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## Estimation of a variable rate of depreciation: A dummy variable approach

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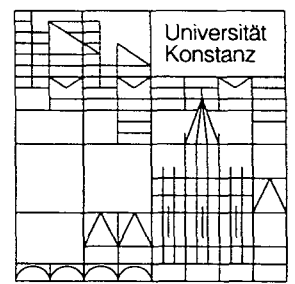
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# **Estimation of a Variable Rate of Depreciation: A Dummy Variable Approach**

**Ingmar Prucha<sup>1</sup>**

## **Abstract**

Many important economic problems require measures of physical and R&D capital. Except for some recent studies, there have been relatively few contributions in the literature that provide econometric estimates for the depreciation rates of physical and R&D capital. One reason for the relative paucity of such econometric studies may be that if the depreciation rates are viewed as unknown parameters, or as a functions of unknown parameters, then the corresponding stocks of physical and R&D capital are also unobserved, resulting in a formidable estimation problem. In the past econometric estimates of the depreciation rates of physical and R&D capital were typically obtained from self-programmed estimation algorithms. In this note we introduce an approach that permits the econometric estimation of constant and variable rates of depreciation of physical and R&D capital using standard econometric packages.

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# 1 Introduction<sup>1</sup>

Many important economic problems require measures of capital. For example, physical capital plays a pivotal role in studies on production and cost, and in studies on the sources of productivity and output growth. Also, knowledge capital approximated by R&D capital has been recognized as an important contributor to productivity growth. Furthermore, for policy purposes it is important to be able to distinguish between net and replacement investment in these types of capital. Crucial to an analysis of the contributions of physical and R&D capital is the measurement of the stocks of physical and R&D capital, which in turn typically requires measuring their depreciation rates. The conventional procedure for computing estimates of the stocks of physical and R&D capital is the perpetual inventory method; see, e.g., Hulten (1990) for a recent review. While the perpetual inventory method permits the incorporation of detailed information/assumptions (concerning the mean useful life of assets, the retirement distribution centered on that life and on efficiency pattern), these assumptions and hence the implied estimates of the depreciation rates are typically not subjected to rigorous econometric testing. Recently, however, there have been several studies that provide econometric estimates of the depreciation rates of the stocks of physical and/or R&D capital goods; see, e.g., Epstein and Denny (1980), Kokkelenberg(1984), Kollintzas and Choi (1985), Bischoff and Kokkelenberg (1987), Nadiri and Prucha (1994) and Prucha and Nadiri (1994).

One reason for the relative paucity of such econometric studies may be that if the depreciation rates are viewed as unknown parameters, or as a functions of unknown parameters, then the corresponding stocks of physical and R&D capital are also unobserved, resulting in a formidable estimation problem. Of course we can, in principle, always express the capital stocks as functions of the observed current and past investments and the unknown depreciation rates, substitute those functions for the stocks in, say, the production function or a set of factor demand equations, and then estimate the resulting relationships. However, in addition to the resulting complexity of the estimation problem, we then also encounter an implementation problem if we try to use standard econometric packages for this task. Estimation

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routines as programmed in standard econometric packages (such as, e.g., the LSQ procedure in TSP) typically do not permit the specification of functions where the argument list, say, increases with time. However, as discussed in more detail below, this is exactly the case if stocks are generated recursively from some initial stock in the usual way. Therefore, all of the above cited studies that report econometric estimates of depreciation rates have not used standard econometric packages, but found it necessary to program their own estimation routines, which is time consuming and requires specialized skills (or to settle for an approach that does not generate a fully consistent capital stock series). In the following we now show how a reformulation of the estimation problem, using dummy variables, permits the application of standard econometric packages. The approach discussed below will cover a variety of applications including not only constant but also variable depreciation rates, including situations where the depreciation rate of one capital good depends on the stock of some other capital good. While, on the one hand, the observations made below are simple, it still seems that, on the other hand, they were not evident in the past.

## 2 An Illustration of the Problem

In the following we give a simple example that illustrates the practical difficulties in econometrically estimating capital depreciation rates. Suppose a researcher would like to estimate the following production function:

$$Y_t = F(L_t, K_t, R_t, \theta), \quad t = 1, \dots, T, \quad (1)$$

where  $Y_t$  and  $L_t$  denote, respectively, output and labor input in period  $t$ , where  $K_t$  and  $R_t$  denote the end of period  $t$  stocks of two types of capital, e.g., physical and R&D capital, and where  $T$  denotes the sample size. With  $\theta$  we denote the vector of unknown model parameters. For simplicity, assume for the moment that only the depreciation rate of  $K_t$  is unknown and hence  $K_t$  is unobserved, but that  $R_t$  is observed. Assume further, for the moment, that the depreciation rate of  $K_t$ , say  $\delta^K$ , is constant. The capital stock  $K_t$  then accumulates according to the equation:

$$K_t = I_t^K + (1 - \delta^K)K_{t-1}, \quad (2)$$

where  $I_t^K$  denotes gross investment. Solving (2) recursively yields

$$K_t = k_t(I_t^K, \dots, I_1^K, K_0, t, \delta^K) \quad (3)$$

$$= \sum_{i=0}^{t-1} I_{t-i}^K (1 - \delta^K)^i + K_0 (1 - \delta^K)^t.$$

Substitution of this expression into the production function (1) gives:

$$\begin{aligned} Y_t &= F(L_t, k_t(I_t^K, \dots, I_1^K, K_0, t, \delta^K), R_t, \theta) \\ &= G_t(L_t, I_t^K, \dots, I_1^K, K_0, R_t, t, \delta^K, \theta). \end{aligned} \quad (4)$$

Note that  $L_1, \dots, L_T$  and  $I_1, \dots, I_T$  are observed and thus, in principle, we can estimate the production function parameters  $\theta$  and the depreciation rate  $\delta^K$  jointly from (4) by some standard estimation methods for nonlinear models.<sup>2</sup> However, as remarked above, there is a practical difficulty if we would like to apply a standard econometric package such as, e.g., TSP. The argument list of  $k_t(\cdot)$  and hence of the function  $G_t(\cdot)$  depends on  $t$ . However, as remarked above, standard econometric packages will typically not permit the specification of a function where the argument list changes with  $t$ ; they can hence not be employed directly to estimate (4).

### 3 A Dummy Variable Approach

In the following we now show how for a fairly wide class of models of capital accumulation we can ‘artificially’ re-write the recursive solution for the capital stock such that the argument list does not depend on  $t$ . We first consider the case where only  $K_t$  is unobserved and accumulates according to the equation:

$$K_t = I_t^K + K_t^o, \quad K_t^o = f(Z_t, \theta, \alpha) + a(Z_t, \theta, \alpha)K_{t-1}, \quad (5)$$

where  $K_t^o$  has the interpretation of the capital stock left over at the end of period  $t$  from the initial stock  $K_{t-1}$ , and where  $f(\cdot)$  and  $a(\cdot)$  are real valued functions that depend (possibly) on some vector of observed variables  $Z_t$  and (possibly) on the parameter vectors  $\theta$  and  $\alpha$ .<sup>3</sup> The parameter vector  $\alpha$

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<sup>2</sup>For a survey of estimation methods for dynamic nonlinear models see, e.g., Pötscher and Prucha (1991a,b). We note that if an estimate of the initial stock  $K_0$  is not available we may also treat the initial stock as an unknown parameter.

<sup>3</sup>The depreciation rate of capital is then given by  $\delta_t^K = 1 - K_t^o/K_{t-1}$ , and will in general be variable over time.

was included to allow for parameters that are not included in  $\theta$ . Solving (5) recursively for  $K_t$  yields:<sup>4</sup>

$$\begin{aligned} K_t &= k_t(I_t^K, \dots, I_1^K, K_0, Z_t, \dots, Z_1, \theta, \alpha) \\ &= \sum_{i=0}^{t-1} (I_{t-i}^K + f_{t-i}) \prod_{j=0}^{i-1} a_{t-j} + K_0 \prod_{j=0}^{t-1} a_{t-j}, \end{aligned} \quad (6)$$

where we use the abbreviations  $f_t = f(Z_t, \theta, \alpha)$  and  $a_t = a(Z_t, \theta, \alpha)$ .

The motivation for the specification in (5) is that it contains various models that have appeared in the literature as special cases:<sup>5</sup> By setting  $f(Z_t, \theta, \alpha) \equiv 0$  and  $a(Z_t, \theta, \alpha) \equiv \alpha$  equation (5) covers the case of a constant depreciation rate with  $\delta^K = 1 - \alpha$ , which has, e.g., been considered in Nadiri and Prucha (1994). By setting  $f(Z_t, \theta, \alpha) \equiv 0$  and  $a(Z_t, \theta, \alpha) = 1 - Z_t' \alpha$  we obtain a model where the depreciation rate depends linearly on  $Z_t$ . Suppose  $Z_t = (1, t)'$ , then the depreciation rate is a linear function of  $t$ , or if  $Z_t = (1, CU_t)'$ , where  $CU_t$  denotes a measure of capacity utilization, then we obtain the specification considered by Bischoff and Kokkelenberg (1987). The models considered in Epstein and Denny (1980), Kollintzas and Choi (1985) and Prucha and Nadiri (1994) are also readily seen to be special cases of (5) with  $a(Z_t, \theta, \alpha) = \alpha$  and where  $f(Z_t, \theta, \alpha)$  represents some function of prices and outputs.

We now define the following dummy variable:

$$D_t = \begin{cases} 1 & 1 \leq t \leq T, \\ 0 & -T \leq t < 1. \end{cases}$$

We furthermore assign arbitrary values to  $Z_t$  for  $t < 1$  such that  $f_t$  and  $a_t$  remain well defined. Then we can rewrite (6) as<sup>6</sup>

$$\begin{aligned} K_t &= k(I_t^K, \dots, I_{t-T+1}^K, K_0, Z_t, \dots, Z_{t-T+1}, D_t, \dots, D_{t-T+1}, \theta, \alpha) \\ &= \sum_{i=0}^{T-1} (I_{t-i}^K + f_{t-i}) D_{t-i} \prod_{j=0}^{i-1} a_{t-j} + K_0 \prod_{j=0}^{T-1} a_{t-j}^{D_{t-j}}. \end{aligned} \quad (7)$$

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<sup>4</sup>In the following expression we adopted the convention that  $\prod_{j=0}^{-1} a_{t-j} = 1$ .

<sup>5</sup>The dimensionality of  $\alpha$  varies in the subsequent examples, but will be obvious from the context.

<sup>6</sup>In the following expression we have used implicitly  $x^0 = 1$  for any real number  $x$ .



We note that the number of arguments of  $k(\cdot)$  no longer depends on  $t$ .<sup>7</sup> Thus the expression in (7) can be programmed in standard econometric packages and then substituted into some other equation such as, e.g., (1) for purposes of estimation. If  $a_t \equiv a$ , then the product expressions in (7) can be simplified to  $\prod_{j=0}^{i-1} a_{t-j} = a^i$  and  $\prod_{j=0}^{T-1} a_{t-j}^{D_{t-j}} = a^t$ . Of course, if  $R_t$  is also unobserved, accumulates analogously to (5), and there is no interaction between  $K_t$  and  $R_t$  in the accumulation equations, then the same approach can be applied for  $R_t$ .

We next consider the case where both  $K_t$  and  $R_t$  are unobserved, and where the accumulation equation for  $K_t$  depends both on  $K_{t-1}$  and  $R_{t-1}$ . In particular, we consider the case where  $K_t$  and  $R_t$  accumulate according to the following equations:

$$K_t = I_t^K + K_t^o, \quad K_t^o = f(Z_t, \theta, \alpha) + aK_{t-1} + cR_{t-1}, \quad (8a)$$

$$R_t = I_t^R + R_t^o, \quad R_t^o = g(Z_t, \theta, \alpha) + bR_{t-1}. \quad (8b)$$

The function  $g(\cdot)$  is defined analogously to  $f(\cdot)$ , and  $a$ ,  $b$ , and  $c$  represent scalars that may be elements of  $\alpha$  or  $\theta$ . The above accumulation equations cover again various models of interest. They contain, e.g., the set of accumulation equations considered in Prucha and Nadiri (1994) as a special case.<sup>8</sup> It can be checked that the recursive solution for  $K_t$  and  $R_t$  is now given by:

$$K_t = \sum_{i=0}^{t-1} \left[ (I_{t-i}^K + f_{t-i})a^i + (I_{t-i}^R + g_{t-i}) \frac{(a^i - b^i)c}{(a-b)} \right] + K_0 a^t + R_0 \frac{(a^t - b^t)c}{(a-b)}, \quad (9a)$$

$$R_t = \sum_{i=0}^{t-1} (I_{t-i}^R + g_{t-i})b^i + R_0 b^t, \quad (9b)$$

where we use the abbreviations  $g_t = g(Z_t, \theta, \alpha)$ . Of course, the above expression is only well defined for  $a \neq b$ , however  $(a^t - b^t)/(a - b) = \sum_{s=0}^{t-1} a^{t-1-s} b^s$ .

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<sup>7</sup>Of course, alternatively we could write  $K_t = \sum_{s=1}^T k_s D_t^s$  with  $D_t^s = 1$  for  $t = s$  and zero otherwise, and where  $k_s$  is defined in (6). However the resulting expression for  $K_t$  would then typically be too complex for being practically implementable.

<sup>8</sup>We could have further generalized the specification by allowing  $a$ ,  $b$ , and  $c$  to vary over time and/or for  $K_{t-1}$  to also appear in the expression for  $R_t^o$ . However, it seems that then the algebraic expressions for the recursive solution will in general become too complex to be practically implementable. An alternative, but more complex approach for the case of a single capital good with a constant depreciation rate is discussed in Prucha (1995).

The latter expression is also appropriate for  $a = b$ , but more complex to implement. The solution in (9) is best found by rewriting (8) in matrix notation and by observing that the following diagonalization holds:

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} = Q\Lambda Q^{-1}, \quad Q = \begin{bmatrix} a & -bc/(a-b) \\ 0 & b \end{bmatrix}, \quad \Lambda = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

and thus  $A^i = Q\Lambda^i Q^{-1}$ . Analogously to before, using the dummy variable  $D_t$ , we can rewrite (9) as

$$K_t = \sum_{i=0}^{T-1} \left[ (I_{t-i}^K + f_{t-i})D_{t-i}a^i + (I_{t-i}^R + g_{t-i})D_{t-i}\frac{(a^i - b^i)c}{(a-b)} \right] \quad (10a)$$

$$+ K_0a^t + R_0\frac{(a^t - b^t)c}{(a-b)},$$

$$R_t = \sum_{i=0}^{T-1} (I_{t-i}^R + g_{t-i})D_{t-i}b^i + R_0b^t. \quad (10b)$$

Again, in this formulation the size of the argument lists on the r.h.s. of (10) no longer depends on  $t$ , and hence the expressions can be programmed in standard econometric packages.

Of course, the above outlined dummy variable approach is not limited to two capital goods, but can be also applied to several capital goods, given their accumulation equations are of the form (5) and/or (8). To test the feasibility of the dummy variable approach we have applied the approach to re-estimate both the model in Nadiri and Prucha (1994) and that in Prucha and Nadiri (1994) using TSP. Originally the results reported in those papers have been obtained via a self-programmed FORTRAN estimation program. In both cases we were able to duplicate the previously obtained results.

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