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Working Paper

Stability issues of endogenous growth models

Diskussionsbeiträge - Serie II, No. 270

Provided in Cooperation with:

Department of Economics, University of Konstanz

Suggested Citation: Koch, Karl-Josef (1995) : Stability issues of endogenous growth models, Diskussionsbeiträge - Serie II, No. 270, Universität Konstanz, Sonderforschungsbereich 178 - Internationalisierung der Wirtschaft, Konstanz

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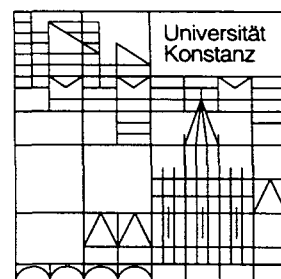
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Sonderforschungsbereich 178
„Internationalisierung der Wirtschaft“

Diskussionsbeiträge



Juristische
Fakultät

Fakultät für Wirtschafts-
wissenschaften und Statistik

Karl-Josef Koch

**Stability Issues of
Endogenous Growth Models**

22. JUNI 1995 Weltwirtschaft
Kiel

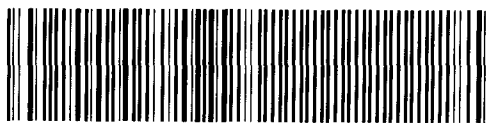
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Serie II - Nr. 270

Mai 1995

Stability Issues of Endogenous Growth Models

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April 15 1995

preliminary version
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Abstract

In the analysis of endogenous growth models often projections are used which map the balanced growth path into a single point. This is done in order to draw conclusions about the stability of the system or the determinacy of transition paths. This procedure hinges on the homogeneity of the models under consideration, whereas the possible existence of a balanced growth path or a more general asymptotic attractor does not. Furthermore, exponential asymptotic stability cannot be proved on the basis of a model reduced by projection.

We show that one has to be careful in drawing conclusions from the projected system in order to judge upon the behavior near the balanced growth path. And with two standard examples of endogenous growth we demonstrate an analytical method to study exponential stability in the full model.

1 Introduction

The analysis of endogenous growth models typically leads to a system of partial differential equations together with a set of initial conditions for the state of the model and transversality conditions. Furthermore, this system is characterized by the fact that it does not have a stationary solution. Instead, the type of solution the attention is focussed on is a path of sustainable balanced growth. Not always the optimal solution of such an endogenous growth model is uniquely determined. Moreover, local or global stability of the balanced growth path is not trivial to analyze.

The examination of these questions is the subject of series of recent papers (e.g. Mulligan and Sala-I-Martin (1991, 1993), Caballè and Santos (1993), Benhabib and Perli (1994), Devereux and Lapham (1994) and others). Some general questions of stability and uniqueness are answered only partially and still open to question. They are related to the changes of coordinates which are used in the analysis. In traditional models of optimal growth as they were first considered by Cass (1965) and Koopmans (1965) it was natural to inspect per capita measures of capital stock, consumption and so forth. The justification is twofold: it is the pure fact that a per capita version of the dynamical system is well defined and apart from that it is the exogeneity of labor growth. Even constant returns to scale production functions do not always allow for such a reduction. A different transformation may be necessary (e.g. in the Lucas model discussed in section 4.2). With labor growing at a known constant rate the change of endogenously accumulated capital over time can be investigated by a projection of the state of the economy to a single variable. But, if a number of variables all grow with endogenous rates the picture becomes more complicated. Even if the ratios of certain variables approach exogenously given constants the time path may show substantial fluctuations with non diminishing amplitudes! Stability of a projected system of ratios of variables only means that in the long run the differences between growth rates converge to zero. If none of these growth rates is exogenously given and constant this obviously does not imply convergence of growth rates. Even if all growth rates were exogenous there could be qualitative aspects in the asymptotic behavior which cannot be detected after projection. Regardless of the (trivial) asymptotic convergence of growth rates and the convergence in the projected model the full dynamical system may move away monotonically from the balanced growth path. But it is the endogeneity of growth rates off the balanced

growth path which creates the possibility of cycles even if the projected model is stable.

The aim of this paper is to shed some light on the mathematical aspects of these problems and to apply the resulting technique to two standard models of endogenous growth. Section 2 and 3 of the paper provide the reader with some technical tools which then are applied to two proto types of models of endogenous growth in section 4. Section 5 concludes with some general remarks and section 6 contains the proofs of some formal details of the paper.

2 Analysis of Balanced Growth

In order to build up a solid foundation for the analysis let us first define basic concepts.

Throughout this section of the paper let $\dot{x} = F(x)$ denote a partial differential equation on an open set $W \subset \mathbb{R}_+^n$.

Definition 1 *An invariant set of F is a subset $\mathcal{I} \subset W$ such that any solution $x(t)$ of F with $x(0) \in \mathcal{I}$ stay in \mathcal{I} for all admissible $t > 0$.*

A special kind of invariant sets are curve along which the growth rates of all variables are constant. We call this a balanced growth path.

Definition 2 *A balanced growth path (BGP) of F with growth rates $\tilde{g} = (g_1, \dots, g_n) \in \mathbb{R}_+^n$, $\tilde{g} \neq 0$, is an invariant set*

$$C = \{x \in W \mid F_i(x) = g_i x_i \forall i\}$$

Obviously, such a balanced growth path is bent if it has - at least - two different g_i . Otherwise it is an affine set, a line. By an obvious exponential change of coordinates every balanced growth path can be transformed into a linear one.

Lemma 1 *Assume C is a BGP of F with growth rates $\tilde{g} = (g_1, \dots, g_n)$ and at least one g_i different from 0. Let g be a common multiple of all $g_i \neq 0$ and $\mu_i = g/g_i$ if $g_i \neq 0$ and $\mu_i = 1$ else. Define new coordinates $y = H(x)$ by $y_i = x_i^{\mu_i}$ for all i . Then H maps $\dot{x} = F(x)$ into $\dot{y} = G(y)$ and C into linear balanced growth path with growth rate g . At any point $y = H(x)$ on L the Jacobian of \dot{y} is of following form:*

$$\partial G(y) = \tilde{G} + \partial H(x) \partial F(x) \partial H(x)^{-1}$$

where \tilde{G} is the diagonal matrix with entries $\tilde{G}_{ii} = g - g_i$.

The proof this lemma is straight forward, but given in the appendix for the sake of completeness. At least in many cases this change of coordinates doesn't affect stability questions. A solution $x(t)$ of F will converge to C if and only if the induced solution $y(t) = H(x(t))$ of G converges to $L = H(C)$. In order to give this statement a precise meaning let us now define a concept of convergence to a balanced growth path. It will be based on the distance between point x on a path $x(t)$ and the line L , which is defined to be the minimum of $\|x - y\|$ for all y on L .

Definition 3 *A path $x(t)$ converges to a balanced growth path C , if for every $\varepsilon > 0$ there is a t_0 such that for all $t > t_0$ the distance between $x(t)$ and C is smaller than ε .*

A linear balanced growth path of $\dot{y} = G(y)$ is of the form

$$L = \{y = b + \tau a | \tau > 0\}$$

for some constant vectors a, b with $b_i = 0$, if y_i grows along \tilde{L} , and $a_i = 0$, if y_i remains constant along \tilde{L} . Obviously, for every $y \in L$ the vector b is an eigenvector of the Jacobian $\partial G(y)$ with eigenvalue g .

Definition 4 *A linear balanced growth path L is called hyperbolic, if for every $y \in L$ all eigenvalues of the Jacobian $\partial G(y)$ have real parts different from zero.*

The eigenvalues of the Jacobian at y characterize the type of local stability of the system. Hyperbolicity of balanced growth paths implies that the eigenvalues may change along the balanced growth path, but never can change their sign. In other words, the type of stability is the same along L . To avoid some asymptotic difficulties we make a further assumption which is fulfilled in most of the economic applications.

Assumption 1 *Assume, the spectrum of the Jacobian matrix is constant along the balanced growth path.*

Typically, there will be positive and negative eigenvalues apart from g itself. This establishes the saddlepoint character of optimal control. The following theorem provides the basis for stability considerations near a balanced growth path.

Theorem 1 (Stable manifold of a linear balanced growth path)

Let $L = \{y = b + \tau a | \tau > 0\}$ be a hyperbolic balanced growth path of a differential equation $\dot{y} = G(y)$ with growth rate $g > 0$, n_u eigenvalues with positive real part in addition to g , and n_s ones with negative real part. Then there is a neighbourhood U of L with the following properties:

- 1) There is a $n_s + 1$ -dimensional C^1 manifold \mathcal{M}^s containing L which is invariant under the flow of G .
- 2) Every solution $y(t)$ of the differential equation which starts on \mathcal{M}^s in U converges exponentially to L .
- 3) No other solution converges to L .

We call \mathcal{M}^s the stable manifold of L .

This theorem is a special case of a very general result on dynamical systems (cf. Hirsch, Pugh and Shub 1977, chapter 6 or Irwin 1980). The proof is based on the following ideas: Locally the differential equation can be approximated by the Jacobian system. Hyperbolicity implies that the generalized eigenvectors of the Jacobian span the whole space. We can group the eigenvalues in the following way: The growth rate g is a (positive) eigenvalue with eigenvector, say b , collinear with the balanced growth path. The generalized eigenvectors of the eigenvalues with negative real part together with b span a subspace of dimension $n_s + 1$, say M^s . And on the other hand, the eigenvectors of eigenvalues with positive real part (including g) span a subspace of dimension $n_u + 1$, say M^u . The intersection $M^s \cap M^u$ is the one dimensional subspace containing b , i.e. it is parallel to the balanced growth path. Standard arguments can now be applied to show the existence of two manifolds both containing the balanced growth path and tangent to M^s and M^u , respectively. Moreover, these manifolds \mathcal{M}^s and \mathcal{M}^u are invariant sets of the differential equation, and coming to the central statement of the theorem, \mathcal{M}^s contains exactly those solutions $y(t)$ which converge exponentially towards the balanced growth path if time goes to $+\infty$, whereas \mathcal{M}^u contains the solutions which converge to the balanced growth path if time goes to $-\infty$.

The dimension $n_s + 1$ of the stable manifold \mathcal{M}^s characterizes the type of local stability of the *BGP*. It may be equal to 1 in case of which the *BGP* is totally unstable. If it equals 2 (locally) a solution of the control problem converging to the *BGP* exists and is unique, and if it exceeds 2 there is a continuum of solutions with this property.¹

Remark 1 *Statement 2 and 3 of theorem 1 imply that by a further change of coordinates the BGP may be turned into a center manifold. (cf. Hale 1980).*

Figure 1 illustrates the geometry of the situation. The following example is de-

¹At this point we suppose that on the *BGP* the transversality conditions of the control problem are satisfied. It is not necessarily the case that they are not satisfied off the stable manifold \mathcal{M}^s .

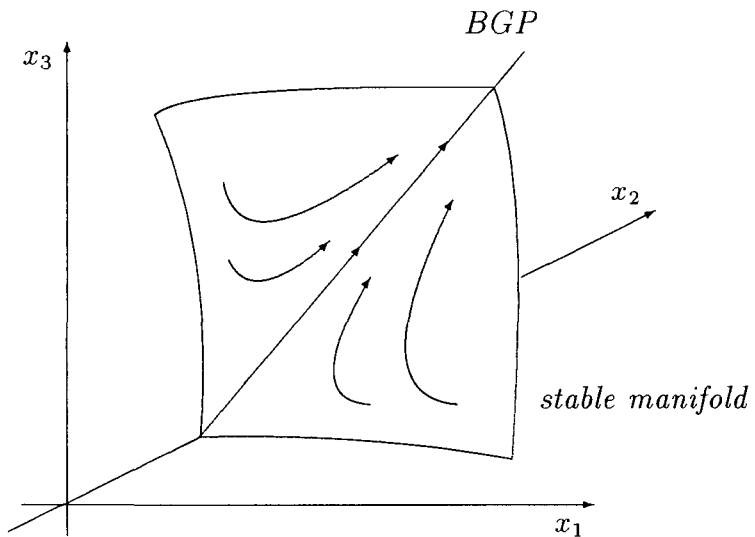


Figure 1: Local Stability of a Balanced Growth Path

signed to illustrate the theorem without being derived from an economic model of intertemporal optimization.

Example 1 Let a, b, c be positive constants, $\eta \neq 0$ a constant with $|c - \eta| > 0$ and define a hyperbolic, linear partial differential equation on \mathbb{R}_{++}^3 by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = F(x_1, x_2, x_3) = \begin{pmatrix} ax_3 + \eta x_1 \\ -1 + bx_2 \\ cx_3 + \eta x_3 \end{pmatrix}$$

The constant Jacobian has eigenvalues $\lambda_1 = \eta$, $\lambda_2 = b$, and $\lambda_3 = c + \eta$, all different from zero. Hence the system is hyperbolic. Corresponding eigenvectors are $x^1 = (1, 0, 0)$, $x^2 = (0, 1, 0)$, and $x^3 = (a, 0, c)$. The line $\{(0, 1/b, 0) + \tau(a, 0, c) | \tau > 0\}$ is the *BGP* of F we want to analyze. Along the line *BGP* the growth rate is constantly equal to $g = c + \eta$.

Assume $\eta < 0$. Then the plane $M = \{(x_1, 1/b, x_3) | x_1, x_3 > 0\}$ is the stable manifold \mathcal{M}^s . If η is positive the *BGP* is unstable (and \mathbb{R}_{++}^3 is the unstable manifold).

This particular example was introduced here, because it allows to clarify another issue. As a matter of fact, the flow within M may be 'converging' to the *BGP* in a weaker sense even if $\eta > 0$. We are going to explain this concept in the next section.

3 Projections

In many economic application we don't consider the dynamics around the balanced growth path as such, but the behavior of a projection of this system. The traditional, neoclassical growth model is based on constant returns to scale production functions and it is natural to describe the economy in per capita terms only. The balanced growth path of such a model by this simplification maps into a steady state of a lower dimensional system. It is important to see that this procedure is perfectly acceptable, as long as we are interested in in the economic situation of a representative agent in such an economy. Nevertheless, the projection eliminates some information and, to pin down the point we want to make, in models of endogenous growth this information may be of some importance to properly judge upon the long run behavior of the economy.

There is a further issue which should be clarified at the beginning: a projection may fail to induce a dynamical system in the new variables! In general it depends on both the differential equation and the projection mapping itself whether a differential equation in the projected variables is well defined.

Consider a projection mapping

$$\pi : U \rightarrow V \subset R^m .$$

Then $y = \pi(x)$ implies $\dot{y} = \partial\pi_x(x)\dot{x} = \partial\pi_x(x)F(x)$. The right hand side is a function from U into R^m . The projection induces a differential equation in y if this function factorizes into

$$\partial\pi_x(x)F(x) = G(\pi(x)) .$$

If this is the case,

$$\dot{y} = G(y) = \partial\pi(F(\pi^{-1}(y))) F(\pi^{-1}(y))$$

is the differential equation on $V \subset R^m$ we are looking for. In other words, we must find out whether a function G exists such that the following diagram commutes:

$$\begin{array}{ccc} R^n & \xrightarrow{F} & R^n \\ \downarrow \pi & & \downarrow \partial\pi \\ R^m & \xrightarrow{G} & R^m \end{array}$$

For the typical application we have in mind the projection will be a central projection of a subset of variables whereas other variables remain unchanged. From the origin this subspace is mapped onto a plane where one coordinate of x is constantly equal to 1. Such a projection properly chosen maps a balanced growth path of an n dimensional space into a single point of an $m = n - 1$ dimensional space. The fibers of such a projection are rays through the origin of the subspace. For the induced dynamics to be well defined it is therefore necessary and sufficient that the original dynamical system is constant along these rays. One should realize at this point that the existence of a linear balanced growth path is not sufficient for the existence of a projection which satisfies this condition. Nonetheless, the examples of endogenous growth models discussed in the literature all allow this sort of dynamic simplification.

Let us now turn towards the question of asymptotic stability. Recall example 1 and consider the following projection

$$(y_1, y_2) = \pi(x_1, x_2, x_3) = (x_1/x_3, x_2)$$

As the dynamic system in x was linear the projection induces a differential equation in y :

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} a - cy_1 \\ -1 + by_2 \end{pmatrix}$$

Obviously this system has a saddlepoint at $\bar{y} = (a/c, 1/b)$ with stable manifold $\{y_2 = 1/b\}$. Notice that the induced system is completely independent of the parameter η , whereas saddlepoint stability of the original system required η to be negative. In other words, this example demonstrates that saddlepoint stability of a system induced by projection does not imply the corresponding property for the original system. But not all information gets lost by the projection. Consider a solution $y(t)$ on the stable manifold of \bar{y} . By definition of a stable manifold it must converge to the point \bar{y} . The preimage of every point on this stable manifold is a ray in the manifold $M = \{(x_1, 1/b, x_3) | x_1, x_3 > 0\}$ defined above. In particular, the *BGP* as a ray in M is mapped to \bar{y} .

Now consider any point $x(0) \in M$. The solution $x(t)$ starting in $x(0)$ converges to the *BGP* if and only if $\eta < 0$. But, as $\pi(x(t))$ converges to \bar{y} the sequence of rays in M through $x(t)$ converges to the *BGP*. In other words, the asymptotic cone of $x(t)$ in M converges to the *BGP* independently of the sign of η . The geometrical explanation for the possible lack of convergence is clear: the rays in M through $x(t)$ converge to the *BGP* because difference between eigenvalues

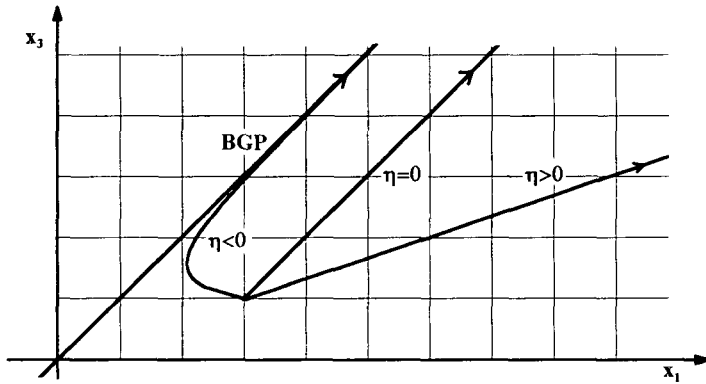


Figure 2: Convergence in terms of the Asymptotic Cone

$\lambda_1 - \lambda_3 = -c$ is negative independent of η . But on these rays x_1 and x_3 grow with a speed increasing with η . If η were equal to 0 the solution $x(t)$ would move parallel to the *BGP*.² If η is positive the motion along the ray is too fast to allow for a convergence of $x(t)$ to the *BGP*. Figure 2 illustrates the difference between convergence of the asymptotic cone and the - proper - convergence to the *BGP*. The asymptotic cone of all three orbits shown is the *BGP* although only one of them converges properly to the *BGP*. The possible dominance of the divergent process over the convergence of rays is disguised by the projection onto the induced system. An important aspect of economic stability analysis therefore may get lost by the projection. In a traditional growth model like for example in the Ramsey-Cass-Koopmans model of optimal growth the interesting questions can be posed directly in per capita terms, i.e. in terms of the system induced by the projection. In endogenous growth models the projection mapping the balanced growth path into a steady state in general does not generate a per capita version. Hence the analysis of the induced model may yield unsatisfactory answers to questions of asymptotic stability of a balanced growth path.

²This is the non-hyperbolic case we have excluded because a slight perturbation of the linear example could either be stable or unstable.

4 Economic Applications

4.1 Romer's Model

Let us consider Romer's model of endogenous technological change as a first prototype (Romer 1990). A very short description of the model should be sufficient for our purpose. Assume that a final product Y is produced from a unskilled labor L , from human capital H_Y and an aggregate of intermediate products X . Capital K is used to manufacture intermediates and the productivity of capital depends on the degree of differentiation A . A can be increased by $R\&D$ activities at a rate proportional to the use of human capital in this sector, H_A . Output which is not consumed increases the stock of capital, and the consumption path C is derived by intertemporal optimization. The total amount of labor L_0 and human capital H_0 is constant. Of course, all these relations have to be formalized in a particular parametric form.

Concisely the model can be described by a reduced form for the production function

$$Y = \gamma^{\alpha+\beta-1} L_0^\beta H_Y^\alpha A^{\alpha+\beta} K^{1-\alpha-\beta}$$

and a system of differential equations in (K, A, C, H_A) derived from the first order conditions of intertemporal optimization:

$$\dot{K} = Y - C \tag{1}$$

$$\dot{A} = \sigma H_A A \tag{2}$$

$$\dot{C} = \frac{1}{\theta} C \left((1 - \alpha - \beta) \frac{Y}{K} - \rho \right) \tag{3}$$

$$\dot{H}_A = -H_Y \left\{ \frac{\alpha + \beta}{1 - \alpha} + \frac{\alpha + \beta}{1 - \alpha} \sigma H_0 - \frac{1 - \alpha - \beta}{1 - \alpha} \frac{C}{K} + \frac{\alpha + \beta}{\alpha} \sigma H_Y \right\} \tag{4}$$

The parameters used must satisfy certain constraints to guarantee the existence of a balanced growth path (with positive growth rate):

$$\mathcal{P} = \left\{ (\alpha, \beta, \theta, \rho, \sigma, H_0) \in (0, 1)^4 \times \mathbb{R}^2 \mid \frac{\alpha}{\alpha + \beta} < \frac{\sigma H_0}{\rho} < \frac{1}{1 - \theta} \right\} \tag{5}$$

It is straight forward to check that the transversality conditions of the control problem are satisfied on the balanced growth path.

In order to analyze the stability of the model we compute the Jacobian matrix

of this system (using the shorthand $\zeta = \alpha + \beta$)

$$J = \begin{pmatrix} (1-\zeta)\frac{Y}{K} & \zeta\frac{Y}{A} & -1 & -\alpha\frac{Y}{H_Y} \\ 0 & \hat{A} & 0 & \sigma A \\ -\frac{1}{\theta}\zeta(1-\zeta)\frac{CY}{K^2} & \frac{1}{\theta}\zeta(1-\zeta)\frac{CY}{KA} & \hat{C} & -\frac{1}{\theta}(1-\zeta)\alpha\frac{CY}{KH_Y} \\ -\frac{1-\zeta}{1-\alpha}\frac{CH_Y}{K^2} & 0 & \frac{1-\zeta}{1-\alpha}\frac{H_Y}{K} & \hat{H}_Y + \frac{1-\zeta}{\alpha}\sigma H_Y \end{pmatrix}$$

The balanced growth path (BGP) is characterized by $g = \hat{K} = \hat{A} = \hat{C} = \hat{Y}$ and $0 = \hat{H}_A = \hat{H}_Y$. Along the balanced growth path the Jacobian therefore reduces to

$$J = \begin{pmatrix} (1-\zeta)\frac{Y}{K} & \zeta\frac{Y}{A} & -1 & -\alpha\frac{Y}{H_Y} \\ 0 & g & 0 & \sigma A \\ -\frac{\zeta(1-\zeta)}{\theta}\frac{CY}{K^2} & \frac{\zeta(1-\zeta)}{\theta}\frac{CY}{KA} & g & -\frac{\alpha(1-\zeta)}{\theta}\frac{CY}{KH_Y} \\ -\frac{1-\zeta}{1-\alpha}\frac{CH_Y}{K^2} & 0 & \frac{1-\zeta}{1-\alpha}\frac{H_Y}{K} & \frac{\zeta}{\alpha}\sigma H_Y \end{pmatrix}$$

Let us try to simplify this Jacobian further by manipulating it without changing the determinant nor the characteristic polynomial. To do so we multiply the second row by K/A , the third row by K/C , the fourth row by $\alpha K/(\zeta H_Y)$ and columns two, three and four by the respective reciprocals. Mathematically speaking, we express the Jacobian in transformed coordinates such that an eigenvector corresponding to the balanced growth path is given by $\pi = (1, 1, 1, 0)'$. The change in the last coordinate only simplifies terms.

$$\tilde{J} = \begin{pmatrix} (1-\zeta)\frac{Y}{K} & \zeta\frac{Y}{K} & -C/K & -\zeta\frac{Y}{K} \\ 0 & g & 0 & \frac{\zeta}{\alpha}\sigma H_Y \\ -\frac{\zeta(1-\zeta)}{\theta}\frac{Y}{K} & \frac{\zeta(1-\zeta)}{\theta}\frac{Y}{K} & g & -\frac{\zeta(1-\zeta)}{\theta}\frac{Y}{K} \\ -\frac{\alpha(1-\zeta)}{(1-\alpha)\zeta}\frac{C}{K} & 0 & \frac{\alpha(1-\zeta)}{(1-\alpha)\zeta}\frac{C}{K} & \frac{\zeta}{\alpha}\sigma H_Y \end{pmatrix}$$

Now we use the eigenvector $\pi = (1, 1, 1, 0)'$ to factorize the characteristic polynomial of J according to Lemma 2 with the help of the following transformation matrix:

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad H^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The transformation $\tilde{B} = H^{-1}\tilde{J}H$ subtracts the first row of \tilde{J} from its second and third and replaces the first column by the vector $(g, 0, 0, 0)'$.

$$\tilde{B} = \begin{pmatrix} g & b \\ 0 & B \end{pmatrix}$$

where b is row vector and B the 3×3 matrix

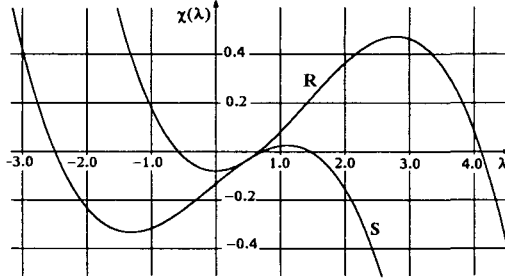
$$B = \begin{pmatrix} g - \zeta \frac{Y}{K} & \frac{C}{K} & \frac{\zeta}{\alpha} \sigma H_Y + \zeta \frac{Y}{K} \\ \frac{\zeta(1 - \zeta - \theta) Y}{\theta K} & \frac{Y}{K} & -\frac{\zeta(1 - \zeta - \theta) Y}{\theta K} \\ 0 & \frac{\alpha(1 - \zeta) C}{(1 - \alpha)\zeta K} & \frac{\zeta}{\alpha} \sigma H_Y \end{pmatrix}$$

Accordinging to the balanced growth conditions of K and H_A , equations (1) and (4) respectively, we can substitute C/K by $Y/K - g$ and $(\zeta/\alpha)\sigma H_Y$ by $(1 - \zeta)Y/K - g$.

$$B = \begin{pmatrix} g - \zeta \frac{Y}{K} & \frac{Y}{K} - g & \frac{Y}{K} - g \\ \frac{\zeta(1 - \zeta - \theta) Y}{\theta K} & \frac{Y}{K} & -\frac{\zeta(1 - \zeta - \theta) Y}{\theta K} \\ 0 & \frac{\alpha(1 - \zeta)}{(1 - \alpha)\zeta} \left(\frac{Y}{K} - g \right) & (1 - \zeta) \frac{Y}{K} - g \end{pmatrix}$$

The missing eigenvalues of J now are the ones of B . To apply theorem (3) to this matrix we need at least the sign of the trace and the determinant of B . The trace is easy compute

$$Tr_B = \left(g - \zeta \frac{Y}{K} \right) + \frac{Y}{K} + \left((1 - \zeta) \frac{Y}{K} - g \right) = 2(1 - \zeta) \frac{Y}{K} > \frac{2\zeta\sigma}{\alpha} H_Y > 0$$



□

Figure 3: The characteristic polynomial of the Jacobian matrix. R refers to the reduced balanced growth version and S to the steady state version of the model.

The sign of Det_B is much more tricky to check. The sign pattern of B looks as follows:

$$B = \begin{pmatrix} +/- & + & + \\ +/- & + & -/+ \\ 0 & + & + \end{pmatrix}$$

To see this, notice that only the signs of three entries of B are not obvious, $b_{1,1}$ and $b_{2,1} = -b_{2,3}$. Indeed, these entries may be positive or negative. Regardless of the sign, we can develop $det(B)$ along the third row using $\xi = \alpha(1 - \zeta)/((1 - \alpha)\zeta) > 0$ as a shorthand:

$$det(b) = b_{2,1} b_{1,2} (\xi(b_{1,1} + b_{1,2}) - b_{3,3}) + b_{1,1} b_{2,2} b_{3,3}$$

As $b_{1,1} + b_{2,1}$ is positive, $det(B)$ is clearly negative, if $b_{2,1} \leq 0$ and $\xi(b_{1,1} + b_{1,2}) - b_{3,3} > 0$. Of course, this constellation remains to be asserted.

So far we have been unable to check this property, but we could not find a counterexample even by checking a wide range of parameters with a computer. Hence we conjecture that the determinant of B always is negative. Figure 3 illustrates the situation for a particular choice of parameters. In addition to the characteristic polynomial $\chi_B(\lambda)$ the picture shows the graph of the characteristic polynomial of the Jacobian one gets from the projection of the *BGP* into a steady state by considering the ratios $K/A, C/A$ apart from H_A . We see that the negative eigenvalue determining the stability of the *BGP* is much closer to zero than the negative eigenvalue of the steady state version of the model. To make a correct

statement let \mathcal{P}' be the subset of \mathcal{P} where $Det_B < 0$. The stable manifold theorem then implies for Romer's model:³

Theorem 2 *Assume the parameters are in \mathcal{P}' . Then there is a unique balanced growth path (BGP) and a neighborhood U of the (BGP) with the following property: If (C^0, H_A^0) is the optimal choice of controls for the initial state (K^0, A^0) and $(K^0, A^0, C^0, H_A^0) \in U$, the optimal path converges asymptotically to the balanced growth path.*

And we add

Conjecture 1 *\mathcal{P}' seems to be equal to \mathcal{P} .*

The stability result in this model is the strongest one can get. Optimal control always leads asymptotically to balanced growth - provided balanced growth is possible. The proviso, on the other hand, reveals the weakest aspect of this model: Given the other parameters of the model, human capital must be provided within strict limits. Too low supply of this fixed resource leads to a collapse of the economy in the long run, whereas too much human capital will let future returns too large. For ever the economy will be investing without ever extracting consumption from current production.

4.2 Lucas' Model

Another example turns out to be more interesting: the model discussed by Benhabib and Perli (1993), Mulligan and Sala-i-Martin (1993), and others. It is based on the seminal paper by Lucas (1988). Assume a final product is produced from physical and human capital, k and h , respectively. The stock of human capital can be split into a share u used for production in this sector and $(1-u)$ used to increase human capital. Due to human capital spill over effects there are increasing returns to scale in the production sector. Intertemporal utility of consumption c with constant elasticity of substitution σ and discount rate ρ is to be maximized, and solutions can be computed based on the current value Hamiltonian⁴

$$\mathcal{H} = \frac{c^{1-\sigma} - 1}{1-\sigma} + \lambda_1(A k^\beta h^{1-\beta} h^\gamma u^{1-\beta} - c) + \lambda_2(\delta(1-u)h)$$

³Uniqueness of the optimal solution of this model is easy to show by Arrow's criterion.

⁴For a more detailed description c.f. Benhabib and Perli (1993).

This yields the following system of differential equations:

$$\begin{aligned}\dot{k} &= Ak^\beta h^{1-\beta+\gamma} u^{1-\beta} - c \\ \dot{h} &= \delta(1-u)h \\ \dot{c} &= \left(\frac{A\beta}{\sigma} k^{\beta-1} h^{1-\beta+\gamma} u^{1-\beta} - \frac{\rho}{\sigma} \right) c \\ \dot{u} &= \left(\frac{\delta(\beta-\gamma)}{\beta} u + \frac{\delta(1-\beta+\gamma)}{\beta} - \frac{c}{k} \right) u\end{aligned}$$

It is obvious after Lemma 1 that the following transformation yields a linear balanced growth path

$$G(h) = h^{((1-\beta+\gamma)/(1-\beta))} \quad \text{and hence} \quad \hat{G}(h) = \frac{1-\beta+\gamma}{1-\beta} \hat{h}$$

To avoid extra notation we don't introduce a new symbol for $G(h)$! Using a couple of short hands for parameter combinations

$$\psi := -\delta \frac{1-\beta+\gamma}{1-\beta}, \quad \phi := \frac{\beta}{\sigma} - 1, \quad \xi := \frac{\rho}{\sigma}, \quad \text{and} \quad \eta := \delta \frac{\beta-\gamma}{\beta}$$

the system is transformed into

$$\dot{k} = Ak^\beta (hu)^{1-\beta} - c \tag{6}$$

$$\dot{h} = -\psi(1-u)h \tag{7}$$

$$\dot{c} = (A(\phi+1)k^{\beta-1}(hu)^{1-\beta} - \xi)c \tag{8}$$

$$\dot{u} = \left(\frac{\delta}{\beta} - \eta + \eta u - \frac{c}{k} \right) u \tag{9}$$

The growth rates of variables are

$$\begin{aligned}\mu_k &= \beta A(uh/k)^{1-\beta} - c/k & \mu_h &= -\psi(1-u) \\ \mu_c &= (\phi+1)A(uh/k)^{1-\beta} - \xi & \mu_u &= \delta/\beta - \eta + \eta u - c/k\end{aligned}$$

By definition we have on a *BGP*

$$\mu_k = \mu_h = \mu_c = \mu; \quad \mu_u = 0$$

From the differential equation we deduce for the *BGP* (if it exists)⁵

$$\mu = \frac{\psi(\rho - \delta)}{\psi(\beta - \sigma) + \delta(\beta - \gamma)} \tag{10}$$

⁵In terms of Benhabib and Perli's notation q equals (c/k) and x corresponds to (k/h) .

$$u = 1 + \frac{\mu}{\psi} \quad (11)$$

$$\frac{c}{k} = \frac{\delta}{\beta} - \eta + \eta u \quad (12)$$

$$\frac{k}{h} = \left(\frac{\beta A}{\sigma \mu + \rho} \right)^{\frac{1}{1-\beta}} u \quad (13)$$

Parameter sets of $\theta = (A, \beta, \gamma, \delta, \rho, \sigma)$ which give existence to a *BGP* with different stability properties are subsets of $R_{++}^4 \times R_+ \times (0, 1)$:

$$\Theta_1 = \{\theta \mid 0 < \rho < \delta \text{ and } \sigma > 1 + \rho/\psi\}$$

$$\Theta_2 = \{\theta \mid \delta < \rho < -\psi \text{ and } \sigma < 1 + \rho/\psi\}$$

with separating surface

$$\Theta_3 = \{\theta \mid \rho = \delta \text{ and } \sigma = 1 + \rho/\psi = \gamma/(1 - \beta + \gamma)\}$$

Benhabib and Perli use the following projection to map the *BGP* into a steady state

$$x = k/h \quad \text{and} \quad q = c/k$$

and continue to use u . The Jacobian of the projected system in the steady state is

$$C = \begin{pmatrix} J_{11}^* & \frac{x}{u}(J_{11}^* + \psi u) & -x \\ 0 & \eta u & -u \\ J_{11}^* \phi_x^q & -J_{11}^* \phi_x^q & q \end{pmatrix}$$

where $J_{11}^* = A(\beta - 1)(x/u)^{\beta-1}$. The analysis of local stability of this steady state is done by Benhabib and Perli. They find uniqueness and stability of the solution if $\theta \in \Theta_1$, but possible multiplicity and instability for subsets of Θ_2 . In the following we want to demonstrate again that the analysis should be refined for the *BGP*.

The Jacobian of the system at an arbitrary admissible state is:

$$J = \begin{pmatrix} \beta(\mu_k + \frac{c}{k}) & (1 - \beta)(\mu_k + \frac{c}{k})\frac{k}{h} & -1 & (1 - \beta)(\mu_k + \frac{c}{k})\frac{k}{u} \\ 0 & \mu_h & 0 & -\mu_h \frac{h}{1 - u} \\ -(1 - \beta)(\mu_c + \xi)\frac{c}{k} & (1 - \beta)(\mu_c + \xi)\frac{c}{h} & \mu_c & (1 - \beta)(\mu_c + \xi)\frac{c}{u} \\ \frac{c}{k} \frac{u}{k} & 0 & -\frac{u}{k} & \mu_u + \eta u \end{pmatrix}$$

The Jacobian of the full (transformed) system on the BGP is given by:⁶

$$J^* = \begin{pmatrix} \tilde{J}_{11}^* & \frac{1-\beta}{\beta} \frac{k}{h} \tilde{J}_{11}^* & -1 & \frac{1-\beta}{\beta} \frac{k}{u} \tilde{J}_{11}^* \\ 0 & \mu & 0 & -\mu \frac{h}{1-u} \\ \tilde{J}_{31}^* \frac{c}{k} & -\tilde{J}_{31}^* \frac{c}{h} & \mu & -\tilde{J}_{31}^* \frac{c}{u} \\ \frac{c}{k} \frac{u}{k} & 0 & -\frac{u}{k} & \eta u \end{pmatrix}$$

where $\tilde{J}_{11}^* = \beta(\mu + c/k)$ and $\tilde{J}_{31}^* = (\beta - 1)(\mu + \xi)$. Notice that the Jacobian is not constant along the balanced growth path. Nonetheless, the eigenvalues are constant and only the eigenvectors change, as we will see below. The BGP determines an eigenvector $\pi^t = (1, h/k, c/k, 0)$ where k, h, c are taken from a vector on the BGP. We can use π to factorize the Jacobian and rescale the variables at the same time in order to simplify terms of the remaining matrix. Setting

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ h/k & h/k & 0 & 0 \\ c/k & 0 & c/k & 0 \\ 0 & 0 & 0 & u/k \end{pmatrix}$$

we get

$$H^{-1} J^* H = \begin{pmatrix} \mu & a^t \\ \mathbf{0} & B \end{pmatrix}$$

with

$$\begin{aligned} a^t &= \left(\frac{1-\beta}{\beta} \tilde{J}_{11}^*, \quad 0, \quad \frac{1-\beta}{\beta} \tilde{J}_{11}^* \right) \\ B &= \begin{pmatrix} \mu - \frac{1-\beta}{\beta} \tilde{J}_{11}^* & \frac{c}{k} & -\mu \frac{u}{1-u} - \frac{1-\beta}{\beta} \tilde{J}_{11}^* \\ -\tilde{J}_{31}^* - \frac{1-\beta}{\beta} \tilde{J}_{11}^* & \mu + \frac{c}{k} & -\tilde{J}_{31}^* - \frac{1-\beta}{\beta} \tilde{J}_{11}^* \\ 0 & -\frac{c}{k} & \eta u \end{pmatrix} \\ &= \begin{pmatrix} \beta\mu - (1-\beta)c/k & c/k & \beta\mu - (1-\beta)c/k + \psi \\ (1-\beta)(\xi - c/k) & \mu + c/k & (1-\beta)(\xi - c/k) \\ 0 & -c/k & \eta u \end{pmatrix} \end{aligned}$$

The eigenvalues of this matrix B are different from the eigenvalues of the Jacobian C of the steady state version of the model: we demonstrate the implications for stability with the following example. To do so we compare the eigenvalues of C and B at parameter values close to a situation where a Hopf bifurcation of C occurs.

⁶One may substitute the ratio $(1-\beta)/\beta$ by $\frac{\eta-\delta/(\sigma(\phi+1))}{\psi}$ in order to eliminate β from the Jacobian. But, it would complicate notations, and hence we refrain from doing so.

A	β	γ	δ	ρ	σ
1	0.75	0.3	0.049	0.05	0.42

implies for balanced growth

μ	k/h	c/k	u
0.007971014493	0.06315942029	36175.62507	0.9260573795

This yields for the scheme of Theorem 3 of matrix B and C , respectively:

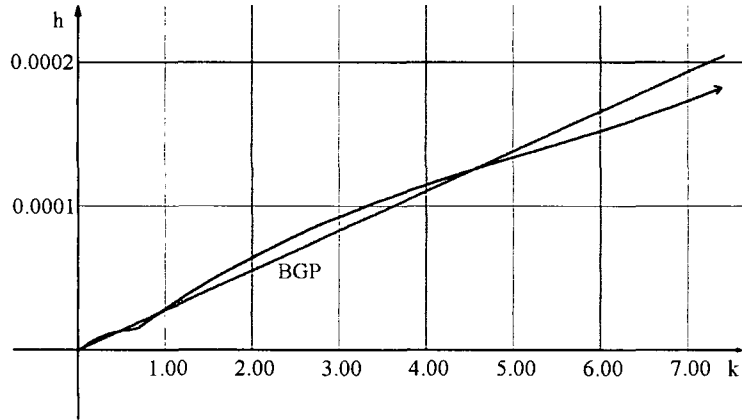
$$C: -1, 0.07260289856, 0.0003489955761, 0.00003349094300, \\ -, +, -, +, -, +,$$

$$B: -1, 0.1607529412, -0.005232500434, 0.001017762640, \\ -, +, -, +, -, +,$$

C has one positive and a pair of conjugate complex eigenvalues with real part smaller than but very close to zero, but a determinant clearly different from zero. The system is on the fringe of transition between a cyclically stable and a cyclically unstable steady state. In other words the asymptotic cone of the BGP is on the fringe to change from a line, i.e. the BGP itself, to a full dimensional cone containing the growth paths spiralling away from the BGP . By two figures we illustrate the instability of an optimal path in such a situation of this example. Figure 4 allows to compare the coordinates (k, h) along an optimal trajectory with the balanced growth path. Although the eigenvalues of the projected system imply stability the full system moves away from the BGP . Figure 5 shows the evolution of state and control variables of the same solution along the time axis. The see the solution moving along the balanced growth with extremely long waves the amplitudes of which do not deminish over time.

5 Conclusion

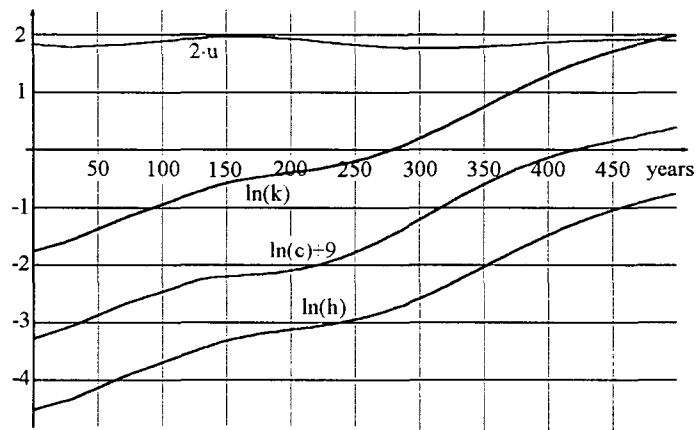
Along Romer's example of an endogenous growth model we analysed the asymptotic stability of the balanced growth path applying the method explained in the paper. The purpose of this exercise was to point out the difference between the stability concept for the balanced growth path itself opposed to the stability of the steady state of ratios of variables. In the example it obvious that the ratio of consumption or capital over the measure for the degree of deversification of



□

Figure 4: The Trajectory in State Space

□



□

Figure 5: Path with cycles

intermediate products are not the variables we should be interested in. It is the change of consumption and capital itself we primarily want to describe.

On the other hand, the type of saddlepoint stability is simple in some examples. In Romer's differentiated intermediate product model (1990) the uniqueness of optimal control is confirmed correctly by both approaches. But this is due to the absence of bifurcations in this example. If we consider a different type of model like Lucas' human capital model (1988) the necessity of analyzing the stability of the *BGP* itself becomes more important. In such a model a change of the parameter values may turn a situation with unique control into one with multiple solutions and asymptotically stable ones into unstable ones. The proper bifurcation points of such a model cannot be deduced from the steady state version of the dynamical system.

There is a more general objection against the use of projection methods. They are applicable only to a narrow class of dynamical models. The examples of the so called 'Endogenous Growth' literature are of this class, but the core of the new approach to economic growth urgently requires more general structures of modelling. The investigation of interactions between different engines of growth which endogenously generate growth impacts is a promising line of research. But, it has to be put on a structurally robust basis. And there is no doubt that elementary projection methods will play no major role in such a framework.

6 Appendix

First we give the

Proof of Lemma 1

The chain rule of differentiation yields \dot{y} as stated in the Lemma. The growth rates are transformed according to $\hat{y}_i = \mu_i \hat{x}_i$. Let I denote the index set of those components with $g_i \neq 0$ along C . Assume $x \in C$ and $y = H(x)$. Then the growth rate of y_i is equal to 0 if i is not in I . Otherwise $\hat{y}_i = \mu_i g_i = g$. Hence $H(C)$ is a line.

The Jacobian of \dot{y} results from chain and product rule for matrix differentiation using the fact that $\partial H(x)$ is diagonal. The second derivative of $H_i(x_i)$ equals $(\mu_i - 1)/x_i$ times the first derivative. With \tilde{M} being the diagonal matrix with entries $(\mu_i - 1)/x_i$ we get therefore

$$\begin{aligned}
\partial G(y) &= (\tilde{M} \partial H(x) F(x) + \partial H(x) \partial F(x)) \cdot \partial H(x)^{-1} \\
&= (\tilde{G} \partial H(x) + \partial H(x) \partial F(x)) \cdot \partial H(x)^{-1} \\
&= \tilde{G} + \partial H(x) \partial F(x) \partial H(x)^{-1}
\end{aligned}$$

□

The eigenvalues of the Jacobian apart from the balanced growth rate can be computed with the help change of coordinates. Let π be a vector spanning the *BGP*. Then π is an eigenvector of the Jacobian and without loss of generality we can assume that $\pi_1 = 1$. Now a change of coordinates which replaces the unit vector $(1, 0, \dots, 0)$ by π simplifies the Jacobian and factorizes the characteristic polynomial.

Lemma 2 *Let $\pi = (1, \pi_2, \dots, \pi_n)^t$ be an eigenvector of a matrix J with eigenvalue g and H a coordinate transformation of the form*

$$H = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \pi_2 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \pi_n & 0 & \cdots & 1 \end{pmatrix} \quad \text{with inverse} \quad H^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\pi_2 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -\pi_n & 0 & \cdots & 1 \end{pmatrix}$$

Then $H^{-1} J H$ is of the form

$$\begin{pmatrix} g & a^t \\ 0 & B \end{pmatrix}$$

for an $(n-1) \times (n-1)$ matrix B and an $(n-1)$ vector a . Hence, the transformation factorizes the characteristic polynomial

$$\chi_J(\lambda) = (g - \lambda) \chi_B(\lambda)$$

Furthermore the eigenvectors of J are of the form

$$\begin{pmatrix} 0 \\ x \end{pmatrix} + v \pi$$

where either $x = 0$ and v is arbitrary or (λ, x) , is a pair of eigenvalues and -vectors of B and v is determined by $v = a^t x / (\lambda - g)$ if $\lambda \neq g$ or v is arbitrary if $\lambda = g$ and $a^t x = 0$.⁷

⁷The usual embedding into the field of complex numbers yields the corresponding formula for pairs of complex conjugate eigenvalues and -vectors. In this case λ, v and the vector x are complex, and the real and the imaginary part of the eigenvectors span the plane the complex eigenvalue λ operates on.

Proof of Lemma 2

The inverse of H can be checked straight forwardly. JH is equal to J apart from the first column which is replaced by $g\pi$. $H^{-1}g\pi = g(1, 0, \dots, 0)^t$ is the first column of $H^{-1}JH$ as stated in the lemma. The factorization of the characteristic polynomial is an immediate consequence of the form of the conjugate matrix $H^{-1}JH$. The relation between eigenvectors of B and J can be checked in a straight forward manner. Assume, $Bx = \lambda x$ for a real eigenvalue λ . Then, the first component v of an eigenvector $y = (v, x)^t$ of $H^{-1}JH$ has to solve $gv + a^t x = \lambda v$. If $\lambda = g$ this holds only if and only if $a^t x = 0$, and then for arbitrary v . Otherwise, the condition can be solved for v as stated in the lemma. The corresponding eigenvector of J is equal to $Hy = (0, x^t)^t + v\pi$. \square

Of course there are alternative transformations yielding the same factorization but a different form of the reduced matrix B . For example, one may think of projecting the base vectors apart from π into the orthogonal complement of π . Then B is the Jacobian constraint to the orthogonal complement of π . The computational effort for this transformation is greater, but afterwards it may be easier to check the signs of the eigenvalues.

The stability of the balanced growth path can now be studied by the characteristic polynomial $\chi_B(\lambda)$ of B .

In many of our applications the dimension n of the dynamic system is equal to 4. Then the characteristic polynomial of the reduced Jacobian B is of the form

$$\chi_B(\lambda) = -\lambda^3 + Tr_B \lambda^2 - M_B \lambda + Det_B$$

where Tr denotes the trace of the matrix and M_B stands for the sum of the principal minors. Applying Routh's theorem (c.f. Gantmacher 1960) to this situation we get:

Theorem 3 *The number of eigenvalues of B with positive real parts is equal to the number of variations of sign in the following cheme of real numbers*

$$-1 ; Tr_B ; -M_B + \frac{Det_B}{Tr_B} ; Det_B$$

For example, there are two eigenvalues with real parts if $Tr_B > 0$ and $Det_B < 0$.

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