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Sonderforschungsbereich 178 "Internationalisierung der Wirtschaft"

Diskussionsbeiträge



Juristische Fakultät Fakultät für Wirtschaftswissenschaften und Statistik

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Risk-Value Efficient Portfolios and Asset Pricing

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Risk-Value Efficient Portfolios

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and Asset Pricing

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Serie II - Nr. 354

September 1997

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Abstract

Portfolio choice is usually modelled by von Neumann-Morgenstern utility. Risk-value models are more general and permit the derivation of risk-value efficient frontiers. A behaviorally based risk measure with an endogenous or exogenous benchmark is used to derive efficient portfolios and to analyse the implied equilibrium asset pricing. In risk-value models a richer set of sharing rules is obtained than in a von Neumann-Morgenstern world. Linear sharing rules are obtained only for quadratic risk functions. If the risk function is modelled by a negative HARA-function, then sharing rules are convex or concave relative to each other. Hence, agents buy and sell portfolio insurance motivating trade in options. Asset pricing, however, is similar to that in a von Neumann-Morgenstern world.

Introduction

In his pioneering work *Harry Markowitz* (1959) used mean and variance as primitives for decision making under risk and derived the set of mean-variance efficient portfolios. Since variance is a particular measure of risk which has been critized frequently, later on portfolio choice models have been based on expected utility. Thereby the set of risk measures has been widely augmented, but the tradeoff between risk and value (= mean) is completely determined by the utility function so that only an optimal portfolio instead of a set of risk-value efficient portfolios can be derived. This may be considered a strength of expected utility, but equally a weakness since the utility function does not allow to separate the measurement of risk from the tradeoff between risk and value. There is no evidence that risk perception is intertwined with the tradeoff in a unique manner. Hence a more general approach is desirable which allows for this separation. This would also be in line with the separate analysis of risk and value common in financial practice.

Recently, important advances in modelling decision making under risk have been presented. Our approach is based on these advances and on empirical findings about people's risk perception. We refer to the work about risk-value models in decision theory. In a risk-value model, risk and value are taken as primitives. In order to come up with an evaluation of an alternative, the decision maker first separately derives value and risk of this alternative. In a second step, value and risk are combined into an overall preference.

In this paper, we are only dealing with the first step. Once the risk measure is defined, value and risk can be determined for every alternative. Then, in the Markowitz tradition, alternatives can be screened for risk-value efficiency with variance being replaced by another risk measure. As a result, a risk-value efficient frontier is derived. While value is always defined as

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the expected outcome, the essential question relates to the definition of the risk measure. Risk is defined as the expected value of the transformed deviation of the outcome from a benchmark. An example for a risk measure is a monotonically decreasing convex function with positive deviations from the benchmark reducing risk and negative deviations increasing risk. This reflects the practitioners' view that negative deviations constitute "risk" and positive deviations "chances". Such a risk function can be considered the negative of an increasing concave utility function. Jia and Dyer (1996) have investigated risk functions being the negative of a standard utility function. If we take the standard utility function used in finance, reflecting risk aversion and having a positive third derivative [see Kraus & Litzenberger (1976)], i.e. having positive prudence [see Kimball (1990), (1993)], we get a risk function which is decreasing and convex and has a negative third derivative. The risk measure derived from this function is able to capture some major behavioral findings about people's risk perception. The benchmark for risk measurement can be given exogenously or taken as endogenous. In the latter case, we assume value to be the benchmark postulating that risk should be zero if the outcome is non-random, irrespective of the level of this outcome. Since value is also a primitive in risk-value models, this endogenous benchmark assures that value is not counted twice through an impact on both primitives. This is also in the Markowitz tradition.

The purpose of this paper is threefold. *First*, using a risk-value model, for every investor the risk-value efficient set of portfolios in a perfect, complete capital market is derived. This allows us to determine the shape of the investors' relative sharing rules. Two definitions of sharing rules will be used. The *absolute* sharing rule is the function which relates the investor's portfolio payoff to the aggregate payoff, i.e. the exogenously given payoff to all investors. The *relative* sharing rule is the function which relates an investor's payoff to the payoff of another investor. *Tobin (1958)* has shown the separation property for investors who measure risk by the

variance; then every investor buys the risk-free asset und a portfolio of risky assets such that the composition of this portfolio is independent of the required portfolio return. Therefore all relative sharing rules are linear. Later on following the idea of two fund-separation *Cass and Stiglitz* (1970) showed a similar result for expected utility maximisers using a utility function of the HARA-class. Their relative sharing rules are also linear.

In risk-value models where risk is measured by a negative utility function, relative sharing rules are linear if and only if this function is quadratic. Otherwise more complicated sharing rules are obtained. This is explained in risk-value models with an endogenous benchmark by the impact of portfolio choice on this benchmark. In risk-value models with an exogenous benchmark it is explained by the existence of two constraints (the budget constraint and the expected return constraint) instead of one under which risk has to be minimized. Therefore the shape of the relative sharing rule changes when the required portfolio return or the initial endowment changes.

If we further restrict the risk measure to be derived from a HARA-utility function with a positive third derivative, the risk measure is still able to model major empirical findings on risk perception. A positive third derivative implies a strictly convex first derivative. Hence if the risk function is the negative HARA-function, then marginal risk is a positively sloped, strictly concave function. The shape of an investor's relative sharing rule then depends on the required expected portfolio return and on his initial endowment. Changing these parameters, the new sharing rule is either strictly convex or concave in the previous one. The intuition can be explained as follows. Suppose the required expected portfolio return or the initial endowment are raised. Then a linear transformation of the sharing rule raising its slope would have a strong impact on risk. The payoff in the "low states" (where aggregate payoff is low) would be strongly reduced relative to the mean and this reduction would have an even stronger impact on risk because the marginal risk function is strictly concave. Therefore it is better to purchase more claims in the "low states" and fewer claims in the "high states". This implies a strictly convex relative sharing rule.

Second, the paper yields implications for the absolute sharing rules in equilibrium. Assuming homogeneous expectations of investors, Tobin's separation result implies that each investor has a linear absolute sharing rule; *Rubinstein (1974)* has proved the same under the assumption that all investors maximize expected utility using the same type of HARA-utility function. In our risk-value model, this is in general not true. Suppose that all investors use the same measure of risk. Since the shape of the relative sharing rule depends on the investor's initial endowment and on the required portfolio return, also absolute sharing rules will have different shapes. The absolute sharing rules are linear if and only if all investors use quadratic risk functions, i.e. use variance as a risk measure.

Nonlinear relative sharing rules relate our model to the literature on portfolio insurance [Leland (1980), Brennan and Solanki (1981), Benninga and Blume (1985), Franke, Stapleton and Subrahmanyam (1995), Grossman and Zhou (1996), Benninga and Mayshar (1997)]. Leland defines portfolio insurance as a portfolio policy which leads to a convex absolute sharing rule. The intuition behind his concept is that a convex absolute sharing rule gives the investor a fairly high payoff when the aggregate payoff is low (= low state). Thus, a convex sharing rule should lead to a higher payoff in low states as compared to a linear or concave sharing rule. Our results show that an investor with a risk function of the HARA-type who demands a higher expected portfolio return or has a higher initial endowment compared to another investor buys a sharing rule which is strictly convex relative to that of the other investor. Thus, the first investor tends to buy portfolio insurance from the second investor. This is possible through trading of options. It may appear counterintuitive that a more aggressive investor, i.e. an investor who demands a higher expected portfolio return, and a richer investor buy portfolio insurance instead of selling it. This result is generated by the risk function which makes risk very sensitive to small payoff changes in the low states.

Third, we derive the implications of our model for asset pricing. The capital asset pricing model developed by *Sharpe (1964)*, *Lintner (1965)* and *Mossin (1966)* is based on a mean-variance approach. Later on a variety of models have been developed which generalise the CAPM. *Rubinstein (1974)* developed an asset pricing model based on HARA-utility. The elegance of these models derives from the fact that in equilibrium every investor buys the risk-free asset and a share of the market portfolio. Thus an investor's absolute sharing rule is linear. Therefore, a representative investor exists.

Various empirical studies have questioned the validity of the CAPM [see, e.g., *Fama* and *French (1992)*]. In one stream of literature the assumptions about preferences have been revised. Higher moments of the return distribution have been considered [*Kraus and Litzen-* berger (1976)] as well as lower partial moments¹. Also investors have been assumed to have heterogeneous preferences so that a representative investor does not exist [see *Franke, Stapleton* and Subrahmanyam (1995), Grossman and Zhou (1996), Benninga and Mayshar (1997)]. Focusing instead on market imperfections and informational asymmetries is typical of the market microstructure literature [see, e.g. *Kyle (1985)*].

Our equilibrium analysis is based on a perfect and complete market. We obtain similar asset pricing results as in a von Neumann-Morgenstern world. Every investor's payoff is decreasing and convex in the probability-deflated price for state-contingent claims which, in turn, is decreasing and convex in the aggregate payoff. Since our model does not constrain the tradeoff between value and risk, it cannot be ruled out a priori that the marginal benefit of a payoff can be negative in some states implying negative prices for state-contingent claims. Some very weak assumptions guarantee positivity of these prices, however. Then asset pricing is hardly distinguishable from that in a von Neumann-Morgenstern world even though a representative investor does not exist.

The paper is organised as follows. In section 1, we will review some of the theoretical and empirical research in decision theory on how to measure risk. Based on this, we will derive some general properties a risk measure should have. In section 2, risk-value efficient individual portfolios are derived. Equilibrium is analysed in section 3, first for a rather general class of risk functions, second for negative HARA functions. Sharing rules of different investors are compared and implications for asset pricing are derived. Section 4 summarises the main results.

1 Risk Measurement

1.1 Background²

Traditionally, choice is modelled using expected utility (EU). If e denotes the random payoff of an alternative and u the concave utility function, then a Taylor expansion yields

$$E[u(e)] = u(\bar{e}) + \sum_{j=2}^{\infty} \frac{u^{j}(\bar{e})}{j!} \left[(e - \bar{e})^{j} \right].$$

 \bar{e} is the expected payoff and $u^{i}(\cdot)$ the j-th derivative of the utility function. Hence expected utility can be decomposed into value \bar{e} and risk measured by all the central moments of the payoff distribution. These moments are derived using value as an endogenous benchmark. The tradeoff between value and risk is defined by the utility function and its derivatives.

This decomposition of expected utility into value and risk is of little use unless the higher moments converge quickly to zero or all the infomation in the higher moments is contained in a few moments or the higher derivatives of the utility function are zero. Thus, under normality of payoffs or quadratic utility, a functional based on value and variance exists which ranks the alternatives consistent with EU.

The separation of value and risk is quite popular in finance. Investors usually talk about the risk of an investment strategy which then is evaluated against the expected return. Thus decisions are made by judging risk and return separately and trading off both components. Taking risk and value separately allows different investors to have different traedeoffs even though their risk measures may be the same. This is relevant in group decision making.

Variance as a widely used measure of risk with value being the endogenous benchmark, however, is not appropriate for non-normal payoffs which can be seen most easily by evaluating the following set of alternatives³:

$$A = (.5, 10 \ \text{$\ $}; .5, -10 \ \text{$\ $}) \qquad B = (.2, 20 \ \text{$\ $}; .8, -5 \ \text{$\ $}) \qquad C = (.8, \ 5 \ \text{$\ $}; .2, -20 \ \text{$\ $}).$$

All three alternatives have the same expected value (EV(A) = EV(B) = EV(C) = 0 \$)and equal variance (Var(A) = Var(B) = Var(C) = 100). When subjects are asked for preference judgments, most subjects are not indifferent. In general, they prefer B over A over C: they dislike the relatively large loss potential in alternative C. This preference is compatible with the general wisdom that decision makers prefer positive over negative skewness [see *Kraus and Litzenberger (1976)*]. In addition, variance as a measure of risk implies that all investors have the same perception of risk, a claim not supported by empirical studies [*E. Weber and Milliman* (1997)].

Quite a number of different approaches have been presented to model decision behavior not compatible with EU. One stream of research generalizes EU by weakening the axioms of EU but still taking preference as a primitive. Prospect theory [*Kahneman and Tversky* (1979, 1992)] might be the most prominent representative of this class of models [see *Machina* (1987) and Weber and Camerer (1987) for reviews]. At this point, however, we hardly know anything about decomposing these models into value and risk⁴. We will therefore concentrate on a second development in decision theory which generalizes the expected utility approach by directly defining preference as a function of some measure of value and some measure of risk, thus taking value and risk as primitives [*Coombs (1969)*]. Value is again the expected value of the alternative, and risk is mapped by a function which models the risk judgments of decision makers.

To model risk judgments, different theories have been proposed and tested [see Brachinger and Weber (1997) for an overview]. An exponential model [Sarin (1984)] and a power function model [Luce and E. Weber (1986)] were found to fit the data quite well [Keller et al. (1986) and E. Weber and Bottom (1990)]. These models define risk as the expected value of a function of the outcomes or of the deviations of the outcome from an exogenous benchmark (which may be equal to zero). This captures the widespread notion that positive deviations of the payoff from a benchmark are perceived as gains which reduce risk while negative deviations are perceived as losses which increase risk. There are two principle ways for defining a benchmark: exogenously or endogenously. An exogenous benchmark is given by the investor, e.g., it can be the minimum level the investor wants to achieve⁵. An endogenous benchmark depends on the characteristics of the payoff distribution. The most prominent endogenous benchmark is the expected value of the payoffs as in variance and other moments. We will consider risk functions which use the expected value of the payoff distributions as an endogenous benchmark. This benchmark definition has some important advantages. This definition implies that the risk measure is location free, i.e. risk does not change if the return distribution is shifted by adding or subtracting a positive number⁶. This is a desirable property since the expected payoff is already used as a primitive in the preference function. Otherwise a risk-value model would involve some double counting of the expected payoff. In addition, the risk function can be defined easily in a manner such that the risk of a sure amount is zero.

Another important reason to consider risk-value models as models for preference is that recently there is a lot of work on the relation of these models with traditional EU^7 . This work shows to which extent risk-value models are more general than EU models and to which extent they are compatible with EU models. An example of a compatibility condition for a special risk measure and a special risk-value preference function was recently presented by *Jia and Dyer* (1996). They define a standard measure of risk to be

$$Risk(e) = -E[u(e-\bar{e})]$$
(1)

where u denotes a von Neumann-Morgenstern utility function and \bar{e} equals value. This risk measure is general enough to include risk measures which were found to describe people's risk perception, e.g. the exponential risk measure. Jia and Dyer showed that an expected utility model can also be written as a risk-value model as defined before if and only if a rather strong condition, called risk independence, holds. Basically, this condition requires that lotteries with the same expected payoff can be preference ranked by their risk measures such that this ranking does not change when the expected payoff changes.

In the following, we go beyond expected utility and allow for more general preferences by using a general risk-value model. In a first step, we derive desirable properties of the risk measure.

1.2 Properties of a Risk Measure

Using a risk-value framework to model preferences, the risk measure has to be described in more detail. As stated above, risk will be measured as the expectation of a function of the deviation of a random variable e from a benchmark \bar{e} . In the case of an exogenous benchmark \bar{e} is a

given number, in the case of an endogenous benchmark \bar{e} is the expected value E (e). Thus, the risk of a payoff e can be written as

$$\operatorname{Risk}(\mathbf{e}) = \mathbf{E}[\mathbf{F}(\mathbf{e} - \bar{\mathbf{e}})] - \mathbf{F}^0 . \tag{2}$$

For simplicity we define the deviation $\hat{e} := e - \bar{e}$. We may⁸ substract F^0 .

As a next step we define properties of the risk function F. We will assume that F has three key properties. After discussing these properties, we will show that those risk measures which were found to describe people's risk perception also obey these properties.

- i) Outcomes above the benchmark reduce risk and outcomes below increase risk, $F(\hat{e}_1) > F(0) > F(\hat{e}_2)$ for $\hat{e}_1 < 0 < \hat{e}_2$. More restrictive, we require monotonicity: A higher payoff will contribute less to risk than a lower payoff, thus F' < 0.
- ii) Mean preserving spreads increase risk, thus F'' > 0.
- iii) The sensitivity to a mean preserving spread is larger in the loss domain (relative to the expected value) than in the gain domain. Requiring monotonicity then implies that the sensitivity decreases if the payoff increases; thus we require F''' < 0.

Proof : Let y, z be two states with the same payoff and the same probability p; in both states the payoff deviates from the expected value by Δ . Their contribution to the total risk of the portfolio is then given by $2pF(\Delta)$. Now replace the payoff deviation Δ in the states y and z by a mean preserving spread around Δ , that is: the deviation from the expected payoff is $\Delta - \alpha$ in state y and $\Delta + \alpha$ in state z ($\alpha > 0$). Notice that the expected payoff is not changed and hence it does not influence the risk contribution of the other states. The new contribution of the states y, z to the total risk is $pF(\Delta-\alpha) + pF(\Delta+\alpha)$. The risk increase, denoted RI, is then given by: $RI_{\alpha}(\Delta) = pF(\Delta-\alpha) + pF(\Delta+\alpha) - 2pF(\Delta)$

and the strict convexity of F is equivalent to $RI_{\alpha}(\Delta) > 0$ for all Δ and all $\alpha > 0$. Furthermore $RI'_{\alpha}(\Delta) = pF'(\Delta - \alpha) + pF'(\Delta + \alpha) - 2pF'(\Delta) < 0$ for F' concave. Therefore the risk increase RI decreases in Δ for F''' < 0.

As proposed by *Jia and Dyer (1996)*, risk functions with F' < 0, F'' > 0, and F''' < 0 directly correspond to utility functions with u' > 0, u'' < 0 and u''' > 0 [*Kraus and Litzenberger (1976)*]. Hence the well-known relationships between risk aversion and risk premia apply also to these risk functions [*Jia and Dyer (1996)*]. u''' > 0, u'' < 0 implies positive prudence (i.e. - u'''/u'' > 0) such that background risk like the α above raises expected marginal utility [*Kimball (1990)*]. If prudence is, in addition, declining in the payoff, then background risk makes the decision maker more averse to other risks [*Kimball (1993)*].

Standard models for risk perception, e.g. the exponential model, have the properties noted above. Note that variance neither fulfills property i) nor property iii). The three properties imply that in the example in section 1.1 alternative B is the least risky and alternative C with the lowest possible outcome is the most risky. Risk judgments found in a number of empirical studies, see, e.g., *Keller et al (1986)*, were identical to the risk ranking implied by these properties. There is also a variety of studies that show that people judge alternatives with potential catastrophic outcomes as being especially risky, see, e.g., *Slovic (1987)*.

In the following sections we will derive results using properties i) - iii). At the end, to get further results, we need more detailed information about the risk measure. At that point we will assume that the risk function belongs to the set of negative HARA-functions. The HARA - class has been widely used in financial economics. In classical portfolio theory, *Cass and Stiglitz* (1970) have shown that the HARA-class of utility functions implies two fund-separation. *Ru-binstein* (1974) has used this result to derive an equilibrium in which every investor buys the

risk-free asset and a share of the market portfolio. Every investor, thus, has a positively sloped linear absolute sharing rule, i.e., his portfolio payoff is a linear function of the aggregate payoff. As the HARA-class of utility functions is the only class which implies two fund-separation, it is instructive to find out what sharing rules this class of functions implies if used as risk functions.

We consider negative HARA-functions with properties i) - iii). They include the exponential function, an important function to describe people's risk judgments. The negative HARA - class is defined as

$$F(\hat{e}) = -\frac{1-\gamma}{\gamma} \left(A + \frac{\hat{e}}{1-\gamma}\right)^{\gamma}$$
(3)

where $\gamma \in \mathbb{R} \setminus \{0, 1\}$; $A \in \mathbb{R}^+$. We can extend this class of functions to the case $\gamma = 0$ where we obtain : $F(\hat{e}) = -\ln (A + \hat{e})$; and to the case $\gamma = -\infty$ where we get: $F(\hat{e}) = \exp(-B\hat{e})$ with B > 0. For $\gamma > -\infty$ the domain of F is constrained by $(A + \hat{e}/(1-\gamma)) > 0$. A has to be sufficiently high positive. For $\gamma < 1$, we need inf $\hat{e} > -A(1-\gamma)$; for $\gamma > 1$, we need sup $\hat{e} < -A(1-\gamma)$. We have

$$F'(\hat{e}) = -\left(A + \frac{\hat{e}}{1 - \gamma}\right)^{\gamma - 1} \quad (\Rightarrow F' < 0) \tag{4}$$

$$F''(\hat{e}) = \left(A + \frac{\hat{e}}{1 - \gamma}\right)^{\gamma - 2} \qquad (\Rightarrow F'' > 0)$$
(5)

$$F'''(\hat{e}) = \frac{\gamma - 2}{1 - \gamma} \left(A + \frac{\hat{e}}{1 - \gamma} \right)^{\gamma - 3}$$
(6)

As we require $F'''(\hat{e}) < 0$, $\gamma < 1$ or $\gamma > 2$ is implied. Therefore, we will only consider functions with $\gamma < 1$ or $\gamma > 2$.

2 Efficient Portfolios

2.1 Risk-Value Models With an Endogenous Benchmark

In order to derive risk-value efficient portfolios, we assume a perfect and complete capital market. A state of nature is defined by the exogenously given aggregate payoff ε ; i.e. the sum of payoffs to all investors⁹. ε is assumed to be a continuous positive variable $\varepsilon \in \mathbb{R}^+$.

Define:

 p_{ϵ} := probability density of ϵ ,

- e_{ϵ} := number of claims contingent on state ϵ , purchased by the investor; each claim pays off 1 \$ if and only if state ϵ obtains,
- π_{ϵ} := pricing kernel; it denotes the price for a claim contingent on state ϵ divided by the state probability density,

 W_0 := the investor's initial endowment (wealth), $W_0 > 0$.

 $\pi_{\epsilon} = \pi(\epsilon) \in \mathbb{R}^+$, $\forall \epsilon$, is assumed; moreover, $\pi(\epsilon)$ is assumed to be twice continuously differentiable, $\forall \epsilon \cdot \hat{e} = e - \bar{e}$ is defined on the range $[\hat{e}, \bar{e}]$ on which F (\hat{e}) and the first three derivatives are finite and, in addition, F'(\hat{e}) < 0, F''(\hat{e}) > 0 and F'''(\hat{e}) < 0. This implies that E [F(\hat{e})] is finite, E [F'(\hat{e})] < 0, E [F''(\hat{e})] > 0 and E [F'''(\hat{e})] < 0 and finite. To simplify notation, the state index ϵ will be dropped unless necessary for clarity of exposition.

A risk-value efficient portfolio minimizes the risk subject to the constraint that the expected gross portfolio return $E(e)/W_0$ does not fall below some exogenously given value R^{*}. The prices for state-contingent claims are assumed to be positive and exogenously given. Then an efficient portfolio is the solution to the following problem.

Minimize $E[F(\hat{e})] - F^0$ (7)

subject to the budget constraint : $E[e\pi] = W_0$ (8)

and the return constraint:
$$E(e)/W_0 \ge R^*$$
. (9)

Varying R^* parametrically allows us to derive all risk-value efficient portfolios. In the following the function

$$f(\hat{e}) := -\frac{\partial F(\hat{e})}{\partial \hat{e}}$$
(10)

will be of special importance. From the properties of the risk function F (F' < 0, F'' > 0, and F''' < 0), we immediately get: f > 0, f' < 0, f'' > 0.

Using this notation we can write the first order condition for a solution to the minimization problem (7) - (9) with an endogenous benchmark $\bar{e} = E$ (e) (η is the Lagrange-multiplier of constraint (8) and λ the Lagrange-multiplier of constraint (9)) as¹⁰

$$-p_{\varepsilon}f(\hat{e}_{\varepsilon}) + p_{\varepsilon}E[f(\hat{e})] = \eta p_{\varepsilon}\pi_{\varepsilon} + \frac{\lambda p_{\varepsilon}}{W_{0}}; \quad \forall \ \varepsilon.$$
(11)

Since E (π) = 1 / R_f with R_f being the gross risk-free interest rate, aggregation across all states yields

$$0 = \frac{\eta}{R_f} + \frac{\lambda}{W_0}$$
(12)

Since raising R^{*} raises the risk of the efficient portfolio, $\lambda > 0$ so that, by (12), $\eta < 0$. Substituting λ in equation (11) and dividing by p_{ϵ} yields

$$-E[f(\hat{e})] + f(\hat{e}_{\varepsilon}) = \frac{\eta}{R_{f}} \left[1 - R_{f} \pi_{\varepsilon} \right]; \quad \forall \varepsilon.$$
(13)

Defining θ_{ϵ} := (1- R_f π_{ϵ}) this equation can be rewritten as :

$$-E[f(\hat{e})] + f(\hat{e}_{\varepsilon}) = \frac{\eta}{R_{f}} \theta_{\varepsilon}; \quad \forall \varepsilon.$$
(14)

The investor's efficient portfolio is characterised by equation (14) and constraints (8) and (9). π_{ϵ} is the probability deflated price and $R_{f} \pi_{\epsilon}$ is the probability deflated forward price of a state ϵ -contingent claim. θ is the difference between the forward price of the risk-free claim, 1, and the probability deflated forward price of a state ϵ -contingent claim. θ_{ϵ} is negative if the probability deflated forward price of a state ϵ -contingent claim exceeds the forward price of the risk-free claim. θ has zero expectation. The higher θ_{ϵ} , the cheaper are the state ϵ -contingent claims, and the more state ϵ -contingent claims the investor buys because a higher level of these claims raises $-f(\hat{\epsilon}_{\epsilon})$.

Now, consider the Lagrange multiplier λ . If the required expected gross portfolio return R^* equals or is smaller than the gross risk-free rate, then $\lambda = 0$ and risk is zero. All endowment is invested in the risk-free asset, (that is e_e is the same for every state). λ grows with the required expected return $R^*(R^* > R_f)$, because the objective function is strictly convex. An increase in the required expected return raises risk and also the deviations of $f(\hat{e})$ from its mean; thus, by equation (14), λ must grow. Since $\lambda = dE[F(\hat{e})]/dR^*$ given efficient portfolios, it follows that the risk-value efficient frontier is strictly convex as shown in figure 1.

Figure 1

2.2 Risk-Value Models With an Exogenous Benchmark

Now consider risk functions with an exogenous benchmark \bar{e} ; $\bar{e} \stackrel{>}{<} 0$. Then in the first order condition (11) for a risk-value efficient portfolio the second term disappears since portfolio choice has no effect on the benchmark. Hence the first order condition reads after dividing by p_{e}

$$- f(\hat{e}_{\epsilon}) = \eta \pi_{\epsilon} + \frac{\lambda}{W_0}; \quad \forall \epsilon.$$
 (15)

Taking expectations yields

$$-E[f(\hat{e})] = \frac{\eta}{R_f} + \frac{\lambda}{W_0}$$
(16)

so that subtraction of (15) from (16) leads to

$$-E[f(\hat{e})] + f(\hat{e}_{\varepsilon}) = \frac{\eta}{R_{f}} \theta_{\varepsilon} ; \quad \forall \varepsilon.$$
(17)

Compare this first order condition with (14), the first order condition for an endogenous benchmark model. The surprising result is that the first order conditions (14) and (17) look precisely the same although the optimal portfolios are different because of the different benchmarks. The transformed deviation $f(\hat{e}_{e})$ minus the expected transformed deviation must be strictly proportional to - θ_{e} regardless of whether the benchmark is exogenous or endogenous. The formal identity between (14) and (17) indicates that strong similarities between the sharing rules with endogenous and those with exogenous benchmarks should exist. Fig. 1 is also true for risk-value models with exogenous benchmarks.

2.3 Expected Utility Models

To pinpoint the difference between risk-value models and the classical state preference model based on a von Neumann-Morgenstern utility function u(e), consider the classical portfolio choice problem

$$Max E[u(e)] \tag{18}$$

s.t. the budget constraint (8).

Let μ denote the Lagrange-multiplier of the budget constraint. Then the first order condition is:¹¹

$$u'(e_{\epsilon}) = \mu \pi_{\epsilon}; \quad \forall \epsilon.$$
 (19)

A comparison of the first order conditions reveals two important differences. First, in the classical model marginal utility is strictly proportional to the price for state-contingent claims (equation (19)). In the risk-value models strict proportionality of the state-dependent marginal risk term f (\hat{e}_{ϵ}) and the price for state-contingent claims is lost (equations (14), (17)). This has an impact on the investor's sharing rule. Second, "marginal risk" depends on the benchmark, whereas such an influence does not exist in expected utility models.

A third difference between risk-value with an endogenous benchmark and classical models relates to satiation. In the classical model, marginal utility is always positive by assumption. In the risk-value model, the investor may be worse off if she receives an additional payoff in some state with a high portfolio return. Consider as an example the preference function $P(\bar{e}, risk)$ with $\alpha > 0$ and $\bar{e} = E(e)$,

$$P(\tilde{e}, risk) = \alpha \bar{e} - risk, \qquad (20)$$

with risk as defined in equation (2). Differentiate the preference function with respect to e_{ϵ} . This yields :

$$\frac{\partial P}{\partial e_{\varepsilon}} = p_{\varepsilon} \left[\alpha + f(\hat{e}_{\varepsilon}) - E[f(\hat{e})] \right]$$
(21)

If the portfolio payoff is random, there must exist a state ε with $f(\hat{e}_{\varepsilon}) < E[f(\hat{e})]$ (for example, the state ε with the lowest π_{ε}). Then, given a sufficiently small α , we have $\partial P/\partial e_{\varepsilon} < 0$. Hence an increase in the state- ε -portfolio return may reduce the investor's welfare which contradicts the usual assumption of non-satiation. Negative prices can also appear in the

traditional CAPM. For this case, *Nielsen (1992)* derives conditions which bound the risk aversion, thus restricting the trade-off between expected value and risk. Instead of restricting α , we argue that in case all prices for state-contingent claims, π_{e} , are positive, then the investor always chooses his/her optimal portfolio such that he/she never reaches or crosses satiation. This follows from the fact that in our world a risk free-asset exists, and from the observation that the investor can always purchase fewer claims contingent on these critical states, invest the saved money in the risk-free asset and, thereby, increase his/her welfare.

3 Equilibrium

3.1 Equilibrium Prices in Risk-Value Models

Next equilibrium in the capital market will be investigated. We do not investigate the existence of an equilibrium, instead we assume existence. If there exist multiple equilibria, we analyse any one of them. Every investor chooses a risk-value efficient portfolio. All investors are assumed to have homogeneous expectations. First, equilibrium prices will be derived, second, individual sharing rules will be analysed. Individual investors are indexed by i. Hence all investor-dependent variables have to be indexed by i. Then condition (14) for an endogenous benchmark reads

$$-E[f_i(\hat{e}_i)] + f_i(\hat{e}_{i\epsilon}) = \frac{\eta_i}{R_f} \theta_{\epsilon}; \quad \forall \epsilon, i.$$
(22)

Condition (17) for an exogenous benchmark is the same. Unless stated otherwise, the following results hold for efficient portfolios with an endogenous and with an exogenous benchmark. $R_i^* > R_f$ will be assumed throughout so that $\lambda_i > 0$ and $\eta_i < 0$. Then we can derive a proposition on investors' portfolio choices which immediately leads to a basic result on the pricing of state-contingent claims.

Proposition 1:

- a) For every investor, his/her optimal payoff is decreasing and convex in the probabilitydeflated price for state-contingent claims.
- b) The probability-deflated price for state-contingent claims is decreasing and convex in the aggregate payoff.

Proof:12

- a) Differentiate equation (22) with respect to θ . Recall, $f_i(\hat{e}_i) > 0$, $f_i'(\hat{e}_i) < 0$, and $f_i''(\hat{e}_i) > 0$. Hence it follows from (22) that e_i is an increasing and convex function in θ . As $\theta = 1 R_f \pi$, e_i is a decreasing, convex function in π . This proves statement a).
- b) In order to prove statement b), note that the sum of the payoffs of individual investors is the aggregate payoff. Hence statement a) implies that the aggregate payoff is decreasing and convex in π . Therefore π must be decreasing and convex in the aggregate payoff

Figure 2 illustrates Proposition 1. For very high aggregate payoffs the market price for state-contingent claims could be negative. This would violate the common assumption of arbitrage-free markets. A negative market price for some state-contingent claims requires, however, that every investor crosses satiation in that state, a possibility which can be safely ignored.

Figure 2

It should be noted that the results of Proposition 1 are also obtained if every investor has a von Neumann-Morgenstern utility function with positive prudence. This is important since it means that there is no need for a von Neumann-Morgenstern world to obtain the pricing function depicted in figure 2b). This pricing function is obtained under different sets of assumptions and, thus, is of a fairly general nature. Therefore it is hardly possible to distinguish empirically asset pricing in a risk-value world from that in an expected utility world.

3.2 Individual Relative Sharing Rules

In this section we first derive some general results on individual relative sharing rules in a riskvalue world and, second, illustrate our analysis by using negative HARA-functions as risk functions. This allows us to compare our results with those derived from classical HARA-utility theory.

a) Some General Results

An absolute sharing rule relates investor i's payoff e_i to the aggregate payoff ε . Proposition 1 states $de_i/d\pi < 0$ and $d\pi/d\varepsilon < 0$. Hence it follows that $de_i/d\varepsilon > 0$. The positive slope of the absolute sharing rule does not come as a surprise. This is still in line with classical theory. The major difference between our model and classical theory is reflected in the shapes of the sharing rules.

Equation (22) implies for two investors i and j

$$\frac{-E[f_i(\hat{e}_i)] + f_i(\hat{e}_{i\epsilon})}{-\eta_i} = \frac{-E[f_j(\hat{e}_j)] + f_j(\hat{e}_{j\epsilon})}{-\eta_j}; \quad \forall \epsilon.$$
(23)

Hence marginal risk of investor i, adjusted for η_i , must equal that of investor j in every state. In the classical portfolio model, marginal utility of investor i, adjusted for μ_i , must equal that of investor j for every state.

$$\frac{u_i'(e_{i\epsilon})}{\mu_i} = \frac{u_j'(e_{j\epsilon})}{\mu_i} ; \quad \forall \epsilon.$$
 (24)

The important difference between conditions (23) and (24) is that in a risk-value model marginal risk is given by the difference $f(\hat{e}_{\epsilon}) - E[f(\hat{e})]$, i.e. the difference between a state-dependent and a state-independent term. In classical theory, all effects are captured by only one state-dependent term, marginal utility $u'(e_{\epsilon})$.

Define

 $s_i := E[f_i(\hat{e}_i)] / - \eta_i$ to be investor i's sharing constant.

Then equation (23) can be written as

$$-s_{i} + \frac{f_{i}(\hat{e}_{i\epsilon})}{-\eta_{i}} = -s_{j} + \frac{f_{j}(\hat{e}_{j\epsilon})}{-\eta_{i}}; \quad \forall \epsilon.$$
(25)

Equation (25) shows that the sharing constants create an inhomogeneity which has strong implications for the investors' sharing rules. The difference between these sharing constants is an additional factor determining the sharing rules. Therefore, our model generates a larger varie-ty of sharing rules than classical theory. In order to gain some insight into the mechanics of the model, we analyse the impact of changes in initial endowment and in the required expected payoff on an investor's sharing constant. This amounts to a comparative statics exercise. For simplicity of notation, we drop the index i in Lemma 1.

<u>Lemma 1:</u> Consider risk-value efficient portfolios under the condition $R^* > R_f$. Then, given constant prices of state-contingent claims, the sharing constant s declines when -- the initial endowment W_0 increases, or

-- the required expected portfolio return R* increases.

<u>Proof:</u> See Appendix A.

From Lemma 1 it is apparent that the sharing constants differ across investors. Therefore the sharing constants create an additional source of differences among the shapes of individual sharing rules. This is illustrated by Proposition 2.

<u>Proposition 2</u>: In an equilibrium with risk-value models every investor has a linear (absolute) sharing rule if and only if every investor uses a quadratic function F (ê).

<u>Proof:</u> See Appendix B.

Proposition 2 provides a strong result about the shape of sharing rules. In a risk-value world the sharing rules are linear if and only if every investor uses variance or a related quadratic risk measure. For expected utility equilibrium models, *Rubinstein (1974)* has shown that linear sharing rules are obtained whenever all investors have a HARA-utility function with the same γ . For this class risk tolerance is linear.

If each investor has a linear absolute sharing rule, then all relative sharing rules are also linear, and vice versa. Proposition 2 shows, that in general we do not have linear absolute sharing rules. We therefore want to analyze the sharing rule of investor i relative to that of investor j, i.e. the relative sharing rule $e_i(e_i)$.

Proposition 3: In an equilibrium with risk-value models the following statements are equivalent:

- -- Investor i's sharing rule is strictly convex [linear] [strictly concave] relative to that of investor j.
- -- The coefficient of absolute prudence of investor i's risk function, adjusted for de_i/de_j, is everywhere greater than [equal to] [smaller than] the coefficient of absolute prudence of investor j's risk function.

$$-\eta_{i} \frac{f_{i}^{''}(\hat{e}_{i})}{[f_{i}^{'}(\hat{e}_{i})]^{2}} > [=] [<] -\eta_{j} \frac{f_{j}^{''}(\hat{e}_{j})}{[f_{j}^{'}(\hat{e}_{j})]^{2}}; \quad \forall e_{j}.$$
(26)

<u>Proof:</u> See Appendix C.

Proposition 3 gives necessary and sufficient conditions for the shape of an investor's sharing rule relative to that of another one. The shape depends on the investors' coefficients of absolute prudence given efficient portfolios.

These conditions are different from those obtained by *Leland (1980)*, who has investigated in classical portfolio theory the conditions under which an investor's absolute sharing rule $e_i(\varepsilon)$ is convex or concave. He defines purchasing portfolio insurance as purchasing a convex absolute sharing rule. The intuition behind this definition is that a convex sharing rule should lead to relatively high payoffs in low states (i.e. states with low aggregate payoff).

Leland's approach can be modified easily to obtain relative sharing rules. Differentiating the log of equation (24) with respect to e_i yields

$$-\frac{u_i''(e_i)}{u_i'(e_i)}\frac{de_i}{de_j} = -\frac{u_j''(e_j)}{u_j'(e_j)}$$

Therefore Leland concludes that investor i's sharing rule is convex [concave] relative to that of investor j if investor j's absolute risk aversion increases faster [slower] than that of investor i in e_j . As an example, consider two investors i and j with the same utility function, but investor i has a higher initial endowment than investor j. Then it can be shown that investor i's sharing rule is convex [linear] [concave] relative to investor j's sharing rule if and only if the risk tolerance -u'(e)/u''(e) is convex [linear] [concave] in e.

This is a strong result in classical portfolio theory. We cannot obtain a similar result in a risk-value model because the sharing constants matter. More insight will be gained in the following when we analyse the negative HARA-functions as a specific example of a risk function.

b) HARA-Based Risk Functions

In this section, we derive more specific results assuming that the risk function F belongs to the negative HARA-class. F (\hat{e}) is given by equation (3) with $\gamma > 2$ or $\gamma < 1$. We first show that investor i's sharing rule relative to that of investor j is either concave, linear or convex.

<u>Proposition 4:</u> Consider two investors i and j who measure risk by a negative HARA-function with γ being the same for both. Then the following statements are equivalent:

- -- Investor i's sharing rule is strictly convex [linear] [strictly concave] relative to that of investor j.
- -- Investor i's sharing constant is smaller than [equal to] [greater than] that of investor j.

Proof: See Appendix D.

Proposition 4 illustrates our earlier claim that the inhomogeneity in the optimality condition (14) resp. (17) generates room for a larger variety of sharing rules. While in the classical world with HARA-utility all investors have linear sharing rules, in our world one investor's sharing rule relative to another one's is concave, linear or convex¹³. It is the difference between the sharing constants of both investors which determines convexity, linearity and concavity. If the difference is zero, then linearity obtains as in the classical portfolio model. More insight can be gained by analysing the investor's sharing rule in terms of the gross return R rather than the payoff e. Since $R := e/W_0$, the risk function can also be written as $[\hat{R} := R - \bar{R} \text{ with } \bar{R} = E(R)$ for an endogenous benchmark resp. \bar{R} being the exogenous benchmark]

$$E[F(\hat{R})] = -\frac{1-\gamma}{\gamma} E\left(\frac{A}{W_0} + \frac{\hat{R}}{1-\gamma}\right)^{\gamma}.$$

Hence the ratio A/W_0 is crucial for risk measurement given γ . As $A/W_0 + \hat{R}/(1-\gamma) = [A - \bar{R}W_0/(1-\gamma)]/W_0 + R/(1-\gamma)$, $[A - \bar{R}W_0/(1-\gamma)]/W_0 = A^*/W_0$ turns out to be relevant. Changes in the benchmark \bar{R} have the same effect on the risk measure as changes in A. Hence

$$E[F(\hat{R})] = -\frac{1-\gamma}{\gamma} E\left(\frac{A^*}{W_0} + \frac{R}{1-\gamma}\right)^{\gamma}.$$
 (27)

The coefficient of risk aversion, -f'(R)/f(R) increases monotonically in W_0/A^* . Therefore we denote W_0/A^* as the investor's risk sensitivity. Risk sensitivity is exogenous [endogenous] if the benchmark is exogenous [endogenous]. Somewhat counterintuitive, ceteris paribus risk sensitivity is higher for a richer investor, i.e. an investor with a higher initial endowment. Also it is higher for a more aggressive investor, i.e. an investor, who demands a higher expected portfolio return, provided that he uses this as an endogenous benchmark. Empirically little is known about risk sensitivity. Is it smaller or higher for wealthier people? Do people with smaller risk sensitivity prefer portfolios with higher expected returns?

Minimizing risk (27) subject to the budget constraint $E(R\pi) = 1$ and the required return constraint $E(\tilde{R}) \ge R^*$ yields an efficient absolute return sharing rule $R(\epsilon)$. Since the initial endowment only shows up in risk sensitivity, it has no effect on the efficient sharing rule, given γ and risk sensitivity. Therefore it is straightforward to get further insights into the return sharing rules. <u>Proposition 5:</u> Consider two investors i and j who have HARA-risk functions with the same γ .

- a) Suppose that the risk sensitivity, W_0/A^* , is higher for investor i and/or she demands a higher expected portfolio return. Then investor i's sharing rule is strictly convex relative to that of investor j.
- b1) Suppose that for both investors the risk sensitivity, W_0/A^* , is the same, but investor i demands a higher expected portfolio return. Then there exists some portfolio return R^1 such that

$$R_i < [=] [>] R_i$$
 for $R_i < [=] [>] R^1$.

b2) Suppose that the risk sensitivity, W_0/A^* , is higher for investor i but both investors demand the same expected portfolio return. Then there exist R^0 and R^{00} with $R^0 < R^{00}$ such that

$$R_i > R_j$$
 for $R_j < R^0$ and $R_j > R^{00}$,
 $R_i < R_i$ for $R^0 < R_i < R^{00}$.

Proof: See Appendix E.

Figure 3

Proposition 5 is illustrated in Figure 3. From Proposition 5a) it follows that if two investors i and j have the same expected portfolio return and the same risk sensitivity, then their portfolio returns are the same in every state. If investor i has a higher risk sensitivity and/or demands a higher expected portfolio return, then her sharing rule is convex relative to that of the other investor. Hence a more risk sensitive investor tends to buy portfolio insurance while a less risk sensitive investor tends to sell it. The special cases b1) and b2) will help to understand the intuition behind this result.

Proposition 5 b1) says that an investor i who ceteris paribus demands a higher expected portfolio return than investor j chooses a portfolio such that in the low states $(R_j < R^1)$ his portfolio return is lower and in the high states $(R_j > R^1)$ it is higher. Hence, a higher expected return forces him to take more risk. Now suppose, he would choose a sharing rule which is linear relative to that of the other investor but has higher slope (such as in the classical portfolio analysis). This would raise his risk dramatically because his return would fall strongly in the low states and the strict concavity of his marginal risk function would reinforce the impact on risk. Therefore, he rebalances his portfolio by buying more claims in the low states yielding a convex sharing rule relative to investor j. This result may be counterintuitive since one may feel that investors who demand a higher expected return are more aggressive and, therefore, should sell portfolio insurance.

Proposition 5 b2) states that, given the same expected return, the more risk sensitive investor chooses a portfolio such that his return is higher in the very low and in the very high states, but lower in between. Thus, the more risk sensitive investor cuts back large negative deviations of his return from the expected return in the very low states and, thereby, reduces his risk. Moreover, he raises positive deviations in the very high states which also reduces his risk. In order to obtain the same expected return, he has to accept lower returns in the states in between in which the deviations are small anyway and, thus, have a small impact on risk. Again, this result may appear counterintuitive since the richer investor would buy portfolio insurance. The reason is that the risk aversion of an investor increases with his endowment.

Franke, Stapleton and Subrahmanyam (1995) obtain a related result in a classic utility approach where investors have HARA-utility but face background risk. Background risk destroys linearity of the sharing rules. In that world a poor investor suffers more from a given level of background risk than a rich investor and, therefore, takes less marketable risk, irrespective of

differences in their A's. This implies that the poor investor tends to buy a *convex* sharing rule. Similarly, *Grossman and Zhou (1996)* analyze the equilibrium for two investors who maximize expected utility. Suppose that both optimization problems are identical except that the second investor also makes sure that his random wealth never falls below a given floor. Then this investor buys options from the other one. *Benninga and Mayshar (1997)* prove a related result for an equilibrium with two investors who maximize their expected utility without a floor. Both have constant relative risk aversion, but at different levels. Then the investor with higher relative risk aversion buys options from the other investor.

3.3 Individual Absolute Sharing Rules

So far relative sharing rules of investors have been investigated. The absolute sharing rule relates an investor's payoff or return to the aggregate payoff. Proposition 6 provides a result about absolute sharing rules.

<u>Proposition 6</u>: Assume that every investor uses a HARA-risk function and that γ is the same for every investor.

a) Then the absolute sharing rule $R_i(\varepsilon)$ of every investor i is either strictly convex, linear or strictly concave if the pricing functional can be written as

$$\theta(\varepsilon) = c - \frac{1}{b} \left(a + \frac{b\varepsilon}{1-\gamma} \right)^{\gamma-1}$$
(28)

with a, b and c suitably chosen.

b) Condition (28) is equivalent to the condition that the harmonic mean of

$$\frac{R_{f}}{-\eta_{i} W_{0i}} \left(\frac{A_{i}}{W_{0i}} + \frac{R_{i}}{1-\gamma} \right)^{\gamma-2}, \quad i = 1, ..., I, \quad \text{equals} \quad I \left(a + \frac{b\epsilon}{1-\gamma} \right)^{\gamma-2}.$$

Proof: See Appendix F.

If the pricing functional is given by equation (28), then the efficient absolute sharing rule $R_{j}(\varepsilon)$ of a (fictitious) investor j is linear, similar to the classical utility approach. Since every investor's sharing rule is convex, linear or concave relative to any other investor's sharing rule, it must also be convex, linear or concave with respect to investor j's sharing rule and, hence, with respect to the aggregate payoff ε .

Proposition 6b) shows that the pricing functional (28) is equivalent to a rather strong condition. Therefore it appears unlikely that all absolute sharing rules are linear, convex or concave. However, condition (28) is a sufficient, not a necessary condition for convexity, linearity or concavity of all absolute sharing rules $R_j(\varepsilon)$. If, for example, only two investors 1 and 2 exist such that $R_1(R_2)$ is convex, then $R_1(\varepsilon)$ must be convex and, hence, $R_2(\varepsilon)$ must be concave.

Even if condition (28) holds, our model is richer than EU-theory with HARA-preferences. Our model can explain the trade of options whereas EU-theory with HARA-preferences cannot.

4 Conclusion

The paper considers portfolio choice and asset pricing in a world where investors' preferences are modelled by a risk-value approach. We consider risk functions with an exogenous and with an endogenous benchmark. Both models yield very similar results. The risk-value approach is more general than the expected utility approach since it does not constrain the tradeoff between value and risk. Moreover, the risk-value approach appears to be more in line with the observable separation of value and risk in decision making. The risk-value approach allows to derive the set of efficient portfolios while the expected utility approach only leads to an optimal portfolio. We have defined properties the risk measure should have. These properties are verified using empirical findings on risk judgments and related to those of utility functions.

After presenting general results for efficient portfolio choice and asset pricing, we consider individual sharing rules. In risk-value models the first order conditions for efficient portfolios display an additional term, the sharing constant. This constant generates a larger variety of sharing rules. In an expected HARA-utility world all relative and absolute sharing rules are linear. In a risk-value world with the risk function being a negative HARA-function with an endogenous or exogenous benchmark, these sharing constants determine whether an investor's sharing rule is convex, linear or concave relative to that of another investor. Highly risk sensitive investors tend to buy portfolio insurance from less risk sensitive investors. This is in line with intuition. Counterintuitively, the more aggressive investors, i.e. those who demand a higher expected portfolio return, and the richer investors tend to buy portfolio insurance. Since buying and selling portfolio insurance means buying and selling options, this paper adds another aspect to the sparse academic explanations for option trading.

Although asset pricing appears to be similar in expected utility and in risk-value models, in future research some of the assumptions made in this paper should be tightened to get more specific results which allow to distinguish more clearly between asset prices in a risk-value framework and those in an expected utility framework. This would provide additional empirical clues as to models used for decision making under risk.

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Appendix A: Proof of Lemma 1

First, we consider an increase in W_0 to bW_0 holding the expected return R^* constant; b > 1. Then η changes to a η ; a > 0. Define $\hat{e}_1 := (e - \bar{e})$ for the initial endowment W_0 and define $\hat{e}_b := (e - \bar{e})$ for the initial endowment bW_0 . Then we need to show that

$$\frac{\mathrm{E}[f(\hat{\mathbf{e}}_{1})]}{-\eta} > \frac{\mathrm{E}[f(\hat{\mathbf{e}}_{b})]}{-a\eta}$$
$$a \mathrm{E}[f(\hat{\mathbf{e}}_{1})] > \mathrm{E}[f(\hat{\mathbf{e}}_{b})]. \qquad (29)$$

From equation (22) it follows that $\forall \epsilon$,

or

$$- \operatorname{E}[f(\hat{e}_{b})] + f(\hat{e}_{b\epsilon}) = \frac{a\eta}{R_{f}} \theta_{\epsilon} = a\left(-\operatorname{E}[f(\hat{e}_{1})] + f(\hat{e}_{1\epsilon})\right).$$

As the mean absolute deviation between payoffs across states has to grow with W_0 , the monotonicity of f implies that also the mean absolute deviation $|E[f(\hat{e})] - f(\hat{e})||$ has to grow. Hence a > 1. Now assume, by contradiction, that inequality (29) is not true. Then the last equation implies

$$f(\hat{e}_{b\epsilon}) \ge a f(\hat{e}_{1\epsilon}) ; \quad \forall \epsilon.$$

As a > 1 and f > 0, this implies

$$f(\hat{e}_{b\epsilon}) > f(\hat{e}_{1\epsilon}) ; \quad \forall \epsilon.$$

Since f' < 0, it follows that $\hat{e}_{b\epsilon} < \hat{e}_{1\epsilon}$ and hence

$$\hat{\mathbf{e}}_{\mathbf{b}\boldsymbol{\varepsilon}} < \hat{\mathbf{e}}_{\mathbf{l}\boldsymbol{\varepsilon}}; \quad \forall \boldsymbol{\varepsilon},$$

which contradicts the budget constraint (8). Therefore inequality (29) must be true.

Second, we consider an increase in \mathbb{R}^* so that \hat{e}_1 changes to \hat{e}_{b^0} . Then η changes to $a^0 \eta$. Hence the sharing constant decreases if inequality (29) holds with a and b being replaced by a^0 and b^0 . Therefore the same method by which the first part of Lemma 1 has been proved can be applied here

Appendix B: Proof of Proposition 2

a) <u>Necessity:</u>

Differentiate (22) with respect to ε ; this yields

$$f_i'(\hat{e}_{i\epsilon}) \frac{de_i}{d\epsilon} = \frac{\eta_i}{R_f} \frac{d\theta}{d\epsilon}$$
(30)

Divide (22) by (30):

$$\frac{1}{de_i/d\epsilon} \left(\frac{E[f_i(\hat{e}_i)]}{-f_i'(\hat{e}_{i\epsilon})} - \frac{f_i(\hat{e}_{i\epsilon})}{-f_i'(\hat{e}_{i\epsilon})} \right) = \frac{\theta_{\epsilon}}{d\theta/d\epsilon}$$

In analogy to EU-models, define the risk tolerance τ as

$$\tau_i(\hat{e}_i) \equiv \frac{f_i(\hat{e}_i)}{-f_i'(\hat{e}_i)}$$

$$\tau(\varepsilon) \equiv \frac{\theta_{\varepsilon}}{-d\theta/d\varepsilon}$$
, then

$$\frac{1}{\mathrm{d} \mathbf{e}_{i}/\mathrm{d} \mathbf{\epsilon}} \left(\frac{\mathrm{E}[f_{i}(\hat{\mathbf{e}}_{i})]}{-f_{i}'(\hat{\mathbf{e}}_{i\epsilon})} - \tau_{i}(\hat{\mathbf{e}}_{i\epsilon}) \right) = -\tau(\mathbf{\epsilon})$$
(31)

Now suppose a linear absolute sharing rule for every investor: $\hat{e}_i = \alpha_i + \beta_i \epsilon$, so that $d\hat{e}_i / d\epsilon = \beta_i$. Differentiate (31) with respect to ϵ and obtain

$$\frac{\mathrm{E}[f_{i}(\hat{\mathbf{e}}_{i})]}{[f_{i}'(\hat{\mathbf{e}}_{i\epsilon})]^{2}} \quad f_{i}''(\hat{\mathbf{e}}_{i\epsilon}) - \tau_{i}'(\hat{\mathbf{e}}_{i\epsilon}) = -\tau'(\epsilon)$$
(32)

with
$$\tau_i'(\hat{e}_{i\epsilon}) = -1 + \frac{f_i(\hat{e}_{i\epsilon}) f_i''(\hat{e}_{i\epsilon})}{[f_i'(\hat{e}_{i\epsilon})]^2}$$

Consider two investors i and j with the same function F (\hat{e}), but different endowments and different required returns. Independent variations in these parameters require independent variations in α and β . However, given F_i(\hat{e}) amd τ ' (ϵ), equation (32) does not permit independent variations of α and β unless $f_i''(\hat{e}_{i\epsilon}) = 0$. Therefore F (\hat{e}) must be quadratic.

b) Sufficiency:

Rubinstein (1974) has shown that quadratic utility which is equivalent to mean-variance models implies two fund-separation and, hence, linear sharing rules \blacksquare

Appendix C: Proof of Proposition 3

Differentiating equation (25) with respect to e_i yields

$$\frac{f_{i}'(\hat{e}_{i})}{-\eta_{i}} \frac{de_{i}}{de_{j}} = \frac{f_{j}'(\hat{e}_{j})}{-\eta_{j}}.$$
(33)

Hence de_i /de_j is a constant if and only if $f'_{j}(\hat{e}_{j})/f'_{i}(\hat{e}_{i})$ is a constant. Then investor i's sharing rule is linear relative to that of investor j. Investor i's sharing rule is strictly convex [concave] relative to that of investor j if $f'_{j}(\hat{e}_{j})/f'_{i}(\hat{e}_{i})$ is strictly increasing [decreasing] in e_j and, hence, in the aggregate payoff ϵ . This result can be restated using the coefficient of the negative third to the second derivative of the F-function, $-f''_{i}(\hat{e}_{i})/f'_{i}(\hat{e}_{i})$. This coefficient is called the coefficient of absolute prudence of the investor's risk function; it is the analogue to Kimball's coefficient of absolute prudence. Multiplying equation (33) by - 1, taking logs and differentiating with respect to e_i yields

$$\frac{d\ln(de_{i}/de_{j})}{de_{j}} = -\frac{f_{j}''(\hat{e}_{j})}{-f_{i}'(\hat{e}_{i})} + \frac{f_{i}''(\hat{e}_{i})}{-f_{i}'(\hat{e}_{i})} \frac{de_{i}}{de_{j}}$$
(34)

This proves the equivalence of the first two statements in Proposition 3. Substituting de_i/de_j in equation (34) from equation (33) shows that $dln (de_i/de_j)/de_j > [=] [<] 0$ if and only if (26) holds

Appendix D: Proof of Proposition 4

For any HARA-function,
$$\frac{f_i''(\hat{e}_i)}{[f_i'(\hat{e}_i)]^2} = \frac{\gamma - 2}{\gamma - 1} / f_i(\hat{e}_i)$$
.

Hence, by the last statement of Proposition 3, $e_i(e_j)$ is strictly convex [linear] [strictly concave] if and only if

$$\frac{f_{i}(\hat{e}_{i})}{-\eta_{i}} < [=] [>] \frac{f_{j}(\hat{e}_{j})}{-\eta_{j}} ; \quad \forall e_{j} .$$

Hence, by equation (25), $e_i (e_j)$ is strictly convex [linear] [strictly concave] if and only if investor i's sharing constant is smaller than [equal to] [greater than] that of investor j. This proves Proposition 4

Appendix E: Proof of Proposition 5

a) First, we prove statement a). Consider the minimisation of the objective function (27)
 s. t. E[Rπ] = 1 and E(R) ≥ R^{*}.

Hence for two investors i and j with $A_{0i} / W_{0i} = A_{0j} / W_{0j}$ and $R_i^* = R_j^*$ it follows that the efficient portfolio returns are the same. By Lemma 1, investor i's sharing constant decreases when his initial endowment increases or when she demands a higher expected portfolio return. Then, by Proposition 4, her sharing rule is strictly convex relative to that of investor j.

b1) Now we prove statement b1). As $R_i(R_j)$ is strictly convex, the curves $R_i(R_j)$ and $R_j(R_j)$ can intersect at most twice. They have to intersect at least once, otherwise the budget constraint E [R π] = 1 cannot hold for both. Hence we have to show that two intersections are impossible. Equation (33) yields in the HARA-case, starting from objective function (27),

$$\frac{dR_i}{dR_j} = \left(\frac{A_i^*}{W_{0i}} + \frac{R_i}{1-\gamma}}{\frac{A_j^*}{W_{0j}} + \frac{R_j}{1-\gamma}}\right)^{2-\gamma} \frac{\eta_i}{\eta_j}$$

At an intersection $R_i = R_j$ so that $W_{0i}/A_i^* = W_{0j}/A_j^*$ implies that the bracketed term equals 1. Hence the slope dR_i/dR_j at an intersection is unique. Therefore convexity of $R_i(R_j)$ rules out two intersections. Given one intersection at $R_j = R^1$, a higher expected portfolio return can be obtained only if in the states $R_j < R^1$ relatively expensive claims are sold and in the states $R_j > R^1$ relatively cheap claims are bought. Thus $dR_i/dR_j > 1$ for $R_j = R^1$.

b2) Finally we prove statement b2). The strict convexity of $R_i(R_j)$ implies that $R_i(R_j)$ and $R_j(R_j)$ can intersect at most twice. $R_i^* = R_j^*$ requires at least one intersection. Suppose there exists one intersection only. As $\pi(\varepsilon)$ is monotonically decreasing, purchasing claims in some low states $\varepsilon < \varepsilon^*$ and selling claims in some high states $\varepsilon > \varepsilon^*$ such that this transaction is self-financing implies a lower expected return. Thus, one intersection contradicts $R_i^* = R_j^*$ so that two intersections are implied

Appendix F: Proof of Proposition 6

a) <u>Proof of Proposition 6a</u>): It follows from condition (28) that

$$\frac{\mathrm{d}\theta}{\mathrm{d}\varepsilon} = \left(a + \frac{b}{1-\gamma}\varepsilon\right)^{\gamma-2}.$$
(35)

Differentiating equation (22) with respect to ε yields in the HARA-case starting from objective function (27), together with equation (35),

$$\left(\frac{A_{i}}{W_{0i}} + \frac{R_{i}}{1-\gamma}\right)^{\gamma-2} \frac{dR_{i}}{d\varepsilon} = \frac{-\eta_{i}}{R_{f}} \left(a + \frac{b\varepsilon}{1-\gamma}\right)^{\gamma-2}.$$
(36)

Now consider a (fictitious) investor j. Then $dR_j/d\epsilon$ is constant if and only if (\bar{c} is a positive constant)

$$\frac{A_{j}^{*}}{W_{0j}} + \frac{R_{j}}{1-\gamma} = \bar{c} \left(a + \frac{b\epsilon}{1-\gamma} \right) .$$
(37)

It is always possible to choose A_j^* , W_{0j} , R_j^* and \bar{c} such that the linear absolute sharing rule (37) exists. Such a sharing rule is unique and, therefore, efficient.

From the previous results we know that every investor i has a sharing rule which is either strictly convex, linear or strictly concave relative to that of the (fictitious) investor. As this investor has a linear absolute sharing rule, investor i's absolute sharing rule can only be strictly convex, linear or strictly concave.

b) <u>Proof of Proposition 6b</u>): Solve equation (36) for $dR_i/d\epsilon$, multiply by W_{0i} and aggregate across all investors. As $\sum_i W_{0i} dR_i/d\epsilon = 1$, it follows that

$$\left(a + \frac{b\varepsilon}{1-\gamma}\right)^{\gamma-2} = \left[\sum_{j} \frac{-\eta_{i}W_{0i}}{R_{f}} \left(\frac{A_{j}}{W_{0j}} + \frac{R_{j}}{1-\gamma}\right)^{2-\gamma}\right]^{-1}$$

The right hand side of this equation is the harmonic mean of $\frac{R_f}{-\eta_i W_{0i}} \left(\frac{A_j^*}{W_{0j}} + \frac{R_j}{1-\gamma}\right)^{\gamma-2}$, divided by I

Footnotes

- 1. Hogan and Warran (1974) and *Bawa and Lindenberg (1977)* developed a mean-lower partial capital asset pricing model which was extended by *Lee and Rao (1988)* and *Harlow and Rao (1989)*.
- 2. See Sarin and Weber (1993) and E. Weber (1997) for reviews on risk measurement.
- The vector (p, x \$; 1-p, x' \$) denotes an alternative which has a p-chance of getting x \$ and a 1-p-chance of getting x' \$.
- 4. Few new models can easily be decomposed into value and risk. However, these models are rather tangential in the recent discussion in decision theory, see *Sarin and Weber* (1993).
- 5. Note, that risk measures with an exogenous benchmark are equivalent to risk measures without any benchmark: F(e) = F(a + (e a)), with F being the risk measure, e the payoff and a the exogenous benchmark.
- 6. The location-dependent models mentioned above can be easily extended to become location free, see, e.g. *Weber (1990)* for an extension of *Sarin (1984)*.
- 7. See Bell (1988), (1995a,b), Jia and Dyer (1995, 1996) and Sarin and Weber (1993).
- 8. For an endogenous benchmark, $F^0 \equiv F$ ($\hat{e} = 0$). Then risk is zero if the payoff is constant. For an exogenous benchmark, subtraction of F^0 cannot assure zero risk for different levels of a constant payoff.
- 9. The market is said to be complete if for every ε^0 there exists a claim which pays off 1 \$ if $\varepsilon > \varepsilon^0$ and zero otherwise [see *Nachman (1988)*].
- 10. In order to obtain the first order condition as an equality, $f(\hat{e}) \rightarrow 0$ for $\hat{e} \rightarrow \overline{\hat{e}}$ and $f(\hat{e}) \rightarrow \infty$ for $\hat{e} \rightarrow \hat{\underline{e}}$ is required. In addition, the solution of the minimization problem exists and is unique if a) for every e satisfying the contraints (8) and (9) the value of the objective

function is finite and b) for every λ , η and e satisfying the first order condition (11). E [e π] is finite [Back and Dybvig (1993)].

- 11. The conditions mentioned in footnote 9 apply in an analogue manner here.
- 12. By the implicit function theorem, $\hat{e}_i(\varepsilon)$ is twice continuously differentiable since $f_i(\hat{e}_i)$ and $\theta_{\varepsilon} = \theta(\varepsilon)$ are twice continuously differentiable.
- 13. Thus the conjecture of Weber (1990) that all investors have linear sharing rules, is wrong.

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Figures

Figure 1



<u>Fig. 1</u>: The risk-value efficient frontier depicts the minimal portfolio risk as a function of the required expected portfolio return R^* diminished by the risk free rate R_f .

Figure 2



- a) The investor's payoff is decreasing and convex in the price for statecontingent claims.
- b) The price for state-contingent claims is decreasing and convex in the aggregate payoff.

Figure 3





b1) Investor i demands a higher expected portfolio return. His sharing rule is convex relative to that of investor j such that R_i and R_j intersect once.

b2) Investor i has a higher risk sensitivity.His sharing rule is convex relative to that of investor j such that both intersect twice.