows that

$$\iota'G(I - \theta\iota') = \iota'(I - \theta\iota') = 0,$$

i.e., that the rows of  $G(I - \theta\iota')$  (hence those of  $GM$ ) are linearly dependent. Therefore  $GM$  has rank  $\bar{n} - 1$  and nullity  $n - (\bar{n} - 1) = n - \bar{n} + 1 \geq 2$ . The null space of  $GM$  contains that of  $M$ , and  $n - \bar{n}$  dimensions in addition; it follows that it contains vectors  $\rho$  whose elements are not all equal to one another, i.e., that (38) is satisfied.

Now define, for such a  $\rho$  satisfying (38),

$$(41) \quad \lambda^k = \omega_k \iota + \varepsilon_k \rho \quad (\omega_k > 0, \varepsilon_k \neq 0)$$

where

$$(42) \quad \sum_{k=1}^K \omega_k = 1, \quad \sum_{k=1}^K \varepsilon_k = 0.$$

Scaling down the  $\varepsilon_k$ s by a common factor (if necessary), it is clear that they can be chosen so that (36) is satisfied; similarly, for sufficiently small  $\varepsilon_k$ s we have from (41)

$$(43) \quad (I-A)^{-1} \Gamma^{-1} \text{diag}(\Gamma \theta) \lambda^k = (I-A)^{-1} \{ \omega_k \theta + \varepsilon_k \Gamma^{-1} \text{diag}(\Gamma \theta) \rho \} > 0 \quad (k = 1, 2, \dots, K),$$

hence (34) is satisfied. Finally we verify that

$$(44) \quad M \lambda^k = \varepsilon_k M \rho \neq 0 \quad \text{and} \quad G M \lambda^k = \varepsilon_k G M \rho = 0,$$

hence (35) and (33) are satisfied. Q.E.D.

In interpreting this theorem the following points should be kept in mind. (a) If  $\bar{n} = 1$  (which in particular will be the case if  $n = 2$ ), the conclusion of the theorem is of course trivial, since if trade is balanced and all goods are grouped into one industry, obviously all trade is intra-industry trade. (b) If  $\bar{n} > n/2$ , then necessarily at least one of the  $n$  commodities will be left unaggregated. The theorem therefore implies that this commodity is not traded. If the theorem is to be consistent with the existence of trade in every category, obviously one must have  $\bar{n} \leq n/2$ . From (a) and (b) it follows that if the theorem is to be interesting one must have  $n \geq 4$ . Since in the real world  $n$  is of the order of tens of thousands, and even at the crudest levels of aggregation (say, 1-digit SITC)  $\bar{n} \geq 10$ , the result can claim to be of practical interest. (c) Note from (43) that allowance for trade in intermediate products widens the possibilities for intra-industry trade.

There is also an interesting interpretation in terms of the criterion of similarity of factor endowments. If  $\lambda^k = \omega_k \iota$  for  $0 < \omega_k < 1$ , then from (18) and (32)  $y^k = \omega_k P^{-1} \theta Y$  (net output equals consumption) and country  $k$  will not trade. The form (41) chosen for  $\lambda^k$  ensures that this will not be the case; but for small  $\varepsilon_k$  it



may be interpreted as saying that countries' factor endowments are "similar". As the  $\varepsilon_k$ s increase in absolute value, sooner or later an inequality will be violated (say (34); world equilibrium would have to be computed in a different way, but it is safe to say that there would be no reason to assume that all trade would continue to be intra-industry trade. In this sense it may be said that a strong relationship appears to hold between similarity of factor endowments and the phenomenon of intra-industry trade.

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