

A Service of

ZBW

Leibniz-Informationszentrum Wirtschaft Leibniz Information Centre for Economics

Albert, Max

Working Paper Expectations and adjustment dynamics in the specificfactors model: Correction and extension

Diskussionsbeiträge - Serie II, No. 246

Provided in Cooperation with: Department of Economics, University of Konstanz

Suggested Citation: Albert, Max (1995) : Expectations and adjustment dynamics in the specific-factors model: Correction and extension, Diskussionsbeiträge - Serie II, No. 246, Universität Konstanz, Sonderforschungsbereich 178 - Internationalisierung der Wirtschaft, Konstanz

This Version is available at: https://hdl.handle.net/10419/101477

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



WWW.ECONSTOR.EU

Sonderforschungsbereich 178 "Internationalisierung der Wirtschaft"

Diskussionsbeiträge



Juristische Fakultät Fakultät für Wirtschaftswissenschaften und Statistik

Max Albert

Expectations and Adjustment Dynamics in the Specific-Factors Model —Correction and Extension

Expectations and Adjustment Dynamics in the Specific-Factors Model - Correction and Extension

Max Albert*

Serie II - Nr. 246

Januar 1995

* I am indebted to Jürgen Meckl and Michael Redford for valuable comments.

, ,

Abstract

This paper corrects the analysis, presented in Albert and Meckl (1991), of adjustment in a dynamic specific-factors model with endogenous capital stocks, and extends it to a multi-sector setup.

Errata

On p. 9

Replace the second and third paragraph by the following paragraphs:

- If the actual wage is *higher* than the long-run wage, EE gross investment I_j^{EE} is lower than LRE gross investment I_j^{LRE} , since LRE anticipate a sharper decrease in wages than EE.
- If, on the other hand, the actual wage is *lower* than the long-run wage, EE gross investment is higher than LRE gross investment, since LRE anticipate a sharper increase in wages than EE.

On p. 15

Replace the third sentence of the second paragraph by the following sentence: A conservative bias, i.e. a bias in the direction of SE, makes adjustment too fast, while expectations biased in the direction of LRE make adjustment too slow.

1 Introduction

The present paper discusses adjustment in a multi-sector version of the dynamic specific-factors (SF) model introduced by Albert (1989). Temporary equilibrium is described by the static equilibrium of the traditional SF model. The paper considers a small open economy facing constant prices and a constant rate of interest; all goods are traded freely. Capital accumulation proceeds under convex costs of adjustment. The focus is on analyzing the role of expectation formation for adjustment dynamics.

The present paper is, as the title indicates, an extension of a previous paper (Albert and Meckl 1991). The proof of global stability under rational expectations in Albert and Meckl (1991), while basically correct, contains a faulty argument; correction of this argument allows for an extension of the proof to any number of goods. Some overlap between both papers serves to make the present paper self-contained.

The paper is organized as follows. Section 2 reviews the basic model. An analysis of adjustment under extrapolative expectations follows in section 3. Section 4 discusses stability under rational expectations. Section 5 offers some concluding remarks.

2 The Model

Firm Behavior

The center piece of the model is the well-known dynamic model of a firm representing one sector of the economy. The firm is a price taker on all relevant markets and operates under convex costs of adjustment (cf. Treadway 1969, Söderström 1976). It is assumed that at each point t in time the firm tries to maximize the present value of net returns NR over an infinite horizon. The relevant maximization problem is given in (1); indices pertaining to the sector are dropped at this stage.

$$\max_{I(s),L(s)} \int_{t}^{\infty} NR(s) \cdot e^{i \cdot (t-s)} \, ds \tag{1}$$

s.t. $NR(s) \stackrel{\text{def}}{=} p \cdot x(s) - w^{\mathbb{F}}(s) \cdot L(s) - h(I(s))$ x(s) = f(K(s), L(s)) $\dot{K}(s) = I(s) - \delta \cdot K(s), \quad I(s) \ge 0$

 δ is a constant rate of depreciation; i > 0 is the interest rate at which the firm can borrow or lend financial capital. The firm's level of production x can be described by a production function f with arguments capital (K) and labor (L); we assume f to be linearly homogeneous, concave, and strictly quasi-concave. w^{E} is the expected wage rate. Assuming a cost-minimizing choice of L, we may write $p \cdot x - w^{E} \cdot L$ as $r^{E} \cdot K$, where r^{E} is the expected marginal value product (rental rate) of capital determined by the expected wage rate and the price p. h denotes the costs connected with gross investment I. Since all goods are traded at given world market prices, it is immaterial which and how many goods enter the production function for gross investment. We assume homogeneity and decreasing returns of this production function; h then is strictly convex, as assumed above. In order to guarantee smooth adjustment and to exclude corner solutions and hysteresis effects, which are of no interest in the present context, it is furthermore assumed that h(I) is defined for all $I \geq 0$ and that

$$h(0) = 0$$
, $Dh(0) = 0$, $Dh(\infty) = \infty$.

Under these assumptions, the non-negativity constraint for I will actually never be active.

(1) is to be solved for different forms of expectation formation. The first-order conditions of (1) are valid for all forms. The current-value Hamiltonian is given by

$$H(K, I, \lambda, \lambda_0) = \lambda_0 \cdot [r^{\mathcal{E}} \cdot K - h(I)] + \lambda \cdot [I - \delta \cdot K], \qquad (2)$$

where λ_0 is a constant. The necessary condition for the choice of I maximizing the Hamiltonian is

$$\lambda_0 \cdot Dh(I^{opt}) = \lambda. \tag{3}$$

Inserting the optimal value $I^{opt}(\lambda_0, \lambda(t))$ into the Hamiltonian yields the maximized Hamiltonian $H^{max}(K, \lambda, \lambda_0)$. From H^{max} we get the necessary conditions by application of the following theorem (cf. Feichtinger and Hartl 1986: ch. 2, esp. 39-44).

Theorem 1 (Necessary Conditions) The necessary conditions for an optimal solution of problem (1) are given by (3) and

$$\dot{\lambda} = i \cdot \lambda - D_1 H^{max}(K, \lambda, \lambda_0) = (i + \delta) \cdot \lambda - \lambda_0 \cdot r^E.$$
(4)

Furthermore the vector $[\lambda_0, \lambda(t)]$ must never vanish, i.e. if λ_0 equals zero, $\lambda(t)$ must not become zero (even in the limit), and if $\lambda(t)$ either actually becomes zero or goes to zero in the limit, λ_0 cannot be zero.

 $\lambda_0 = 0$ corresponds to the case of an infinite investment demand, since with finite investment demand the necessary optimality condition (3) would be violated. This case can be ruled out because infinite gross investment makes net returns NR negative for all times, which is suboptimal since I = 0 guarantees nonnegativity. Thus $\lambda_0 = 0$ is ruled out and it it possible to set $\lambda_0 = 1$ as usual. Solving (4) for $\lambda(t)$ yields

$$\lambda(t) = A \cdot e^{(i+\delta) \cdot t} + \int_t^\infty r^{\mathcal{E}}(s) \cdot e^{(i+\delta) \cdot (t-s)} \, ds$$

provided the integral exists. For now it is assumed that investors expect w and therefore r to converge to some long-run value. This implies

$$\lim_{t\to\infty}\int_t^\infty r^E(s)\cdot e^{(i+\delta)\cdot(t-s)}\,ds=\frac{r^*}{i+\delta}\,,$$

where r^* is the expected long-run value of r. The stable solution to problem (1) therefore is

$$\lambda(t) = \int_{t}^{\infty} r^{E}(s) \cdot e^{(i+\delta) \cdot (t-s)} \, ds \,. \tag{5}$$

Optimality of the stable solution can be shown by applying the following theorem (Feichtinger and Hartl 1986: 42-43).

Theorem 2 (Sufficient Conditions) If a solution $\overline{\lambda}(t)$, $\overline{K}(t)$ satisfies the firstorder conditions, if H^{max} is concave in K, and if the transversality condition

$$\lim_{t \to \infty} e^{-i \cdot t} \cdot \bar{\lambda}(t) \cdot \left[K(t) - \bar{K}(t) \right] \ge 0$$
(6)

holds for all admissible K(t), then the solution $\bar{\lambda}(t), \bar{K}(t)$ is optimal.

Since H is linear in K, H^{max} is also linear and thus concave in K. Hence (5) is an optimal solution of (4); the optimal λ is the expected present value of one unit of capital.¹

¹Although it seems to be obvious that the stable solution is the only optimum, this is not

Expectation Formation

We first determine investment under extrapolative expectations (EE).² Note that expectations are formed with respect to the development of the wage rate; all the other variables relevant to the firm are either assumed to be fixed or under the control of the firm. We consider the simplest form of EE given by

$$\ddot{w}^{\scriptscriptstyle EE}(s) = -\beta \cdot \dot{w}(s), \qquad 0 \le \beta \le \infty.$$

The solution of this second-order differential equation is

$$w^{EE}(s) = [w(t) - w^*] \cdot e^{-\beta \cdot (s-t)} + w^*, \ s \ge t,$$
(7)

where w^* is the long-run value of w. Later we will assume w^* to be the (unique) steady-state value of w in the *n*-sector model. This means that we assume a kind of long-run rationality of expectations, which is required for long-run consistency of EE.

(7) implies that the expected wage path runs between the present and the long-run wage. Given goods prices, wage expectations translate into expecations of the rental rate via the zero-profit condition b(w, r) = p, where b is the unit-cost function.³ Given p, r^{E} is a non-increasing and convex function of w^{E} . Thus (7) implies that the expected path of the rental rate of capital is always between the present rental rate r(t) and the long-run rate r^{*} . Taking this result into account, we get from (5) and (7) an analoguous result for the expected present value of capital under EE:

$$\lambda^{\mathcal{E}\mathcal{E}}(t) = \theta(t) \cdot \frac{r(t)}{i+\delta} + (1-\theta(t)) \cdot \frac{r^*}{i+\delta}, \qquad (8)$$

easy to prove. Note that the necessary conditions can be violated for a countable number of points in time, since only piecewise continuity is required for the solutions (cf. Feichtinger and Hartl 1986: ch. 2). This means that the constant A in the general solution for λ may jump. Thus one cannot argue that a single choice for $A \neq 0$ leads to inferior results; one has to show that even countably infinite changes in A cannot compensate for deviations from the stable solution. Uniqueness could be proved if H^{max} was strictly concave in K; unfortunately, this is not the case.

²In Albert & Meckl (1991) the slightly misleading term "adaptive" is used instead of "extrapolative".

³The zero-profit condition just states that wages and capital rentals exhaust revenue. Thus the condition holds even if the rental rate of capital is not given but maximized.

where $\theta \in [0, 1]$. With $\beta = 0$, and $\theta = 1$ accordingly, we have the special case of SE, where the firm expects the rental rate r to stay at its present value. $\beta = \infty$, and $\theta = 0$ accordingly, yields the case of LRE, where the firm expects the rental rate to jump to its steady-state value. For the subsequent analyses, it is immaterial whether β in (7) is constant or not. Moreover, it is immaterial whether expectations in different sectors are based on different values for β . Therefore we generalize the notion of EE such that all kinds of expectation formation compatible with (8) and with an arbitrary time path of θ in [0, 1] are included.

Let us briefly state the consequences of the above analysis for the resulting differential equation. With $\theta = 1$, we get SE investment as

$$I^{SE}(t) = m(\lambda^{SE}(t)) = m\left(\frac{r(t)}{i+\delta}\right), \qquad (9)$$

where $m(\lambda) \stackrel{\text{def}}{=} Dh^{-1}(\lambda)$. The function *m* will be used throughout the paper with the same definition. With $\theta = 0$, we get LRE investment as

$$I^{LRE}(t) = m(\lambda^{LRE}(t)) = m\left(\frac{r^*}{i+\delta}\right).$$
(10)

By (3), investment under EE is a weighted average of SE and LRE investment.

Rational expectations (RE) imply that the firm correctly anticipates changes in the wage rate and in the rental rate of capital. Thus we cannot solve for the firm's investment without knowledge of the time path of r. The following solution results from (5) and the condition $r^{E} = r$; it is based on the assumption that the integral converges:

$$\lambda^{RE}(t) = \int_{t}^{\infty} r(s) \cdot e^{(i+\delta) \cdot (t-s)} ds$$

$$I^{RE}(t) = m(\lambda^{RE}(t)) \qquad (11)$$

$$\dot{K}(t) = I^{RE}(t) - \delta \cdot K(t)$$

The Multi-Sector Model

We assume that each of the model's n sectors can be represented by a firm of the kind analyzed above. This yields 2n equations describing expectation formation and investment. To this we add the equation

$$\sum_{i=1}^{n} L_i = L \tag{12}$$

describing the assumption that sectoral labor demands, summed up, are equal to a fixed total labor supply. Sectoral labor demand at any given moment depends on the current stock of capital and is determined by the usual optimality condition

$$w = p_i \cdot D_2 f_i(K_i, L_i), \qquad (13)$$

where f_i is sector *i*'s production function. We get rid of all equations pertaining to labor reallocation by making use of the gross domestic product (GDP) function of the SF model which is defined by

$$y(\mathbf{p}, \mathbf{K}, L) \stackrel{\text{def}}{=} \max_{L_i} \left\{ \sum_{i=1}^n p_i \cdot f_i(K_i, L_i) : \sum_{i=1}^n L_i = L \right\},$$
(14)

where $\mathbf{p} \stackrel{\text{def}}{=} (p_1, \ldots, p_n)^{\mathrm{T}}$ and $\mathbf{K} \stackrel{\text{def}}{=} (K_1, \ldots, K_n)^{\mathrm{T}}$. This function describes the result of the labor reallocation process at every point in time; it is concave in \mathbf{K} and yields the rental rates of capital as $D_{n+i}y(\mathbf{p}, \mathbf{K}, L) = r_i$.⁴

As noted before, the rate of interest i as well as p are exogenous, i.e. if they change at all this is not anticipated even under RE. The dynamic system is given by

$$\lambda_{i}(t) = \int_{t}^{\infty} r_{i}^{E}(s) \cdot e^{(i+\delta) \cdot (t-s)} ds$$

$$I_{i}^{opt}(t) = m_{i}(\lambda_{i}(t)) \qquad (15)$$

$$\dot{\mathbf{K}}(t) = \mathbf{I}^{opt}(t) - \boldsymbol{\delta} \cdot \mathbf{K}(t),$$

where

$\delta \stackrel{\mathrm{\tiny def}}{=}$	$ \left(\begin{array}{cccc} \delta_1 & 0 & \cdots & 0 \\ 0 & \delta_2 & \cdots & 0 \end{array}\right) $	
	$\begin{array}{c} \vdots \vdots \ddots \vdots \\ 0 & 0 & \cdots & \delta_n \end{array}$	

and $\mathbf{I} \stackrel{\text{def}}{=} (I_1, \ldots, I_n)^{\mathrm{T}}$.

⁴Except for very special cases, i.e. (locally) identical production functions in different sectors, y will also be *strictly* concave in **K**.

3 Extrapolative–Expectations Adjustment

Global Stability under Static and Long-Run Expectations

Systems resulting from SE, LRE, EE and RE have the same steady state, since in a steady state these kinds of expectation formation coincide. This steady state can be described by the following maximization problem:

$$g(\mathbf{p}, L, i) \stackrel{\text{def}}{=} \max_{\mathbf{K}} \left\{ y(\mathbf{p}, \mathbf{K}, L) - \sum_{j=1}^{n} \frac{i+\delta_{i}}{\delta_{i}} \cdot h_{i}(\delta_{i} \cdot K_{i}) \right\}$$
(16)

The *n* first-order conditions of (16) are given by

$$D_{n+i}y(\mathbf{p},\mathbf{K},L) = (i+\delta_i) \cdot Dh_i(\delta_i \cdot K_i).$$
(17)

These conditions are identical to the steady-state conditions of the dynamic system (15) as can easily be verified. In the following we consider the maximand

$$V(\mathbf{K}) \stackrel{\text{def}}{=} y(\mathbf{p}, \mathbf{K}, L) - \sum_{j=1}^{n} \frac{i + \delta_{i}}{\delta_{i}} \cdot h_{i}(\delta_{i} \cdot K_{i})$$

in (16). $V(\mathbf{K})$ is strictly concave in \mathbf{K} due to the concavity of y in \mathbf{K} and the strict convexity of the adjustment-cost functions. Thus the solution of (17), and hence the steady state, exists, is unique and the global maximum of $V(\mathbf{K})$.

The function $V(\mathbf{K})$ plays an important role in the analysis of the system's stability. In the case of SE, we prove global stability by using $V(\mathbf{K})$ as a Liapunov function. The time derivative of this function under the dynamic system is given by

$$\dot{V}(\mathbf{K}) = DV(\mathbf{K}) \cdot \dot{\mathbf{K}} = \sum_{i=1}^{n} \left[r_i - (i + \delta_i) \cdot Dh_i(\delta_i \cdot K_i) \right] \cdot \left[I_i^{SE} - \delta_i \cdot K_i \right].$$
(18)

Since $I^{SE}_{i} = m_i(\lambda_i^{SE})$ and $\lambda_i^{SE} = r_i/(i + \delta_i)$, (18) is strictly positive everywhere except for the steady state, where it is zero. Therefore $V(\mathbf{K})$ will always converge to its unique maximum under the dynamic system. Since the maximum is identical with the steady state, this shows that the system is globally stable.

According to (10), LRE investment is constant and equal to its steady-state rate. Thus adjustment to the steady state is globally stable. Integration of the equations of motion yields

$$K_i(t) = [K_i(0) - K_i^*] \cdot e^{-\delta_i \cdot t} + K_i^*, \qquad (19)$$

where an asterisk denotes steady-state values. If rates of depreciation are identical, i.e. if $\delta_i = \delta$ for all *i*, we have the case of constant and intersectorally identical speeds of adjustment; then and only then the LRE system moves on a straight line.

Elements of Capital–Stock Dynamics

Surprisingly, stability of the dynamic system under EE can be proved with elementary considerations. However, the analysis is restricted to the case of identical rates of depreciation: $\delta_i = \delta$ for all *i*.

It is well-known from the SF model that the iso-wage surfaces in K-space are hyperplanes. With a given wage rate the vector of sectoral labor intensities $\mathbf{q} \stackrel{\text{def}}{=} (q_1, \ldots, q_n)^{\text{T}}$ is determined. This allows us to write (12) as

$$\mathbf{q}^{\mathrm{T}}\mathbf{K} = \sum_{j=1}^{n} \frac{K_j}{k_j(w, p_j)} = L$$
(20)

which is the equation of a *n*-dimensional hyperplane cutting all the axes at positive values, since its normal vector \mathbf{q} has only positive components. k_j is the capital intensity in sector j; via the zero-profit conditions, it depends on the wage rate and on p_j but on nothing else. Now consider dynamic adjustment. If

$$\mathbf{q}^{\mathrm{T}}\dot{\mathbf{K}} = \mathbf{0}\,,\tag{21}$$

the system stays on the same iso-wage plane, i.e. the wage rate does not change locally: $\dot{w} = 0$. If the left-hand side of (21) is positive, however, the system goes on to a higher iso-wage plane. Substituting for \dot{K}_j according to (15) yields

$$\mathbf{q}^{\mathrm{T}}\mathbf{I} = \delta \cdot L \tag{22}$$

instead of (21). Thus we get the following result:

$$\mathbf{q}^{\mathrm{T}}\mathbf{I} > \delta \cdot L \Leftrightarrow \dot{w} > 0$$

$$\mathbf{q}^{\mathrm{T}}\mathbf{I} = \delta \cdot L \Leftrightarrow \dot{w} = 0$$

$$\mathbf{q}^{\mathrm{T}}\mathbf{I} < \delta \cdot L \Leftrightarrow \dot{w} < 0$$

(23)

Economically, this implies that wages go up (fall) if a certain weighted average of gross investment over all sectors is high (low) enough. This analysis is the point of departure for a simple proof of global stability for EE adjustment.

Global Stability

EE are defined by the condition that the expected present values of capital are always between the values under LRE and under SE. As already shown above, both SE and LRE generate globally stable adjustment and monotonic wage adjustment. For the more general case of EE, we can distinguish three cases.

- If the actual wage is higher than the long-run wage, EE gross investment I_j^{EE} is higher than SE gross investment I_j^{SE} , since EE anticipate falling wages while SE assume that wages stay high.
- If, on the other hand, the actual wage is *lower* than the long-run wage, EE gross investment is lower than LRE gross investment, since LRE anticipate a sharper increase in wages than EE.
- If the actual wage is equal to the long-run wage, LRE and EE come to the same thing as SE: the wage is expected to remain constant. Gross investment then coincides for all three.

This analysis implies the following results (where w^* again denotes the long-run wage):

$$\mathbf{q}^{\mathrm{T}}\mathbf{I}^{LRE} \geq \mathbf{q}^{\mathrm{T}}\mathbf{I}^{EE} \geq \mathbf{q}^{\mathrm{T}}\mathbf{I}^{SE} > \delta \cdot L \Leftrightarrow w > w^{*}$$
$$\mathbf{q}^{\mathrm{T}}\mathbf{I}^{LRE} = \mathbf{q}^{\mathrm{T}}\mathbf{I}^{EE} = \mathbf{q}^{\mathrm{T}}\mathbf{I}^{SE} = \delta \cdot L \Leftrightarrow w = w^{*}$$
$$\mathbf{q}^{\mathrm{T}}\mathbf{I}^{SE} \leq \mathbf{q}^{\mathrm{T}}\mathbf{I}^{EE} \leq \mathbf{q}^{\mathrm{T}}\mathbf{I}^{LRE} < \delta \cdot L \Leftrightarrow w < w^{*}$$

As a comparison of (24) with (23) shows, EE inherit the property of monotonic wage adjustment from SE and LRE. This proves global stability of EE: On the long-run iso-wage plane, stability is guaranteed, since the EE system is identical to the SE or LRE system; the system moves on a straight line and never leaves the long-run iso-wage plane. For the rest of the phase space, convergence to the long-run wage follows from the fact that the wage rises (falls) whenever it is lower (higher) than the long-run wage.

The economic reason behind global stability of EE (including the extreme cases of SE and LRE) is provided by the fact that expectations are *qualitatively rational*: Expectations are rational with respect to the direction of a change in wages. This property of the model suffices to guarantee global stability although there is considerable scope for *systematic* errors in expectation formation. In view of this result it comes as no surprise that RE are just a special case of EE in this model, sharing the properties of global stability and monotonic wage adjustment with the other forms of expectation formation. This is shown in the next section.

4 Rational–Expectations Adjustment

We analyse RE adjustment in four steps. First we show that stability of adjustment is a sufficient condition for an intertemporal social optimum. Then we demonstrate the local existence of a stable path. The next step is a proof that stable RE adjustment is a special case of EE adjustment; this implies that RE wage adjustment is monotonic, too. Last, we show the existence of a stable path for all points in the interior of K-space. Since stable paths are optimal, this proves global existence of a socially optimal and stable RE path.

Optimality of Stable Adjustment

The dynamic system under RE is identical to a system resulting from an optimalcontrol problem for the entire economy. Optimal adjustment in a model of a small open economy means optimization of the present value of total net revenues:

$$\max_{\mathbf{I}} \left\{ \int_{t}^{\infty} \left[y(\mathbf{p}, \mathbf{K}, L) - \sum_{i=1}^{n} h_{i}(I_{i}) \right] \cdot e^{i \cdot (t-s)} \, ds : \dot{\mathbf{K}} = \mathbf{I} - \boldsymbol{\delta} \cdot \mathbf{K} \right\}$$
(25)

The Hamiltonian of (25) is

$$H(\mathbf{K}, \mathbf{I}, \boldsymbol{\lambda}, \lambda_0) = \lambda_0 \cdot \left[y(\mathbf{p}, \mathbf{K}, L) - \sum_{i=1}^n h_i(I_i) \right] + \sum_{i=1}^n \lambda_i \cdot \left[I_i - \delta_i \cdot K_i \right], \quad (26)$$

where $\lambda \stackrel{\text{def}}{=} (\lambda_1, \dots, \lambda_n)^{\text{T}}$. By arguments analoguous to those used in the discussion of firm behaviour, it can be shown that λ_0 can be set equal to unity. Furthermore the necessary conditions are identical with the differential equations of the RE system. By (the multi-dimensional analogue of) theorem 2 we now prove that stability is a sufficient condition for optimality. We note that the Hamiltonian H and therefore the maximized Hamiltonian H^{max} is always concave in **K**. We consider an admissible trajectory $\tilde{\mathbf{K}}(t)$ that fulfills the necessary conditions. If the corresponding costate vector $\tilde{\mathbf{\lambda}}(t)$ behaves such that for all the

other admissible trajectories $\mathbf{K}(t)$ the transversality condition

$$\lim_{t \to \infty} \bar{\boldsymbol{\lambda}}^{\mathrm{T}}(t) \cdot \left[\mathbf{K}(t) - \bar{\mathbf{K}}(t) \right] \cdot \mathrm{e}^{-i \cdot t} \ge 0$$
(27)

is fulfilled, then the path is optimal according to theorem (2). However, under the assumption that the solution under consideration is stable, the transversality condition is obviously fulfilled: On a stable path $\bar{\lambda}(t)$ is non-negative and bounded, since it is equal to the present value of a unit of capital; $\mathbf{K}(t)$ is of course non-negative for all t; and $\bar{\lambda}^{\mathrm{T}}\bar{\mathbf{K}} \cdot e^{-i \cdot t}$ vanishes in the long run. Thus stability is sufficient for optimality as in the problem of the single firm.

Local Stability

We linearize the system (15) around the steady state; this results in the following linear approximation:

$$\begin{bmatrix} \dot{\mathbf{K}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \mathbf{J} \cdot \begin{bmatrix} \mathbf{K} - \mathbf{K}^* \\ \boldsymbol{\lambda} - \boldsymbol{\lambda}^* \end{bmatrix}$$
(28)

The matrix **J** is a partitioned matrix consisting of four $n \times n$ submatrices:

$$\mathbf{J} = \begin{pmatrix} \mathbf{A} \ \mathbf{B} \\ \mathbf{C} \ \mathbf{D} \end{pmatrix}$$
(29)

The submatrices are

$$\mathbf{A} = \begin{pmatrix} -\delta_1 & 0 & \cdots & 0 \\ 0 & -\delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\delta_n \end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix} Dm_1(\lambda_1^*) & 0 & \cdots & 0 \\ 0 & Dm_2(\lambda_2^*) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Dm_n(\lambda_n^*) \end{pmatrix}$$
$$\mathbf{C} = \begin{pmatrix} i + \delta_1 & 0 & \cdots & 0 \\ 0 & i + \delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & i + \delta_n \end{pmatrix}$$

and $\mathbf{D} = D_K^2 y(\mathbf{p}, \mathbf{K}, L)$. The following proof shows that the 2*n* eigenvalues of **J** are all real with *n* eigenvalues positive and *n* negative. Thus there exists an *n*-dimensional hyperplane in which the system has a stable node. This hyperplane is tangential to the stable manifold of the original system at the steady state.

Proof: The matrix $\bar{\mathbf{J}} = \mathbf{J} - \mu \cdot \mathbf{I}$ again is a partitioned matrix, where the submatrices \mathbf{A} and \mathbf{C} are modified by substracting μ from the main diagonal. We denote the modified matrices by $\bar{\mathbf{A}}$ and $\bar{\mathbf{C}}$. The eigenvalues are the solutions of $|\bar{\mathbf{J}}| = 0$ in μ . Applying the formula for determinants of partitioned matrices,

$$\left| \bar{\mathbf{J}} \right| = \left| \mathbf{B} \right| \left| \mathbf{D} - \bar{\mathbf{A}} \mathbf{B}^{-1} \bar{\mathbf{C}} \right|$$

we get the result that $|\bar{\mathbf{J}}|$ is proportional to $|s \cdot \mathbf{A} - \mathbf{G}|$ with $s = i^2/4 - (\mu - i/2)^2$. **G** is the Hessian matrix of optimization problem (16), which yields the steady state. **G** is negative definite almost everywhere by our assumption of strict concavity of problem (16)'s target function $V(\mathbf{K})$.

If $s \ge 0$, the matrix $s \cdot \mathbf{A} - \mathbf{G}$ is positive definite and thus regular. In order to fulfill the condition $|s \cdot \mathbf{A} - \mathbf{G}| = 0$ it is necessary that s < 0; therefore we have $0 < i^2/4 < (\mu - i/2)^2$ which implies that the eigenvalues are real. Obviously there are *n* pairs of eigenvalues which lie symmetrically around i/2. Since i > 0, one of the eigenvalues of a pair is greater than *i*, the other being smaller than zero; thus we have *n* positive and *n* negative eigenvalues.

This ensures the existence of a saddle-point at the steady state.

Monotonicity of Wage–Rate Adjustment

In this section we will show that on stable adjustment paths the wage rate behaves monotonically. Again we have to restrict the analysis to the case of identical rates of depreciation. Let us consider a stable RE path. The present value of a unit of capital in sector i is equal to

$$\lambda_i^{RE}(t) = \int_t^\infty r_i(s) \cdot e^{(i+\delta) \cdot (t-s)} \, ds \tag{30}$$

as stated in section 2. Assume for a moment that it is true what we intend to prove: that the wage rate, and therefore r_i , behaves strictly monotonically under RE. Under these circumstances we have the following properties of $\lambda_i^{RE}(t)$ in the three sections of the K-space defined above:

I. If the wage rate is equal to its long-run value, we have

$$\frac{r_i(t)}{i+\delta} = \lambda_i^{\text{SE}}(t) = \lambda_i^{\text{RE}}(t) = \lambda_i^{\text{LRE}}(t) = \frac{r_i^*}{i+\delta} \,.$$

All adjustment paths coincide.

II. If the wage rate is above its long-run value, r_i will grow over time. SE are too pessimistic, and we have

$$\lambda_i^{\text{SE}}(t) < \lambda_i^{\text{RE}}(t) < \lambda_i^{\text{LRE}}(t) \, .$$

III. If the wage rate is below its long-run value, r_i will fall over time. SE are too optimistic, and we have

$$\lambda_i^{\scriptscriptstyle SE}(t) > \lambda_i^{\scriptscriptstyle RE}(t) > \lambda_i^{\scriptscriptstyle LRE}(t) \, .$$

Strict inequalities follow from the assumption of strict monotonicity and hold for every finite t. This shows that except for region I, where stability is no problem, RE investment in all sectors will be strictly between SE and LRE investment. Given our as yet unproven assumption of strictly monotonic behavior of the wage rate, RE adjustment is a special case of EE adjustment.

The crucial point, then, is to establish the monotonicity of the wage rate under RE which we have assumed up to now. Local analysis shows that the equilibrium is a stable node in the stable manifold. Therefore we know that at least the last part of the path shows a strictly monotonic behavior of the wage rate. We now argue backwards from the equilibrium to show that if the last part of the path has this property, the whole path must have it. We prove this by deriving a contradiction from the assumption of non-monotonicity.

Consider the last extremal point of the wage rate before the steady state. At this point the path is tangential to some iso-wage plane. After this point of tangency the wage rate behaves monotonically on the path. At the point of tangency, however, the strict inequalities above pertaining to the present values of capital units must hold, since (i) the wage rate now is different from its longrun value and (ii) will change monotonically. Hence the direction of movement is transversal to the iso-wage plane, which contradicts our assumption that we have an extremal point. Therefore extremal points cannot occur on stable paths.

Existence of Stable Paths

As has been mentioned above, it is difficult to prove global stability for RE adjustment. The nature of the problem is easily explained. Consider a control problem with a single state variable x and a corresponding costate variable μ . Assume there is a saddle-point equilibrium. Let the stable manifold in phase space be the graph of a function $\mu(x)$ with $\lim_{h\to+0} \mu(\bar{x}+h) = \infty$ such that for values $x \leq \bar{x}$ there exists no stable manifold. Thus on the x-axis there exists a region where the costate variable μ can be chosen such that adjustment is stable; this region is bounded from below by \bar{x} . In multi-dimensional problems it is difficult to explore the region in state-space where stable choices of costate variables are possible; local stability only shows that such a region exists but tells nothing about its extension. Liapunov functions, which solve this problem, are difficult to find, since one needs a Liapunov function for *optimal* trajectories.

The argument demonstrating the monotonicity of wage-rate adjustment, which is already to be found in Albert and Meckl (1991), is not sufficient for global existence of stable paths. However, it can easily be extended to a sufficient argument. Note that in the example of non-existence it is necessary that the costate variable must go to infinity when the border of the region in state space where a stable path exists is approached. In the model of this paper, this means that the present value of units of capital in at least one sector must go to infinity. This is impossible in the interior of K-space, since the present value is always in the interval $[r_i/(i + \delta), r_i^*]$, where r_i^* is fixed and r_i is finite. Thus there is no boundary between a region in K-space where stable RE paths exist and a region where they do not exist. The basin of attraction of the steady state therefore covers \mathbb{R}_{++}^n . Independently from the starting point, there always exists an optimal path converging to the steady state. However, this result depends again on the assumption of identical rates of depreciation.

This completes our analysis of the RE system. Two appendices contain additional material on the system that might be useful for further extensions of the analysis.

5 Conclusions

The analysis of the model's dynamic behavior stresses the role of expectation formation, covering a range of expectation regimes, formally defined as a generalization of extrapolative expectations (EE). This range includes rational (RE), static (SE), and long-run (LRE) expectations as special cases. The usual assumptions of SE or RE are both rather unrealistic; in many cases it seems more reasonable to assume that agents correctly anticipate the sign, positive or negative, of a change in prices but not the actual extent or absolute value of the change. In order to find a range of expectation regimes that fulfill this requirement we started with a simple form of EE. With this kind of EE, the expected present value of one unit of capital, λ^{SE} , is a weighted average of the expected present values under SE and LRE, λ^{SE} and λ^{LRE} . The range of expectations considered contains all kinds of expectation formation for which this is also true, i.e. for which $\lambda^{S} \in [\lambda^{SE}, \lambda^{LRE}]$. It is one of the main results that, with appropriate simplifications, RE belong to this spectrum.

Adjustment proceeds qualitatively alike for all kinds of expectations considered. RE yield optimal adjustment. A conservative bias, i.e. a bias in the direction of SE, makes adjustment too slow, while expectations biased in the direction of LRE make adjustment too fast. This seems to be a replication of the results of Mussa (1978). However, it is important that speed of adjustment has to be measured by the speed of wage adjustment. This point does not emerge from Mussa's analysis. If one looks at net investment, there is no clear-cut message concerning the effects of biases in expectations on the speed of adjustment. It depends on the way the speed of adjustment is measured whether Mussa's results generalize or not.

The analysis of EE and RE adjustment with rates of depreciation differing across sectors remains an unsolved problem. Obviously small differences will not destroy global stability. That much follows from continuity considerations. However, big differences may affect the system's behavior more dramatically.

A The Isokine–Surfaces of the Control Problem

This appendix contains information on the $(\dot{\lambda} = 0)$ - and $(\dot{K} = 0)$ -surfaces of the system resulting from section 4's control problem.

The $(\dot{\lambda} = 0)$ -Surface

The $(\dot{\lambda} = 0)$ -surface is given by $(\lambda^*(\mathbf{K}), \mathbf{K})$ with $\lambda^*(\mathbf{K}) \stackrel{\text{def}}{=} D_K y(\mathbf{p}, \mathbf{K}, L)/(i+\delta)$. The surface has the following properties:

- All K on the same iso-wage hyperplane are mapped to the same point λ^* .
- If $K_i \to \infty$ for at least one *i*, it follows that $\lambda_i^* \to 0$ for all *j*.
- If $K_i > 0$ for at least one *i*, it follows that $\lambda_j^* < \infty$ for all *j*.
- If $\mathbf{K} \to \mathbf{0}$, it follows that $\lambda_j^* \to \infty$ for all j.

The surface's intersection with λ -space is a line with positive slopes because there is one λ for every w and each λ_i^* falls with rising w. The $(\dot{\lambda} = 0)$ -surface has dimension n, since there are n-1 vectors perpendicular to λ -space spanning the iso-wage surface.

The $(\dot{\mathbf{K}} = \mathbf{0})$ -Surface

The $(\mathbf{K} = \mathbf{0})$ -surface is given by $(\lambda^*(\mathbf{K}), \mathbf{K})$ with $\lambda_i^*(K_i) = Dh_i(\delta_i \cdot K_i)$. The surface is the graph of a diffeomorphism from \mathbb{R}^n_+ onto itself and therefore has dimension n. The projection of the surface into the (λ_i, K_i) -plane is a curve with positive slope starting at the origin.

B A Liapunov Function of the Control Problem

This appendix contains further information on global behavior of the solutions of section 4's control problem (25). Independently from the stability proof of section 4, it is possible to show that in certain cases every optimal bounded path is stable. Thus closed orbits and bounded chaotic attractors are ruled out.

The Liapunov Argument

If an optimum exists, we can define an optimum-value function $W(\mathbf{K}(t))$ that gives the maximum value of the integral as a function of capital stocks at time t:

$$\max_{\mathbf{I}} \left\{ \int_{t}^{\infty} \left[y(\mathbf{p}, \mathbf{K}, L) - \sum_{i=1}^{n} h_{i}(I_{i}) \right] \cdot e^{i \cdot (t-s)} \, ds : \, \dot{\mathbf{K}} = \mathbf{I} - \boldsymbol{\delta} \cdot \mathbf{K} \right\}$$
(31)

а. c

The proof proceeds from two assumptions. First, the value function (31) is assumed to be defined everywhere. This just means that we assume that the problem has a solution for every **K**. Second, the value function is assumed to be twice continously differentiable. Starting from these assumptions, it is shown by a Liapunov argument that every bounded optimal path conververges to the steady state. We first prove a lemma and then proceed to the Liapunov argument.

Lemma If the value function (31) exists and is twice continuously differentiable, it follows that $\dot{\boldsymbol{\lambda}}^{\mathrm{T}} \dot{\mathbf{K}} < 0$ for all solutions $\boldsymbol{\lambda}$, \mathbf{K} of the problem but the steady state itself.

Proof: We show that the value function is strictly concave if it exists. From this the lemma follows by a standard argument (cf. Feichtinger and Hartl 1986: 148). Concavity follows from (i) the linearity of the dynamic constraints and (ii) the concavity of the function under the integral. (i) guarantees that, given two time paths $I^{1}(t)$ and $I^{2}(t)$ for investment, the path

$$\mathbf{I}^{3}(t) \stackrel{\text{def}}{=} \alpha \cdot \mathbf{I}^{1}(t) + (1-\alpha) \cdot \mathbf{I}^{2}(t)$$

with $0 < \alpha < 1$ has the following property: Given the time paths $\mathbf{K}^{1}(t)$, $\mathbf{K}^{2}(t)$ and $\mathbf{K}^{3}(t)$ of capital resulting from the three investment paths, we always have

$$\mathbf{K}^{3}(t) = \alpha \cdot \mathbf{K}^{1}(t) + (1 - \alpha) \cdot \mathbf{K}^{2}(t).$$

Because of (ii), the third path leads to a function value higher than the α -weighted average of the function values of paths 1 and 2 at every point of time. Since the integration operator is linear, the same holds for the integrals over the three paths. Now assume that paths 1 and 2 are optimal paths. The integrals over these paths are then given by the value function as $W(\mathbf{K}^1)$ and $W(\mathbf{K}^2)$. The integral over the third path, which we denote by S, is a *lower* bound for $W(\mathbf{K}^3)$, since the third path need not be optimal; thus we have $S \leq W(\mathbf{K}^3)$. We have already established that $S > \alpha \cdot W(\mathbf{K}^1) + (1 - \alpha) \cdot W(\mathbf{K}^2)$; it follows that

$$W(\alpha \cdot \mathbf{K}^{1} + (1 - \alpha) \cdot \mathbf{K}^{2}) = W(\mathbf{K}^{3}) > \alpha \cdot W(\mathbf{K}^{1}) + (1 - \alpha) \cdot W(\mathbf{K}^{2})$$

showing that the value function is strictly concave.

The following argument uses the Liapunov function

$$V(\mathbf{K}, \boldsymbol{\lambda}) \stackrel{\text{def}}{=} y(\mathbf{p}, \mathbf{K}, L) - \frac{i+\delta}{\delta} \cdot \sum_{i=1}^{n} g_i(\lambda_i) + \frac{i+\delta}{\delta} \sum_{i=1}^{n} \lambda_i \cdot (m_i(\lambda_i) - \delta K_i), \quad (32)$$

where

$$\begin{array}{l}
 m_i(\lambda_i) \stackrel{\text{def}}{=} Dh_i^{-1}(\lambda_i) \\
 g_i(\lambda_i) \stackrel{\text{def}}{=} h_i(m_i(\lambda_i))
\end{array}$$
(33)

and therefore

$$Dg_i(\lambda_i) \equiv \lambda_i \cdot Dm_i(\lambda_i).$$
 (34)

Below it is shown that the time-derivative of $V(\mathbf{K}, \boldsymbol{\lambda})$ under the RE system is

. .

$$\dot{V}(\mathbf{K}(t), \boldsymbol{\lambda}(t)) = \frac{i}{\delta} \cdot \dot{\boldsymbol{\lambda}}^{\mathrm{T}} \dot{\mathbf{K}}, \qquad (35)$$

which by the lemma above is negative outside the steady state. According to a variant of Liapunov's theorem (cf. Feichtinger and Hartl 1986: 150), this means that, if a trajectory has any cluster points at all, they must be in the set of all points where $\dot{V} = 0$. In the present case this set is just the steady state. Thus all optimal trajectories are either stable or unbounded. While every bounded optimal trajectory is stable, the existence of unbounded optimal trajectories is not ruled out by the Liapunov argument. However, it is quite likely that one may find an additional argument filling the gap. In this case, the argument of this appendix would form part of an alternative proof of global stability. However, the proof presupposes existence of the value function.

Properties of the Liapunov Function

The Liapunov function (32) has the following properties:

$$DV(\mathbf{K}, \boldsymbol{\lambda}) = \left(-\dot{\boldsymbol{\lambda}}^{\mathrm{T}}, \frac{i+\delta}{\delta} \cdot \dot{\mathbf{K}}^{\mathrm{T}}\right)$$

$$D_{K}^{2}V(\mathbf{K}, \boldsymbol{\lambda}) = D_{K}^{2}y(\mathbf{p}, \mathbf{K}, L)$$

$$D_{\lambda}^{2}V(\mathbf{K}, \boldsymbol{\lambda}) = \frac{i+\delta}{\delta} \cdot \begin{pmatrix} Dm_{1}(\lambda_{1}) & 0 & \cdots & 0 \\ 0 & Dm_{2}(\lambda_{2}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Dm_{n}(\lambda_{n}) \end{pmatrix}$$

$$D_{K\lambda}V(\mathbf{K}, \boldsymbol{\lambda}) = D_{\lambda K}V(\mathbf{K}, \boldsymbol{\lambda}) = -(i+\delta) \cdot \mathbf{I}$$
(36)

Here I denotes the n-dimensional unit matrix. Time-derivatives under the system are given by

$$\dot{V}(\mathbf{K}(t), \boldsymbol{\lambda}(t)) = \frac{i}{\delta} \cdot \dot{\boldsymbol{\lambda}}^{\mathrm{T}} \dot{\mathbf{K}}
\ddot{V}(\mathbf{K}(t), \boldsymbol{\lambda}(t)) = i \cdot \dot{V} - \dot{\mathbf{K}}^{\mathrm{T}} \cdot D_{K}^{2} V \cdot \dot{\mathbf{K}} + \frac{\delta}{i+\delta} \cdot \dot{\boldsymbol{\lambda}}^{\mathrm{T}} \cdot D_{\lambda}^{2} V \cdot \dot{\boldsymbol{\lambda}}.$$
(37)

Because $D_K^2 V$ is negative definite and $D_\lambda^2 V$ is positive definite, we have $\ddot{V} > i \cdot \dot{V}$, and the sign of \ddot{V} cannot be determined a priori if $\dot{V} < 0$. Obviously the function V has a saddle-point at DV = 0, and the isokines discussed in appendix A are the loci where the maxima with respect to **K** given λ and the minima with respect to λ given **K**, respectively, are to be found.

References

- Albert, M. (1989), Capital Mobility and Costs of Adjustment, Discussion Paper 96, SFB 178/Konstanz.
- Albert, M. and J. Meckl (1991), Expectations And Adjustment Dynamics in a Two-Sector Model of a Small Open Economy, Discussion Paper 132, SFB 178/Konstanz.
- Feichtinger, G. and R. F. Hartl (1986), Optimale Kontrolle ökonomischer Prozesse, Berlin-New York, de Gruyter.
- Mussa, M. (1978), Dynamic adjustment in the Heckscher-Ohlin-Samuelson model, Journal of Political Economy 86, 775-791.
- Söderström, H. T. (1976), Production and Investment under Costs of Adjustment—A Survey, Zeitschrift für Nationalökonomie 36, 369-388.
- Treadway, A. B. (1969), On Rational Entrepreneurial Behavior and the Demand for Investment, *Review of Economic Studies* 36, 227-239.