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Working Paper

The capacity constrained mining firm under various tax regimes

Diskussionsbeiträge - Serie II, No. 49

Provided in Cooperation with:

Department of Economics, University of Konstanz

Suggested Citation: Meyer, Anke; Weihs, Claus (1988) : The capacity constrained mining firm under various tax regimes, Diskussionsbeiträge - Serie II, No. 49, Universität Konstanz, Sonderforschungsbereich 178 - Internationalisierung der Wirtschaft, Konstanz

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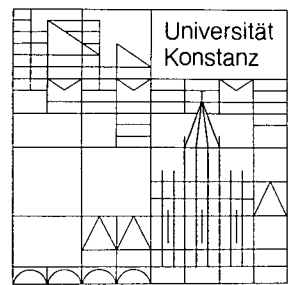
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Sonderforschungsbereich 178
„Internationalisierung der Wirtschaft“

Diskussionsbeiträge



Juristische
Fakultät

Fakultät für Wirtschafts-
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**The Capacity Constrained Mining Firm
under Various Tax Regimes**

THE CAPACITY CONSTRAINED MINING FIRM
UNDER VARIOUS TAX REGIMES

Anke Meyer, Claus Weihs

Serie II - Nr. 49

Ag 2763/88 Weltwirtschaft
Kl. 1

April 1988

The Capacity Constrained Mining Firm
under Various Tax Regimes*

Anke Meyer* and Claus Weihs[§]

Draft, February 1988

Abstract

This paper presents a model of resource extraction based on two of its key aspects: investment and taxation.

A mining firm is considered which may be in an environment of perfect competition or have a monopoly. It plans to extract from a fixed stock of resources. Before extraction begins, the firm must invest in capital goods, the amount of which depends on the desired maximum quantity of extraction. The net present value of the resource extracted is then maximized, i.e. profits after taxes minus capital costs after taxes and depreciation.

The necessary conditions for an optimal time path of extraction are derived from results well-known in control theory. Such a path starts with a phase of constant extraction at the capacity limit, followed by a phase in which the amounts of resource extracted decrease. The reaction of such a path to changes in tax and depreciation rates is analyzed. Only an income tax with immediate write-off of capital costs keeps the extraction path the same for all tax rates. If no depreciation is allowed or straight-line depreciation applies, different tax rates result in different capacity limits and lengths of extraction periods.

Continuous extraction paths satisfying the necessary conditions are optimal not only if the capital stock function is convex, but also if the function is concave and steep in relation to the profit function. In both cases, the company will react to a higher tax rate or lower present value of depreciation allowances by choosing a smaller capacity and a longer extraction phase.

Finally, the existence of such optimal extraction paths is discussed with an example.

* This paper is based on a chapter of the doctoral dissertation of A. Meyer (1988), providing the results in a concise form and including generalizations made by C. Weihs. We wish to thank Michael Rauscher (University of Konstanz) for his helpful comments.

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0. Introduction

The days when one could make a fortune simply by spreading around a little baksheesh and then drilling a hole in an Arabian desert or forcing Indians into Latin American mines are long gone. Nowadays most resource projects are characterized by high capital intensity -- most importantly, high set-up costs -- and compliance with stiff regulations, i.e. taxation and governmental controls.

Despite the upswing in resource economics after the first oil crisis, economic theory was slow to address these aspects of resource extraction and not very thorough. True, the number of taxation models has grown,¹ and those dealing with investment can also be found,² but models combining the two are lacking.

The model developed in this paper starts from Campbell's (1980) investment model and incorporates different forms of income taxation.³ According to Campbell (1980), a resource-extracting firm's maximization of the net present value of profits is restricted through the fixed resource stock and capacity constraints limiting the maximum quantity extractable in each period. Making certain assumptions with respect to the profit function, Campbell shows that in the first phase it is optimal to extract exactly the maximum quantity, and that in the second phase, the extraction profile follows the well-known Hotelling (1931) result that quantities decline over time. Campbell does not assume any limit to exist for the amount of investment per period. If such a limit is assumed, the first phase is no longer described by a constant amount of resource extracted but by a rising amount (cf. Puu 1977).

1 For an overview cf. e.g. Dasgupta/Heal (1979) and Lewis/Slade (1985).

2 Cf. Puu (1977), Campbell (1980), Crabbé (1982), Siebert (1982), Lasserre (1985) and Hartwick/Kemp/Long (1986).

3 Meyer (1988) also analyzes royalties and other contract forms like production sharing and service contracts.

In this paper, Campbell's model is extended by analyzing the influence of various income tax rates and depreciation systems upon the extraction profile, capacity limit and length of extraction period. The results can be applied to resource firms in a context of perfect competition as well as with a monopoly.

1. The model

Consider a firm with a homogeneous resource deposit, R_0 , at its disposal. All the necessary investment is assumed to have been made before the extraction is begun. This is possible only if there is practically no upper limit to investment per period and if capital goods are durable so that replacement investment is unnecessary. Moreover, investment is assumed to be unusable for other projects. In particular, the scrap value of capital goods is assumed to be zero. The volume of the capital stock, K_0 , to be built up before extraction can start is assumed to be determined by the projected maximum quantity, q_m , of extraction per period:

$$(1) \quad K_0 = f(q_m), \quad f' > 0, \quad f(0) = 0.$$

For the moment, no further restrictions apply to the shape of the capital stock function. A convex function $f(q_m)$, i.e. $f'' \geq 0$, assumes that progressively increasing amounts of capital goods have to be provided to increase maximum capacity. In contrast, a function $f(q_m)$, which is concave in at least a certain region, i.e. $f'' \leq 0$, assumes economies of scale exist in the build-up of the capital stock.⁴

The optimization problem of the firm thus consists of choosing capacity limit q_m , stopping point T , the resource R_T left in situ after extraction, and the resource quantities $q_t \leq q_m$ to be

⁴ Westphal (1971, 37) mentions decreasing costs in capacity construction in plants manufacturing petroleum, chemicals, steel, cement and aluminium.

extracted during period $0 \leq t \leq T$ such that the net present value of resource extraction, i.e.

$$(2) \quad N := (1-\tau) \int_0^T e^{-rt} \pi(q_t) dt - (1-\tau D(T)) p_K f(q_m)$$

is maximized under the following restrictions:

$$(3) \quad \begin{aligned} \dot{R}_t &:= dR/dt = -q_t, \\ 0 &\leq q_t \leq q_m, \\ R_0 &= \bar{R}_0 \quad (\geq 0), \\ R_T &\geq 0, \end{aligned}$$

where $\pi(q_t)$ with $\pi' > 0$, $\pi'' \leq 0$ and $\pi(0) = 0$ is the operating profit,
 p_K is the exogenous price of capital good K_0 ,
 $r > 0$ is the exogenous discount rate,
 R_t is the resource in situ in period t ,
 $0 \leq \tau < 1$ is the exogenous income tax rate applicable to the project,
 $0 \leq D \leq 1$ is a factor representing the present value of depreciation allowances caused by \$1 of investment; D may depend upon T (cf. (3d), (3e)).

Note that the first restriction in (3) simply reflects the fact that in every period the resource in situ R_t is diminished per period by the amount q_t of resource extracted.

The optimization problem (2), (3) can cover various tax systems, in particular:

- (3a) $\tau=0, D=0$: no tax, no depreciation,
 (3b) $0 < \tau < 1, D=0$: income tax, no depreciation,
 (3c) $0 < \tau < 1, D=1$: income tax, immediate write-off,
 (3d) $0 < \tau < 1, D = \frac{\Omega}{r+\Omega} (1 - e^{-(r+\Omega)T})$: income tax, declining-balance depreciation;

since

$$\int_0^T e^{-rt} [\pi(q_t)(1-\tau) + \tau\Omega e^{-\Omega t} \cdot p_k f(q_m)] dt - p_k f(q_m) =$$

$$= (1-\tau) \int_0^T e^{-rt} \pi(q_t) dt + \left(1 - \frac{\tau\Omega}{r+\Omega} (1-e^{-(r+\Omega)T})\right) p_k f(q_m) ,$$

where Ω is an exogenous depreciation parameter,

(3e) $0 < \tau < 1$, $D = (1 - e^{-rT})/rT$: income tax, straight-line depreciation;

since

$$\int_0^T e^{-rt} [\pi(q_t)(1-\tau) + \tau p_k f(q_m)/T] dt - p_k f(q_m) =$$

$$= (1-\tau) \int_0^T e^{-rt} \pi(q_t) dt + \left(1 - \frac{\tau}{rT} (1 - e^{-rT})\right) p_k f(q_m) .$$

The pattern of competition is not pre-fixed by the formulation of the problem, since the form of the operating profit function is left open. In what follows, we use

$$(4) \quad \pi(q_t) = p_t q_t - c(q_t) \cdot q_t ,$$

where $c(q_t)$ with $c' \geq 0$, $c'' \geq 0$ and $c(0) = 0$ represents variable unit costs, and p_t is the market price of the resource.

Since p_t can be assumed to be exogenous or dependent on q_t ($p_t = p_t(q_t)$), perfect competition as well as monopoly are covered by the model. In the most general case, (4) leads to

$$(5) \quad \pi'(q_t) = (p'(q_t) - c'(q_t)) \cdot q_t + p_t - c(q_t)$$

$$\pi''(q_t) = (p''(q_t) - c''(q_t)) \cdot q_t + 2(p'(q_t) - c'(q_t)) .$$

Special cases considered here will be:

$$(4a) \quad p' = 0 \quad (\text{perfect competition}) \quad \text{with} \quad c' > 0, \quad c'' > 0;$$

(4b) $c=0$, no variable costs (, but monopoly).

Note that in a context of perfect competition the profit function will be strictly concave, if $c' > 0$ and $c'' > 0$. In general, π is assumed to be only concave. Finally, profits are assumed to be independent of the resource in situ, R_t , and capital stock, K_0 . The latter implies that variable and capital inputs cannot be substituted (cf. Campbell (1980, 351), Lasserre (1985, 181p)).

Let us now start discussing the solution of problem (2), (3) by giving necessary conditions of Takayama (1985, 656-660). The "non-discounted" Lagrange function of the problem can be written as

$$(6) \quad L(q_m, q_t) = (1-\tau)\pi(q_t) - \Gamma_t q_t + \mu_{1t} q_t + \mu_t (q_m - q_t) \quad ,$$

where Γ_t is the shadow price of the resource in situ and μ_{1t} and μ_t are Lagrange multipliers corresponding to the second restriction in (3). R_T , the amount of the resource in situ at time T , the final time T itself, and q_m , the capacity limit, are to be determined by optimization, but are assumed to be constant. They are called "control parameters".

Let us now assume that

$$(7) \quad q_t > 0 \quad \text{for } t \in [0, T), \text{ which implies that } \mu_1 = 0 \quad \text{and} \quad q_m > 0 \quad .$$

This restricts the model to the effective extraction period (cf. (17)).

If we let $(q_t, 0 \leq t \leq T, R_T, T, q_m)$ be a solution to (2), (3) and (7), Γ_t, μ_t and θ exist, do not simultaneously vanish and satisfy the conditions:

$$(8)(a) \quad (1-\tau)\pi'(q_t) - \Gamma_t - \mu_t = 0 \quad ,$$

$$(b) \quad \dot{\Gamma}_t = r\Gamma_t \quad ,$$

$$(c) \quad \mu_t (q_m - q_t) = 0 \quad , \quad \mu_t \geq 0 \quad ,$$

$$(d) \quad \theta R_T = 0 \quad , \quad \theta \geq 0 \quad \text{constant} \quad ,$$

$$(e) \quad -\theta + e^{-rT} \Gamma_T = 0 \quad ,$$

$$(f) \quad -\frac{\partial D}{\partial T} \cdot \tau p_K f(q_M) - e^{-rT} L(q_M, q_T) = 0 \quad ,$$

$$(g) \quad (1-D\tau) p_K f'(q_M) = \int_0^T \mu_t e^{-rt} dt \quad .$$

Conditions (e), (f) and (g) are the transversality conditions corresponding to the control parameters R_T , T and q_M , respectively. Conditions (a), (f) and (g) depend on the tax system chosen.

In what follows, (8) is used to develop statements about the paths of prices and extraction in the optimum, and about the reaction of such paths to changes in tax rates and depreciation systems. Let us start with some assumptions:

(9) Let $\Gamma_0 > 0$, i.e. let the resource in situ be scarce, and let

(10) q be continuous and
 $q = g(p)$, $g' < 0$, $g'' \geq 0$ (demand function).

$\Gamma_0 = 0$ would imply that $\Gamma_t = 0$, $t > 0$, using (8b), and thus the resource would be available for free all the time. This is implausible for non-renewable resources. Moreover, a continuous extraction path may be justified by technical restrictions, which do not allow subunits to be cut off, e.g. when using a single giant wheel excavator in open pit mining. Note that the form of the demand function g is not assumed to be known, but only that its derivatives have the indicated sign. This specification allows both perfect competition and monopoly. With perfect competition, according to Siebert (1982), q is assumed to be dependent via g on exogenous p , and with monopoly, $p = p(q) = g^{-1}(q)$ is the inverse demand function.

Let us now consider two cases:

(11)(i) $q_t = q_M$ for $t \in [t_0, t_1]$, and

(ii) $q_t < q_m$ for $t \in (t_0, t_1)$.

In case (11)(i) one obtains

$$0 = \dot{q}_t = g'(p_t) \cdot \dot{p}_t \text{ and thus } \dot{p}_t = 0 \text{ for } t \in (t_0, t_1).$$

In case (11)(ii) one obtains instead for all $t \in (t_0, t_1)$

$$\mu_t = 0 \text{ because of (8)(c),}$$

$$(12) \quad \dot{\pi}'(q_t) := \dot{\pi}'(q_t) / \pi'(q_t) = \dot{\Gamma}_t = r, \text{ because of (8)(a) and (b)}$$

(Hotelling rule),

$$(13) \quad \dot{q}_t = \dot{\pi}'(q_t) / (\pi''(q_t) + 1/g'(p(q_t))) < 0 \quad , \text{ because of}$$

(5) and (10), for profits (4a),

$$(14) \quad \dot{q}_t = \dot{\pi}'(q_t) / \pi''(q_t) < 0 \quad , \text{ for profits (4b),}$$

$$(15) \quad 0 < \dot{p} = r + [\dot{q}_t(2c' + q_t c'') - r(c + c'q_t)] / p_t < r \quad ,$$

for profits (4a),

$$(16) \quad 0 < \dot{p} = r - \dot{H}(q) \quad \text{for profits (4b), where}$$

$$H(q) := 1 + 1/\Phi(q) \quad , \text{ and}$$

$$\Phi(q) := \frac{p(q)}{q} \cdot \left(\frac{\partial p}{\partial q} \right)^{-1}$$

is the price elasticity of demand.

Thus, if the resource is extracted at the capacity limit, q_m , optimality excludes price changes in the corresponding time period. Moreover, in the case of perfect competition (4a), the price has to rise at a rate lower than the discount rate to achieve optimal extraction $q_t < q_m$. Optimality thus restricts the possible paths of the exogenous market price in the case of perfect competition. With monopoly case (4b), the path of the endogenous price depends on the price elasticity of demand. Only

if this elasticity increases with demand, does the price rise at a rate lower than the discount rate. In this case the monopolist will extract more slowly than a firm under perfect competition (with no variable costs) (cf. also Dasgupta/Heal (1979, 327)).

Let us now concentrate on the discussion of the paths of the control, q_t , and the values of the control parameters R_T , T and q_m , which are possible for optimal solutions to the control problem (2), (3), (4), (7), (9), (10). First, the paths q_t will be discussed. From (8c), the following paths are possible:

- (17)(a) $\mu_t \geq 0$ and $q_t = q_m$ for $t \in [0, T]$
 (b) $\mu_t = 0$ and $q_t \leq q_m$ for $t \in [0, T]$
 (c) $\mu_t = 0$ and $q_t < q_m$ for $t \in [0, t_1]$
 $\mu_t \geq 0$ and $q_t = q_m$ for $t \in [t_1, T]$
 (d) $\mu_t \geq 0$ and $q_t = q_m$ for $t \in [0, t_1]$
 $\mu_t = 0$ and $q_t < q_m$ for $t \in (t_1, T]$.

Note that once $q_t < q_m$, q_t will fall (cf. (13), (14)) and thus q_t will later never be equal q_m . This excludes $q_t = q_m$ in unconnected intervals, but it also excludes (17c). (17a) can be taken as a special case of (17d) ($t_1 = T$). (17b) is impossible since $\mu_t = 0$, $0 \leq t \leq T$, contradicts (8g), because $f' > 0$ (cf. (1)). Thus, the only possible scenario is (17d). Note that (17d) is true regardless of which tax rate τ and depreciation factor D are valid. q_t decreases at the same rate at equal levels of q , again for each τ , D (cf. (13), (14)). Note that time t , where a certain level of q is reached, may be different for different τ , D . This may result from different time points t_1 at which production drops below capacity limit q_m . Thus, period t_1 is an extra parameter to be determined in addition to the control parameters R_T , T , q_m . The transversality conditions (8e, f, g) will be used for this purpose.

First, (8e) and (8d) imply that

$$(18) \quad R_T = 0, \quad \text{since } \Gamma_0 > 0 \text{ (cf. (9)) and } \dot{\Gamma}_t = r\Gamma_t \text{ (cf. (8b)) and} \\ \text{thus } \Gamma_T > 0.$$

This means that the deposit must be exhausted at final time T .

Utilizing an idea of Campbell (1980), optimal values for the parameters t_1 , T , q_m and q_T are determined by the following four identities:

$$(19) \quad \pi'(q_m) - \pi'(q_T) e^{-r(T-t_1)} = 0 \quad ,$$

$$(20) \quad t_1 q_m + \int_{t_1}^T q_t dt = R_0 \quad ;$$

$$(21) \quad p_k f'(q_m) \cdot \frac{1-D\tau}{1-\tau} = \pi'(q_m) \cdot \frac{(1-e^{-rt_1})}{r} - t_1 \pi'(q_T) \cdot e^{-rT} \quad ,$$

$$(22) \quad e^{-rT} \cdot ((1-\tau)\pi(q_T) - \Gamma_T q_T) = - \frac{\partial D}{\partial T} \cdot \tau p_k f(q_m) \quad ,$$

Obviously, (19) follows from (12), (20) is the summation condition implied by (3) and (18), and (21) and (22) follow from (8g) and (8f).

Let us first discuss (22). From (8a) one obtains

$$(23) \quad e^{-rT} \cdot (1-\tau)(\pi(q_T) - \pi'(q_T) \cdot q_T) = - \frac{\partial D}{\partial T} \cdot \tau p_k f(q_m) \quad , \text{ and}$$

$$(24) \quad \pi(q_T) - \pi'(q_T) \cdot q_T = \begin{cases} c'(q_T) \cdot q_T^2 \geq 0 & \text{in case (4a)} \\ -p'(q_T) \cdot q_T^2 \geq 0 & \text{in case (4b)} \end{cases}$$

For the depreciation allowances (3a)-(3e) one obtains:

$$(25) \quad \frac{\partial D}{\partial T} = \begin{cases} 0 & \text{for the cases (3a), (3b), (3c),} \\ \Omega e^{-(r+\Omega)T} > 0 & \text{for case (3d) (declining-} \\ & \text{balance depreciation)} \\ \frac{e^{-rT} \cdot (1-rT) - 1}{rT^2} < 0 & \text{for case (3e) (straight-line} \\ & \text{depreciation)} \end{cases}$$

On account of (24) and (25), equation (23) will never be fulfilled for declining-balance depreciation (3d). Thus, problem (2), (3), (4), (7), (9), (10) has no solution in this case. Indeed, with this method there are positive amounts of depreciation for an indefinite time. The firm would therefore have an interest in artificially extending the life of the mining project.

On the other hand, straight-line depreciation (3e) is not applicable in practice, since final time T is not known by the government at the beginning of extraction. Indeed, T in (3e) is in practice replaced by a policy parameter T_0 . Usually, depreciation for tax purposes does not coincide with economic depreciation. Instead, T_0 is chosen to be less than T to provide investment incentives. Therefore, D does not depend upon the actual length of the extraction period even for straight-line depreciation. Since for declining-balance depreciation no solution to our problem exists, we concentrate upon the case where

$$(26) \quad \partial D / \partial T = 0 \quad .$$

Here, (23) and (24) would imply

$$(27) \quad q_T = 0 \quad .$$

In other words, nothing will be extracted in the final period, and the special case (17a) of scenario (17d) is excluded since $q_m > 0$ (cf. (7)).⁵

Now that q_T is fixed, let us use the implicit equation system (19), (20) and (21) to characterize optimal values for t_1 , T and q_m . In order to let the Hotelling rule (12) define a one-to-one relation between $\pi'(q_t)$ and q_t , it must be assumed for this analysis that

$$(28) \quad \pi''(q_t) < 0 \quad , \text{ i.e. } \pi \text{ is strictly concave.}$$

⁵ Crabbé (1982) discusses the case $c'(q)=0$ under perfect competition, where scenario (17a) is possible (cf. (23), (24)).

After some transformations, one obtains

$$(29) \quad T - t_1 - \ln[\pi'(0)/\pi'(q_m)] / r = 0 \quad ,$$

$$(30) \quad t_1 q_m + \int_{t_1}^T \pi'^{-1}[\pi'(0) \cdot e^{-r(\tau-t)}] dt - \bar{R}_0 = 0 \quad ,$$

$$(31) \quad p_k f'(q_m) \cdot \frac{1-D\tau}{1-\tau} - \pi'(q_m) \cdot \frac{1-e^{-r t_1}}{r} + t_1 \pi'(0) \cdot e^{-r\tau} = 0 \quad ,$$

Since parameters r , R_0 and p_k are assumed to be uncontrollable (cf. (3)), we have a system of 3 nonlinear equations in 5 unknowns, t_1 , T , q_m , τ and D . In what follows, we will discuss, whether this system can be locally solved for t_1 , T and q_m

- as a function of τ , given D , and
- as a function of D , given τ .

The local reaction of t_1 , T and q_m upon variations of τ and D , will also be analyzed.

Let us first discuss variations of τ (for given D). If the Jacobian matrix corresponding to t_1 , T , q_m of system (29)-(31) is nonsingular at a point where the equations are satisfied, the implicit function theorem implies that

$$(32) \quad - \frac{dt_1}{d\tau} + \frac{dT}{d\tau} + \frac{1}{r} \frac{\pi''(q_m)}{\pi'(q_m)} \frac{dq_m}{d\tau} = 0$$

$$(33) \quad q_m \cdot \frac{dT}{d\tau} + t_1 \cdot \frac{dq_m}{d\tau} = 0$$

$$(34) \quad -rt_1 \pi'(0) \cdot e^{-r\tau} \cdot \frac{dT}{d\tau} + \left[p_k f''(q_m) \cdot \frac{1-D\tau}{1-\tau} - \pi''(q_m) \cdot \frac{1-e^{-rt_1}}{r} \right] \cdot \frac{dq_m}{d\tau} =$$

$$= p_k f'(q_m) \cdot \frac{D-1}{(1-\tau)^2} .$$

From (33) and (34) one obtains

$$(35) \quad \left[p_k f''(q_m) \cdot \frac{1-D\tau}{1-\tau} - \pi''(q_m) \cdot \frac{1-e^{-rt_1}}{r} + r \frac{t_1^2}{q_m} \pi'(q_m) \cdot e^{-rt_1} \right] \frac{dq_m}{d\tau}$$

$$=: A \cdot \frac{dq_m}{d\tau} = p_k f'(q_m) \cdot \frac{D-1}{(1-\tau)^2} .$$

Since $q_m > 0$ (cf. (7)), the time points t_1 and T are also positive. Thus, using (35), (33) and (32), one obtains for $D < 1$:

$$(36) \quad \text{If } A > 0, \text{ then } \frac{dq_m}{d\tau} < 0, \quad \frac{dT}{d\tau} > 0, \quad \frac{dt_1}{d\tau} > 0 .$$

If $A = 0$, (35) has no solution, since $f' > 0$ (cf. (1)).

$$\text{If } A < 0, \text{ then } \frac{dq_m}{d\tau} > 0, \quad \frac{dT}{d\tau} < 0, \quad \frac{dt_1}{d\tau} < 0 .$$

For $D=1$ (immediate write-off) one obtains:

$$(37) \quad \text{If } A > 0 \text{ or } A < 0, \text{ then } \frac{dq_m}{d\tau} = 0, \quad \frac{dT}{d\tau} = 0, \quad \frac{dt_1}{d\tau} = 0 .$$

If $A = 0$, (35) does not determine $dq_m/d\tau$.

If D varies (τ is given), one obtains

$$(38) \quad A \cdot \frac{dq_m}{dD} = p_k f'(q_m) \cdot \frac{\tau}{1-\tau} \quad \text{instead of (35),}$$

and (32) and (33) with d/dD instead of $d/d\tau$. Then, one obtains:

$$(39) \quad \text{If } A > 0, \quad \text{then } \frac{dq_m}{dD} > 0, \quad \frac{dT}{dD} < 0, \quad \frac{dt_1}{dD} < 0.$$

If $A = 0$, (38) has no solution.

$$\text{If } A < 0, \quad \text{then } \frac{dq_m}{dD} < 0, \quad \frac{dT}{dD} > 0, \quad \frac{dt_1}{dD} > 0.$$

Intuitively, the implications of $A > 0$ are most appealing. In this case, the capacity limit is reduced and the extraction period is extended when the tax rate, τ , increases, and capacity is increased and the extraction period shortened, when the present value D of depreciation allowances increases. In the special case of $D=1$ (immediate write-off), the optimal path will not react to tax rate changes, since capital costs are reduced by the same factor as profits for all tax rates τ . For $A=0$ the parameters t_1 , T and q_m are not uniquely determined by the system (32), (33) and (34) when τ or D is varied. Unfortunately, the case $A < 0$ leads to unexpected results and must be discussed more thoroughly.

First note that $A \leq 0$ is impossible if

$$(40) \quad f''(q_m) \geq 0.$$

Here not only the operating profit $\pi(q_t)$ but also the negative capital cost function $-(1-D\tau)p_k f(q_m)$ is concave and the necessary conditions (8) are indeed sufficient, at least for a given T (cf. Long/Vousden (1977, 25-27)).

However, if

$$(41) \quad f''(q_m) < 0, \text{ there is a possibility of } A \leq 0.$$

$A < 0$ is then equivalent to

$$(42) \quad p_k |f''(q_m)| (1-D\tau) - (1-\tau) \int_0^{t_1} |\pi''(q_t)| \cdot e^{-rt} dt > \\ > (1-\tau) r t_1^2 \pi'(q_m) e^{-r t_1} / q_m = (1-\tau) t_1^2 / \int_{t_1}^T \frac{dt}{|\pi''(q_t)| \cdot e^{-rt}} > 0$$

Thus, to obtain $A < 0$, the decrease in marginal capital costs (after taxation and depreciation) has to be greater than the discounted decrease in marginal profits (after taxation) in period $[0, t_1]$. This means that if the capital cost function is flat in relation to the discounted profit function, one cannot exclude the case $A < 0$. In what follows, the relation of the concavity of the net present value, N , and the sign of A will be discussed.

Let us consider continuous extraction paths, q_t , with the properties (17d) (decreasing extraction after a period of maximum extraction), (29) for all $t \geq t_1$ (discounted marginal profits constant and $q_T = 0$), and (30) (summation condition with $R_T = 0$). These paths are of the form

$$(43) \quad q_t = q_m, \quad t \in [0, t_1], \\ = (\pi')^{-1}(\pi'(0) e^{-r(\tau-t)}), \quad t \in [t_1, T],$$

satisfying summation condition (30).

Note that the paths (43) satisfy all necessary conditions except the transversality condition (31) for q_m . Here (31) will be shown to be equivalent to the first order condition for an optimal net present value N for all paths (43), and $A > 0$, $A = 0$ and $A < 0$ will be shown to be equivalent to local strict concavity, linearity and strict convexity of N in the path identified by (31). Thus, $A \leq 0$ does not lead to a local maximum of N , ruling

out the economically counter-intuitive parts of results (36), (37) and (38).

Let us now sketch the proof of these results.

Obviously, the continuity of paths (43) and the summation condition (30) lead to

$$(44) \quad q_m(t_1, T) = (\pi')^{-1}(\pi'(0)e^{-r(\tau-t_1)}), \quad \text{and}$$

$$(45) \quad F(t_1, T(t_1)) := t_1 q_m(t_1, T) + \int_{t_1}^T (\pi')^{-1}(\pi'(0)e^{-r(\tau-t)}) dt - R_0 = 0$$

Thus, both q_m and T can be expressed as (at least implicit) functions of t_1 , and the net present value N only depends on parameter t_1 :

$$(46) \quad N(t_1) = (1-\tau) \left[\pi(q_m(t_1))(1-e^{-rt_1})/r + \int_{t_1}^{\tau(t_1)} e^{-rt} \cdot \pi((\pi')^{-1}(\pi'(0)e^{-r(\tau(t_1)-t_1)})) dt \right] - (1-D\tau)pkf(q_m(t_1)) .$$

Taking

$$(47) \quad \frac{dq_m}{dt_1} = r \cdot \frac{\pi'(q_m)}{\pi''(q_m)} \cdot \left(1 - \frac{dT}{dt_1}\right) = -q_m / \left(t_1 - \frac{\pi''(q_m) q_m}{\pi'(q_m) r}\right) ,$$

$$(48) \quad \frac{dT}{dt_1} = t_1 / \left(t_1 + \pi''(q_m)e^{-rt_1} \int_{t_1}^{\tau} \frac{dt}{e^{-rt} \cdot \pi''(q_t)}\right) \\ = t_1 / \left(t_1 - \frac{\pi''(q_m) q_m}{\pi'(q_m) r}\right) ,$$

critical paths may be obtained by solving

$$(49) \quad \frac{dN}{dt_1} = \left(1 - \frac{dT}{dt_1}\right) \frac{\pi'(q_m)}{\pi''(q_m)} \cdot \left[(1-\tau)\pi'(q_m) \cdot e^{-rt_1}(e^{rt_1}-1-rt_1) - (1-D\tau)p_k r f'(q_m) \right]$$

= 0, which is equivalent to (31). Moreover

$$(50) \quad \frac{d^2N}{dt_1^2}(t_1) = \left(1 - \frac{dT}{dt_1}\right) \frac{\pi'(q_m)}{\pi''(q_m)} \cdot \left[(1-\tau) \left(\pi''(q_m) \cdot \frac{dq_m}{dt_1} \cdot (1 - e^{-rt_1} - rt_1 e^{-rt_1}) + \pi'(q_m) r^2 t_1 e^{-rt_1} \right) - (1-D\tau)p_k r f''(q_m) \cdot \frac{dq_m}{dt_1} \right] \text{ for solutions } t_1 \text{ of (49).}$$

Utilizing (47), one can show

$$(51) \quad \frac{d^2N}{dt_1^2}(t_1) \begin{matrix} > \\ < \end{matrix} 0, \text{ if and only if } A \begin{matrix} < \\ = \\ > \end{matrix} 0$$

for solutions t_1 of (49), which completes the results.

Altogether,

(52) a solution $(R_t, q_m, T, q_t, t \in [0, T])$ of (17d), (12), (18), (27), (29)-(31) is a relative maximum of the problem (2), (3), (4), (7), (9), (10), (26), (28), if and only if $A > 0$. Such a solution takes the form

$$q_t = q_m, \quad t \in [0, t_1], \\ = (\pi')^{-1}(\pi'(0))e^{-r(\tau-t)}, \quad t \in [t_1, T],$$

and one obtains

$$(i) \quad \frac{dq_m}{d\tau} < 0, \quad \frac{dT}{d\tau} > 0, \quad \frac{dt_1}{d\tau} > 0 \quad \text{for } 0 \leq D < 1, \\ (ii) \quad \frac{dq_m}{d\tau} = 0, \quad \frac{dT}{d\tau} = 0, \quad \frac{dt_1}{d\tau} = 0 \quad \text{for } D = 1,$$

$$(iii) \quad \frac{dq_m}{dD} > 0, \quad \frac{dT}{dD} < 0, \quad \frac{dt_1}{dD} < 0 \quad \text{for } 0 \leq \tau \leq 1.$$

Note that the condition $A > 0$ is weaker than condition $f''(q_m) \geq 0$, the local convexity of the capital cost function.

Finally, the existence of optimal extraction paths is discussed with an example.

2. Example (cf. Hotelling (1931) and Campbell (1980))

(53) Let $p_t = w(1 - e^{-sq_t})/q_t$ with $w, s > 0$, the inverse demand function in the monopoly case. For (4b) this results in the profit function

$$\pi(q_t) = w(1 - e^{-sq_t}), \quad w, s > 0, \quad \text{with}$$

$$\pi'(q_t) = wse^{-sq_t} > 0 \quad \text{and} \quad \pi''(q_t) = -ws^2e^{-sq_t} < 0.$$

Let the capital stock K be of the same form as the profit function π :

$$K = f(q_m) = v(1 - e^{-zq_m}), \quad v, z > 0, \quad \text{with}$$

$$f'(q_m) = vze^{-zq_m} > 0 \quad \text{and} \quad f''(q_m) = -vz^2e^{-zq_m} < 0.$$

Thus, the profit function π as well as the cost function f are assumed to approach an upper limit w or v , when q_t or q_m approach infinity. The height w , v of such limit as well as the speed s , z with which it is approached, are free and can be used to control the sign of A . Here the equation system (29), (30), (31) takes the form:

$$(54) \quad sq_m = r(T - t_1)$$

$$(55) \quad t_1 q_m + \frac{r}{2s} (T - t_1)^2 = R_0$$

$$(56) \quad p_K v z \cdot e^{-z q_m} \cdot (1-D\tau) = (1-\tau) w s \cdot \left[\frac{e^{-r t_1} - 1}{r} - t_1 \right] \cdot e^{-r T},$$

and

$$(57) \quad A = -p_K v z^2 \cdot e^{-z q_m} \cdot \frac{1-D\tau}{1-\tau} + w s^2 \cdot \left[\frac{e^{r t_1} - 1}{r} + \frac{t_1^2}{T-t_1} \right] \cdot e^{-r T} < 0$$

can be shown to be equivalent to

$$(58) \quad \frac{z}{s} > 1 + \frac{T t_1}{T-t_1} \left[\frac{e^{r t_1} - 1}{r} - t_1 \right].$$

Note that (54) is valid for all $t \in [t_1, T]$ and thus

$$(59) \quad q_t = \frac{r}{s} \cdot (T-t)$$

is linear in the second phase of extraction.

$$(60) \quad \text{Now, let } r=0.1, \quad R_0=100, \quad p_K=1, \\ s=0.5, \quad v/w = 5, \quad \text{and} \\ (1-D\tau) = 2(1-\tau), \quad 0.5 \leq \tau \leq 1.$$

Extraction paths for these constants, which satisfy conditions (54) and (55) are of the form (cf. Figure 1):

$$(61) \quad q_t = q_m, \quad t \in [0, t_1] \\ = r/s \cdot (T-t) = 0.2(T-t), \quad t \in [t_1, T] \quad \text{with}$$

$$(62) \quad T = \sqrt{t_1^2 + 2sR_0/r} = \sqrt{t_1^2 + 1000}.$$

Thus,

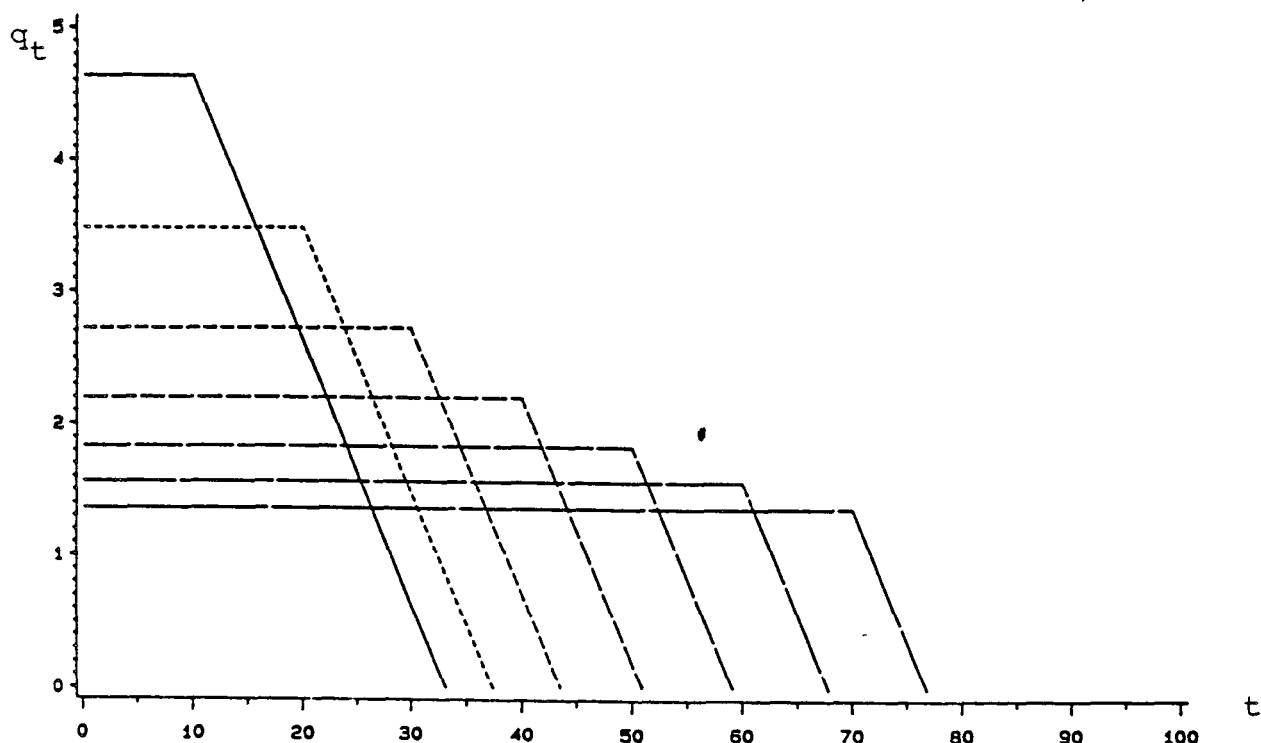
$$(63) \quad \frac{dT}{dt_1} = t_1/T, \quad \frac{dq_m}{dt_1} = -q_m/T.$$

The net present value N takes the form:

$$(64) \quad N(t_1) = (1-\tau) \frac{W}{r} (e^{-(z/s)r(\tau-t_1)} - (e^{rt_1+r(T-t_1)})e^{-rT})$$

with $T=T(t_1)$ as in (62).

Figure 1: Extraction paths $q(t)$ for different t_1



For this N , the desired properties are easy to show:

$$(65) \quad \frac{dN}{dt_1}(t_1) = 0, \quad \text{if and only if (56) applies, and}$$

$$(66) \quad \frac{d^2N}{dt_1^2}(t_1) = 0, \quad \text{if and only if } A = 0,$$

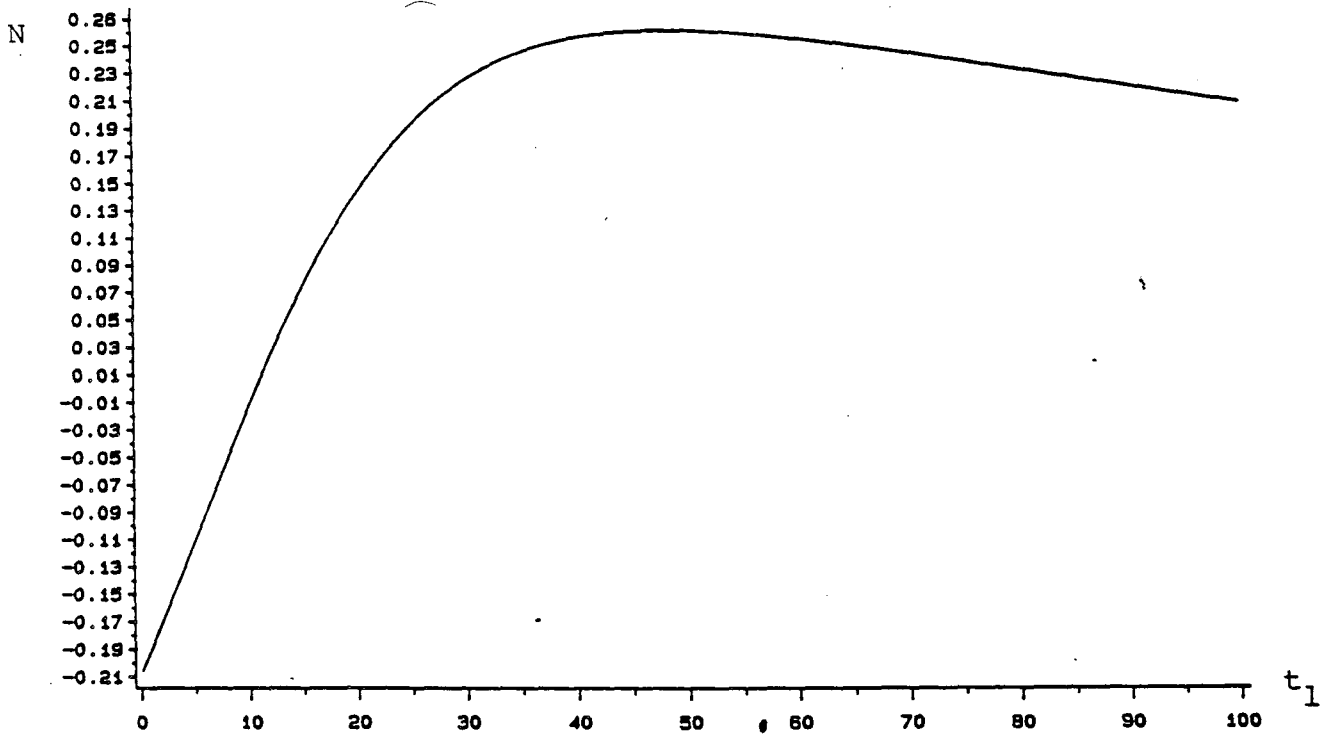
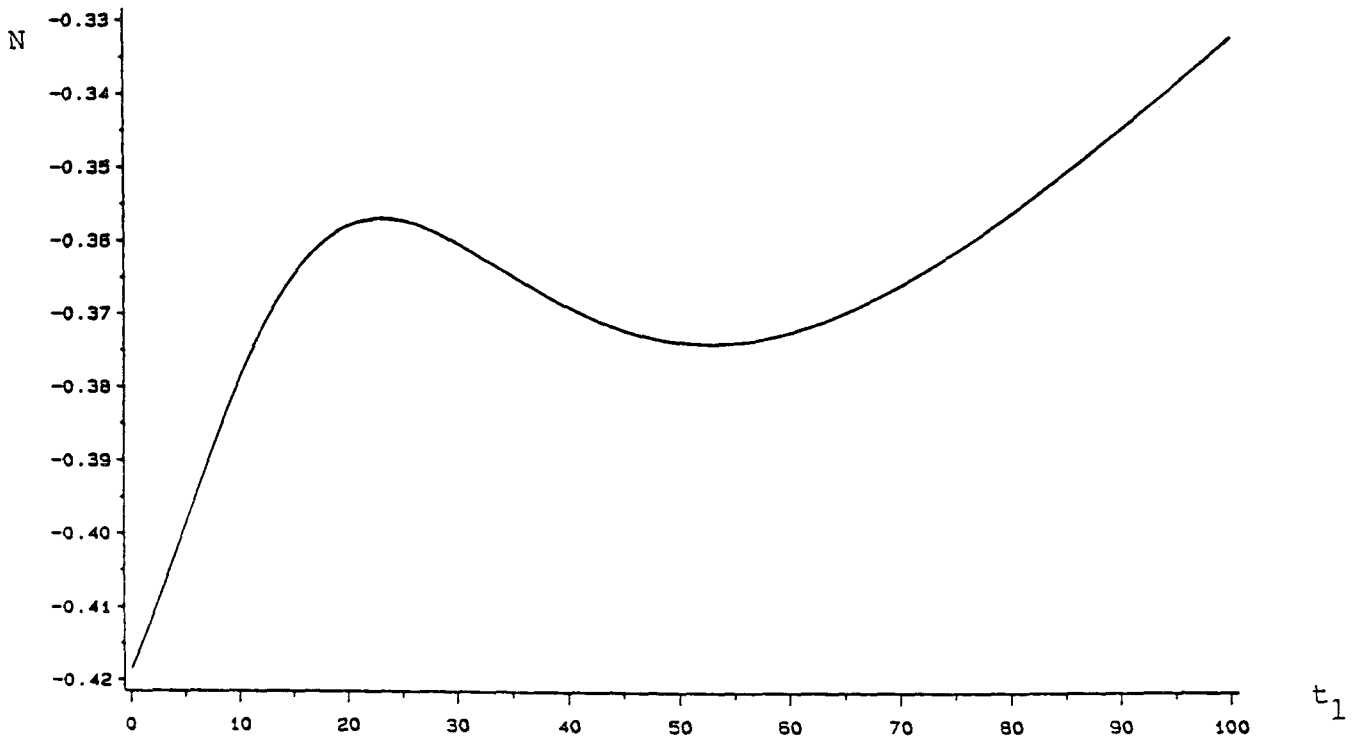
for all solutions t_1 of (65).

Let us consider the solution of (65) for different z (cf. Table 1).

Table 1: Critical paths for different z/s

z/s	t_1	T	q_m	A	N
0.75	47.8	57.3	1.9	>0	>0
1	-	-	-	-	(<0)
1.5	23.2	39.2	3.2	>0	<0
	52.6	61.4	1.8	<0	<0

Note that for $z=s$, there is no critical path and N is approaching zero from below for $t_1 \rightarrow \infty$. For $z=0.75 \cdot s$ there is one global maximum at $t_1=47.8$ resulting in a positive net present value. But for $z=1.5 \cdot s$ there is a local minimum at $t_1=52.6$ as well as a local maximum at $t_1=23.2$, both resulting in a negative net present value. The net present value N as a function of t_1 is displayed in Figure 2a (for $z/s=0.75$) and Figure 2b (for $z/s=1.5$) for the tax rate $\tau=0.75$, and therefore $D=2/3$ from (53). Thus, we have found examples of convex negative capital cost functions for both $A>0$ and $A<0$, and even for the case where no solution to the necessary conditions exists.

Figure 2a: Net present value $N(t_1)$ for $z/s = 0.75$ Figure 2b: Net present value $N(t_1)$ for $z/s = 1.5$ 

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