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# Working Paper How to compute perfect foresight equilibria

Diskussionsbeiträge - Serie II, No. 150

**Provided in Cooperation with:** Department of Economics, University of Konstanz

*Suggested Citation:* Keuschnigg, Christian (1991) : How to compute perfect foresight equilibria, Diskussionsbeiträge - Serie II, No. 150, Universität Konstanz, Sonderforschungsbereich 178 - Internationalisierung der Wirtschaft, Konstanz

This Version is available at: https://hdl.handle.net/10419/101656

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# Sonderforschungsbereich 178 "Internationalisierung der Wirtschaft"

Diskussionsbeiträge



Juristische Fakultät Fakultät für Wirtschaftswissenschaften und Statistik

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How to Compute Perfect Foresight Equilibria

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Serie II — Nr. 150 September 1991

# HOW TO COMPUTE PERFECT FORESIGHT EQUILIBRIA

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Serie II - Nr. 150

September 1991

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\*I thank Peter J. Wilcoxen for introducing me to the algorithm and providing a Fortran code, and Gilg Seeber for supporting me with his invaluable computational expertise. The paper is part of a project on "An Applied General Equilibrium Model for Austria" which is supported by the Austrian National Bank under grant no. 3327. I started this work at the University of Innsbruck, continued it as a research fellow of the Alexander von Humboldt – Foundation at Bonn, and completed it as a member of the Sonderforschungsbereich 303 of the Deutsche Forschungsgemeinschaft at the University of Bonn. I gratefully acknowledge financial support from all institutions.

#### Abstract

The paper reviews an algorithm for the iterative solution of rational expectations models. It shows in detail how general equilibrium models with perfect foresight and intertemporally optimizing behavior must be set up to be solved numerically with the algorithm. Three examples of intertemporal equilibrium models with increasing complexity are given. An extensive sensitivity analysis with respect to changes in the control parameters tests the efficiency of the algorithm in solving the examples.

Die Arbeit stellt einen Algorithmus für die iterative Lösung von Modellen mit perfekter Voraussicht vor. Im besonderen wird gezeigt, wie allgemeine Gleichgewichtsmodelle mit perfekter Voraussicht und intertemporalem Optimierungsverhalten der Akteure formuliert werden müssen, damit sie mit dem Algorithmus numerisch gelöst werden können. Dies wird an drei Beispielen mit zunehmendem Komplexitätsgrad veranschaulicht. Eine ausführliche Sensitivitätsanalyse in bezug auf die Kontrollparameter testet die numerische Effizienz bei der Lösung der Beispiele.

# 1 Introduction

When incorporating dynamics in applied general equilibrium models, one may either assume static expectations or perfect foresight of agents. From a computational viewpoint, static or myopic expectations have the advantage that a single sequence of temporary flow equilibria can be computed that are independent from future equilibrium prices. Such a recursive approach still allows for capital accumulation over time and eventually leads to balanced growth of the model economy. While the concept of perfect foresight may be not the most realistic approch, any other formulation of expectations is not compatible with the notion of an intertemporal equilibrium. Except in a steady state equilibrium, none of the agents decisions is optimal ex post since the actual equilibrium prices on a transition path are generally different from what was expected. Hence, agents continually err with respect to their price projections and would like to revise their decisions ex post. With perfect foresight, price expectations are selffulfilling and the agents decision making is optimal ex ante and ex post. Consistent welfare analysis of policy experiments becomes possible [see Keen (1990) on welfare analysis in intertemporal models]. Furthermore, postulating perfect foresight allows for a more attractive intertemporal formulation of consumption and investment decisions. It allows to analyze interesting intertemporal phenomena such as agents reactions to anticipated preannounced policy changes or temporary policy changes.

However, the perfect foresight assumption complicates quite substantially numerical solution of equilibrium growth paths. Since agents use information on future equilibrium prices to implement their current decisions, current and future equilibria become intertemporally connected. A simultaneous approach is required to compute perfect foresight equilibria. Indeed, Codsi, Pearson and Wilcoxen (1991) form a large simultaneous equation system by stacking together all the equations in different time periods. The whole system of equations including the intertemporal ones is solved simultaneously. In relying on a linearized representation of the equation system, their method produces true solutions to the nonlinear model. The dimensionality of the system increases progressively with the number of static equations and the number of time periods to be solved. An alternative approach which was originally introduced in the macroeconomic rational expectations literatur by Fair and Taylor (1983), seperates the problem into two simpler tasks. Conditional on a guess for the foreward looking (expected) variables one computes a sequence of temporary equilibria giving rise to a sequence of actual values of the ex-

pected variables. A linear combination of initial guesses and actual values is computed to arrive at an improved guess for the expected variables. This procedure is iterated with expectations hopefully converging to a fixed point. The advantage of the method is that the dimensionality of the equation system to be solved is limited to the number of static equilibrium conditions. However, the system has to be solved many times to converge to the perfect foresight solution.

Recently, Wilcoxen (1989) modified the Fair-Taylor technique by introducing an improved method for revising guesses of expected variables. Wilcoxen's method basically exploits the dynamic structure of the problem in computing revisions and, thus, improves convergence to the fixed point for expectations. In the appendix, the paper describes the more sophisticated generalized Fair Taylor algorithm due to Wilcoxen. The main text shows in detail how intertemporal equilibrium models must be set up to be applicable to the algorithm. Hence, it should be useful to applied general equilibrium modellers who want to incorporate foreward looking behavior in their models.

The paper discusses in section 2 the general problem to be solved and describes the basic idea of the solution method. A detailed description of Wilcoxen's algorithm is relegated to an appendix. Section 3 analyses in detail the nature of intertemporal investment and savings decisions which form the main building blocks of any intertemporal equilibrium model. Section 4 contains three examples of intertemporal equilibrium models with increasing complexity. Specifically, it shows how these models must be formulated such that the proposed algorithm is applicable. It introduces the notion of temporary equilibria conditional on guesses for expected variables and explains how the guesses are revised when a sequence of temporary equilibria is computed. It discusses the properties of temporary equilibria that are important for the solution procedure: Walras' Law and homogeneity.

The first example given in section 4 discusses the most rudimentary model of neoclassical decentralized growth for which an exact solution for the transition paths is known [see Sargent (1987)]. Many authors have analyzed intertemporal policy issues in aggregate dynamic equilibrium models, e.g. Brock and Turnovsky (1981), Abel and Blanchard (1983), Judd (1985), Sinn (1987) and Gavin (1990) which is, of course, an extremely incomplete list of examples. For our demonstration purposes, the second example builds on the model of Abel and Blanchard (1983) with independent intertemporal consumption and investment plans [see also Bovenberg (1989)] but adds variable labor supply as in Judd (1987) and Lucas (1990). The third example contains a large scale equilibrium model. Jorgenson and Yun (1986a,b) were among the first to give an empirical analysis of U.S. tax reform using an econometrically estimated applied intertemporal equilibrium model [see also Jorgenson and Yun (1990) and Jorgenson and Wilcoxen (1990a,b)]. The parameters of applied general equilibrium models are often generated by calibration methods as an alternative to econometric estimation. This is justified by a lack of time series data for detailed disaggregate models. Relying on calibration methods, Goulder and Summers (1989) and Goulder and Eichengreen (1989) have developed intertemporal equilibrium models with perfect foresight for the evaluation of tax and trade policies. Keuschnigg and Kohler (1991) set up an applied intertemporal model of this kind for Austria emphasizing the model features of a small open economy.

# 2 Iterative Perfect Foresight Solutions

Generally, the basic structure of applied intertemporal equilibrium models with perfect foresight is essentially a nonlinear difference equation system in the endogenous dynamic variables. General equilibrium in period t is conditional on a set of stock (predetermined, state) variables  $K_t$ . Stocks represent accumulated past flows such as investment or savings and are therefore the result of accumulation decisions in equilibria prior to period t. While the stock variables provide exogenously given initial conditions, the period t solution also depends on a set of expected (foreward looking, jumping, non-predetermined) variables  $E_t$  which contain the information pertaining to the perfectly anticipated future equilibria of the system. Since they depend on the future, history cannot give initial conditions for them. If the system is well behaved, however, the endogenous variables must satisfy some non-explosion (transversality) conditions which may be taken as terminal conditions of the expected variables. Therefore, the computation of perfect foresight equilibria requires to solve a two point boundary value problem: the solution must pass through the initial conditions for the state variables and the terminal constraints for the expected variables given by the non-explosion or transversality conditions.

In applied intertemporal models, expected variables typically represent sums of discounted future values,  $E_t = \sum_{s=t}^{\infty} e(p_s)p_s/p_t$ . If prices  $p_s$  are stated in terms of present value or futures prices, discounting is done by the intertemporal price ratio  $p_s/p_t$ . Taking the differences gives the equation of motion  $E_{t+1} = [E_t - e(p_t)]p_t/p_{t+1}$  which dictates the evolution over time of the expected variables. In intertemporal general equilibrium, the following equations must be satisfied

(a) 
$$K_{t+1} = I(p_t, K_t, E_t),$$
  
(b)  $E_{t+1} = [E_t - e(p_t)]p_t/p_{t+1},$  (1)  
(c)  $\zeta(p_t, K_t, E_t) = 0, \quad \zeta(p_{t+1}, K_{t+1}, E_{t+1}) = 0.$ 

The equations show how two consecutive temporary equilibria are connected in a fully intertemporal perfect foresight equilibrium. The first equation gives the endogenous accumulation decision conditional on previously accumulated stocks  $K_t$ , expectations  $E_t$  and equilibrium prices  $p_t = p(K_t, E_t)$  which solve the temporary excess demand system  $\zeta(p_t, K_t, E_t) = 0$ . In a perfect foresight equilibrium, the evolution of the expected variables  $E_t$  must satisfy the second equation which requires knowledge of the equilibrium prices in both periods,  $p_t$  and  $p_{t+1}$ . The dynamic system formed by the two types of endogenous dynamic variables is saddle point stable. If the initial values of the expected variables are not chosen appropriately, sequential solution of the system in (1) produces divergent paths for the dynamic variables. If the system is determinate, however, the appropriate initial values of the expected variables are uniquely determined by the requirement of dynamic stability.<sup>1</sup>

The values of expected variables contain information on the nature of future equilibria. The correct initial values of the expected variables could be determined, for example, by solving for the equilibria in all periods simultaneously in one stroke [see the paper by Codsi, Pearson and Wilcoxen (1991) referred to in the introduction]. For sufficiently disaggregated models and many transition periods, the simultaneous approach requires the ability to compute solutions to a simultaneous equation system with extremely high dimension. An alternative solution strategy is to break up the problem into separate smaller tasks.

<sup>&</sup>lt;sup>1</sup>In intertemporal equilibrium models, agents solve dynamic optimization problems contingent on given price paths. Loosely speaking, explosive or implosive paths of the dynamic variables would eventually force agents to violate some of the necessary conditions for dynamic optimization contradicting the notion of an intertemporal equilibrium.

(a) 
$$K_{t+1} = I(p_t, K_t, E_t),$$
  
(b)  $A_{t-1} = e_{t-1} + E_t p_t / p_{t-1},$   
(c)  $\zeta(p_t, K_t, E_t) = 0.$ 
(2)

At each date in time, the value of the state variable  $K_t$  provides an initial condition which is exogenous to the period t problem. If the value of the expected variable  $E_t$ were known, one could solve for a temporary equilibrium  $\zeta(p_t, K_t, E_t) = 0$  to obtain the equilibrium accumulation decision from equation (2a) and determine the optimal state in the next period. This suggests a numerical solution concept: guess a vector  $E = (E_1, \ldots, E_T)'$  of the expected variables and compute from (2a) the accumulation of the state variables for all transition periods. Except in a perfect foresight equilibrium, the guessed values  $E_{t-1}, E_t$  for the expected variables will not satisfy (2b) which indicates errors in forecasting the future. From period t = 2 onwards, equation (2b) is used to compute new actual values of the expected variables,  $A_1, \ldots, A_{T-1}$ . Computing a sequence of solutions of (2) generates a vector of revised values for the expectations which may be used as guesses for the next pass through. However, equation (2b) does not provide an update for the last period of the transition phase. For a certain class of problems, the terminal values of the expected variables may be taken from separate steady state calculations. In any case, if the problem is properly set up and the transition phase is long enough, the exact terminal values of the expected variables will be unimportant. The model solution early in the transition phase is insensitive to variations in terminal constraints, the solution trajectories exhibit the Turnpike property.

Given a vector E containing the guesses, an evaluation of the system (2) during all of the transition periods t = 1, ..., T provides a vector of actuals A(E). A perfect foresight solution constitutes a fixed point E = A(E) of the vector valued function A(.) defined by (2). Fair and Taylor (1983) developed a simple technique for solving perfect foresight equilibria which iterates on

$$E^{i+1} = \mu_0 A(E^i) + (1 - \mu_0) E^i.$$
(3)

Index *i* denotes the *i*-th iteration. Convergence of this iterative procedure may be proved only for the very simplest models. To improve convergence numerically,  $\mu_0$  may be chosen between zero and unity. Recently, Wilcoxen (1989) generalized this technique

by exploiting the intertemporal structure of the problem to the benefit of more rapid convergence. Wilcoxen's generalized version of the Fair Taylor algorithm is presented in the appendix.

# 3 Savings and Investment

### 3.1 Intertemporal Theory of Consumption and Labor Supply

Allowing for exogenous growth, some variables contain trends.<sup>2</sup> It is convenient to consider the detrended economy. The wage rate w, for example, is detrended from wage trend growth due to labor productivity enhancing technological progress X. The aggregate labor force is devided by the population size N, hence, L measures labor supply per capita. Therefore, overall wage income wL is measured in labor efficiency units and stays constant in a balanced growth equilibrium of the detrended economy. After eliminating any exogenous growth trends, the flow budget identity is

$$F_s = F_{s-1}(1+r_s)/(1+\bar{g}) + w_s L_s - p_s C_s.$$
(4)

Subindices denote time. Choosing a discrete time framework requires a careful dating convention. All transactions are meant to take place at the beginning of period, and the same holds for stocks. The stock of financial wealth at the beginning of period s - 1,  $F_{s-1}$ , earns an interest income  $r_s F_{s-1}$  at the beginning of period s. Consumption C of the commodity is available at a price p. Financial wealth, consumption and wage income wL are measured in labor efficiency units NX.

It is assumed that the economy is populated by agents with an operative altruistic bequest motive towards their heirs. Since they fully internalize the interests of future generations, household sector behavior may be thought of as resulting from the decisions of a single infinitely lived family planning over the entire future [see Barro (1974) and Weil (1987)]. Intertemporal preferences of the infinitely lived family are represented by the Benthamite utility functional [see Sinn (1987), pp. 26–28 for a detailed discussion]

$$U_t = \sum_{s=t}^{\infty} (1+\rho)^{t-s} N_s u(C_s X_s, h_s).$$
(5)

<sup>&</sup>lt;sup>2</sup>Population is assumed to grow at an exogenous constant rate n,  $N_t = (1 + n)^t N_0$  while the state of productivity enhancing technological progress increases at an exogenous rate x,  $X_t = (1 + x)^t X_0$ . Hence, labor efficiency units  $N_t X_t$  grow exogenously at rate  $\bar{g} = (1 + n)(1 + x) - 1$ .

Future per capita momentary utility (felicity) is weighted by family size N and is deflated by the subjective rate of discount  $\rho$  to calculate the present value of lifetime utility. Felicity is in terms of per capita consumption  $C_sX_s$  and leisure  $h_s = 1 - L_s$ . In a steady state, per capita leisure stays constant, but per capita consumption steadily increases with the productivity growth rate x. As King, Plosser and Rebelo (1987) show, the most general functional form compatible with such steady state characteristics is

$$u(CX,h) = \frac{(CX)^{1-1/\gamma}}{1-1/\gamma} v(h).$$
(6)

Momentary utility features multiplicative separability in consumption and leisure. The parameter  $\gamma$  expresses the intertemporal elasticity of substitution in consumption. Concavity of the felicity function u(.) requires that  $\gamma > 0$ . Furthermore, v must be increasing and concave if  $\gamma > 1$  and decreasing and convex if  $\gamma < 1$ . Given (6), life time utility is equivalently expressed as

$$U_t = N_t X_t^{(1-1/\gamma)} \sum_{s=t}^{\infty} \beta^{s-t} u(C_s, h_s), \quad \beta \equiv [(\frac{1+n}{1+\rho})(1+x)^{(1-1/\gamma)}].$$
(7)

The constant  $N_t X_t^{(1-1/\gamma)}$  can be ignored as it is exogenous and does not influence the results of maximization. We actually assume that the momentary utility function is of the form  $u_t = u[v(C, h)]$  where u(v) satisfies constant intertemporal elasticity of substitution at rate  $\gamma$  and v(C, h) is linear homogeneous with unitary elasticity of substitution between consumption and leisure. While Cobb Douglas may seem restrictive, it is in fact the most general form compatible with wage trend growth. Furthermore, homotheticity is a useful assumption since it allows to separate optimization into separate stages. In writing the flow budget identity

$$F_s = F_{s-1}(1+r_s)/(1+\bar{g}) + w_s - M_s^{\nu}, \quad M_s^{\nu} \equiv p_s C_s + w_s h_s, \tag{8}$$

one recognizes a static subproblem of maximizing within period utility by optimally spending a given budget  $M_t^v$  on consumption and leisure,

$$v_t = \max_{C_t, h_t} \{ v(C_t, h_t) \quad s.t. \quad p_t C_t + w_t h_t \le M_t^v \}.$$
(9)

Because of linear homogeneity, Marshallian demand functions as well as indirect utility are linear in expenditures,  $C_t = c(p_t, w_t)M_t^v$ ,  $h_t = h(p_t, w_t)M_t^v$  and  $v_t = M_t^v/p^v(p_t, w_t)$ . Inversion of indirect utility gives the expenditure function  $M_t^{\nu} = p^{\nu}(p_t, w_t)v_t$ . Hence, the within period problem defines full consumption  $v_t$  which is available at a price index  $p_t^{\nu}$ .

The basic intertemporal nature of the problem is to choose a sequence of consumption bundles with a properly specified restriction on borrowing so as to maximize lifetime utility. While one does not want to rule out temporary indebtedness, households must be prevented to accumulate debt at such a rate that interest obligations can only be serviced by new debt (no-Ponzi-game condition). The dynamic optimization problem of the family consists of maximizing (7) subject to the equation of motion (8), the initial condition  $F_{t-1} = F^0$ , and the terminal condition  $\lim_{T\to\infty} F_T \ge 0$  ensuring solvency at the end of lifetime. It is solved by setting up the Lagrangean

$$\mathcal{L}_{t} = \sum_{s=t}^{\infty} \beta^{s-t} \Big\{ u(v_{s}) + \mu_{s} \Big[ F_{s-1}(\frac{1+r_{s}}{1+\hat{g}}) + w_{s} - p_{s}^{v}v_{s} - F_{s} \Big] \Big\}.$$
(10)

We have introduced the current value multiplier  $\mu$  which is the shadow value of financial wealth. The first order necessary conditions for an (interior) optimum are

(a) 
$$u'(v_s) = \mu_s p_s^v,$$
  
(b)  $\mu_s = \mu_{s+1}\beta(1+r_{s+1})/(1+\bar{g}),$   
(c)  $F_s = F_{s-1}(1+r_s)/(1+\bar{g}) + w_s - p_s^v v_s,$   
(d)  $\lim_{T \to \infty} \mu_T \beta^{T-t} F_T = 0.$ 
(11)

Combining conditions (a) and (b) gives the Euler equation which shows how full consumption is allocated across periods to derive maximum utility,

$$u'(v_s) = \left[ \left( \frac{1+r_{s+1}}{1+\bar{g}} \right) \frac{p_s^v}{p_{s+1}^v} \right] \beta u'(v_{s+1}).$$
(12)

The left hand side gives the utility cost of forgoing one consumption unit at the beginning of period s. Given the interest on savings and price changes, the consumer can buy a quantity of consumption equal to the square bracket at the end of period s which yields a marginal utility gain  $u'(v_{s+1})$ . Hence, the Euler equation compares the marginal utility cost with the discounted utility gain from transferring a unit of consumption from this to the next period. Repeated application of the Euler equation shows how the consumer trades off the gains and losses from shifting consumption between any arbitrary two periods.

The next steps show how full consumption in any period depends on future life time income. Applying repetitively condition (c) yields

$$F_{t-1} = \sum_{s=t}^{T} (p_s^{v} v_s - w_s) R_{t,s} + F_T R_{t,T}, \quad R_{t,s} = \prod_{u=t}^{s} (\frac{1+\bar{g}}{1+r_u}), \quad R_{t,t-1} \equiv 1.$$
(13)

Similarly condition (b) gives  $\mu_t = \beta^{T-t} R_{t+1,T}^{-1} \mu_T$ . Using this in the transversality condition (d), one obtains  $\lim_{T\to\infty} \mu_t F_T R_{t+1,T} = \mu_t \frac{1+r_t}{1+\bar{g}} \lim_{T\to\infty} F_T R_{t,T} = 0$ . Since for an interior solution the current shadow value of financial wealth in terms of marginal utility of consumption  $\mu_t$  must be positive and finite, the discounted value of terminal financial wealth in (13) must vanish. Hence, the transversality condition restricts the consumer to satisfy the intertemporal budget constraint,

$$\sum_{s=t}^{\infty} p_s^{\nu} v_s R_{t+1,s} = \mathcal{W}_t, \qquad \mathcal{W}_t \equiv F_{t-1} \frac{1+r_t}{1+\bar{g}} + H_t^e, \qquad H_t^e \equiv \sum_{s=t}^{\infty} w_s R_{t+1,s}.$$
(14)

The present value of future spending on full consumption must not exceed total lifetime wealth which includes financial wealth plus expected human wealth  $H_t^e$ . A closed form solution for current consumption is obtained by using the isoelastic functional form for felicity. Hence, the Euler equation relates full consumption in different periods by

$$v_s = [(p_t^v/p_s^v)\beta^{s-t}R_{t+1,s}^{-1}]^{\gamma}v_t.$$
(15)

After substituting out future consumption in the intertemporal constraint, one can solve for present expenditures

$$M_t^{\nu} = p_t^{\nu} v_t = (\Omega_t^e)^{-1} \mathcal{W}_t, \qquad \Omega_t^e \equiv \sum_{s=t}^{\infty} \beta^{\gamma(s-t)} [(p_s^{\nu}/p_t^{\nu}) R_{t+1,s}]^{1-\gamma}.$$
 (16)

The "consumption function" relates expenditures on present full consumption to lifetime wealth  $W_t$ . The factor  $(\Omega_t^e)^{-1}$  is the marginal propensity to consume out of total wealth and depends on the price of present relative to future consumption. As a check for consistency, evaluate  $\Omega$  for the logarithmic utility function  $u(v) = \ln(v)$  with  $\gamma = 1$ . One obtains from (16)  $\Omega = 1/(1-\beta)$  and  $M_t^v = (1-\beta)W_t$ . Consumption is a constant fraction of life time wealth [see Sargent (1987), p. 23]. Knowing the current budget, the demands for leisure and the consumption commodity is obtained from the solution of (9).

Hence, we calculate current consumption and leisure conditional on expectations  $H_t^e$ and  $\Omega_t^e$ . Computation of temporary equilibria generates actual values of expected variables that may deviate from the initial guesses for  $H_{t-1}^e$  and  $\Omega_{t-1}^e$ . Actual values are obtained by taking the differences of the present value definitions of the expected variables,

$$H_{t-1} = w_{t-1} + \left(\frac{1+\bar{g}}{1+r_t}\right) H_t^e,$$
  

$$\Omega_{t-1} = 1 + \beta^{\gamma} \left[\frac{p_{t-1}^{\nu}}{p_{t-1}^{\nu}} \frac{1+\bar{g}}{1+r_t}\right]^{1-\gamma} \Omega_t^e.$$
(17)

### 3.2 Intertemporal Theory of Investment

We first start with a simple partial equilibrium model of production subject to adjustment costs in investment. The assumption of adjustment costs gives rise to uniquely determined neoclassical investment demand in the spirit of Tobin's (1969) Q-theory of investment. Abel (1980) and Hayashi (1982) gave a rigorous neoclassical interpretation. The theory predicts that investment increases if the market valuation of an incremental unit of new capital exceeds replacement costs. Since it introduces partial immobility of physical capital across sectors and slows the intertemporal responsiveness of investment, the adjustment cost framework is an important element in the newer work on applied general equilibrium modelling [see Bovenberg (1989) and Goulder and Summers (1989), among others]. To correct for exogenous growth trends, all variables are expressed again in labor efficiency units.

The organization of production consists of static and dynamic aspects, and it is convenient to treat them separately and to perform optimization in two stages. The static problem is one of choosing optimal input levels of factors that are truly variable in the short run. Following virtually all of the applied general equilibrium literature, we treat labor and intermediates as variable in the short run. Restricting our interest to labor demand for the moment, the profit function in labor efficiency units is, therefore,

$$\Pi(p,w,K) = \max_{L} \left\{ pf(K,L) - wL \right\}.$$
(18)

The variables stand for output price p, capital stock in labor efficiency units K, labor input per capita L, and wage rate net of its productivity growth component w. Optimal labor input must satisfy

$$pf_L(K,L) = w.$$
<sup>(19)</sup>

The necessary condition for an optimum defines a short run labor demand function  $L^d = L^d(p, w, K)$ . Assuming linear homogeneity of the production technology and invoking the envelope theorem gives the following properties of the short run profit function,

$$\frac{\partial \Pi}{\partial K} = pf_K, \qquad K \frac{\partial \Pi}{\partial K} = pf_K K = \Pi.$$
 (20)

Given short run profits and assuming that investment is financed via retained earnings, cash flow of the producing unit in labor efficiency units is

$$\chi_{s} = \Pi(p_{s}, w_{s}, K_{s}) - p_{s}\phi(I_{s}, K_{s}) - p_{s}^{I}I_{s}.$$
(21)

Investment is I and  $p^{I}$  denotes the acquisition price of capital goods. Due to adjustment costs, investment additionally involves a loss in output. The installation function  $\phi(I, K)$  is assumed linear homogeneous in its arguments and convex in investment. It satisfies  $\phi(0, K) = \phi_{I}(0, K) = 0$ ,  $\phi_{I}(I, K) \gtrsim 0$ ,  $\phi_{II}(I, K) > 0$ ,  $\phi_{K}(I, K) < 0$ ,  $\phi_{IK}(I, K) < 0$ . Investment takes place at the beginning of the period and capital must be purchased one period in advance of its productive use. Expressing in labor efficiency units, capital accumulates according to

$$(1+\bar{g})K_{t+1} = K_t + I_t.$$
(22)

Investors are willing to hold equities only if it guarantees a rate of return equal to the available return on alternative assets which is the prevailing market rate of interest  $r_t$ . The return on equities consists of dividends plus capital gains, all measured in efficiency units. Defining the value of the firm ex dividend, the following arbitrage relation must hold,

$$r_{t+1} = \frac{(1+\bar{g})(\chi_{t+1}+V_{t+1}) - V_t}{V_t}.$$
(23)

With the rates of return of assets being all equal, investors are indifferent between different types of assets and will invest their savings in whatever is offered on the asset market. Portfolio decisions are passive. Solving the arbitrage equation for  $V_t$  shows the ex dividend value of the firm to correspond to the present value of future dividend payments to the owners<sup>3</sup>. Production organizing agents must discount with the market rate of interest to satisfy the arbitrage equation. If they apply a lower discount rate they will not

<sup>&</sup>lt;sup>3</sup>In solving for  $V_t$  one must impose the condition  $\lim_{T\to\infty} V_T R_{t+1,T} = 0$  which rules out eternal speculative bubbles but requires equity values to reflect the fundamental returns of the firm.

be able to sell the equities on the asset market. If they apply a higher discount rate they will inevitably attract all the financial wealth of households and thereby create excess demand for the equities in the firm. With the discount factor defined in (13), firm value is

$$V_t = \sum_{s=t+1}^{\infty} \chi_s R_{t+1,s}.$$
 (24)

The main behavioral assumption concerning intertemporal investment decisions is value maximization. In addition to the ex dividend value, total wealth of the owners includes the dividend at the beginning of period t. Optimality requires to choose a path of future investments that maximizes total financial wealth  $\chi_t + V_t$  of the owners subject to (21), (22), the initial capital stock  $K_t = K^0$  and non-negativity constraints  $K_s > 0$  for all dates  $s = t, \ldots, \infty$ . Differentiating the Lagrangean

$$\mathcal{L}_{t} = \sum_{s=t}^{\infty} \left\{ \Pi(p_{s}, w_{s}, K_{s}) - p_{s}\phi(I_{s}, K_{s}) - p_{s}^{I}I_{s} + q_{s}[K_{s} + I_{s} - (1 + \bar{g})K_{s+1}] \right\} R_{t+1,s}$$
(25)

gives the set of necessary first order conditions

(a) 
$$q_s = p_s^I + p_s \phi_I(I_s, K_s),$$
  
(b)  $(1 + r_{s+1})q_s = [\Pi_K(K_{s+1}) - p_{s+1}\phi_K(I_{s+1}, K_{s+1}) + q_{s+1}]$   
(c)  $(1 + \bar{g})K_{s+1} = I_s + K_s,$   
(d)  $\lim_{T \to \infty} (1 + \bar{g})K_{T+1}q_T R_{t+1,T} = 0.$ 
(26)

The optimality conditions have intuitive interpretations. Condition (a) implies that new capital is installed until the effective acquisition cost consisting of the price of capital goods plus marginal adjustment costs equals the marginal benefit of investment reflected in the shadow value of productive capital. An economic interpretation of q is seen by solving (b),<sup>4</sup>

$$q_t = \sum_{s=t+1}^{\infty} \left[ \prod_{K_s} - p_s \phi_{K_s} \right] \prod_{u=t+1}^{s} \left( \frac{1}{1+r_u} \right).$$
(27)

The shadow value q expresses the benefit of a marginal increase in the state variable in terms of the objective function. In the present case it is the present value of the incremental future income stream (including savings from reductions in adjustment costs)

<sup>&</sup>lt;sup>4</sup>The transversality condition (d) implies  $\lim_{T\to\infty} q_T R_{t+1,T} = 0$  for  $K_{T+1} > 0$  and ensures existence.

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that is due to the investment of an additional unit of capital. Hence, conditions (a) and (b) express the simple criterion that the present value of incremental future income must exceed the effective acquisition costs of new capital for investment to be profitable. Given the assumptions on the installation function  $\phi$ , the optimality condition (26a) may be inverted to give neoclassical investment demand

$$I_{s} = I(q_{s}^{e}, K_{s}) = \phi_{I}^{-1}[(q_{s}^{e} - p_{s}^{I})/p_{s}, K_{s}].$$
(28)

The shadow value q, also referred to as Tobin's "q", is the forward looking variable and contains the information on the future. We denote this by attaching an upper index e to this expected variable. According to Tobin's theory, investment increases with the rate at which the expected market value  $q^e$  of productive capital exceeds the acquisition price of capital.

Substituting the investment demand function into the equation of motion for the capital stock in (26c) and using (26b) as an updating equation produces the difference equation system (2a,b) which we may try to solve using the iterative Fair – Taylor technique. Using the shadow price as an expected variable, however, is an extremely unstable procedure. For a given sequence of prices, it is clear from (28) that the trajectory for the capital stock conditional on an arbitrary guess for  $q_s^e$  explodes or implodes. This is a consequence of saddle point stability in (q, K) space and may be demonstrated easily by drawing the phase diagram [see Blanchard and Fischer (1989), pp. 63]. If the expected variable is not exactly chosen at its equilibrium value on the stable manifold, the dynamic system diverges away from the solution. Either the capital stock becomes negative which violates the non-negativity constraints, or it explodes and thereby violates the transversality condition of dynamic optimization.

The transversality condition belongs to the set of necessary conditions for a firm optimum. Exploiting the information contained in it, Hayashi (1982) derived a relationship between the marginal and average value of capital equipment. According to Hayashi's theorem, productive capital in place at the end of the period times the shadow value of new capital gives the ex dividend value of the production unit,

$$V_t = (1 + \bar{g}) K_{t+1} q_t.$$
(29)

To prove this, note that in an optimum the firm must satisfy the arbitrage condition (26b) in each future period. Multiplying both sides by  $K_{t+1}$  and expanding the r.h.s. with

 $q_{t+1}I_{t+1}$  gives

$$(1 + r_{t+1})K_{t+1}q_t = [K_{t+1}\Pi_{K_{t+1}} - p_{t+1}\phi_{K_{t+1}}K_{t+1} - q_{t+1}I_{t+1} + q_{t+1}(I_{t+1} + K_{t+1})]$$
  

$$= [\Pi_{t+1} - p_{t+1}(K_{t+1}\phi_{K_{t+1}} + I_{t+1}\phi_{I_{t+1}}) - p_{t+1}^II_{t+1} + (1 + \bar{g})K_{t+2}q_{t+1}]$$
  

$$= [\chi_{t+1} + (1 + \bar{g})K_{t+2}q_{t+1}].$$
(30)

The second equality is obtained by inserting the optimality conditions (26a,c) and using the homogeneity of the production technology. Finally, homogeneity of the installation function and the definition of dividends in (21) yield the last equality. The relation must hold between any two consecutive periods. Dividing by  $(1 + r_{t+1})$ , multiplying by  $(1 + \bar{g})$  and solving forward yields

$$(1+\bar{g})K_{t+1}q_t = \sum_{s=t+1}^T \chi_s R_{t+1,s} + (1+\bar{g})K_{T+1}q_T R_{t+1,T}.$$
(31)

Taking the limit  $T \to \infty$ , using the transversality condition (26d) and the definition of firm value in (24) proves (29).

The relation between total value of the firm and the shadow value of the capital stock is extremely useful for the numerical implementation. Using firm value as the expected variable in (26a), investment demand is implicitly defined by

$$p_s\phi_I(I_s, K_s) = [V_s^e/(I_s + K_s) - p_s^I] \quad \Longleftrightarrow \quad I_s = I(p_s; K_s; V_s^e).$$
(32)

A too high guess of the expected variable induces higher than equilibrium investment. On the other hand, too rapid capital accumulation reduces the implicitely defined shadow value  $q_s^e = V_s^e/(I_s + K_s)$  on the right hand side and thereby slows investment in the following periods. Quite intuitively, this prevents any degenerate capital accumulation paths even if the guess for the expected variable is off the mark. The updating equation is now (23) which is easily shown to be equivalent to (26b) by linear homogeneity of the production and installation technologies and (29). The iterative Fair – Taylor technique is now applied to the dynamic system

(a) 
$$K_{t+1} = [K_t + I(K_t, V_t^e)] \frac{1}{1+\bar{g}},$$
  
(b)  $V_{t-1} = [\chi(K_t, V_t^e) + V_t^e] \frac{1+\bar{g}}{1+r_t}.$ 
(33)

# 4 Computable General Equilibrium Models

### 4.1 The Basic Infinite Horizon Model

We now use the main building blocks developed in the preceding two sections to construct intertemporal equilibrium models with perfect foresight. The first model is extremely simple, but is particularly useful to test the numerical procedures since its solution is known from the example given in Sargent (1987), pp. 24. Consider a stationary economy with zero growth rates, fixed labor supply and capital being adapted freely without any adjustment costs.

**Production:** In the absence of adjustment costs, all terms involving the installation function and its derivatives are zero in (26). Specifically, the shadow price of capital is equal to the commodity price. According to Hayashi's (1982) theorem, the value of the firm must be equal to the value of the desired capital stock at the current commodity price. Evaluating (29) in the no growth case, investment demand is

$$I_t = I(p_t; K_t; V_t^e) = V_t^e / p_t - K_t.$$
(34)

Investment is a function of the current commodity price, the beginning of period capital stock and the expected value of the firm. Labor is paid its competitive wage according to (19). Dividends are defined in (21),  $\chi_t = p_t f_t - w_t - p_t I_t$ .

**Consumption:** Given fixed labor supply equal to unity, the composite good  $v_t$  coincides with consumption  $C_t$  and its price index with the commodity price. Furthermore, assuming a logarithmic felicity function, intertemporal household optimization yields the consumption function [see the discussion following (16)]

$$C_t = (1 - \beta) \mathcal{W}_t / p_t, \qquad \mathcal{W}_t = \chi_t + V_t^e + H_t^e.$$
(35)

In the absence of any other assets, the consumer holds only equities, his beginning of period t financial wealth is therefore  $(1 + r_t)V_{t-1} = \chi_t + V_t$  which is the cum dividend value of the firm.

**Equilibrium:** A perfect foresight equilibrium consists of sequences of prices and quantities that clear all markets, are consistent with optimizing behavior of agents and

satisfy all budget constraints and initial conditions. To avoid the simultaneous computation of market clearing in all dates, our solution strategy separates the problem into two simpler tasks: First, compute sequences of temporary equilibria for given guesses on expected variables. Second, iterate the paths of expected variables until the perfect foresight requirement is fulfilled. Hence, we must compute many sequences of temporary equilibria.

Consider first the nature of **temporary equilibria**. They depend on previously accumulated stocks of physical capital and are also parametric on expectations. The conditions for market clearing are

(a) 
$$\zeta^{C}(p_{t}; K_{t}; E_{t}) = I_{t} + C_{t} - f(K_{t}) = 0,$$
  
(b)  $\zeta^{K}(p_{t}; K_{t}; E_{t}) = \chi_{t} + w_{t} - p_{t}C_{t} = 0.$ 
(36)

Given fixed labor supply, output supply is exogenous in any period t since it depends on the previously accumulated capital stock and on fixed and fully employed labor. Available output changes over time as a consequence of capital accumulation. Given some guesses for the expected variables  $E_t = \{H_t^e, V_t^e\}$ , demand in the commodity market stems from the behavioral functions explaining consumption and investment. Since investment is internally financed, there is no effective demand for funds in the capital market. In the absence of any other demand for savings, current household savings must be zero in a capital market equilibrium.

The temporary excess demand system satisfies Walras' Law,

$$p_t \zeta_t^C + \zeta_t^K = 0. \tag{37}$$

Walras' Law follows almost trivially from substituting the definitions of the excess demands and the dividend equation. Market clearing in one market implies via fulfillment of the (flow) budget constraints of all agents market clearing in the other market. Finding temporary equilibria, therefore, reduces to computing the zeros of only one excess demand function in the present case.

It is also easily shown that the temporary excess demand system satisfies certain homogeneity properties that allow to choose a convenient price normalization. Commodity excess demand is homogeneous of degree zero in  $\{p_t; H_t^e, V_t^e\}$ , since investment and consumption demand functions are homogeneous of degree zero in these variables. One needs



### Fig. 1: Computation of Temporary Equilibrium

to include also  $\{H_t^e, V_t^e\}$  for zero homogeneity since the excess demand system must be homogeneous of degree zero in all prices, present and future. The present value terms  $\{H_t^e, V_t^e\}$  are homogeneous of degree one in future commodity prices and wage rates. The excess demand function for capital is homogeneous of degree one since it is stated in terms of products of quantities and prices.

Figure 1 shows how temporary equilibria are computed conditional on capital stocks and expectations regarding future prices. In a fully **intertemporal equilibrium** expectations of agents turn out to be selffulfilling at any date. Agents need not revise their plans and expectations regarding the future in response to realizations of current equilibrium prices. Starting with vectors of initial guesses for the expected variables  $E_t = \{H_t^e, V_t^e\}$ , one computes a sequence of temporary equilibria and thereby generates revisions for the expected variables,

$$H_{t-1} = w_{t-1} + H_t^e / (1+r_t),$$
  

$$V_{t-1} = (\chi_t + V_t^e) / (1+r_t).$$
(38)

This is like a single evaluation of the vector valued function A(E) defined in (2). Treating revisions and initial guesses as in section 2, one arrives at an improved guess for the next iteration. The perfect foresight solution is obtained as soon as guesses and realizations of the expected variables converge to a fixed point E = A(E), i.e.actuals computed from (38) must coincide with the guessed values in each period,  $H_{t-1} = H_{t-1}^e$  for example.

**Price normalization:** The last step in the computational procedure is price normalization. Note that the sequence of interest rates  $\{r_t\}$  actually is never changed during computations of temporary equilibria<sup>5</sup>. The proposed solution procedure implicitely changes the real consumption based interest rates via changing commodity price levels. This is motivated by the fact that in a Walrasian equilibrium system spot prices can be redefined as present value or futures prices by discounting with the interest factors [see Burmeister (1980) and Varian (1984)]. The numerical solution procedure solves for commodity prices relative to a given sequence of 'nominal' interest rates. Homogeneity of the excess demand system allows to renormalize prices. In this simple model, the commodity is a convenient numeraire. Expressing all prices and value terms relative to the commodity price, one obtains transformed variables  $\tilde{V}_t^e = V_t^e/p_t$ ,  $\tilde{H}_t^e = H_t^e/p_t$  and  $\tilde{p}_t = 1$  in all periods. Price normalization also redefines interest rates

$$1 + \tilde{r}_t = (1 + r_t) p_{t-1} / p_t.$$
(39)

If households sell one commodity at the beginning of period t-1 at a price  $p_{t-1}$ , they can obtain  $(1 + r_t)p_{t-1}/p_t$  units next period. After normalization with spot prices equal to unity, agents must be able to perform the same intertemporal trade which defines a consumption based real rate of interest  $\tilde{r}_t$  according to (39) and leaves the Euler equation (12) unaffected. Price normalization does not affect temporary equilibria because of the homogeneity properties of the excess demand system. It does not interfere with the updating relations either. Devide (38) by the spot price to obtain

$$\tilde{H}_{t-1} = \tilde{w}_{t-1} + \tilde{H}_t / [(1+r_t)p_{t-1}/p_t], 
\tilde{V}_{t-1} = (\tilde{\chi}_t + \tilde{V}_t) / [(1+r_t)p_{t-1}/p_t].$$
(40)

Using the definition of real interest  $\tilde{r}_t$  shows that the updating relations still hold after normalization if they do so before.

<sup>&</sup>lt;sup>5</sup>Before starting the iterative computations, an initial sequence of interest rates could be obtained from the steady state version of the Euler equation (12) with constant spot prices of the commodity,  $1 = (1 + r)\beta$ . Setting the commodity price unity and substituting the marginal productivity condition (26b) defines the stationary capital stock which also determines all other steady state values.

The example is particularly valuable since it possesses an analytical solution [see Sargent (1987), pp. 24]. To conform to Sargent's example, let fixed labor input be unity and assume a Cobb Douglas production function  $f(K_t) = AK_t^{\alpha} - K_t$ . Hence, under the stated assumptions, we have from (19) and (26b)  $w_t = (1 - \alpha)AK_t^{\alpha}$  and  $1 + r_{t+1} = \alpha AK_{t+1}^{\alpha-1}$ with the commodity price unity in each period. Using dynamic programming methods, Sargent (1987) derives from the value function of the problem a policy function that gives the exact solution for the transition path of capital,  $K_{t+1} = \beta \alpha AK_t^{\alpha}$ . Since commodity prices are set to unity in each period, the solution trajectories for the expected variables<sup>6</sup> are  $V_t = p_t K_{t+1} = \alpha \beta AK_t^{\alpha}$  and  $H_t = (1 - \alpha)AK_t^{\alpha}/(1 - \beta)$ . Using the production function, the factor prices and the policy function, dividends are  $\chi_t = (1 - \beta)\alpha AK_t^{\alpha}$ . Substituting appropriately into (38), the updating relations turn out to be equivalent to the policy function. Hence, if iteration of the sequences of temporary equilibria converges at all, it must converge to the solution given by the policy function derived in Sargent (1987).

As a numerical exercise, consider the following experiment. Let the initial capital stock be  $K_1 = .8K_{SS}$  and fix the terminal values of the expected variables at  $H_T = 1.1H_{SS}$  and  $V_T = .8V_{SS}$ . The solution for the capital stock and the commodity price paths is depicted in figure 1 and demonstrates the Turnpike property. The model converges rapidly to the steady state equilibrium but deviates from it some periods before the terminal date since the terminal values of the expected variables are fixed and differ from their steady state values. The system is saddlepoint stable. The speed of convergence to the steady state is determined by the stable eigenvalues [of the underlying linearized system] while the influence of the terminal values of the expected variables on the transition path is determined by the explosive eigenvalues [see Blanchard and Fischer (1989), pp. 100].

In table 1, the same solution is computed with different combinations of the control parameters  $\mu_0$ ,  $\mu_1$ ,  $\mu_2$  of the generalized Fair Taylor algorithm. The table shows how many evaluations of the vector valued function A(E) are required to arrive at a perfect foresight solution E = A(E). The model is computed for T = 100 periods. Each evaluation of the vector valued function A(E) including the computation of the Jacobians  $J_{11}$  and  $J_{12}$  requires to compute 108 temporary equilibria. It turns out that the original Fair Taylor method  $\mu_0 = 1$ ,  $\mu_1 = 0$ ,  $\mu_2 = 0$  is quite inefficient for this specific example. The Generalized Fair Taylor method obtains a solution for the expected variables after a

<sup>&</sup>lt;sup>6</sup>Human wealth is the present value of future wage earnings. Using the derivatives of the production function and the transition function for capital, one obtains  $w_s/(1+r_s) = \beta w_{s-1}$  which implies  $w_{t+i}/[\prod_{u=t+1}^{t+i}(1+r_u)] = \beta^i w_t$ . Therefore, human wealth is  $H_t = w_t[1+\beta+\beta^2+\ldots]$ .

single evaluation of A(E), if the Jacobians are not dampened ( $\mu_1 = 1, \ \mu_2 = 1$ ). The next iteration is already a fixed point.

### 4.2 An Aggregate Model of Savings and Investment

**Production:** This section fully incorporates the framework of section 3.2. which gives rise to well defined supply and demand functions for output, capital and labor. Conditional on the capital stock in place, production organizing agents determine labor demand  $L^d(p_t, w_t; K_t)$  and output supply  $f(K_t, L_t^d)$  by solving the short run profit maximization problem stated in (18). Investment demand  $I^d(p_t; K_t; V_t^e)$  is explained in (32) as a function of the commodity price, the capital stock in place and of expected firm value. Since there is only one commodity, the capital goods price coincides with the commodity price. The choice of investment from intertemporal value maximization plus short run profit maximization determines the level of dividends in (21). The expectations of firm values are revised using the arbitrage equation (23) if they are not consistent with an intertemporal equilibrium.

Consumption and labor supply: Households derive consumption demands and labor supply from maximization of life time utility. A no Ponzi game condition is imposed on households to avoid that they permanently roll over their debt and never satisfy interest obligations. As shown in section 3.1, the present value of life time expenditures on consumption and leisure must not exceed full wealth. According to (16), households designate a fraction of full wealth for present consumption and leisure,  $M_t^v = W_t/\Omega_t^e$ . The within period budgets are allocated among consumption and leisure to maximize current utility in (9) giving rise to demand functions  $C_t^d = c(p_t, w_t)M_t^v$  and  $h_t^d = h(p_t, w_t)M_t^v$ . Labor supply is  $L_t^s = 1 - h_t^d$ . Current savings are obtained from (8) by substituting the definition of financial wealth  $F_{t-1} = V_{t-1}$ . Household savings must be zero in equilibrium since investment is financed internally.<sup>7</sup> Dividends are paid out only after subtracting the costs of investment. Expectations of human wealth  $H_t^e$  and of the marginal propensity to consume  $(\Omega_t^e)^{-1}$  are revised according to (17) if they are not selffulfilling.

<sup>&</sup>lt;sup>7</sup>Capital market equilibrium in the previous period gives  $F_{t-1} = V_{t-1}$ . From the arbitrage condition (23) we have  $V_{t-1}(1+r_t)/(1+\bar{g}) = \chi_t + V_t$  where the r.h.s. gives the cum dividend value of owning firms as a result of the period t equilibrium. Writing (8) as  $F_t = V_t + S_t$  with  $S_t = \chi_t + w_t L_t - p_t C_t$  shows that equilibrium in the capital market must hold simultaneously in terms of flows,  $S_t = 0$ , and desired stocks  $F_t = V_t$ .

**Temporary equilibrium:** In any period t, an equilibrium is conditional on previously accumulated stocks of physical capital  $K_t$  and on expectations  $E_t = \{\Omega_t^e, H_t^e, V_t^e\}$ . The market clearing conditions are

(a) 
$$\zeta^{C}(p_{t}, w_{t}; K_{t}; E_{t}) = I_{t}^{d} + C_{t}^{d} - [f(K_{t}, L_{t}^{d}) - \phi(I_{t}^{d}, K_{t})] = 0,$$
  
(b)  $\zeta^{L}(p_{t}, w_{t}; K_{t}; E_{t}) = L_{t}^{d} - L_{t}^{s} = 0,$   
(c)  $\zeta^{K}(p_{t}, w_{t}; K_{t}; E_{t}) = \chi_{t} + w_{t}L_{t}^{s} - p_{t}C_{t}^{d} = 0.$ 
(41)

The temporary excess demand system satisfies Walras's Law,  $p_t \zeta_t^C + w_t \zeta_t^L + \zeta_t^K = 0$ . Finding temporary equilibria, therefore, reduces to computing the zeros of only two excess demand functions by iterating on  $\{p_t, w_t\}$ . Furthermore, by now obvious reasons, the temporary excess demand system is homogeneous in  $\{p_t, w_t; H_t^e, V_t^e\}$ . Excess demands for labor and commodities are homogeneous of degree zero while excess capital demand is homogeneous of degree one. Again, we require  $\{H_t^e, V_t^e\}$  to obtain homogeneity since the excess demand system must be homogeneous in all prices, present and future. The marginal propensity to consume,  $\Omega^{-1}$ , is a function of relative prices only and hence is homogeneous of degree zero. Zero homogeneity of the excess demand system allows convenient price normalizations such as choosing the commodity price to be unity in each period.

Intertemporal equilibrium: In a perfect foresight equilibrium expectations of agents turn out to be correct at any date. Agents need not revise their plans and expectations regarding the future in response to realizations of current equilibrium prices. Starting with vectors of initial guesses for the expected variables  $E_t = \{\Omega_t^e, H_t^e, V_t^e\}$ , one computes a sequence of temporary equilibria and thereby generates revisions for the expected variables,

$$\Omega_{t-1} = 1 + \Omega_t^e \beta^{\gamma} [(p_t^v / p_{t-1}^v)(1+\bar{g})/(1+r_t)]^{1-\gamma},$$

$$H_{t-1} = w_{t-1} + H_t^e (1+\bar{g})/(1+r_t),$$

$$V_{t-1} = (\chi_t + V_t^e)(1+\bar{g})/(1+r_t).$$
(42)

This is like a single evaluation of the vector valued function A(E) defined in (2). Treating revisions and initial guesses as in section 2, one arrives at an improved guess for the next iteration. The perfect foresight solution is obtained as soon as guesses and realizations of the lead variables converge to a fixed point E = A(E). **Price normalization:** Computation of an intertemporal equilibrium changes commodity prices and wage rates relative to a given sequence of interest rates. Economic interpretation of the equilibrium paths is easier and more transparent with a price normalization that sets the commodity price to unity in each period. This is also the normalization chosen in virtually all theoretical work relying on variants of this aggregate equilibrium model. Let a tilde denote transformed prices,  $\tilde{p}_t = 1$  and  $\tilde{w}_t = w_t/p_t$  for example. Because of homogeneity of the excess demand system, temporary equilibria are invariant to price renormalization  $\{\tilde{p}_t, \tilde{w}_t, \tilde{H}_t, \tilde{V}_t\}$ . The updating relations for the expected variables also hold with the transformed variables if they did so at the original prices. For a proof, see first how the interest rates are redefined. The Euler equation (12) shows how agents trade consumption between periods t - 1 and t. Such intertemporal trades must not be affected from price normalization,

$$\left(\frac{1+r_t}{1+\bar{g}}\right)\frac{p_{t-1}^{\nu}}{p_t^{\nu}} = \left(\frac{1+\tilde{r}_t}{1+\bar{g}}\right)\frac{\tilde{p_{t-1}}}{\tilde{p_t}}.$$
(43)

Since  $\tilde{p_t} = p_t^v/p_t$ , the transformed interest rate is defined by  $1 + \tilde{r}_t = (1 + r_t)p_{t-1}/p_t$ . Assume now that the updating relations hold exactly, i.e. we have a perfect foresight equilibrium. Then, by simple expansion,

$$\Omega_{t-1} = 1 + \Omega_t^e \beta^{\gamma} \Big[ \frac{\tilde{p^v}_t}{\tilde{p^v}_{t-1}} \frac{p_t}{p_{t-1}} \Big( \frac{1+\bar{g}}{1+r_t} \Big) \Big]^{1-\gamma}.$$
(44)

Substituting the redefined interest rate reveals that the updating relation also holds at the transformed prices. Note that  $\Omega$  is not redefined since it contains only relative prices whereas the other two expected variables  $\{H^e, V^e\}$  are rescaled.

Computational performance: The numerical experiment is more difficult now. Let the initial capital stock be  $K_1 = .8K_{SS}$ . Let the initial guesses of the expected variables be constant vectors  $\Omega_t = .9\Omega_{SS}$ ,  $H_t = 1.2H_{SS}$  and  $V_t = .8V_{SS}$  for all periods including the terminal period T. Table 2 reports the state of convergence after fifty vector iterations when the actual for the terminal period is computed via  $A_T = A_{T-s}$ . This imposes a restriction that the expected variables stay constant in a steady state. The table demonstrates that the efficacy of the algorithm may be much improved by revising the last period according to  $A_T = A_{T-10}$  instead of  $A_T = A_{T-1}$  or  $A_T = A_{T-5}$ . In a way this exploits the turnpike property of the model: the system approaches the steady state and deviates from it some periods before the terminal period if the terminal values of the expected variables deviate from their steady state values. Obviously,  $A_{T-10}$  reveals more information concerning steady state values than  $A_{T-1}$  does. With the terminal values revised according to  $A_T = A_{T-10}$ , it seems best to assign high values to both parameters  $\mu_1$  and  $\mu_2$ .

### 4.3 A Large Scale Intertemporal Equilibrium Model

This section presents a disaggregate version of the basic intertemporal equilibrium model with different commodities indexed by i = 1, ..., n. Each one is produced in an intertemporally optimizing production sector with capital, labor and intermediates from other sectors as inputs. An input output matrix models interindustry details.

**Production:** Production technology is of a fixed coefficient nature in intermediates and value added,

$$Y^{i} = \min\left\{\frac{f^{i}}{a_{0i}}, \frac{m_{1i}}{a_{1i}}, \frac{m_{2i}}{a_{2i}}, \dots, \frac{m_{ni}}{a_{ni}}\right\}.$$
(45)

Gross output in sector i is  $Y^i$  and the elements  $a_{ji}$  of the input output matrix **A** are fixed coefficients indicating the quantitity of commodity j needed to produce a unit of commodity i. Production of gross output  $Y^i$  at least cost requires exactly  $f^i = a_{0i}Y^i$ value added and  $m_{ji} = a_{ji}Y^i$  intermediates. The short run profit function in sector i is therefore

$$\Pi^{i} = p^{i} Y^{i} - w L^{i} - (\sum_{j} a_{ji} p^{j}) Y^{i}.$$
(46)

Marginal value added achieves an effective price (marginal revenue)

$$\tilde{p}^{i} = [p^{i} - (\sum_{j} a_{ji} p^{j})] / a_{0i}.$$
(47)

Given a fixed coefficient input output technology, the effective price of type i value added is a linear function of all commodity prices. With this definition, the short run profit function reads

$$\Pi^{i}(w, \vec{p}, K^{i}) = \max_{L^{i}} \left\{ \tilde{p}^{i} f(K^{i}, L^{i}) - w L^{i} \right\}.$$
(48)

A linear homogeneous technology  $I(I^{d_1}, \ldots, I^{d_n})$  is available to produce a composite capital good from the *n* commodies. All sectors employ the same technology. Hence, the

capital composite is obtained from

$$I = \max_{\{I^{d_i}\}} \{ I(I^{d_1}, \dots, I^{d_n}) \ s.t. \ \sum_j p^j I^{d_j} \le M^I \}.$$
(49)

The solution gives the capital composite I which is available at a price index  $p^{I}$ . The budget for investment spending is  $M^{I} = p^{I}I = \sum_{j} p^{j}I^{dj}$ , and  $I^{dj}$  are the derived Marshallian demands for commodity j for investment purposes. The capital composite is demanded by all sectors, hence  $I = \sum_{i} I^{i}$ . Each sector pays dividends equal to

$$\chi_{s}^{i} = \Pi^{i}(K_{s}^{i}) - \tilde{p}_{s}^{i}\phi^{i}(I_{s}^{i}, K_{s}^{i}) - p_{s}^{I}I_{s}^{i}.$$
(50)

Optimal sectoral capital accumulation is derived from value maximization discussed in detail in section 3.2. The neoclassical demand for investment subject to adjustment costs is determined from the optimality condition (26a). Use (29) to eliminate the shadow value q and obtain

$$\tilde{p}_{s}^{i}\phi_{I}^{i}(I_{s}^{i},K_{s}^{i}) = (V_{s}^{ie}/(I_{s}^{i}+K_{s}^{i})-p_{s}^{I}).$$
(51)

This implicitely defines sectoral investment demand for the composite capital good,  $I(V_s^{ie}, K_s^i)$ . The investment decision completely determines all supplies and demands of the sectoral production entities conditional on capital stocks and expectations of future equilibria.

Consumption and labor supply: The intertemporal analysis of the consumption sector is completely identical to the preceding example except that current consumption is additionally allocated between different commodities. The within period utility function  $u_s = u(v[C(C^1, \ldots, C^n), h])$  is assumed to be homothetic. Therefore, u(.) is a strictly concave function of linear homogeneous and strictly quasiconcave subutility functions v[.]and c(.). This allows multistage budgeting where each separate static suboptimization problem gives a derived consumption composite and a corresponding price index. On the lowest level,

$$C = \max_{\{C^{j}\}} \{C(C^{1}, \dots, C^{n}) \ s.t. \ \sum_{j} p^{j} C^{j} \le M^{c}\}.$$
 (52)

The solution gives Marshallian demand functions  $C^j = c^j(p^1, \ldots, p^n)M^c$  for individual commodities, a consumption composite C defined by the indirect utility function and a corresponding price index  $p^c$ . Inverting the indirect utility function gives the expenditure function  $M^c = p^c C = \sum_j p^j C^j$ . By household duality, the budget  $M^c$  is the cost minimizing level of expenditures to secure a utility  $C = C(C^1, \ldots, C^n)$ . On the next level, the problem is to trade off leisure against the consumption composite given a budget  $M^v$ ,

$$v = \max_{\{C,h\}} \{v(C,h) \ s.t. \ p^{c}C + wh \le M^{v}\}.$$
(53)

The solution defines a price index  $p^{v}$  and a composite v of commodity consumption and leisure. By household duality, the minimal expenditures to obtain utility v = v[C, h]is  $M^{v} = p^{v}v = p^{c}C + wh$ . The last stage of optimization is explained in section 3.1 and deals with optimal intertemporal consumption choice. Accumulation of financial wealth is now

$$F_{t} = F_{t-1}(\frac{1+r_{t}}{1+\bar{g}}) + w_{t} - \tau_{t} - p_{t}^{v}v_{t} = F_{t-1}(\frac{1+r_{t}}{1+\bar{g}}) + w_{t}L_{t}^{s} - \tau_{t} - \sum_{j} p_{t}^{j}C_{t}^{j}.$$
 (54)

The second equality comes from substituting the definitions of the budgets  $M_t^c$  and  $M_t^v$ . Labor supply is  $L_t^s = 1 - h_t^d$ .

**Government:** We may also specify an exogenously determined demand  $G^j$  for commodifies by a rudimentary government sector. The expenditures are covered by lump sum taxes  $\tau_t = \sum_j p_t^j G_t^j$  in each period.

**Equilibrium:** Temporary equilibrium is conditional on previously accumulated sectoral capital stocks and on expectations  $E_t = \{\Omega_t^e, H_t^e, V_t^{1e}, \ldots, V_t^{ne}\}$ . Defining vectors  $\mathbf{p} = (p^1, \ldots, p^n)'$  for example, the market clearing conditions read now

$$\begin{aligned} \zeta^{C}(w_{t}, \mathbf{p_{t}}; \mathbf{K_{t}}; \mathbf{E_{t}}) &= \mathbf{C_{t}} + \left(\sum_{j} \mathbf{I_{t}^{dj}}\right) + \mathbf{G_{t}} + \mathbf{AY_{t}} - \mathbf{Y_{t}} = \mathbf{0}, \\ \zeta^{L}(w_{t}, \mathbf{p_{t}}; \mathbf{K_{t}}; \mathbf{E_{t}}) &= \left(\sum_{j} L_{t}^{dj}\right) - L_{t}^{s} = \mathbf{0}, \\ \zeta^{K}(w_{t}, \mathbf{p_{t}}; \mathbf{K_{t}}; \mathbf{E_{t}}) &= \left(\sum_{j} \chi_{t}^{j}\right) + w_{t} L_{t}^{s} - \tau_{t} - \mathbf{p_{t}^{\prime}} \mathbf{C_{t}} = \mathbf{0}. \end{aligned}$$
(55)

In a perfect foresight equilibrium, the updating relations (42) must hold. Their number increased to n + 2 since firm values are now distinguished by sectors. As discussed in the previous sections, the temporary excess demand system satisfies the homogeneity property plus Walras' Law. Therefore, one needs to compute temporary equilibria of dimension n + 1 in each period. Homogeneity allows to choose any convenient price normalization which is done the same way as shown in the preceding sections.

Computational performance: In order to evaluate trade and fiscal policies, the applied general equilibrium literature typically performs some comparative dynamic exercises on numerical models. In most cases, the model economy is assumed to be on a steady state growth path which is disturbed by a change in some exogenous government policy parameters. As an example, let the government permanently increase by ten percent its demand for output of sector five which is financed by increases in lump sum taxes to balance the budget in each period. Since the interest of the paper is in efficient numerical solution, we do not bother with the economic interpretation of the exercise for the moment. The solution trajectories of the historical variables (capital stocks) and of some expected variables (firm values) are depicted in figure 3 as deviations from their initial values. The example nicely demonstrates the gradual reallocation of sectoral capital stocks over time as a consequence of shifts in demand.

The numerical computations reported in table 3 start with an initial guess of the expected variables equal to the initial steady state values. The terminal values T = 100 are revised according to  $A_T = A_{T-s}$  with lag s = 5 or s = 10 respectively. According to table 3, it seems best to combine dampening parameters  $\mu_2 = 1$  with  $\mu_1 = .9$  although  $\mu_1 = .6$  and  $\mu_2 = .7$  is also quite efficient. The algorithm obviously does better by setting s = 10 which is intuitively justified by appealing to the turnpike property of neoclassical growth models.

During each evaluation of the vector valued function A(E) the Jacobians  $J_{11}$  and  $J_{12}$  are computed anew. To compute the Jacobian  $J_{11}$ , for example, one must compute two consecutive temporary equilibria for each perturbation of a lead variable which gives  $4 \times n$  equilibria for both Jacobians and n lead variables. Hence, each evaluation of A(E) requires to compute T + 4n = 128 temporary equilibria in our case of n = 7 expected variables. For a large number of expected variables, the computation of the Jacobians in each iteration will become quite expensive. The lower half of table 3 tests convergence of the algorithm when the Jacobians are computed only once in the initial steady state and are not updated during the revision process for expectations. The example suggests that it may have a negligible influence on convergence if the Jacobians are not computed anew in each iteration.

# 5 Conclusions

The paper described an algorithm for solving intertemporal CGE models with infinitely lived agents and perfect foresight. The paper showed how the models must be formulated such that the algorithm is applicable. It emphasized the properties of the temporary excess demand systems that are important for the solution procedure: Walras' Law and homogeneity. The procedure is remarkably stable and allows for quite flexible model formulations. The paper discussed three examples of intertemporal equilibrium models with increasing complexity. The examples given treat the case of a representative household unit planning over an infinite horizon. There is a strong presumption, however, that Blanchard's (1985) model of overlapping generations or Weil's (1989) model of overlapping infinitely lived families may also be solved numerically by the same method since the aggregate dynamics in the predetermined and non-predetermined variables is essentially the same in these models. A numerical application requires a discrete time setup as in Frenkel and Razin (1987).

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# Appendix: The Generalized Fair Taylor Algorithm

The following description of the solution method relies heavily on Wilcoxen (1989). If we have N expectational variables and solve for T time periods, we stack all the expectations in one long vector and get  $E_t = (e_{1t}, e_{2t}, \ldots, e_{Nt})'$  and  $E = (E'_1, E'_2, \ldots, E'_T)'$ . The vector of actualized values A with elements  $a_{nt}$  is defined analogously. For the guessed and derived values  $e_{nt}$  and  $a_{nt}$ , the first index denotes the n-th expectational variable and the second index time period t. Hence, E is of dimension  $(NT \times 1)$ . Assume that the solution for the expectational variables is known to be  $E^* = A(E^*)$ . Starting with a guess E, one may hope to approximate the solution  $E^*$  by a first order Taylor expansion around the guess E,

$$E^* \approx A(E) + \nabla A(E)(E^* - E). \tag{A.1}$$

Denote the Jacobian by  $J \equiv \bigtriangledown A(E)$  for short hand. Starting with a vector of guesses E, one could simply compute the approximate solution by taking (A.1) as an equality,

$$(I-J)E^* = A - JE. \tag{A.2}$$

In principle, the approximate solution  $E^*$  may be computed on the basis of guessed values E, actual values A(E) and the Jacobian evaluated at E. The approximate solution  $E^*$  would lend itself as a new guess. Equation (A.2) may then be iterated until E and  $E^*$  are sufficiently close to each other. However, the dimensions are usually too large for numerical computation. The Jacobian is of dimension  $(NT \times NT)$ , its elements are

$$J = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{1T} \\ J_{21} & J_{22} & \cdots & J_{2T} \\ \vdots \\ J_{T1} & J_{T2} & \cdots & J_{TT} \end{bmatrix}, \qquad J_{ts} = \begin{bmatrix} \frac{\Delta a_{1t}}{\Delta e_{1s}} & \frac{\Delta a_{1t}}{\Delta e_{2s}} & \cdots & \frac{\Delta a_{1t}}{\Delta e_{Ns}} \\ \vdots \\ \frac{\Delta a_{Nt}}{\Delta e_{1s}} & \frac{\Delta a_{Nt}}{\Delta e_{2s}} & \cdots & \frac{\Delta a_{Nt}}{\Delta e_{Ns}} \end{bmatrix}.$$

A block  $J_{ts}$  of the Jacobian is of dimension  $(N \times N)$  and contains the information how a variation of the guesses in period s affects the derived actuals in period t.

Fortunately, Wilcoxen (1989) introduced some simplifying assumptions that enormously reduce the computational requirements but still retain enough of the vital information on the intertemporal structure of A(E) as expressed in the elements of the

 $= \begin{pmatrix} * * 0 - 0 \\ 0 \\ 1 \\ 0 - 0 \\ * \end{pmatrix}$ 

Jacobian. The assumptions are:

(

(i) 
$$J_{ts} = 0, \quad s > t + 1,$$
  
(ii)  $J_{ts} = 0, \quad s < t,$   
(iii)  $J_{tt} = J_{11}, \quad t > 1,$   
(A.3)  
iv)  $J_{t,t+1} = J_{12}, \quad t > 1.$ 

Inspection of (2) motivates assumption (i) which implies that a variation of guesses beyond period t + 1 dos not affect the actuals in period t. Assumption (i) makes the Jacobian almost lower block triangular. According to assumption (ii), a variation of guesses in previous periods s < t does not affect the derived actuals in period t. This assumption is an approximation only since accumulation via (2a) generally carries over some effects to period t. The benefit from assuming (ii) is to make the Jacobian upper block triangular. Finally, (iii) and (iv) assume the effects of a variation of guesses in periods t and t + 1 on actuals in period t to be the same in all periods and equal to  $J_{11}$ and  $J_{12}$ . Hence, only two blocks of the Jacobian need to be computed.

The Jacobians  $J_{11}$  and  $J_{12}$  contain the information on the intertemporal structure of the problem that the algorithm actually exploits in computing revised guesses. For example, the elements of  $J_{12}$  give the influence of a variation of guesses in period 2 on the actuals in period one. Exploiting the assumptions given in (A.3) and writing in expanded notation, (A.2) emerges as

$$\begin{bmatrix} I - J_{11} & -J_{12} & 0 \\ I - J_{11} & \ddots & \\ & \ddots & -J_{12} \\ 0 & & I - J_{11} \end{bmatrix} \begin{bmatrix} E_1^* \\ E_2^* \\ \\ E_T^* \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \\ A_T \end{bmatrix} - \begin{bmatrix} J_{11} & J_{12} & 0 \\ & J_{11} & \ddots \\ & & \ddots & J_{12} \\ 0 & & & J_{11} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ \\ \\ E_T \end{bmatrix}$$

With the below diagonal terms all zero, this system is easily solved by backward substitution. From a computational viewpoint, it is useful to scale the Jacobians by dampening factors  $\mu_1$ ,  $\mu_2$  which make the revision process more gradual and help to keep it monotonic. The solution for  $E^*$  is

(a) 
$$E_T^* = (I - \mu_1 J_{11})^{-1} [A_T - \mu_1 J_{11} E_T],$$
  
(b)  $E_s^* = (I - \mu_1 J_{11})^{-1} [A_s - \mu_1 J_{11} E_s - \mu_2 J_{12} E_{s+1} + \mu_2 J_{12} E_{s+1}^*].$ 
(A.4)

The evaluation of (2) does not produce an update  $E_T$  for the last period of the transition phase. In many economic applications, however, one need not compute (A.4a) as  $E_T$  is fixed and known in advance from separate computation of steady state values. If this is not possible, a revision of the last expectation may come from imposing the steady state restriction  $A_T = A_{T-1}$ . If this is not possible either, one may just fix it at a reasonable value which is legitimated by impealing to the insignificant future condition and the Turnpike property of growth models. Values of expectational variables far in the future will not influence the solution early in the transition phase. The system converges to the balanced growth path and diverges from it only shortly before the terminal period if the terminal values of the expectational variables do not coincide with their steady state values.

With all the approximations involved, the solution in (A.4) will not yet produce the desired self fulfilling expectations that constitute a fixed point E = A(E), but may be taken as the guessed expectations for the next iteration. For the same reason we take a linear combination of the solution in (A.4) and the initial guess in iteration *i* to arrive at the revised guess

$$E^{i+1} = \mu_0 E^{*i} + (1 - \mu_0) E^i.$$
(A.5)

With  $\mu_1 = \mu_2 = 0$ , the solution in (A.4) is  $E^{*i} = A^i = A(E^i)$  and (A.5) specializes to (3) which is the original Fair – Taylor method.

### GENERALIZED FAIR TAYLOR ALGORITHM: FLOW DIAGRAM



	$\mu_1 = 0$	$\mu_1 = .5$	$\mu_1 = .7$	$\mu_1 = .8$	$\mu_1 = .9$	$\mu_1 = 1$	
$\mu_0 = 1$	> 50	> 50	> 50	> 50	> 50	> 50	$\mu_2 = 0$
$\mu_0 = 1$	> 50	> 50	> 50	> 50	> 50	6	$\mu_2 = .5$
$\mu_0 = 1$	> 50	49	46	45	43	5	$\mu_2 = .7$
$\mu_0 = 1$	43	38	35	34	32	5	$\mu_2 = .8$
$\mu_0 = 1$	31	26	24	23	21	4	$\mu_2 = .9$
$\mu_0 = 1$	18	12	10	8	6	2	$\mu_2 = 1$

Table 1: Generalized Fair Taylor Algorithm.Basic Infinite Horizon Model

Parameters are: A = 10,  $\alpha = .25$ ,  $\beta = 1.06^{-1}$ . Convergence is achieved as soon as  $\left[\sum_{s=1}^{T} (A_s - E_s)^2\right]/T < 1E - 8$ .

	$\mu_1 = .6$	$\mu_1 = .7$	$\mu_1 = .8$	$\mu_1 = .9$	$\mu_1 = 1.$	
s = 1	2.471	2.481	2.464	2.467	2.509	$\mu_2 = .6$
s = 1	2.466	2.396	2.328	2.294	2.295	$\mu_2 = .7$
s = 1	2.191	2.037	1.942	1.920	1.947	$\mu_2 = .8$
s = 1	1.576	1.487	1.531	1.623	1.689	$\mu_2 = .9$
s = 1	1.034	1.269	1.463	1.568	—	$\mu_2 = 1.$
	$\mu_1 = .5$	$\mu_1 = .6$	$\mu_1 = .7$	$\mu_1 = .8$	$\mu_1 = .9$	
s = 5	1.355	1.430	1.489	1.521	1.620	$\mu_2 = .6$
s = 5	1.431	1.371	1.185	.978	.938	$\mu_2 = .7$
s = 5	1.148	.730	.281	49	48	$\mu_2 = .8$
s = 5	.065	46	41	37	36	$\mu_2 = .9$
s = 5	40	35	31	25	26	$\mu_2 = 1.$
<i>s</i> = 10	.883	.985	1.071	1.156	1.357	$\mu_2 = .6$
s = 10	.910	.800	.584	.409	.492	$\mu_2 = .7$
s = 10	.454	50	47	44	43	$\mu_2 = .8$
s = 10	46	41	36	32	30	$\mu_2 = .9$
<i>s</i> = 10	37	32	27	22	18	$\mu_2 = 1.$

Table 2: Generalized Fair Taylor Algorithm. Aggregate Model of Savings and Investment

Parameters: Growth rates are n = .01, x = .02, the subjective discount rate is  $\rho = .06$ . Utility u(v) has constant intertemporal elasticity of substitution  $\gamma = .6$ , v(c, h) is C.D. with commodity share .8. Technology is given by a production function  $f(K, L) = K^{\alpha}L^{1-\alpha}$  with capital share  $\alpha = .3$  and an installation function  $\phi(I, K) = \frac{\psi}{2}\frac{I^2}{K}$  with  $\psi = 1$ . The table reports convergence after fifty vector iterations. In case that the algorithm converges earlier we report the number of vector iterations. Convergence is achieved as soon as  $[\sum_{s=1}^{T} (A_s - E_s)^2]/T < 1E - 6$  which would be indicated by a negative number.

	$\mu_1 = .6$	$\mu_1 = .7$	$\mu_1 = .8$	$\mu_1 = .9$	
s = 5	21	26	(.117)	(.600)	$\mu_2 = .7$
s = 5	(.287)	22	24	30	$\mu_2 = .8$
s = 5	(.790)	(.123)	24	23	$\mu_2 = .9$
s = 5	28	23	19	19	$\mu_2 = 1.$
s = 10	14	22	29	(.442)	$\mu_2 = .7$
s = 10	20	14	20	26	$\mu_2 = .8$
s = 10	(.238)	26	14	18	$\mu_2 = .9$
<i>s</i> = 10	25	21	16	12	$\mu_2 = 1.$
s = 5	22	27	(.210)	(.736)	$\mu_2 = .7$
s = 5	(.310)	23	25	(.070)	$\mu_2 = .8$
s = 5	(.817)	(.128)	24	24	$\mu_2 = .9$
s = 5	27	23	19	19	$\mu_2 = 1.$
s = 10	15	23	30	(.578)	$\mu_2 = .7$
s = 10	20	15	21	27	$\mu_2 = .8$
s = 10	(.259)	27	15	19	$\mu_2 = .9$
<i>s</i> = 10	25	21	16	12	$\mu_2 = 1.$

Table 3: Generalized Fair Taylor Algorithm. Disaggregate Model

The table reports the number of vector iterations for convergence which requires  $[\sum_{s=1}^{T} (A_s - E_s)^2]/T < 1E - 6$ . The bracketed entries indicate the closeness of vectors after 30 iterations in cases of slow convergence. The lower half of the table reports convergence in the case where the Jacobians are computed only once in the initial steady state.

Fig.1: Turnpike Property



Fig.2: Transition Paths



Fig.3: Transition Paths



<u>.</u>..

#### WORKSHOP ON

### ISSUES IN INTERNATIONAL ECONOMICS: QUESTIONS TO AND ANSWERS FROM COMPUTABLE GENERAL EQUILIBRIUM ANALYSIS

### JULY 8-9, 1991 UNIVERSITY OF KONSTANZ

#### MONDAY, JULY 8, 1991

10.00 Opening Address

Wolfgang Franz (Dean, University of Konstanz) Bernd Genser (University of Konstanz)

"General Equilibrium Economics: Computation and Application" John Whalley (University of Western Ontario)

#### 11.00 Session 1: International Taxation

"Value-Added Tax Harmonization in the EC" Wolfgang Wiegard/Hans Fehr/Christoph Rosenberg (University of Regensburg)

"Commodity Tax Reform under the Restricted Origin Principle" Andreas Haufler (University of Konstanz)

#### 14.30 Session 2: Factor Supply

"Simulation Models of Labour Supply" Alan Duncan (Institute for Fiscal Studies, London)

"An Intertemporal General Equilibrium Model for Austria with Application to Foreign Trade"

Christian Keuschnigg (University of Bonn)/Wilhelm Kohler (University of Innsbruck)

#### 17.00 Session 3: Environment

"CGE Modelling of International Public Goods: Air Quality and Global Warming" John Piggott (University of New South Wales)

#### TUESDAY, JULY 9, 1991

#### 09.00 Session 4: Imperfect Competition and Growth

"AGE Models in International Economics with Special Reference to Imperfect Competition and Increasing Returns to Scale" Albert Schweinberger (University of Konstanz)

Growth and International Capital Movement: A Simulation Study Karl-Josef Koch (University of Konstanz) (presentation cancelled due to accident)

#### 10.45 Round Table: "The Payoff of Investment in CGE - Modelling"

Chair: Bernd Genser

Introductory Statements: John Chipman (University of Minnesota) Friedrich Schneider (University of Linz)

Closing Address: Bernd Genser