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**Working Paper**

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Working Paper, No. 2014-11

**Provided in Cooperation with:**

Federal Reserve Bank of Atlanta

*Suggested Citation:* Gospodinov, Nikolay; Komunjer, Ivana; Ng, Serena (2014) : Minimum distance estimation of dynamic models with errors-in-variables, Working Paper, No. 2014-11, Federal Reserve Bank of Atlanta, Atlanta, GA

This Version is available at:

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## Minimum Distance Estimation of Dynamic Models with Errors-In-Variables

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Working Paper 2014-11  
August 2014

**Abstract:** Empirical analysis often involves using inexact measures of desired predictors. The bias created by the correlation between the problematic regressors and the error term motivates the need for instrumental variables estimation. This paper considers a class of estimators that can be used when external instruments may not be available or are weak. The idea is to exploit the relation between the parameters of the model and the least squares biases. In cases when this mapping is not analytically tractable, a special algorithm is designed to simulate the latent predictors without completely specifying the processes that induce the biases. The estimators perform well in simulations of the autoregressive distributed lag model and the dynamic panel model. The methodology is used to re-examine the Phillips curve, in which the real activity gap is latent.

JEL classification: C1, C3

Key words: measurement error, minimum distance, simulation estimation, dynamic panel

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The authors acknowledge financial support from National Science Foundation grants SES-0962473 and SES-0962431. The views expressed here are the authors' and not necessarily those of the Federal Reserve Bank of Atlanta or the Federal Reserve System. Any remaining errors are the authors' responsibility.

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# 1 Introduction

Empirical analysis often involves using incorrectly measured data that complicates identification of the behavioral parameters and testing of economic hypothesis.<sup>1</sup> The problem is acute in cross-section and survey data when errors in data collection and reporting are inevitable and new estimation methods continue to be developed. The literature on measurement error in time series data is smaller but the problem is no less important. The real time estimates which underlie economic decisions can differ from the revised estimates that researchers used for analysis. We do not observe variables such as the state of economy, potential output, or natural rate of unemployment, and filtered series are often used as proxies. Except by coincidence, the latent processes will not be the same as the constructed ones with differences that can be correlated over time. Orphanides and van Norden (2002) and Orphanides and Williams (2002) find that misperceptions or measurement errors can be quite persistent. Ermini (1993) shows that allowing for serially uncorrelated measurement errors changes the measure of persistence in consumption growth. Falk and Lee (1990) suggest that measurement errors can explain rejections of the permanent income hypothesis. Nalewalk (2010) shows that the income (GDI) and product (GDP) side of output growth exhibit rather different fluctuations over the past 25 years and that the GDI shows a steeper downturn in 2007-2009 than GDP. Aruoba, Diebold, Nalewaik, Schorfheide, and Song (2013) find that the series filtered from GDP and GDI are less volatile but more persistent than the two contaminated measures. Sargent (1989) allows the data collected to have serially correlated errors and shows that identification of the deep parameters of an accelerator model is affected by how the data are reported.

This paper is concerned with dynamic models that can generically be represented by

$$y_t = W_t' \gamma + V_t, \tag{1}$$

where  $y_t$  is a scalar dependent variable,  $W_t$  is a vector of observed regressors that may include lags of the dependent variable, and  $V_t$  is a latent disturbance term. The crucial feature of the model is that some components of  $W_t$  are mismeasured which causes  $W_t$  and  $V_t$  to become correlated. In consequence, the model (1) contains the regression parameters of interest  $\gamma$  as well as the nuisance parameters  $\delta$ . The correlation between  $V_t$  and  $W_t$  can sometimes be mitigated by resorting to instrumental variables (IVs), such as in purely static models in which measurement errors are known to be uncorrelated over time. More often than not, however, lags of  $W_t$  cannot be used as instruments. This is the case for distributed lag models with autocorrelated measurement errors, and also true of dynamic panel models in which fixed effects are eliminated by demeaning the observables. Not only is the least squares estimator (OLS) biased, but so is the instrumental variable estimator (IV) because the lagged variables are no longer valid instruments.

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<sup>1</sup>Wilcox (1992) discusses the issues in consumption measurements especially at the monthly level.

The goal of this paper is to show that identification of  $\gamma$  is still possible when external instruments are unavailable or that the validity of lagged variables as instruments is in doubt. The possibility of using internal information to turn an unidentified problem into an identifiable one was noted in Goldberger (1972) and Hsiao (1977). Our main insight is that the least squares (OLS) bias depends on the nuisance parameters  $\delta$  that characterize  $W_t$ . While OLS does not consistently estimate  $\gamma$ , there is in general enough information in the serially correlated regressors and the least squares residuals about the measurement error process to permit identification of  $\gamma$ . Persistence in the regressors is important here because the least squares residuals will be serially correlated if the regressors are serially correlated. In a way, our approach is to combine information in several sample estimates whose bias is magnified by persistence of the regressors.

Specifically, we propose to identify  $\theta = (\gamma', \delta')'$  from a vector of *auxiliary statistics*  $\widehat{\psi}(\theta)$  that includes the OLS estimates and moments formed from the least squares residuals. The role of the auxiliary statistics  $\widehat{\psi}(\theta)$  is to provide a mapping from the parameter space of  $\theta$  to the parameter space of an auxiliary model. We study the probability limit of  $\widehat{\psi}(\theta)$  as the sample size grows to infinity. Following Gourieroux, Monfort, and Renault (1993), the mapping  $\psi : \theta \rightarrow \psi(\theta)$  will be referred to as the *binding function*. Local identifiability of  $\theta$  requires that the binding function is locally one-to-one. A sufficient (though not necessary) condition for the latter is that the matrix of partial derivatives  $\partial\psi/\partial\theta$  is of full column rank. The parameter  $\theta$  is globally identifiable over the entire parameter space  $\Theta$  whenever  $\psi$  is invertible on  $\psi(\Theta)$ . In this case, a natural estimator for  $\theta$  is obtained as  $\widehat{\theta} = \psi^{-1}(\widehat{\psi})$ .

In simple models where the mapping between  $\theta$  and the auxiliary statistics can be derived analytically, we have a classical minimum distance (CMD) estimator that is  $\sqrt{T}$  consistent and asymptotically normal, where  $T$  is the sample size. Our CMD estimator is similar in the spirit to the ones proposed in Lewbel (2012) and Erickson (2001) who considered identification of parameters in a linear regression model without additional instruments. Lewbel (2012) uses the fact that under heteroskedasticity of the errors, the product of the regression and measurement error are uncorrelated with an exogenous variable. Erickson (2001) considered identification using higher order moments. Schennach and Hu (2013) also considered identification without side information, but their focus is non- and semi-parametric models. Our emphasis is on combining individually biased estimators without making assumptions about normality or homoskedasticity in a linear regression setting.

In more complex models such as when lags of the dependent variable and the regressors are involved, the binding function will not be tractable. We use Monte-Carlo methods to approximate this mapping. The resulting simulated minimum distance (SMD) estimator is consistent to the extent that the simulator can reconstruct those variations in the contaminated regressors that are

orthogonal to the measurement errors. Our simulation estimator differs from the ones considered in Smith (1993), Gouriéroux, Monfort, and Renault (1993), and Gallant and Tauchen (1996). These estimators treat the predictors as exogenous which can be held fixed in the simulations. The exogeneity assumption is not appropriate in measurement error models because the parameters in the marginal distribution of the covariates and those of the conditional model are not variation free in the sense of Engle, Hendry, and Richard (1983). Thus, even though the correctly measured predictors could have been held fixed, the mismeasured ones cannot. Our SMD estimator is designed to use limited information from the joint distribution of the data to simulate the model.

The paper proceeds as follows. After stating the assumptions for analysis, Section 2 uses the simple regression model to motivate the choice of auxiliary statistics. Section 3 turns to autoregressive distributed lag models with possibly serially correlated measurement errors. Section 4 extends the analysis to dynamic panel models. Classical minimum distance estimation is not feasible in models with more general dynamics as the model-implied moments do not have closed form expressions. This motivates the need for simulation estimation, which is presented in Section 5. The same section presents Monte Carlo simulation evidence and a Phillips-curve application. Last section concludes. Technical proofs are relegated to an Appendix.

As a matter of notation, we use  $\Gamma_x(j) \equiv E[x_t x_{t-j}]$  to generically denote the autocovariance of a covariance stationary mean-zero time series  $x_t$ . We use  $\Gamma_{xy}(j, k) \equiv E[x_{t-j} y_{t-k}]$  to denote the covariance between  $x_t$  and another mean-zero covariance stationary process  $y_t$ . If  $E(x_t) = 0$ ,  $E(x_t^2) = \sigma_x^2 \equiv \Gamma_x(0)$ , and  $\Gamma_x(j) = 0$  for  $j \geq 1$ , then  $x_t$  is a white noise. In this case, we write  $x_t \sim WN(0, \sigma_x^2)$ .

## 2 The Econometric Framework

Consider the autoregressive distributed lag ADL(p,q) model with scalar predictor  $x_t$ :

$$\alpha(L)y_t = \beta(L)x_t + u_t, \tag{2}$$

where  $\alpha(L) = 1 - \sum_{i=1}^p \alpha_i L^i$ ,  $\beta(L) = \sum_{i=0}^q \beta_i L^i$ , and  $L$  is the lag operator. Instead of  $x_t$ , we only observe a contaminated variable  $X_t$ :

$$X_t = x_t + \epsilon_t. \tag{3}$$

Now let  $\alpha = (\alpha_1, \dots, \alpha_p)'$ ,  $\beta = (\beta_0, \beta_1, \dots, \beta_q)'$  and  $\gamma = (\alpha', \beta)'$ . In the context of the general dynamic model in (1),  $W_t = (y_{t-1}, \dots, y_{t-p}, X_t, X_{t-1}, \dots, X_{t-q})'$ , and the error term becomes

$$V_t = u_t - \beta(L)\epsilon_t.$$

As is well known, identification in these models is impossible without further assumptions when the predictors are serially uncorrelated or normally distributed.<sup>2</sup> But as Goldberger (1972, p.996) pointed out, measurement errors in the exogenous regressors need not destroy identification provided that the model is overidentified. One approach is to use additional information from  $N$  variables. Bai and Ng (2010) assume that there are mismeasured indicators  $(X_{1t}, \dots, X_{Nt})'$  of the latent predictor  $x_t$ , each characterized by  $X_{it} = \pi_i x_t + \epsilon_{it}$  with  $\text{cov}(\epsilon_{it}, x_t) = 0$ , and  $N$  is large. The IV approach makes use of  $N = 2$  variables: one  $X_{1t}$  to replace  $x_t$  and a second one  $X_{2t}$  that is correlated with  $X_{1t}$  but uncorrelated with  $\epsilon_{1t}$ . A second approach is to drop the normality assumption. Pal (1980), Dagenais and Dagenais (1997), Lewbel (1997) and Spierdijk and Wansbeek (2011) exploit heteroskedasticity, excess skewness and kurtosis for identification without relying on instruments. Our approach falls in the third category of exploiting the bias in the sample estimates. Assumptions on the latent variables of the model,  $u_t$ ,  $\epsilon_t$ , and  $x_t$ , are as follows.

**Assumption A .**

- (a)  $\{(x_t, \epsilon_t)'\}$  is covariance stationary with  $E(x_t) = 0$ ,  $E(\epsilon_t) = 0$ ,  $E(x_t \epsilon_\tau) = 0$  for every  $(t, \tau)$ .
- (b)  $u_t \sim WN(0, \sigma_u^2)$ . For every  $(t, \tau)$ ,  $E(u_t x_\tau) = 0$  and  $E(u_t \epsilon_\tau) = 0$ .
- (c) The covariance matrix of  $(y_{t-1}, \dots, y_{t-p}, x_t, x_{t-1}, \dots, x_{t-q})$  is nonsingular.
- (d) Measurement error autocovariances  $\Gamma_\epsilon(1), \dots, \Gamma_\epsilon(q^*)$  are well approximated by  $\sigma_\epsilon^2 f(\phi)$  where  $f(\cdot)$  is a function parameterized by  $\phi = (\phi_1, \dots, \phi_m)'$ .

We assume in (a) that the model is dynamically correctly specified and that  $u_t$  is serially uncorrelated. The latent regressor  $x_t$  and measurement error  $\epsilon_t$  are assumed to have zero mean. The intercept is therefore suppressed in (2). The measurement error  $\epsilon_t$  is assumed to be classical, i.e. orthogonal to the latent regressor  $x_t$  at all leads and lags. The true regressor  $x_t$  is observed with error whenever  $\sigma_\epsilon^2 \neq 0$ . The measurement error  $\epsilon_t$  is allowed to be serially correlated. The model disturbance  $u_t$  in (b) is assumed to be white noise and orthogonal to both  $x_t$  and  $\epsilon_t$ . Serial correlation in  $u_t$  and its correlation with  $x_t$  can be seen as due to the omission of relevant lags of  $y_t$  and  $x_t$  in (2). Thus (b) subsumes that all the relevant regressors have been included in the model. Assumption (c) is standard for least squares analysis. Note however that this condition involves latent variables  $x_t, \dots, x_{t-q}$ . As we shall demonstrate next, the dynamics of  $V_t$  depends on the lag order  $q$  which affects the identifiability of  $\alpha$  and  $\beta$ . For  $q = 0$ , identification can be established by leaving the dynamic structure of  $\epsilon_t$  unspecified. For values  $q \geq 1$ , however, this approach leads to a

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<sup>2</sup>See Reiersøl (1950) and Aigner, Hsiao, Kapteyn, and Wansbeek (1984, p.1324) for an overview of identification conditions in measurement error models. Gillard (2010) presents an overview of approaches to handle the errors-in-variables (EIV) problem from different fields.

proliferation of nuisance parameters in the model. In those cases, we reparameterize the model as in (d). While this assumption does not parametrically specify the dynamic structure of  $\epsilon_t$  explicitly, we require that its first  $q^*$  autocorrelations can be parsimoniously approximated by  $m$  parameters, where both  $m$  and  $q^*$  depend on the choice of auxiliary statistics and will be further discussed.

An implication of the presence of measurement error is that  $E(V_t X_{t-j}) \neq 0$  for  $0 \leq j \leq q$ . Moreover,  $\widehat{V}_t$  is generally serially correlated even if  $\epsilon_t$  is white noise. Let  $\delta$  denote the nuisance parameters characterizing  $V_t$ . In the case of white noise measurement errors,  $\delta = (\sigma_u^2, \sigma_\epsilon^2)'$ . When  $\epsilon_t$  is serially correlated,  $\delta = (\sigma_u^2, \sigma_\epsilon^2, \Gamma_\epsilon(1), \dots, \Gamma_\epsilon(q^*))'$  where  $q^*$  depends on auxiliary statistics used. The parameters of the model are  $\theta = (\alpha', \beta', \delta)'$ . The objective of the exercise is to identify and consistently estimate  $\theta$  from suitable choice of auxiliary statistics. While  $\alpha$  and  $\beta$  are the parameters of interest, joint identification of  $\delta$  makes it possible to assess the severity of measurement errors. IV estimators work around the attenuation bias and can only be silent about the properties of  $\epsilon_t$ .

The idea behind our proposed estimator is to exploit the bias in the OLS estimator for identification of  $\theta$ . Specifically, if  $\text{plim}_{T \rightarrow \infty} \widehat{\gamma} - \gamma = \text{bias}(\theta)$  is non-degenerate and

$$\widehat{V}_t = y_t - W_t' \widehat{\gamma} = V_t - W_t' \text{bias}(\theta),$$

then moments formed from  $\widehat{V}_t$  must also be a function of the bias and hence depend on  $\theta$ . We will be specifically interested in  $\widehat{\Gamma}_{\widehat{V}}(j) = \frac{1}{T} \sum_{t=j+1}^T \widehat{V}_t \widehat{V}_{t-j}$  and  $\widehat{\Gamma}_{\widehat{V}X}(j, 0) = \frac{1}{T} \sum_{t=j+1}^T \widehat{V}_{t-j} X_t$ . Their probability limits as  $T \rightarrow \infty$  are, respectively,

$$\Gamma_{\widehat{V}}(j) = \Gamma_V(j) - \left( \Gamma_{VW}(j, 0) + \Gamma_{VW}(0, j) \right)' \text{bias}(\theta) + \text{bias}(\theta)' \Gamma_W(j) \text{bias}(\theta) \quad (4)$$

$$\Gamma_{\widehat{V},X}(j, 0) = \Gamma_{VX}(j, 0) - \Gamma_{WX}(j, 0)' \text{bias}(\theta). \quad (5)$$

Observe that these moments use  $\widehat{V}_t$  instead of  $V_t$  and  $\text{plim } \widehat{\Gamma}_{\widehat{V}}(0) \neq \Gamma_V(0)$ ,  $\text{plim } \widehat{\Gamma}_{\widehat{V}}(1) \neq \Gamma_V(1)$ , and  $\text{plim } \widehat{\Gamma}_{\widehat{V}X}(1, 0) \neq \Gamma_{VX}(1, 0)$ . The biased sample estimates are functions of least squares bias and thus useful for identification of  $\theta$ . To layout the framework for analysis, we use the simple regression model with no lagged dependent variables to motivate the choice of auxiliary statistics. It will be shown that  $\theta$  is identified whether or not measurement errors are serially correlated. We then extend the analysis to allow for lagged dependent variables.

## 2.1 The Simple Regression Model

The scalar regression model

$$y_t = x_t \beta + u_t$$

with  $X_t = x_t + \epsilon_t$  and  $V_t = u_t - \beta \epsilon_t$  is well studied under the assumption that  $x_t$  and  $\epsilon_t$  are *iid*. We consider this model in a time series context when  $x_t$  is serially correlated, and under

different assumptions about  $\epsilon_t$ . In the case when  $\epsilon_t$  is white noise, it holds, as in the *iid* case, that  $\Gamma_X(0) = \Gamma_x(0) + \sigma_\epsilon^2$ . The parameters of this model are

$$\theta_A \equiv (\beta, \sigma_u^2, \sigma_\epsilon^2)'. \quad (6)$$

The least squares estimator has attenuation bias given by

$$\text{plim}_{T \rightarrow \infty} (\hat{\beta} - \beta) = -\frac{\beta \sigma_\epsilon^2}{\Gamma_X(0)} \equiv \text{bias}_A(\theta).$$

Our estimator exploits two features of time series data. The first is the relation between the autocovariance of  $X_t$  (observed) and  $x_t$  (latent). Even though  $X_t$  and  $x_t$  have different autocorrelations, they have the same autocovariance at lag  $j \geq 1$ . That is, if  $\epsilon_t$  is white noise,

$$\Gamma_X(j) = \Gamma_x(j) \quad \text{for all } j \geq 1.$$

The second is the properties of the least squares residuals  $\hat{V}_t = y_t - X_t \hat{\beta} = V_t - X(\hat{\beta} - \beta)$ . By (4), the autocovariances for this model are given by

$$\text{plim}_{T \rightarrow \infty} \hat{\Gamma}_{\hat{V}}(j) = \Gamma_V(j) - \text{bias}_A \left( \Gamma_{VX}(0, j) + \Gamma_{VX}(j, 0) \right) + \text{bias}_A(\theta)^2 \Gamma_X(j).$$

With white noise measurement error,  $\Gamma_V(j)$ ,  $\Gamma_{VX}(0, j)$  and  $\Gamma_{VX}(j, 0)$  are zero for  $j = 1$ . However,  $\text{bias}_A(\theta) \neq 0$  and  $\Gamma_{VX}(0, 0) = -\beta \sigma_\epsilon^2$ . Thus, even though  $V_t$  is serially uncorrelated when  $\epsilon_t$  is white noise,  $\hat{V}_t$  has non-zero autocovariances whenever  $X_t$  is serially correlated. A researcher who is not aware of the presence of measurement error might estimate a model with additional lags of  $X_t$ . The ability to identify  $\sigma_\epsilon^2$  can lead to the choice of a more appropriate dynamic model.

Consider identifying  $\theta_A$  from a vector of three auxiliary statistics:<sup>3</sup>

$$\hat{\psi}_A \equiv \begin{pmatrix} \hat{\beta} \\ \hat{\Gamma}_{\hat{V}}(0) \\ \hat{\Gamma}_{\hat{V}}(1) \end{pmatrix} \xrightarrow{p} \begin{pmatrix} \beta + \text{bias}_A(\theta) \\ \sigma_u^2 + \beta^2 \sigma_\epsilon^2 + \text{bias}_A(\theta) \beta \sigma_\epsilon^2 \\ \text{bias}_A(\theta)^2 \Gamma_X(1) \end{pmatrix} \equiv \psi_A(\theta). \quad (7a)$$

As pointed out in Spierdijk and Wansbeek (2011),  $\Gamma_X(0)$ ,  $\Gamma_Y(0)$  and  $\Gamma_{XY}(0, 0)$  are needed to identify  $\theta_A$ . The proposed  $\hat{\psi}_A$  is implicitly a function of these moments. Its choice is guided by the fact that  $\hat{\beta}$  is a sufficient statistic for  $\beta$  in the absence of measurement error, and the properties of  $\hat{V}_t$  are completely characterized by its second moments. A useful way to think about  $\hat{\psi}_A$  is that its components  $\hat{\beta}$ ,  $\hat{\Gamma}_{\hat{V}}(0)$  and  $\hat{\Gamma}_{\hat{V}}(1)$  are biased for  $\beta$ ,  $\Gamma_V(0)$  and  $\Gamma_V(1)$ , and we exploit the biases to identify  $\theta$ . This contrasts with the IV estimator which works around the correlation between  $X_t$  and  $y_t$  and cannot identify  $\sigma_\epsilon^2$ .

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<sup>3</sup>Note that  $\text{bias}_A \theta \Gamma_X(0) = -\beta \sigma_\epsilon^2$ . Thus,  $\hat{\Gamma}_{\hat{V}}(0) \xrightarrow{p} \sigma_u^2 + \beta^2 \sigma_\epsilon^2 + 2 \text{bias}_A(\theta) \beta \sigma_\epsilon^2 + \text{bias}_A(\theta)^2 \Gamma_X(0) = \sigma_u^2 + \beta^2 \sigma_\epsilon^2 + \text{bias}_A(\theta) \beta \sigma_\epsilon^2$ .



Substituting in  $\text{bias}_A(\theta) = -\beta \frac{\sigma_\epsilon^2}{\Gamma_X(0)}$ , the binding function simplifies to

$$\psi_A(\theta) = \begin{pmatrix} \psi_{A1} \\ \psi_{A2} \\ \psi_{A3} \end{pmatrix} = \begin{pmatrix} \beta - \beta \frac{\sigma_\epsilon^2}{\Gamma_X(0)} \\ \sigma_u^2 + \beta^2 \sigma_\epsilon^2 - \frac{(\beta \sigma_\epsilon^2)^2}{\Gamma_X(0)} \\ \left( \frac{\beta \sigma_\epsilon^2}{\Gamma_X(0)} \right)^2 \Gamma_X(1) \end{pmatrix}. \quad (7b)$$

Given that (7b) is a system of three equations and three unknowns, the parameters are locally identified if the matrix of partial derivatives  $\partial \psi_A / \partial \theta_A$  has a non-zero determinant. To establish global identifiability of  $\theta_A$ , we need to show that the mapping  $\theta_A \mapsto \psi_A(\theta_A)$  is invertible.

**Lemma 1** *Suppose  $y_t = x_t \beta + u_t$ ,  $X_t = x_t + \epsilon_t$ , Assumptions A(a)-(c) hold, and  $\epsilon_t \sim WN(0, \sigma_\epsilon^2)$ . Then:*

- (a)  $(\beta = 0, \sigma_u^2)$  is globally identified from  $\psi_A$ .
- (b)  $(\beta \neq 0, \sigma_u^2)$  is globally identified from  $\psi_A$  if and only if  $\Gamma_X(1) \neq 0$ .
- (c)  $\theta_A = (\beta, \sigma_\epsilon^2, \sigma_u^2)$  is globally identified from  $\psi_A$  if and only if (i)  $\beta \neq 0$  and (ii)  $\Gamma_X(1) \neq 0$ .

Lemma 1, proved in the Appendix, gives the necessary and sufficient conditions for global identification. If  $\beta$  alone is of interest, global identification only requires  $X_t$  to be serially correlated.<sup>4</sup> But  $\beta \neq 0$  is necessary to also identify  $\sigma_\epsilon^2$  because the regression residuals would have no information about  $\beta$  if  $X_t$  has no role in the regression model. Identification strength is determined by  $\Gamma_X(1)$ , which measures the persistence of the mismeasured regressor  $X_t$ .

The question then arises as to whether  $\theta_A$  can be identified at  $\beta = 0$  when the nuisance parameter  $\delta_A = \sigma_\epsilon^2$  is not identifiable. This is possible with additional restrictions on the dynamic structure of the latent process  $x_t$ . For example, suppose that

$$\Gamma_x(j) = \phi^j \Gamma_x(0)$$

for two consecutive values of  $j \geq 1$ , a condition that holds if  $x_t$  has an autoregressive structure. Since  $\epsilon_t$  is white noise, it also holds that  $\Gamma_X(j) = \phi^j \Gamma_x(0)$  for  $j \geq 1$ . From  $\phi = \frac{\Gamma_X(2)}{\Gamma_X(1)}$  when  $j = 2$  and  $\Gamma_X(0) = \Gamma_x(0) + \sigma_\epsilon^2$ , we have

$$\sigma_\epsilon^2 = \Gamma_X(0) - \frac{\Gamma_X(1)^2}{\Gamma_X(2)}.$$

We can use  $\sigma_\epsilon^2$  to assess the severity of measurement error prior to any regression analysis. The result is specific to the fact that  $\epsilon_t$  is a white noise process.

<sup>4</sup>This contrasts with Reiersøl (1950), Pal (1980), Erickson, Jiang, and Whited (2014) in which the identification results exclude the important special case of  $\beta = 0$ . The reason is that they consider identification of the entire parameter vector  $\theta_A = (\beta, \sigma_u^2, \sigma_\epsilon^2)'$ , while parts (a) and (b) of our result apply to  $(\beta, \sigma_u^2)$  alone.

## 2.2 Correlated Measurement Errors

The above analysis for the simple regression model with  $\epsilon_t \sim WN(0, \sigma_\epsilon^2)$  highlights the possibility of combining several biased estimators for identification. Nonetheless,  $X_{t-k}$  could have been used as instrument under the assumptions of the analysis for  $k > q$ . The practical interest of our approach is in situations when lags of  $X_t$  may not be valid instruments. In such cases we will need to augment  $\theta_A$  and  $\hat{\psi}_A$  with additional parameters and statistics. We now show how this can be achieved when  $x_t$  in the simple regression model above is contaminated by a serially correlated measurement error.

Serial correlation in the measurement error has the important implication that  $X_{t-1}$  is no longer a valid instrument. Though longer lags could be valid, they will likely have weak correlation with  $X_t$ . To begin, we define

$$\text{plim}_{T \rightarrow \infty} \hat{\beta} - \beta = -\beta \frac{\sigma_\epsilon^2}{\Gamma_X(0)} \equiv \text{bias}_B(\theta_B)$$

to distinguish it from  $\text{bias}_A(\theta_A)$  even though least squares bias is numerically identical whether or not  $\epsilon_t$  is serially correlated. This distinction is important because  $\theta_B$  will be of higher dimension when  $\epsilon_t$  is correlated as we shall now show.

When  $\epsilon_t$  is serially correlated, the autocovariance of the least squares residuals  $\hat{V}_t = y_t - X_t \hat{\beta}$  is affected by the dynamics of  $\epsilon_t$  in two ways. First,  $V_t = u_t - \beta \epsilon_t$  is serially correlated because  $\epsilon_t$  is serially correlated. As a consequence,  $\Gamma_{VX}(0, 1) = E(u_t - \beta \epsilon_t)(x_{t-1} + \epsilon_{t-1}) = -\beta \Gamma_\epsilon(1) \neq 0$ . Second, with  $\epsilon_t$  serially correlated we now have

$$\Gamma_X(j) = \Gamma_x(j) + \Gamma_\epsilon(j), \quad j \geq 1$$

instead of  $\Gamma_X(j) = \Gamma_x(j)$ . It follows from (4) that

$$\text{plim}_{T \rightarrow \infty} \hat{\Gamma}_{\hat{V}}(1) = \beta^2 \Gamma_\epsilon(1) + 2\beta \Gamma_\epsilon(1) \text{bias}_B(\theta_B) + \text{bias}_B(\theta_B)^2 \Gamma_X(1).$$

Serial correlation in  $\epsilon_t$  thus introduces the additional nuisance parameter  $\Gamma_\epsilon(1)$ , leading to

$$\theta_B \equiv (\beta, \sigma_u^2, \sigma_\epsilon^2, \Gamma_\epsilon(1))'. \quad (8a)$$

Since there is an additional parameter compared to  $\theta_A$ , an additional auxiliary statistic is required. We consider

$$\text{plim}_{T \rightarrow \infty} \hat{\Gamma}_{\hat{V}X}(1, 0) = \frac{1}{T} \sum_{t=1}^T (V_{t-1} - (\hat{\beta} - \beta)X_{t-1})X_t \xrightarrow{p} -\beta \Gamma_\epsilon(1) - \text{bias}_B(\theta) \Gamma_X(1),$$

because its probability limit depends on the same parameters as the limit of  $\Gamma_{\hat{V}}(0)$  and  $\hat{\Gamma}_V(1)$ .<sup>5</sup> The auxiliary statistic

$$\hat{\psi}_B \equiv \left( \hat{\beta}, \hat{\Gamma}_{\hat{V}}(0), \hat{\Gamma}_{\hat{V}}(1), \hat{\Gamma}_{VX}(1, 0) \right)'$$

---

<sup>5</sup>An alternative is  $\hat{\Gamma}_{\hat{V}X}(0, 1)$  which in this case has the same probability limit as  $\hat{\Gamma}_{\hat{V}X}(1, 0)$ .

has binding function given by

$$\psi_B(\theta_B) = \begin{pmatrix} \beta + \text{bias}_B(\theta) \\ \sigma_u^2 + \beta^2\sigma_\epsilon^2 + \text{bias}_B(\theta)\beta\sigma_\epsilon^2 \\ \beta^2\Gamma_\epsilon(1) + 2\beta\Gamma_\epsilon(1)\text{bias}_B(\theta) + \text{bias}_B(\theta)^2\Gamma_X(1) \\ -\beta\Gamma_\epsilon(1) - \text{bias}_B(\theta)\Gamma_X(1) \end{pmatrix} \quad (8b)$$

**Lemma 2** Consider the model  $y_t = x_t\beta + u_t$ ,  $X_t = x_t + \epsilon_t$ . Under Assumptions A(a)-(c):

- (a)  $(\beta = 0, \sigma_u^2)$  is globally identified from  $\psi_B$ ;
- (b)  $(\beta \neq 0, \sigma_u^2)$  is globally identified from  $\psi_B$  if  $\Gamma_x(1) \neq 0$ ;
- (c)  $\theta_B = (\beta, \sigma_u^2, \Gamma_\epsilon(0), \Gamma_\epsilon(1))$  is globally identified from  $\psi_B$  if (i)  $\Gamma_x(1) \neq 0$  and (ii)  $\beta \neq 0$ .

The proof, given in the Appendix, involves inverting the binding function and showing that a unique solution exists. Serial correlation in the latent regressor  $x_t$  is needed. Note that when  $\epsilon_t$  is white noise, as was the case in the previous section, then  $\Gamma_x(1) = \Gamma_X(1)$  and the condition reduces to that in Lemma 1. It is worth pointing out that  $\Gamma_x(1) \neq 0$  is sufficient to globally identify  $\beta$  and  $\sigma_u^2$ , irrespective of whether or not  $\beta = 0$ . However,  $\Gamma_\epsilon(0)$  and  $\Gamma_\epsilon(1)$  can only be identified if  $\beta \neq 0$ . As in the case when  $\epsilon_t$  is white noise, the only way we can learn about the measurement error is by looking at the regressor  $X_t$ .

The assumption that  $\Gamma_x(1) \neq 0$  is not directly testable. However, when  $\beta = 0$ , we can use the fact that  $\Gamma_X(1) = \Gamma_x(1) + \Gamma_\epsilon(1)$ . An IV estimator with  $X_{t-1}$  as instrument has

$$\text{plim } \widehat{\beta}_{IV} = \beta \left( 1 - \frac{\Gamma_\epsilon(1)}{\Gamma_X(1)} \right) = \beta \frac{\Gamma_x(1)}{\Gamma_X(1)}$$

which is zero if and only if  $\Gamma_x(1) = 0$ . Indirect evidence of whether the latent regressor is correlated can be gleaned from the IV estimate, even though it is biased for  $\beta$ .

### 3 Autoregressive Distributed Lag Models

The idea of using the OLS biases to achieve identification extends to more general models. The ADL(p,q) model expressed in terms of the observables is

$$(1 - \alpha_1 L - \dots - \alpha_p L^p)y_t = (\beta_0 + \beta_1 L + \dots + \beta_q L^q)X_t + V_t.$$

The error term  $V_t = u_t - \beta_0\epsilon_t - \dots - \beta_q\epsilon_{t-q}$  is generally serially correlated. For instance, even if  $\epsilon_t$  is white noise,  $V_t$  is an MA(q) process. This serial correlation can confound the determination of the lag orders  $p$  and  $q$ .<sup>6</sup> Furthermore, any  $X_{t-j}$  for  $j \leq q$  is an invalid instrument because it

<sup>6</sup>We assume that  $y_t$  is observed without error. ARMA models when  $y_t$  is observed with error is studied in Komunjer and Ng (2014).

is correlated with  $\epsilon_{t-j}$ . Serial correlation in  $\epsilon_t$  makes it even more difficult to find instruments that are both strong and relevant. While  $X_{t-q+1}$  and further lags are valid instruments, they may only be weakly correlated with the regressors if they are stationary and ergodic as assumed. We consider estimation that does not directly involve internal or external instruments, beginning with the ADL(1,0) model.

### 3.1 ADL(1,0) model

The ADL(1,0) model is represented by  $y_t = \alpha y_{t-1} + \beta x_t + u_t$ , and we observe  $X_t = x_t + \epsilon_t$ . Here,  $\gamma = (\alpha, \beta)$ ,  $W_t = (y_{t-1}, X_t)'$ . Assuming that  $\Gamma_W(0)$  is nonsingular, the least squares bias is given by

$$\text{plim}_{T \rightarrow \infty} \hat{\gamma} - \gamma = \begin{pmatrix} \beta \frac{\sigma_\epsilon^2 \Gamma_{yX}(1,0)}{\Gamma_y(0) \Gamma_X(0) - \Gamma_{yX}(1,0)^2} \\ -\beta \frac{\sigma_\epsilon^2 \Gamma_y(0)}{\Gamma_y(0) \Gamma_X(0) - \Gamma_{yX}(1,0)^2} \end{pmatrix} = \text{bias}_C(\theta).$$

The parameters of the model are  $\theta_C = (\alpha, \beta, \sigma_u^2, \sigma_\epsilon^2(0), \Gamma_\epsilon(1))'$ , and the auxiliary statistics are  $\hat{\psi}_C = (\hat{\alpha}, \hat{\beta}, \hat{\Gamma}_{\hat{V}}(0), \hat{\Gamma}_{\hat{V}}(1), \hat{\Gamma}_{\hat{V}X}(1,0))'$ . Identification requires invertibility of the probability limit  $\psi_C(\theta_C)$  of  $\hat{\psi}_C(\theta_C)$ . The analysis can be simplified by noting that  $\sigma_u^2$  will be identified from  $\Gamma_{\hat{V}}(0)$ . Thus, we only need to consider identification of

$$\theta_C^- = (\gamma, \Gamma_\epsilon(0), \Gamma_\epsilon(1))$$

from

$$\hat{\psi}_C^- = (\hat{\alpha}, \hat{\beta}, \hat{\Gamma}_{\hat{V}}(1), \hat{\Gamma}_{\hat{V}X}(1,0))'$$

As before,  $\theta_C^-$  is globally identified from  $\psi_C^-$  if the binding function  $\psi_C^-(\theta_C^-)$  is invertible. Consider then the system of equations  $\psi_C^-(\theta_C^-) = \psi_C^-$  to solve: by plugging the first two equations into the last two, and pre-multiplying the first two equations by the nonsingular matrix  $\Gamma_W(0)$ , this system is equivalent to:

$$\begin{aligned} \Gamma_{Wy}(0,0) &= \Gamma_W(0) \begin{pmatrix} \psi_{C1}^- \\ \psi_{C2}^- \end{pmatrix} \\ \Gamma_{Wy}(0,0) \Gamma_W(0)^{-1} \Gamma_{Wy}(1,0) &= (\psi_{C1}^- \quad \psi_{C2}^-) \Gamma_W(1) \begin{pmatrix} \psi_{C1}^- \\ \psi_{C2}^- \end{pmatrix} - \psi_{C3}^- \\ \Gamma_{yX}(1,0) &= \psi_{C4}^- + (\psi_{C1}^- \quad \psi_{C2}^-) \Gamma_{WX}(1,0) \end{aligned} \tag{9}$$

The system of 4 equations in 4 unknowns in (9) has the important feature that only the left-hand side of (9) depends on  $\theta$ . The right hand side consists either of  $(\psi_{C1}^-, \dots, \psi_{C4}^-)$  or the elements in  $\Gamma_W(0)$  and  $\Gamma_W(1)$  for which sample estimates are available. Global identifiability of  $\theta_C^-$  from  $\psi_C^-$  holds if it can be established that the system (9) has a unique solution in  $\theta_C^-$ .

**Lemma 3** Consider the model  $y_t = \alpha y_{t-1} + \beta x_t + u_t$ ,  $X_t = x_t + \epsilon_t$ . Under Assumptions A(a)-(c):

- (a)  $(\alpha, \beta = 0)$  is globally identified from  $\psi_C^-$ ;
- (b)  $(\alpha, \beta \neq 0)$  is globally identified from  $\psi_C^-$  if (i)  $\Gamma_x(1) \neq 0$  and (ii)  $\psi_{C3}^- + \psi_{C2}^- \psi_{C4}^- \neq 0$ ;
- (c)  $\theta_C^- = (\alpha, \beta, \Gamma_\epsilon(0), \Gamma_\epsilon(1))$  is globally identified if from  $\psi_C^-$  if (i)  $\Gamma_x(1) \neq 0$ , (ii)  $\psi_{C3}^- + \psi_{C2}^- \psi_{C4}^- \neq 0$ , and (iii)  $\beta \neq 0$ .

Compared to the assumptions in Lemma 2, the additional restriction that  $\psi_{C3}^- + \psi_{C2}^- \psi_{C4}^- \neq 0$  is needed. To understand this restriction, note that

$$\psi_{C3}^- + \psi_{C2}^- \psi_{C4}^- = \Gamma_{\hat{V}}(1) + (\text{plim } T \rightarrow \infty \hat{\beta}) \Gamma_{\hat{V}X}(1, 0).$$

The restriction  $\psi_{C3}^- + \psi_{C2}^- \psi_{C4}^- \neq 0$  ensures that there is independent information in  $\Gamma_{\hat{V}}(1)$  and  $\Gamma_{\hat{V}X}(1, 0)$ .

### 3.2 An Order Condition for the ADL(p,q) Model

For the ADL(p,q) model,  $\gamma = (\alpha_1, \dots, \alpha_p, \beta_0, \dots, \beta_q)$  and  $W_t = (y_{t-1}, \dots, y_{t-p}, X_t, \dots, X_{t-q})$ . The parameters of the model are

$$\theta_C = (\gamma, \sigma_u^2, \Gamma_\epsilon(0), \dots, \Gamma_\epsilon(q+k))',$$

The OLS estimator has asymptotic bias

$$\text{plim } T \rightarrow \infty \hat{\gamma} - \gamma = \Gamma_W(0)^{-1} \Gamma_{VW}(0, 0) \equiv \text{bias}_C(\theta_C)$$

and the auxiliary statistics are  $\hat{\psi}_C(\theta_C) = (\hat{\gamma}, \hat{\Gamma}_{\hat{V}}(0), \dots, \hat{\Gamma}_{\hat{V}}(k), \hat{\Gamma}_{\hat{V}X}(1, 0), \dots, \hat{\Gamma}_{\hat{V}X}(k, 0))'$ .

Exact global identification requires inverting the binding function  $\psi_C(\theta) = \text{plim } T \rightarrow \infty \hat{\psi}_C(\theta_C)$  and establishing that it has a unique solution. This is already a nontrivial problem for the ADL(1,0) model. For the ADL(1,1), there are six nonlinear equations in six unknowns to be solved. However, we can still provide a useful order condition.

The parameters entering  $\text{bias}_C(\theta_C)$  are those appearing in the covariances  $\Gamma_{\hat{V}X}(0, i)$  for  $0 \leq i \leq q$ . Since  $X_t = x_t + \epsilon_t$  and  $V_t = u_t - \beta_0 \epsilon_t - \dots - \beta_q \epsilon_{t-q}$ , the parameters  $\Gamma_\epsilon(i)$  with  $0 \leq i \leq q$  will appear in the probability limit of the moments. This implies that in addition to the  $p + q + 2$  parameters of  $(\gamma, \sigma_u^2)$  in the ADL(p,q) model, there are now  $q + 1$  nuisance parameters. At least  $(p + q + 2) + (q + 1)$  auxiliary statistics are needed. The OLS estimator provides  $p + q + 1$  statistics; the variance of the OLS residuals  $\hat{\Gamma}_{\hat{V}}(0)$  provides another. But we still another need  $q + 1$  auxiliary statistics. By construction of the least squares estimator, we have  $\hat{\Gamma}_{\hat{V}X}(0, i) = 0$ ,  $0 \leq i \leq q$ . So the moments left to consider are  $\hat{\Gamma}_{\hat{V}X}(k, 0)$ ,  $\hat{\Gamma}_{\hat{V}}(k)$ , and  $\hat{\Gamma}_{\hat{V}X}(0, q + k)$  for  $k \geq 1$ . The last one is less desirable unless  $X_t$  is strongly persistent. Thus we focus on  $\hat{\Gamma}_{\hat{V}X}(k, 0)$  and  $\hat{\Gamma}_{\hat{V}}(k)$ .

To determine  $k$ , observe that in the ADL(p,q) model,  $\widehat{\Gamma}_{\widehat{V}_X}(1,0)$  and  $\widehat{\Gamma}_{\widehat{V}}(1)$  depend on parameters of ADL(p,q) ie.  $(\gamma, \sigma_u^2)$ , nuisance parameters already appearing in the OLS bias ie.  $(\Gamma_\epsilon(0), \dots, \Gamma_\epsilon(q))$ , and a new nuisance parameter,  $\Gamma_\epsilon(q+1)$ . Thus, when  $k=1$ , the inclusion of two auxiliary statistics  $\widehat{\Gamma}_{\widehat{V}_X}(1,0)$  and  $\widehat{\Gamma}_{\widehat{V}}(1)$  increases the nuisance parameters by one. This suggests the following simple rule: compare the total number of auxiliary statistics ( $2k$ )  $\widehat{\Gamma}_{\widehat{V}_X}(1,0), \dots, \widehat{\Gamma}_{\widehat{V}_X}(k,0)$  and  $\widehat{\Gamma}_{\widehat{V}}(1), \dots, \widehat{\Gamma}_{\widehat{V}}(k)$ , with the total number of nuisance parameters  $\Gamma_\epsilon(0), \dots, \Gamma_\epsilon(q)$  and  $\Gamma_\epsilon(q+1), \dots, \Gamma_\epsilon(q+k)$  ( $q+1+k$ ). The first has to be larger than the second, i.e.  $2k \geq q+1+k$ . Setting

$$k = q + 1$$

satisfies the rule. The condition works for the simple ADL(0,0) model with correlated errors. Indeed, we have defined  $\theta_B = (\beta, \sigma_u^2, \Gamma_\epsilon(0), \Gamma_\epsilon(1))$  and  $\widehat{\psi}_B = \left(\widehat{\beta}, \widehat{\Gamma}_{\widehat{V}}(0), \widehat{\Gamma}_{\widehat{V}}(1), \widehat{\Gamma}_{VX}(1,0)\right)$  which corresponds to  $k=1$  for that model. The same condition holds for the ADL(1,0). We have used  $\widehat{\psi}_C \equiv \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\Gamma}_{\widehat{V}}(0), \widehat{\Gamma}_{\widehat{V}}(1), \widehat{\Gamma}_{VX}(1,0)\right)$  with  $\theta_C = (\alpha, \beta, \sigma_u^2, \Gamma_\epsilon(0), \Gamma_\epsilon(1))$ .

For higher order ADL(p,q), letting  $k = q+1$  rapidly leads to a large number of new nuisance parameters to include. For example, even in the ADL(1,1) model we would need to identify  $\theta = (\alpha, \beta_0, \beta_1, \sigma_u^2, \Gamma_\epsilon(0), \Gamma_\epsilon(1), \Gamma_\epsilon(2), \Gamma_\epsilon(3))$  from  $\widehat{\psi} = (\widehat{\alpha}, \widehat{\beta}_0, \widehat{\beta}_1, \widehat{\Gamma}_{\widehat{V}}(0), \widehat{\Gamma}_{\widehat{V}}(1), \widehat{\Gamma}_{\widehat{V}}(2), \widehat{\Gamma}_{\widehat{V}_X}(1,0), \widehat{\Gamma}_{\widehat{V}_X}(2,0))$ . But a large number of nuisance parameters would make estimation difficult. To avoid proliferation of nuisance parameters, we use Assumption A(d) which requires that  $\Gamma_\epsilon(1), \dots, \Gamma_\epsilon(q^*)$  are well approximated by  $\phi_1, \dots, \phi_m$ . Thus consider the ADL(p,q) model with  $p+q+1$  parameters  $\gamma$  and nuisance parameters  $\delta = (\sigma_u^2 \quad \sigma_\epsilon^2 \quad \phi)'$ , where  $\phi = (\phi_1, \dots, \phi_m)'$  characterizes  $\Gamma_\epsilon(1), \dots, \Gamma_\epsilon(q^*)$ . If Assumption A(d) holds, a necessary condition for

$$\theta_D = (\gamma', \delta)'$$

to be identified from

$$\widehat{\psi}_D = (\widehat{\gamma}, \widehat{\Gamma}_{\widehat{V}}(0), \dots, \widehat{\Gamma}_{\widehat{V}}(k), \widehat{\Gamma}_{\widehat{V}_X}(1,0), \dots, \widehat{\Gamma}_{\widehat{V}_X}(k,0))' \quad (10)$$

is

$$k \geq (m+1)/2.$$

Now  $q^*$  depends on the choice of auxiliary statistics. Since the residual autocovariances and cross-covariances  $(\widehat{\Gamma}_{\widehat{V}}(0), \dots, \widehat{\Gamma}_{\widehat{V}}(k), \widehat{\Gamma}_{\widehat{V}_X}(1,0), \dots, \widehat{\Gamma}_{\widehat{V}_X}(k,0))$  depend on  $\Gamma_\epsilon(1), \dots, \Gamma_\epsilon(k+q)$  in addition to  $(\gamma, \sigma_u^2, \sigma_\epsilon^2)$ , we need  $q^* = k+q$ .

For example, in the ADL(1,1) model, if we consider  $k=2$  and include  $\widehat{\Gamma}_{\widehat{V}_X}(2,0)$  among others in our auxiliary statistics, we have  $q^* = 3$  autocovariances of  $\epsilon_t$  to consider. However, if we use the same auxiliary statistics as in the simple regression model with correlated measurement errors,

$k = 1$  so  $q^* = 2$ . Assumption A(d) then requires that  $\Gamma_\epsilon(1)$  and  $\Gamma_\epsilon(2)$  are well approximated by  $m = 2k - 1 = 1$  parameter,  $\phi$ . This would be consistent with  $\epsilon_t$  being an AR(1) or an MA(1) process. This suggests the following general rule: choose an order  $k$  in (10). Set

$$m = 2k - 1 \tag{11}$$

and require that  $k + q$  autocovariances of  $\epsilon$ ,  $\Gamma_\epsilon(1), \dots, \Gamma_\epsilon(k + q)$ , be well approximated by  $m$  parameters  $\phi_1, \dots, \phi_m$ .

We should point out that neither of the above order conditions are strictly necessary for identification since additional information relating to heteroskedasticity and skewness of  $\epsilon_t$  can also be exploited. In the spirit of Pal (1980), Dagenais and Dagenais (1997), Lewbel (1997), Spierdijk and Wansbeek (2011), and Erickson and Whited (2000, 2002), higher order moments of  $\epsilon_t$  or of  $X_t$  can also be used to achieve identification. For example, the third moment of  $x_t$  is related to that of  $X_t$  by  $s_X = s_x + s_\epsilon$ . From  $V_t = u_t - \beta\epsilon_t$ , we have  $s_V = \sigma_u^3 s_u - \beta^3 \sigma_\epsilon^3 s_\epsilon$  and  $\widehat{\Gamma}_{\widehat{V}X^2}(0, 0) = \frac{1}{T} \sum_{t=1}^T \widehat{V}_t X_t^2 \xrightarrow{p} s_X \cdot \text{bias}(\theta) - \beta s_\epsilon$ .

Once the order condition is established, we can proceed with a local identification analysis by evaluating the rank of the  $\frac{\partial \psi_G(\theta)}{\partial \theta}$ . Numerical evaluation of the Jacobian matrix reveals the not-so-surprising result that  $\theta_D$  is not identified when  $\beta_0$  or  $\beta_1 = 0$ , and  $x_t$  is not serially correlated. In the ADL(1,1) model,  $\theta_D$  can be identified even when  $x_t$  is not serially correlated provided that  $\beta_0 \neq 0$  or  $\beta_1 \neq 0$ . The reason is that serial correlation in  $\widehat{V}_t$  can also come from autocorrelation in  $y_t$  and  $V_t$ .

## 4 Dynamic Panel Models

The least squares bias in the ADL(p,q) model is due solely to the measurement error in  $X_t$  because  $E(V_t y_{t-j}) = 0$  for  $j = 1, \dots, p$ . This will no longer be the case in the dynamic panel model. Let  $N$  and  $T$  be the number of units and time periods. Consider a panel AR(1) model (or equivalently a PADL(1,0) model) with a scalar exogenous regressor and additive fixed effects:

$$y_{it} = \eta_i + \alpha y_{i,t-1} + \beta x_{it} + u_{it}, \tag{12}$$

for  $i = 1, \dots, N$  and  $t = 1, \dots, T$ . Nickell (1981) showed that  $\widehat{\theta}_{N,T}$  is inconsistent as  $N \rightarrow \infty$  with  $T$  fixed. Estimates can be obtained by either by using instruments or by explicitly correcting for the small  $T$  bias.<sup>7</sup> Kiviet (1995) showed that these bias-corrected estimators tend to have smaller root mean-squared error (RMSE) than the IV estimators. However, the bias corrections are model specific and are invalid when there are additional covariates or lagged independent variables.

<sup>7</sup>Kiviet (1995), Bun and Carree (2005), and Phillips and Sul (2007) approximate the biases using large  $N$  asymptotics. Large  $N$  and  $T$  approximations has been used by Hahn and Kuersteiner (2002) to obtain  $b_{\infty, \infty}(\cdot)$ .

As is well documented, measurement errors in survey and longitudinal data are prevalent.<sup>8</sup> Existing work identifies the parameters from the covariance between the errors in the equations and the measurement errors, or higher order moments of the data.<sup>9</sup> Erickson and Whited (2012) compare high-order moment estimators with dynamic panel estimators and estimators that use lags of the mismeasured regressor. They find that all estimators perform well under correct specification but are biased under misspecification. We suggest a new approach of combining biased estimators.

Let the ideal regressor  $x_{it}$  be possibly contaminated by serially uncorrelated classical measurement errors:

$$X_{it} = x_{it} + \epsilon_{it}.$$

In terms of observables, the dynamic panel model becomes:

$$y_{it} = \eta_i + \alpha y_{i,t-1} + \beta' X_{it} + V_{it}, \quad V_{it} = u_{it} - \beta \epsilon_{it} \quad (13)$$

so the composite error  $V_{it}$  is correlated with  $X_{it}$ , and  $E(V_{it}X_{it}) = -\beta\sigma_\epsilon^2$ .

**Assumption P** For all  $s, t = 1, \dots, T$  and for all  $i = 1, \dots, N$ :

- (a)  $u_{it}$  is iid across  $i$  and  $t$  with  $E(u_{it}) = 0$ ,  $E(u_{it}^2) = \sigma_u^2$ ,  $E(u_{it}y_{i0}) = 0$ ,  $E(u_{it}x_{is}) = 0$ , and  $E(u_{it}\eta_i) = 0$ .
- (b)  $\epsilon_{it}$  is iid across  $i$  and  $t$  with  $E(\epsilon_{it}) = 0$ ,  $E(\epsilon_{it}^2) = \sigma_\epsilon^2$ , and  $E(\epsilon_{it}x_{is}) = 0$ , and  $E(\epsilon_{it}u_{is}) = 0$ .

The disturbances  $u_{it}$  are iid across both time and individuals, and uncorrelated with the initial observations  $y_{i0}$ . While  $u_{it}$  is orthogonal to the individual fixed effect  $\eta_i$  as well as to all values of the explanatory variable  $x_{it}$ , the correlation between  $\eta_i$  and  $x_{it}$  is left unspecified. The measurement error is iid and classical, and uncorrelated with  $u_{it}$ . Thus,  $V_{it} = u_{it} - \beta\epsilon_{it}$  and  $\Gamma_V(0) = \sigma_u^2 + \beta^2\sigma_\epsilon^2$  and  $\Gamma_V(j) = 0$  for  $j \geq 1$ .

As is standard in panel data model with fixed effects, the within-transformation is used to eliminate the unobserved individual effects  $\eta_i$  in (13). For a generic variable  $z$ , let  $\tilde{z}_{it} = z_{it} - \frac{1}{T} \sum_{s=1}^T z_{is}$  denote its demeaned value. We use the notation  $y_{-1}$  to denote the lagged dependent variable and  $\tilde{y}_{i,t-1} = y_{i,t-1} - \frac{1}{T} \sum_{s=1}^T y_{i,s-1}$  its demeaned version. The sample autocovariance of order  $j$  of variable  $z$  is defined as  $\Gamma_{\tilde{z}}(j) = \text{plim}_{N \rightarrow \infty} \hat{\Gamma}_{\tilde{z}}(j)$  where  $\hat{\Gamma}_{\tilde{z}}(j) = \frac{1}{N(T-1)} \sum_{i=1}^N \sum_{t=1+j}^T \tilde{z}_{it} \tilde{z}'_{i,t-j}$ . The cross

<sup>8</sup>The problem appears to be particularly important in investment data. See Erickson and Whited (2000) and Eberly, Rebelo, and Vincent (2009) among others.

<sup>9</sup>Erickson and Whited (2002), Erickson, Jiang, and Whited (2014) exploit non-normality of the data to identify the parameters of the model. See also Biorn (2008), Wansbeek and Koning (1991), Meijer, Spierdijk, and Wansbeek (2012), and Spierdijk and Wansbeek (2011) for discussion of choice of moments.



covariance between generic variables  $z_1$  and  $z_2$  is  $\widehat{\Gamma}_{\tilde{z}_1\tilde{z}_2}(u, v) = \frac{1}{N(T-1)} \sum_{i=1}^N \sum_{t=\max\{u,v\}}^T \tilde{z}_{1,i,t-u} \tilde{z}'_{2,i,t-v}$  and  $\Gamma_{\tilde{z}_1\tilde{z}_2}(u, v)$  is its probability limit as  $N$  tends to  $\infty$ .

The PADL(1,0) model written in demeaned data form becomes

$$\tilde{y}_{it} = \alpha \tilde{y}_{i,t-1} + \beta \tilde{X}_{it} + \tilde{V}_{it}, \quad \tilde{V}_{it} = \tilde{u}_{it} - \beta \tilde{\epsilon}_{it}.$$

Though its structure is similar to the ADL(1,0) model, there is one fundamental difference: the within-transformation induces serial correlation in variables that are initially uncorrelated. This, in particular, will be the case for  $\tilde{V}_{it}$ .

Now let  $\tilde{W}_{it} = (\tilde{y}_{i,t-1}, \tilde{X}_{it})'$  and assume the covariance matrix  $\Gamma_{\tilde{W}}(0)$  is full rank. The least squares dummy variable (LSDV or fixed effect) estimator for  $\gamma = (\alpha, \beta)'$  is the pooled OLS estimator in a regression of  $\tilde{y}_{it}$  on  $\tilde{X}_{it}$ . It has the property that

$$\text{plim}_{N \rightarrow \infty} \hat{\gamma} - \gamma = \Gamma_{\tilde{W}}(0)^{-1} \Gamma_{\tilde{W}\tilde{V}}(0, 0) \equiv \text{bias}_P(\theta_P), \quad (14)$$

where

$$\Gamma_{\tilde{W}\tilde{V}}(0, 0) = \begin{pmatrix} -\sigma_u^2 h_T(\alpha) \\ -\beta \sigma_\epsilon^2 \end{pmatrix}, \quad h_T(\alpha) = \frac{T(1-\alpha) - 1 + \alpha^T}{T(T-1)(1-\alpha)^2}.$$

The fixed- $T$ -Nickell bias is represented by  $h_T(\alpha)$  and is present even in the absence of measurement error.<sup>10</sup> This is in contrast with the ADL(1,0) model considered before where  $E(y_{t-1}V_t) = 0$ . Measurement error introduces a new source of bias to the dynamic panel model because of  $E(X_t V_t) \neq 0$ . However, the measurement error bias and the small- $T$  bias are opposite in sign. Thus, as noted in Biorn (1992), measurement error can alleviate the effect of heterogeneity; estimators that are sensitive to heterogeneity can be more robust to measurement errors.

The LSDV residuals  $\hat{V}_{it} = \tilde{y}_{it} - \hat{\gamma}' \tilde{W}_{it}$  are serially correlated even if  $\tilde{V}_{it}$  is uncorrelated over time. The error  $\tilde{V}_{it}$  is correlated with the demeaned regressors  $\tilde{y}_{it}$ ,  $\tilde{X}_{it}$ , and their lags in a nontrivial manner because of the interaction between the fixed effect bias and measurement error bias. Direct computations show that the autocovariances of  $\hat{V}_{it}$  have probability limits (as  $N \rightarrow \infty$ ) given by

$$\Gamma_{\hat{V}}(0) = \Gamma_{\tilde{V}}(0) - \text{bias}_P(\theta_P)' \Gamma_{\tilde{W}}(0) \text{bias}_P(\theta_P) \quad (15)$$

$$\Gamma_{\hat{V}}(1) = \Gamma_{\tilde{V}}(1) + \text{bias}_P(\theta_P)' \Gamma_{\tilde{W}}(1) \text{bias}_P(\theta_P) - (\Gamma_{\tilde{W}\tilde{V}}(1, 0) + \Gamma_{\tilde{W}\tilde{V}}(0, 1))' \text{bias}_P(\theta_P) \quad (16)$$

with

$$\begin{aligned} \Gamma_{\tilde{V}}(0) &= \sigma_u^2 + \beta^2 \sigma_\epsilon^2, & \Gamma_{\tilde{V}}(1) &= -\frac{1}{T} [\sigma_u^2 + \beta^2 \sigma_\epsilon^2], \\ \Gamma_{\tilde{W}\tilde{V}}(1, 0) &= \begin{pmatrix} -\sigma_u^2 [(T+1)h_T(\alpha) - 1] \\ \frac{1}{T} \beta \sigma_\epsilon^2 \end{pmatrix}, & \Gamma_{\tilde{W}\tilde{V}}(0, 1) &= \begin{pmatrix} -\sigma_u^2 [(T-1)h_T(\alpha) + \frac{T-2}{T} h_{T-1}(\alpha)] \\ \frac{1}{T} \beta \sigma_\epsilon^2 \end{pmatrix}. \end{aligned}$$

<sup>10</sup>The above expressions then reduce to equation (12) in Bun and Carree (2005).

The expression for  $\Gamma_{\widehat{V}}(0)$  reduces to that derived in Bun and Carree (2005) when no measurement errors are present. The lagged autocovariance reflects the cumulated fixed effect bias arising from the correlation between  $\widetilde{y}_{i,t-1}$  and  $\widetilde{V}_{it}$  as well as  $\widetilde{V}_{i,t-1}$ . Note that  $\Gamma_{\widetilde{W}\widetilde{V}}(0,1)$  does not vanish even as  $T \rightarrow \infty$  which we can exploit for identification.

The parameters of the PADL(1,0) model are

$$\theta_P = (\alpha, \beta, \sigma_u^2, \sigma_\epsilon^2)',$$

and we consider the auxiliary statistics

$$\widehat{\psi}_P(\theta_P) = \left( \widehat{\alpha} \quad \widehat{\beta} \quad \widehat{\Gamma}_{\widehat{V}}(0) \quad \widehat{\Gamma}_{\widehat{V}}(1) \right)',$$

whose binding function  $\psi_P(\theta_P) = \text{plim}_{N \rightarrow \infty} \widehat{\psi}_P(\theta_P)$  is given by equations (14)-(16). As before,  $\theta_P$  is identified from  $\psi_P$  is the mapping  $\theta_P \mapsto \psi_P(\theta_P)$  is invertible. To examine invertibility, we first transform the system of equations  $\psi_P(\theta_P) = \psi_P$  into:

$$\begin{aligned} \Gamma_{\widetilde{W}\widetilde{y}}(0,0) &= \Gamma_{\widetilde{W}}(0) \begin{pmatrix} \psi_{P1} \\ \psi_{P2} \end{pmatrix} \\ \Gamma_{\widetilde{y}}(0) &= \psi_{P3} + (\psi_{P1} \quad \psi_{P2}) \Gamma_{\widetilde{W}}(0) \begin{pmatrix} \psi_{P1} \\ \psi_{P2} \end{pmatrix} \\ \Gamma_{\widetilde{W}\widetilde{y}}(0,0)' \Gamma_{\widetilde{W}}(0)^{-1} \Gamma_{\widetilde{W}\widetilde{y}}(1,0) &= -\psi_{P4} + (\psi_{P1} \quad \psi_{P2}) \Gamma_{\widetilde{W}}(1) \begin{pmatrix} \psi_{P1} \\ \psi_{P2} \end{pmatrix} \end{aligned} \quad (17)$$

Note that the system (17) is equivalent to the original set of equations  $\psi_P(\theta_P) = \psi_P$  since it only involves pre-multiplying the first two equations by  $\Gamma_{\widetilde{W}}(0)$  which is nonsingular, and plugging them into the last two equations. As was the case in our analysis of ARX(1,0) in the previous section, only the left hand side terms of (17) depend on  $\theta_P$ . They can be expanded using:

$$\begin{aligned} \Gamma_{\widetilde{W}\widetilde{y}}(0,0) &= \begin{pmatrix} \alpha \Gamma_{\widetilde{y}_{-1}}(0) + \beta \Gamma_{\widetilde{X}\widetilde{y}_{-1}}(0,0) - \sigma_u^2 h_T(\alpha) \\ \alpha \Gamma_{\widetilde{X}\widetilde{y}_{-1}}(0,0) + \beta \Gamma_{\widetilde{X}}(0) - \beta \sigma_\epsilon^2 \end{pmatrix} \\ \Gamma_{\widetilde{y}}(0) &= \alpha^2 \Gamma_{\widetilde{y}_{-1}}(0) + 2\alpha\beta \Gamma_{\widetilde{X}\widetilde{y}_{-1}}(0,0) + \beta^2 \Gamma_{\widetilde{X}}(0) + \sigma_u^2 + \beta^2 \sigma_\epsilon^2 - 2\sigma_u^2 h_T(\alpha) - 2\beta \sigma_\epsilon^2 \\ \Gamma_{\widetilde{W}\widetilde{y}}(1,0) &= \begin{pmatrix} \alpha \Gamma_{\widetilde{y}_{-1}}(1) + \beta \Gamma_{\widetilde{X}\widetilde{y}_{-1}}(0,1) - \sigma_u^2 [(T+1)h_T(\alpha) - 1] \\ \alpha \Gamma_{\widetilde{X}\widetilde{y}_{-1}}(1,0) + \beta \Gamma_{\widetilde{X}}(1) + \beta \frac{1}{T} \sigma_\epsilon^2 \end{pmatrix} \end{aligned}$$

Now define the mapping  $\theta_P \mapsto \Psi_P(\theta_P)$  by letting

$$\Psi_P(\theta_P) = \begin{pmatrix} \Gamma_{\widetilde{W}\widetilde{y}}(0,0) \\ \Gamma_{\widetilde{y}}(0) \\ \Gamma_{\widetilde{W}\widetilde{y}}(0,0)' \Gamma_{\widetilde{W}}(0)^{-1} \Gamma_{\widetilde{W}\widetilde{y}}(1,0) \end{pmatrix}$$

Identification of  $\theta_P$  can now be analyzed in terms of invertibility of  $\Psi_P$ .

**Lemma 4** *Suppose that Assumption P holds. Then  $\theta_P$  is globally identifiable from  $\psi_P$  if and only if the system of equations  $\Psi_P(\theta_P) = \Psi_P$  has a unique solution in  $\theta_P$ . If  $\text{rank} \left( \frac{\partial \Psi_P(\theta_P)}{\partial \theta_P} \right) = 4$ , then  $\theta_P$  is locally identifiable from  $\psi_P$ .*

Due to the presence of  $h_T(\alpha)$ , which is a highly nonlinear function in  $\alpha$ ,  $\psi_P(\theta_P)$  is difficult to invert analytically. Lemma 4 is obtained by first isolating the terms in  $\psi_P(\theta)$  that do not depend on  $\theta_P$ , and then collecting the terms that do depend on  $\theta_P$  into  $\Psi_P(\theta)$ . Local injectivity of  $\psi_P(\theta_P)$  is equivalent to local injectivity of  $\Psi_P(\theta_P)$ . Hence local identification of  $\theta_P$  amounts to checking the rank of the Jacobian matrix,  $\frac{\partial \Psi_P(\theta_P)}{\partial \theta_P}$ .

## 5 Classical and Simulated Minimum Distance Estimation

The above analysis provides conditions under which  $\theta$  can be identified from appropriately defined  $\psi(\theta)$ . This leads naturally to classical minimum distance (CMD) estimation that identifies  $\theta$  from  $\hat{\psi}(\theta)$ . The CMD estimator is generically defined by

$$\hat{\theta} = \text{argmin}_{\theta} J(\theta), \quad J(\theta) = \|\hat{\psi}(\theta_0) - \psi(\theta)\|_{W_T}, \quad (18)$$

where  $W_T$  is a weighting matrix. Provided that the conditions stated in Newey and McFadden (1994) hold,  $\hat{\theta}$  is instrument-free estimator with classical property that

$$\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{d} N\left(0, [\psi_{\theta}(\theta_0) \text{Avar}(\hat{\psi})^{-1} \psi_{\theta}(\theta_0)']^{-1}\right) \equiv N(0, \text{Avar}(\hat{\theta}))$$

where  $\psi_{\theta}(\theta_0)$  is the matrix of derivatives of  $\psi(\theta)$  with respect to  $\theta$  evaluated at  $\theta_0$ .

In simple linear models, the probability limit of  $\hat{\psi}(\theta)$  is easily derived. As shown above for the ADL(0,0) model, the binding function can be inverted to obtain closed-form expressions for  $\psi(\theta)$  in the case of exact identification. But the probability limit of the terms in  $\hat{\psi}(\theta)$  can be tedious and sometimes impossible to calculate. While we can still invert the binding function for the ADL(1,0) model, the exercise was impossible for the ADL (1,1) model despite serious efforts. For the dynamic panel model,  $\Psi_P$  is a complicated function of  $\theta_P$  even in the absence of measurement error. A closed-form solution for  $\theta_P(\theta)$  is out of the question.

Simulated minimum distance (SMD) estimators use Monte-Carlo methods to compute the mapping from  $\theta$  to  $\psi$  and is therefore feasible even when  $\psi(\theta)$  is not analytically tractable. Specifically, let  $(\mathbf{y}^S(\theta), \mathbf{X}^S(\theta))$  denote  $S$  sets of data simulated under an assumed value for  $\theta$ , where each set  $\mathbf{y}^s(\theta)$  is  $T \times 1$  and  $\mathbf{X}^s(\theta)$  is  $T \times K$ . The estimator is

$$\hat{\theta}^S = \text{argmin}_{\theta} J^S(\theta), \quad J^S(\theta) = \|\hat{\psi}(\theta_0) - \psi^S(\theta)\|_{W_T}, \quad (19)$$

where

$$\psi^S(\theta) = \frac{1}{S} \sum_{s=1}^S \widehat{\psi}(\mathbf{y}^s(\theta), \mathbf{X}^s(\theta)) \approx E_{(\mathbf{y}^S(\theta), \mathbf{X}^S(\theta))}[\widehat{\psi}(\mathbf{y}^S(\theta), \mathbf{X}^S(\theta))].$$

As in classical minimum distance estimation, the mapping  $\psi(\theta)$  must be injective; simulations merely provide an approximation to  $\widehat{\theta} = \psi^{-1}(\widehat{\psi})$ . When  $\psi$  is a vector of unconditional moments,  $\widehat{\theta}$  is the simulated method of moments estimator of Duffie and Singleton (1993). When  $\psi$  are the scores of the likelihood, the efficient methods of moments estimator of Gallant and Tauchen (1996) obtains. The indirect inference estimator of Gourieroux, Monfort, and Renault (1993) defines  $\psi$  to be the parameters of an auxiliary regression.<sup>11</sup> Simulation estimators automatically provides second order bias correction if the auxiliary model admits an Edgeworth expansion, though the bias reduction comes at the cost of efficiency, see Gourieroux, Monfort, and Renault (1993). Gourieroux, Phillips, and Yu (2010) showed that indirect inference provides a substantial reduction in bias when the linear dynamic panel model is also the auxiliary model.

As far as we are aware, the only reference to simulation estimation of measurement error models is Jiang and Turnbull (2004) who used indirect inference as a way to adjust the bias in the auxiliary parameters with the help of validation data. Without validation data, simulation estimation cannot be implemented in the standard way. To see why, suppose that the auxiliary model coincides with the regression model. If  $E(V_t X_t) = 0$ , the auxiliary model can be estimated by OLS to yield  $\widehat{\psi}$  and  $\text{plim } \widehat{\psi} = \psi$ . But when  $E(V_t X_t) \neq 0$ ,  $\text{plim } \widehat{\psi} = \psi + \text{bias} \neq \psi$ . If  $X_t$  is fixed in simulations and  $V_t^s$  drawn such that  $X_t \perp V_t^s$ , then by construction,  $\text{plim } \widehat{\psi}^s(\theta) = \psi(\theta)$ . The binding function  $\psi(\theta)$  will not be consistently estimated because the bias term is omitted.

The generic problem is that when  $X_t$  is not weakly exogenous for the parameters, they can not be held fixed in simulations. In the measurement error model, the parameters in the marginal distribution of  $X_t$  and those of the conditional distribution of  $y_t$  given  $X_t$  are not variation free. For the simulation estimation to work, it is necessary that the simulated  $X_t^s$  preserves these relations. One can approximate  $x_t$  by a parametric model such as an AR(2) model  $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t$ . The Yule-Walker equations then yield

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \Gamma_X(1) & \Gamma_X(2) \\ \Gamma_X(0) - \sigma_\epsilon^2 & \Gamma_X(1) \end{pmatrix}^{-1} \begin{pmatrix} \Gamma_X(0) - \sigma_\epsilon^2 - \sigma_w^2 \\ \Gamma_X(1) \end{pmatrix}.$$

The drawback of this approach is that  $\theta$  has to be further augmented to include  $\sigma_w^2$ , and from a computational point of view, the presence of  $\sigma_\epsilon^2$  makes it difficult to control stability of  $x_t$  since  $\phi_1, \phi_2$  can take on values arbitrarily large.

The question then is how to simulate  $X_t$  with the desired properties without fully specifying its dynamic properties. We make use of the fact that covariance stationary processes with identical

<sup>11</sup>The estimators of Gallant and Tauchen (1996) and Gourieroux, Monfort, and Renault (1993) consider pseudo-maximum likelihood estimator of  $\psi$ .

second moments are observationally equivalent. Thus, it is only necessary for the simulated data to match the first and second moment properties the observed data. To motivate our simulator, note that if  $x_t$  were serially uncorrelated, the mean and variance of the simulated data can be preserved by letting  $X_t^s = x_t^s + \epsilon_t^s$  with  $x_t^s = \varphi X_t$  and  $\varphi = (1 - \frac{\sigma_\epsilon^2}{\Gamma_X(0)})^{1/2}$ . By construction, the mean and variance of  $X_t^s$  are equal to the mean and variance of  $X_t$ . But with serially correlated regressors, we will need the simulated regressors to not only preserve the variance, but also the autocovariance structure in the data. For this reason, we propose the following simulation method:

### Algorithm SMD

1. Compute the auxiliary statistics  $\widehat{\psi}$  from the data.
2. For  $s = 1, \dots, S$  and  $t = 1, \dots, T$  and given  $\{\epsilon_t^s\}$  and  $\{u_t^s\}$ :
  - (i) let  $x_t^s = \varphi_1 X_t + \varphi_2 X_{t-1}$ ;
  - (ii) simulate  $X_t^s = x_t^s + \epsilon_t^s$ ;
  - (iii)  $y_t^s = \alpha + \beta x_t^s + u_t^s$ ;
  - (iv) compute  $\widehat{\psi}(\theta^s)$  from the simulated data  $(\mathbf{y}^s, X^s)$ .
- 3 Minimize  $\|\widehat{\psi} - \frac{1}{S} \sum_s \widehat{\psi}^s(\theta)\|_{W_T}$  over  $\theta$ .

The key to Algorithm SMD is Step 2(i) in which we postulate that  $x_t^s$  is linear in  $X_t$  and  $X_{t-1}$ . The constants  $\varphi_1$  and  $\varphi_2$  are chosen to satisfy the pair of equations:

$$\begin{aligned}\Gamma_X(0) - \Gamma_\epsilon(0) &= (\varphi_1^2 + \varphi_2^2)\Gamma_X(0) + 2\varphi_1\varphi_2\Gamma_X(1) \\ \Gamma_X(1) - \Gamma_\epsilon(1) &= (\varphi_1^2 + \varphi_2^2)\Gamma_X(1) + \varphi_1\varphi_2\Gamma_X(0) + \varphi_2\Gamma_X(2).\end{aligned}\tag{20}$$

Step 2(i) thus models  $x_t^s$  as a rescaled but deterministic function of the data  $X_t$ . This method does not directly model the dynamics of  $x_t$  (or of  $x_t^s$ ), but by construction,  $\Gamma_x^s(0) = \Gamma_X(0) - \Gamma_\epsilon(0)$  and  $\Gamma_x^s(1) = \Gamma_X(1) - \Gamma_\epsilon(1)$ . Given estimates of  $\Gamma_X(0)$  and  $\Gamma_X(1)$ , there are two equations in two unknowns. A unique solution for  $\varphi_1$  and  $\varphi_2$  can be obtained by noting that the system in (20) is linear in  $r, \phi$ , where  $r = \varphi_1^2 + \varphi_2^2$  and  $r_2 = \varphi_1\varphi_2$ :

$$\underbrace{\begin{pmatrix} \Gamma_X(0) & 2\Gamma_X(1) \\ \Gamma_X(1) & \Gamma_X(0) + \Gamma_X(2) \end{pmatrix}}_A \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_X(0) - \Gamma_\epsilon(0) \\ \Gamma_X(1) - \Gamma_\epsilon(1) \end{pmatrix}}_B.$$

Assuming that  $A$  is invertible,  $(r_1 \ r_2)' = A^{-1}B$ . Then  $\varphi_1$  and  $\varphi_2$  can be taken as

$$\varphi_1 = \frac{1}{2} [\sqrt{r_1 + 2r_2} + \sqrt{r_1 - 2r_2}], \quad \varphi_2 = \frac{1}{2} [\sqrt{r_1 + 2r_2} - \sqrt{r_1 - 2r_2}].$$

The simulation procedure can be easily adapted to allow for skewed errors or errors with fat tails. This is used in Gospodinov and Ng (2014) to estimate moving-average models without imposing invertibility. The procedure can also be adapted to use in dynamic panel regressions.

One we can simulate the model to obtain draws of  $X_t$ , the arguments in Gourieroux, Monfort, and Renault (1993) can be used to analyze the properties of the SMD estimator. The order condition requires that the dimension of  $\psi$  exceeds the dimension of  $\theta$ . The necessary and sufficient condition for local identification is that  $\frac{\partial \psi(\theta)}{\partial \theta'}$  has full column rank. Once identification is granted, the SMD estimator has standard properties. If the auxiliary parameters  $\hat{\psi}$  are  $\sqrt{T}$ -consistent and asymptotically normal, the simulation estimator is  $\sqrt{T}$ -consistent and asymptotically normal:

$$\sqrt{T}(\hat{\theta}^S - \theta_0) \xrightarrow{d} N\left(0, \left(1 + \frac{1}{S}\right) [\psi_\theta(\theta_0) \text{Avar}(\hat{\psi})^{-1} \psi_\theta(\theta_0)']^{-1}\right) = N(0, \text{Avar}(\hat{\theta}^S)).$$

The asymptotic variance is of the double sandwich form. It depends on the asymptotic variance of the auxiliary statistics, whether the auxiliary model is correctly specified, and is smaller the larger is the number of simulations,  $S$ . Also,  $T \times J(\hat{\theta}^S) \xrightarrow{d} \chi_{\dim(\psi) - \dim(\theta)}^2$  as  $T \rightarrow \infty$  and  $W_T$  is the optimal weighting matrix. The distance metric (DM) statistic for testing the null hypothesis that a constraint is true is  $DM = \frac{TS}{S+1} (J(\hat{\theta}_C^S) - J(\hat{\theta}^S))$  where  $\hat{\theta}_C^S$  is the constrained SMD estimate.

The variance of  $\hat{\theta}^S$  is based on the delta method and can be inaccurate in finite samples. We construct confidence intervals for an element of  $\theta$ ,  $\theta_i$ , by inverting the DM statistics under a sequence of constraints  $H_0 : g(\theta) = 0$ . This approach is also considered in Czellar and Zivot (2008). If  $\eta$  is the significance level of the test and  $q_{1-\eta, df}$  is the  $(1 - \eta)$ -th quantile of the chi-squared distribution with  $df$  degrees of freedom, the  $100(1 - \eta)\%$  confidence interval for  $\theta_i$  is given by the set of values of  $\theta$  satisfying  $DM \leq q_{1-\eta, df}$ .

## 5.1 Simulations

We use 5000 replications to illustrate the properties of the CMD and SMD estimators. For  $t = 1, \dots, T = (200, 500, 1000)$ , the data are generated from the ADL(1,1) model

$$\begin{aligned} y_t &= \alpha y_{t-1} \beta_0 x_t + \beta_1 x_{t-1} + u_{yt}, & u_{yt} &\overset{iid}{\sim} \mathcal{N}(0, \sigma_{uy}^2), \\ x_t &= \rho_x x_{t-1} + u_{xt}, & u_{xt} &\overset{iid}{\sim} \mathcal{N}(0, \sigma_{ux}^2), \\ X_t &= x_t + \epsilon_t, & \epsilon_t &= e_t + \theta e_{t-1}, & e_t &\overset{iid}{\sim} \mathcal{N}(0, \sigma_e^2). \end{aligned}$$

We assume  $(\alpha, \beta) = (0, 1)$  with  $\rho_x = (0.2, 0.5, 0.8)$ . The measurement error process is calibrated such that the signal-to-noise ratio  $R^2 = \frac{\text{var}(x_t)}{\text{var}(X_t)} = 0.7$ . This is achieved by solving  $\sigma_e^2$  from

$$\sigma_e^2(1 + \theta^2) = \frac{1 - R^2}{R^2} \frac{\sigma_{ux}^2}{1 - \rho_x^2}.$$

In the simulations, we let  $\sigma_{uy}^2 = \sigma_{ux}^2 = 1$ . In practice, we do not know if  $\epsilon_t$  is serially correlated or not. We always estimate the model to allow for serial correlation in  $\epsilon_t$  even when  $\epsilon_t$  is white noise. The SMD simulates  $\epsilon_t$  as an AR(1) process even though the true process is MA(1). We only consider the case of exact identification.

We begin with the simple regression model when  $\alpha = \beta_1 = 0$ . As these parameters are not estimated,  $\theta = (\beta_0, \sigma_u^2, \sigma_\epsilon^2, \phi)$  and  $\hat{\psi} = (\hat{\beta}, \hat{\Gamma}_{\hat{v}}(0), \hat{\Gamma}_{\hat{v}}(1), \hat{\Gamma}_{\hat{v}X}(1, 0))'$ . The results are reported in Table 1. In the top panel where  $\epsilon_t$  is white noise,  $X_{t-1}$  is a valid instrument. The estimator is denoted by IV. For comparison purposes, Table 1 also reports the estimates from the infeasible estimator (IDEAL) based on the true (latent) regressor  $x_t$ . As expected, the average of the IDEAL estimates is well centered around the true value of  $\beta$ . The OLS estimates are significantly downward biased when  $X_t$  is used as regressor instead of  $x_t$ . The bias is larger the less persistent is  $x_t$ . The IV estimator gives highly variable estimates when  $\rho_x = 0.2$ . The CMD is more stable than IV. The SMD estimator matches up well with the CMD, showing that simulation estimation of the mapping from  $\theta$  to  $\psi$  did not induce much efficiency loss. The bottom panel shows that when  $\epsilon_t$  is serially correlated, the IV estimates are highly unreliable. The CMD and SMD estimates are similar to the case of white noise measurement error.

The parameters of the ADL(1,1) model are  $\theta = (\alpha, \beta_0, \beta_1, \sigma_u^2, \sigma_\epsilon^2, \phi)$ . The auxiliary statistics are  $\hat{\psi}(\theta) = (\hat{\alpha}, \hat{\beta}_0, \hat{\beta}_1, \hat{\Gamma}_{\hat{v}}(0), \hat{\Gamma}_{\hat{v}}(1), \hat{\Gamma}_{\hat{v}X}(1, 0))'$ . We report the estimated short- and long-run response of  $y_t$  to  $x_t$  as given by  $\hat{\beta}_0$  and  $\frac{\hat{\beta}_0 + \hat{\beta}_1}{1 - \hat{\alpha}}$ . Table 2 reports results for ADL(1,0). This is a special ADL(1,1) model with  $\beta_1 = 0$ , but this constraint is not imposed in the estimation. The estimates exhibit some downward biases that tend to increase with the degree of persistence in  $x_t$ . The estimates are reasonably precise, taking into account that estimates of models with an autoregressive structure tend to be downward biased. Table 3 shows results for the ADL(1,1) model. The estimates are somewhat more precise than those in Table 2 for the ADL(1,0) model.

Results for the panel data model are reported in Table 4 based on  $N = 200$ . Two models are considered: PADL(1,0) and PADL(1,1). We assume that for all  $i$ ,  $\eta_i \sim N(0, 1)$ , and  $x_{it}$  is an AR(1) process with autocorrelation of 0.5. The parameters are again chosen such that  $\sigma_\epsilon^2$  is about 0.7 of  $\Gamma_X(0)$ . The first column is the infeasible estimator based on the latent regressor,  $x$ , and is denoted LSDV- $x$ . The estimates are biased even when  $x$  is correctly observed because of the fixed effect bias. The biases are even larger when  $X$  is used in place of  $x$ . The SMD estimates are closer to the true values.

## 5.2 Application to the Traditional Phillips Curve

Despite the advances in the development and empirical assessment of the New Keynesian Phillips curve, the traditional Phillips curve continues to be widely used for inflation forecasting and pol-

icy analysis, and it remains a topic of active academic research. The traditional Phillips curve considered in Gordon (2013) and Watson (2014) is an autoregressive distributed lag model with exogenous variables  $z$ , which we denote by ADLZ(p,q,s). Specifically,

$$\alpha(L)(1-L)\pi_t = \beta(L)x_t + \zeta(L)z_t + u_t,$$

where  $\pi_t$  is inflation,  $z_t$  is a vector of supply shock variables,  $\alpha(L)$ ,  $\beta(L)$  and  $\zeta(L)$  are lag polynomials of orders  $p$ ,  $q$  and  $s$  respectively, and  $u_t$  is a serially uncorrelated error term. The long-run effect of unemployment on the change in inflation  $\Delta\pi_t$  is

$$\bar{\beta} = \frac{\beta(1)}{\alpha(1)}.$$

An important variable in the traditional Phillips curve is the ‘demand gap’ variable  $x_t$  which is meant to capture the deviation of real activity from its equilibrium. There is no unique measure of real activity. It is quite common in empirical work to use the unemployment rate as a proxy. Feng and Hu (2013) present evidence that the official U.S. unemployment rate underestimates the true unemployment rate due to misclassification of labor force status in the Current Population Survey. Even if real activity is correctly measured, the natural rate of unemployment is not observed. Any constructed ‘gap’ variable used in estimation is potentially subject to measurement error. In what follows, we explicitly account for possible measurement error in the unemployment gap  $x_t = U_t - U_t^*$  and consider different proxies for  $U_t^*$ .

The data, same as in Watson (2014) and available from the author’s website, covers the period 1959:Q3–2013:Q3. The inflation rate  $\pi_t$  is the core inflation (PCE less food and energy) and  $U_t$  is the civilian unemployment rate for all workers ages 16+. The control variables are  $z_t = (Z_{t1}, Z_{t-1,1}, Z_{t-1,2}, Z'_{t,3})'$ , where  $Z_{t1}$  is the relative price of nonpetroleum imports,  $Z_{t,2}$  is change in the relative price of food and energy inflation, and  $Z_{t,3}$  denotes two dummy variables for the 1971–1974 Nixon-era price controls (see Gordon (2013) and Watson (2014)).<sup>12</sup> All series are seasonally adjusted.

We consider six measures of  $U_t^*$ , giving  $X_t(k) = U_t - U_t^*(k)$  for  $k = 1, \dots, 6$ .

$$\begin{aligned} U_t^*(1) &= \min(U_t, \dots, U_{t-11}); & U_t^*(2) &= U_{t-4}; & U_t^*(3) &= BP(U_t, \text{one-sided}); \\ U_t^*(4) &= HPP(U_t, 1600); & U_t^*(5) &= HPP(U_t, 6400). & U_t^*(6) &= \text{constant}. \end{aligned}$$

The first three are ‘real-time gaps’, considered in Stock and Watson (2010), where  $BP$  denotes a band-pass filter. These are one-sided filters that are appropriate for out-of-sample forecasting inflation. In contrast, the last two are based on the Hodrick-Prescott (HP) filter which is a two-sided

<sup>12</sup>More specifically,  $Z_{t1}$  is the difference in the rates of inflation of the GDP deflator for non-petroleum imports and the overall GDP deflator, and  $Z_{t2}$  is the difference between PCE (all-items) inflation and PCE (less food and energy) with different effects pre- and post-1984:Q1.



filter with smoothing parameter  $\lambda$ . For quarterly data, a smoothing parameter of 1600 is quite standard. The larger smoothing parameter of 6400 is used in Gordon (2013). The last gap measure,  $U_t^*$  is assumed constant which is commonly done in practice. All six versions of  $X_t$  are persistent and the condition  $\Gamma_X(1) \neq 0$  is easily satisfied. We allow the measurement error  $\epsilon_t = X_t - x_t$  to be possibly serially correlated. Interestingly, the six gap variables are not particularly strongly correlated with each other. Though  $X_t(4)$  and  $X_t(5)$  have a correlation of 0.98,  $X_t(2)$  and  $X_t(6)$  only have a correlation of 0.34. Most of the remaining correlations are between 0.5 and 0.8.

The CMD estimator is feasible if the binding function can be explicitly derived. The current ADLZ(p,q,s) model has additional predictors  $z_t$ , making the CMD estimator difficult to implement. This is precisely the situation when the SMD estimator is appealing because it can easily accommodate changes to the model. Here, we treat  $z_t$  as exogenous in Algorithm SMD-EIV with  $S = 100$ . Most authors choose  $q \geq 2$ . Because measurement errors can induce serial correlation in the residuals, more lags than necessary might have been included. When we explicitly allow for measurement error, we find  $q = 1$  to be appropriate. The traditional Phillips curve is thus estimating using the ADLZ(1,1,1) model and  $\bar{\beta} = \frac{\beta_0 + \beta_1}{1 - \alpha_1}$ . Confidence intervals for  $\bar{\beta}$  are obtained by inverting the distance metric test as discussed above. The  $100(1 - \eta)\%$  confidence interval for  $\bar{\beta}$  is given by the set of values satisfying  $DM \leq q_{1-\eta,1}$ . The endpoints of the confidence interval (CI) are obtained as

$$\begin{aligned}\bar{\beta}_L &= \inf\{\bar{\beta} \in \Theta^C : \Pr(DM \leq q_{1-\eta,1} \mid H_0) \geq 1 - \eta\}, \\ \bar{\beta}_U &= \sup\{\bar{\beta} \in \Theta^C : \Pr(DM \leq q_{1-\eta,1} \mid H_0) \geq 1 - \eta\}.\end{aligned}$$

Table 5 reports the OLS and SMD estimates of the long-run effect of unemployment gap on inflation,  $\hat{\beta}$  and  $\hat{\sigma}_\epsilon^2$ . The point estimates of  $\bar{\beta}$  are quite similar and for most models and estimators, the long-run estimate is statistically insignificant. The main differences between the OLS and SMD estimators of  $\bar{\beta}$  lie in their assessment of the sampling uncertainty around  $\hat{\beta}$ . Consistent with our simulation results, explicitly recognizing the presence of measurement error produces more imprecise estimates of  $\bar{\beta}$  and wider confidence intervals. For example, while the OLS estimator in the model with  $X_t(6)$  indicates that  $\hat{\beta}$  is statistically different than zero at 10% significance level, allowing for measurement error via our proposed approach renders the SMD estimate insignificant. The measurement error ranges from 3-4% for  $X_t(6)$  and  $X_t(1)$  to 37% for  $X_t(2)$  suggesting that some specifications of the Phillips curve are characterized by a signal-to-noise ratio which is large enough to provide reliable empirical guidance for policy analysis. Similar results are obtained using output gap instead of unemployment gap but the estimated magnitude of the measurement error in these models tends to be larger.

## 6 Conclusion

This paper makes two contributions. First, we show that several biased estimates can jointly identify a model with mismeasured regressors without the need for external instruments. The key is to exploit persistence in the data. Second, we develop a simulation algorithm for situations where the regressors are not weakly exogenous for the parameters of interest. The algorithm can be used in time series and panel data regressions and can accommodate additional regressors. The proposed methodology is especially useful when external instruments are either unavailable or are weak. This can be potentially useful in endogeneity problems.

Table 1: ADL(0,0):  $(\alpha, \beta_0, \beta_1) = (0, 1, 0)$

$$\begin{aligned}\theta &= (\beta_0, \sigma_u^2, \sigma_\epsilon^2, \phi) \\ \widehat{\psi} &= (\widehat{\beta}, \widehat{\Gamma}_{\widehat{V}}(0), \widehat{\Gamma}_{\widehat{V}}(1), \widehat{\Gamma}_{\widehat{V}X}(1, 0))'\end{aligned}$$

$T$	$\rho_x$	$\widehat{\beta}_0$					SE( $\widehat{\beta}_0$ )				
		OLS	IDEAL	IV	CMD	SMD	OLS	IDEAL	IV	CMD	SMD
$\epsilon_t$ White Noise											
200	0.200	0.702	1.006	1.078	1.058	1.089	0.068	0.086	1.151	0.401	0.375
200	0.500	0.699	1.006	1.030	0.960	0.982	0.064	0.079	0.215	0.223	0.220
200	0.800	0.690	1.004	1.013	0.972	0.983	0.062	0.066	0.108	0.129	0.132
500	0.200	0.700	1.000	1.054	1.010	1.037	0.043	0.053	0.451	0.302	0.293
500	0.500	0.699	1.000	1.008	0.967	0.978	0.041	0.049	0.122	0.167	0.166
500	0.800	0.695	1.000	1.004	0.988	0.992	0.040	0.041	0.063	0.075	0.079
1000	0.200	0.699	1.000	1.016	0.975	0.993	0.030	0.037	0.259	0.242	0.238
1000	0.500	0.699	1.000	1.002	0.973	0.978	0.028	0.034	0.083	0.128	0.129
1000	0.800	0.698	1.000	1.001	0.993	0.994	0.028	0.028	0.043	0.050	0.052
$\epsilon_t$ MA(1)											
200	0.200	0.702	1.006	-3.044	1.122	1.123	0.069	0.087	246.882	0.416	0.397
200	0.500	0.700	1.007	1.093	1.004	1.003	0.065	0.083	9.395	0.187	0.189
200	0.800	0.690	1.006	1.031	1.002	1.001	0.064	0.075	0.171	0.100	0.102
500	0.200	0.700	1.000	0.815	1.048	1.042	0.043	0.054	49.865	0.292	0.282
500	0.500	0.699	1.001	1.062	0.998	0.994	0.042	0.052	1.902	0.119	0.120
500	0.800	0.695	1.001	1.009	1.000	0.997	0.042	0.046	0.093	0.061	0.061
1000	0.200	0.700	1.000	-1.57	1.008	0.997	0.030	0.038	154.439	0.206	0.206
1000	0.500	0.699	1.000	1.018	0.996	0.990	0.029	0.036	0.199	0.083	0.085
1000	0.800	0.698	1.000	1.004	0.999	0.996	0.028	0.032	0.061	0.042	0.043

Table 2: ADL(1,0):

$$\begin{aligned}\theta &= (\alpha, \beta_0, \beta_1, \sigma_u^2, \sigma_\varepsilon^2, \phi) \\ \widehat{\psi}(\theta) &= (\widehat{\alpha}, \widehat{\beta}_0, \widehat{\beta}_1, \widehat{\Gamma}_{\widehat{v}}(0), \widehat{\Gamma}_{\widehat{v}}(1), \widehat{\Gamma}_{\widehat{v}X}(1, 0))'\end{aligned}$$

		Estimates of $\beta_0 = 1$ and $\frac{\beta_0 + \beta_1}{1 - \alpha} = 2.5$						Standard Errors					
$T$	$\rho_x$	OLS		CMD		SMD		OLS		CMD		SMD	
$\epsilon_t$ white noise													
200	0.2	0.696	1.858	1.125	2.838	1.137	2.803	0.068	0.261	0.454	5.755	0.438	1.026
200	0.5	0.656	1.996	0.989	2.434	1.018	2.547	0.066	0.224	0.263	1.523	0.264	0.456
200	0.8	0.522	2.129	0.931	2.422	0.936	2.516	0.058	0.179	0.216	0.247	0.170	0.248
500	0.2	0.693	1.858	1.053	2.588	1.079	2.652	0.043	0.166	0.358	0.765	0.366	0.777
500	0.5	0.653	2.000	0.961	2.421	0.988	2.487	0.042	0.143	0.162	0.262	0.177	0.292
500	0.8	0.520	2.144	0.928	2.433	0.928	2.481	0.037	0.113	0.136	0.144	0.111	0.172
1000	0.2	0.693	1.857	1.004	2.486	1.039	2.561	0.030	0.116	0.255	0.531	0.292	0.603
1000	0.5	0.653	2.002	0.956	2.418	0.980	2.471	0.029	0.100	0.108	0.178	0.129	0.217
1000	0.8	0.520	2.151	0.933	2.440	0.931	2.481	0.026	0.077	0.090	0.097	0.078	0.105
$\epsilon_t$ MA(1)													
200	0.2	0.711	1.790	1.105	2.741	1.138	2.803	0.070	0.275	0.466	1.459	0.434	1.115
200	0.5	0.679	1.945	0.941	2.367	1.000	2.504	0.069	0.243	0.251	0.440	0.270	0.501
200	0.8	0.544	2.079	0.907	2.400	0.918	2.504	0.063	0.203	0.199	0.252	0.185	0.289
500	0.2	0.709	1.787	0.981	2.428	1.078	2.640	0.044	0.174	0.355	0.787	0.354	0.783
500	0.5	0.677	1.947	0.901	2.317	0.970	2.455	0.044	0.154	0.156	0.272	0.183	0.317
500	0.8	0.543	2.094	0.899	2.407	0.913	2.482	0.041	0.127	0.125	0.152	0.119	0.168
1000	0.2	0.709	1.785	0.893	2.237	1.030	2.537	0.031	0.122	0.228	0.485	0.276	0.583
1000	0.5	0.677	1.948	0.891	2.308	0.964	2.447	0.030	0.107	0.108	0.191	0.132	0.228
1000	0.8	0.543	2.102	0.900	2.412	0.915	2.482	0.028	0.087	0.085	0.105	0.083	0.114

Table 3: ADL(1,1)

$$\theta = (\alpha, \beta_0, \beta_1, \sigma_u^2, \sigma_\varepsilon^2, \phi)$$

$$\hat{\psi}(\theta) = (\hat{\alpha}, \hat{\beta}_0, \hat{\beta}_1, \hat{\Gamma}_{\hat{v}}(0), \hat{\Gamma}_{\hat{v}}(1), \hat{\Gamma}_{\hat{v}_X}(1, 0))'$$

		Estimates of $\beta_0 = 1$ and $\frac{\beta_0 + \beta_1}{1 - \alpha} = 3.75$						Standard Errors					
$T$	$\rho_x$	OLS		CMD		SMD		OLS		CMD		SMD	
$\epsilon_t$ white noise													
200	0.2	0.715	3.014	0.963	3.629	1.023	3.810	0.071	0.352	0.192	0.564	0.223	0.631
200	0.5	0.706	3.226	0.972	3.685	1.014	3.802	0.069	0.301	0.156	0.385	0.172	0.422
200	0.8	0.605	3.392	0.997	3.741	1.008	3.931	0.063	0.238	0.173	0.267	0.168	1.259
500	0.2	0.713	3.011	0.941	3.590	1.003	3.756	0.045	0.222	0.124	0.355	0.146	0.394
500	0.5	0.703	3.228	0.961	3.675	1.002	3.768	0.044	0.190	0.101	0.239	0.109	0.255
500	0.8	0.603	3.407	0.989	3.736	0.987	3.843	0.041	0.149	0.112	0.164	0.108	0.250
1000	0.2	0.713	3.010	0.936	3.583	1.001	3.747	0.031	0.155	0.085	0.246	0.100	0.273
1000	0.5	0.703	3.230	0.960	3.674	1.000	3.761	0.030	0.132	0.069	0.166	0.075	0.178
1000	0.8	0.603	3.415	0.989	3.735	0.984	3.812	0.028	0.101	0.078	0.113	0.075	0.168
$\epsilon_t$ MA(1)													
200	0.2	0.697	2.892	0.950	3.596	1.018	3.777	0.074	0.375	0.203	0.623	0.234	0.705
200	0.5	0.697	3.146	0.964	3.672	1.010	3.796	0.074	0.334	0.170	0.442	0.185	0.490
200	0.8	0.601	3.334	1.009	3.752	0.997	3.943	0.071	0.280	0.194	0.324	0.187	0.803
500	0.2	0.695	2.884	0.920	3.529	0.992	3.709	0.046	0.236	0.127	0.387	0.154	0.436
500	0.5	0.695	3.144	0.951	3.652	0.995	3.760	0.047	0.210	0.108	0.274	0.116	0.293
500	0.8	0.600	3.349	0.999	3.740	0.985	3.855	0.046	0.175	0.122	0.199	0.114	0.341
1000	0.2	0.694	2.883	0.912	3.513	0.987	3.700	0.032	0.164	0.088	0.269	0.108	0.303
1000	0.5	0.695	3.147	0.948	3.648	0.996	3.753	0.033	0.145	0.074	0.190	0.080	0.203
1000	0.8	0.600	3.357	0.999	3.738	0.985	3.810	0.032	0.119	0.084	0.137	0.080	0.198

Table 4: Panel Data,  $N = 200$ .

$$y_{it} = \eta_i + \alpha y_{it-1} + \beta_0 X_{it} + \beta_1 X_{it-1} + u_{it}$$

$$X_{it} = x_{it} + \epsilon_{it}.$$

		LSDV-x		LSDV-X		SMD	
		Est.	S.E.	Est.	S.E.	Est.	S.E.
$T = 10, \text{PADL}(1,0)$							
$\beta_0$	1.000	1.005	0.022	0.658	0.023	1.063	0.273
$\alpha_1$	0.800	0.722	0.059	0.720	0.066	0.750	0.114
$\sigma_u^2$	1.000	1.172	0.511	1.196	0.406	0.863	0.337
$\sigma_\epsilon^2$	0.756	-	-	-	-	0.534	0.318
$T = 20, \text{PADL}(1,0)$							
$\beta_0$	1.000	1.013	0.019	0.684	0.017	1.055	0.236
$\alpha_1$	0.800	0.765	0.034	0.784	0.034	0.770	0.059
$\sigma_u^2$	1.000	1.327	0.392	1.376	0.311	0.937	0.248
$\sigma_\epsilon^2$	0.756	-	-	-	-	0.555	0.268
$T = 10, \text{PADL}(1,1)$							
$\beta_0$	1.000	0.965	0.038	0.679	0.029	0.994	0.241
$\beta_1$	0.600	0.669	0.062	0.514	0.040	0.568	0.113
$\alpha_1$	0.800	0.744	0.046	0.751	0.047	0.802	0.067
$\sigma_u^2$	1.000	0.945	0.428	1.643	0.402	0.931	0.391
$\sigma_\epsilon^2$	0.756	-	-	-	-	0.466	0.333
$T = 20, \text{PADL}(1,1)$							
$\beta_0$	1.000	0.990	0.019	0.714	0.017	1.037	0.189
$\beta_1$	0.600	0.637	0.039	0.500	0.026	0.600	0.104
$\alpha_1$	0.800	0.778	0.021	0.800	0.020	0.797	0.028
$\sigma_u^2$	1.000	1.232	0.373	2.621	0.442	0.973	0.274
$\sigma_\epsilon^2$	0.756	-	-	-	-	0.579	0.283

Table 5: OLS and SMD Estimates of the traditional Phillips Curve.

	OLS		SMD			$\widehat{\Gamma}_X(0)$
	$\widehat{\beta}$	90% CI of $\bar{\beta}$	$\widehat{\beta}$	90% CI of $\bar{\beta}$	$\widehat{\sigma}_\epsilon^2$	
$X_t(1)$	-0.028	[-0.077, 0.022]	-0.025	[-0.225, 0.139]	0.071	1.727
$X_t(2)$	-0.049	[-0.129, 0.032]	-0.052	[-0.270, 0.072]	0.421	1.128
$X_t(3)$	-0.067	[-0.140, 0.006]	-0.081	[-0.217, 0.056]	0.104	0.802
$X_t(4)$	-0.017	[-0.111, 0.078]	-0.021	[-0.252, 0.184]	0.081	0.587
$X_t(5)$	-0.024	[-0.098, 0.051]	-0.032	[-0.239, 0.145]	0.214	0.857
$X_t(6)$	-0.040	[-0.078, -0.002]	-0.044	[-0.199, 0.079]	0.073	2.583

## A Appendix: Proofs

**Proof of Lemma 1** Note first that since  $\Gamma_X(0) - \sigma_\epsilon^2 = \Gamma_x(0) > 0$ , the equation for  $\psi_{A1}$  implies that  $\beta = 0$  if and only if  $\psi_{A1} = 0$ . Thus,  $\beta = 0$  is directly identified off of  $\psi_{A1}$ . Moreover, we than have  $\sigma_u^2 = \psi_{A2}$ . Thus  $(\beta = 0, \sigma_u^2)$  is identified. For  $\beta \neq 0$ , we have  $\text{sgn}(\beta) = \text{sgn}(\psi_{A1})$ . If  $\Gamma_X(1) \neq 0$ , then from the equation for  $\psi_{A3}$ ,

$$\beta \frac{\sigma_\epsilon^2}{\Gamma_X(0)} = \text{sgn}(\psi_{A1}) \sqrt{\frac{\psi_{A3}}{\Gamma_X(1)}}. \quad (21)$$

Plugging into the solution for  $\psi_{A1}$  and noticing that  $\psi_{A1} = \text{sgn}(\psi_{A1}) |\psi_{A1}|$  then gives

$$\beta = \text{sgn}(\psi_{A1}) \left[ |\psi_{A1}| + \sqrt{\frac{\psi_{A3}}{\Gamma_X(1)}} \right].$$

For  $\sigma_u^2$ , combine (21) with  $\psi_{A2} = \sigma_u^2 + \beta \sigma_\epsilon^2 \psi_{A1}$  to obtain

$$\sigma_u^2 = \psi_{A2} - |\psi_{A1}| \Gamma_X(0) \sqrt{\frac{\psi_{A3}}{\Gamma_X(1)}}.$$

Thus, serial correlation in the regressor  $X$  is sufficient to globally identify  $(\beta \neq 0, \sigma_u^2)$ . Finally, when  $\beta \neq 0$ ,

$$\sigma_\epsilon^2 = \frac{\Gamma_X(0)}{\beta} \text{sgn}(\psi_{A1}) \sqrt{\frac{\psi_{A3}}{\Gamma_X(1)}}$$

where  $\beta$  is defined above. The expression for  $\sigma_u^2$  follows directly from  $\psi_{A2}$ . Thus,  $\Gamma_X(1) \neq 0$  and  $\beta \neq 0$  are sufficient for identification of  $\theta_A$ . To show necessity, suppose  $\Gamma_X(1) = 0$ . For  $\kappa \in (1 - \delta, 1)$  and  $\delta > 0$ , let

$$\begin{aligned} \bar{\beta} &\equiv \kappa \beta \\ \bar{\sigma}_\epsilon^2 &\equiv \frac{1}{\kappa} \sigma_\epsilon^2 + \left(1 - \frac{1}{\kappa}\right) \Gamma_X(0) \\ \bar{\sigma}_u^2 &\equiv \sigma_u^2 + \beta^2 (1 - \kappa) (\Gamma_X(0) - \sigma_\epsilon^2). \end{aligned}$$

Clearly,  $\theta_A = (\beta, \sigma_\epsilon^2, \sigma_u^2)$  and  $\bar{\theta}_A = (\bar{\beta}, \bar{\sigma}_\epsilon^2, \bar{\sigma}_u^2)$  are observationally equivalent since  $\psi_A(\theta_A) = \psi_A(\bar{\theta}_A)$ . It remains to show that  $\beta \neq 0$  is also necessary for identification. This is immediate because if  $\beta = 0$ ,  $\psi_A(\theta_A) = \psi_A(\tilde{\theta}_A)$  with  $\theta_A = (0, \sigma_\epsilon^2, \sigma_u^2)$  and  $\tilde{\theta}_A = (0, \tilde{\sigma}_\epsilon^2, \sigma_u^2)$  for any values of  $\sigma_\epsilon^2$  and  $\tilde{\sigma}_\epsilon^2$ .

**Proof of Lemma 2** Write the binding function as:

$$\psi_B(\theta_B) = \begin{pmatrix} \beta \left(1 - \frac{\Gamma_\epsilon(0)}{\Gamma_X(0)}\right) \\ \beta^2 \Gamma_\epsilon(0) \left(1 - \frac{\Gamma_\epsilon(0)}{\Gamma_X(0)}\right) + \sigma_u^2 \\ \beta^2 \left[ \Gamma_\epsilon(1) - 2\Gamma_\epsilon(1) \frac{\Gamma_\epsilon(0)}{\Gamma_X(0)} + \left(\frac{\Gamma_\epsilon(0)}{\Gamma_X(0)}\right)^2 \Gamma_X(1) \right] \\ -\beta \left( \Gamma_\epsilon(1) - \Gamma_\epsilon(0) \frac{\Gamma_X(1)}{\Gamma_X(0)} \right) \end{pmatrix}.$$



First, consider the case  $\beta = 0$ . Note that  $\psi_{B1} = \beta \frac{\Gamma_x(0)}{\Gamma_X(0)} = 0$ . But  $\Gamma_X(0) - \Gamma_\epsilon(0) = \Gamma_x(0) \neq 0$ . Hence,  $\beta = 0$  if and only if  $\psi_{B1} = 0$ , and  $\beta = 0$  is directly identified off of  $\psi_{B1}$ . For  $\sigma_u^2$ , we have  $\sigma_u^2 = \psi_{B2}$ , so  $(\beta = 0, \sigma_u^2)$  is identified from  $\psi_B$ . Next, we consider the case  $\beta \neq 0$ . In this case  $\psi_{B1} \neq 0$ , and we can solve for  $\beta$  by considering

$$A \equiv \Gamma_X(1)\psi_{B1}^2 + 2\psi_{B4}\psi_{B1} + \psi_{B3}.$$

Using the definition of  $\psi_B$ , this quantity can be computed in two ways:  $A = \beta^2(\Gamma_X(1) - \Gamma_\epsilon(1)) = \beta^2\Gamma_x(1)$  and  $A = \beta(\psi_{B4} + \Gamma_X(1)\psi_{B1})$ . So if  $\Gamma_x(1) \neq 0$ , then  $A \neq 0$  and we use the two expressions for  $A$  to obtain:

$$\beta = \frac{A}{\psi_{B4} + \Gamma_X(1)\psi_{B1}}.$$

For  $\sigma_u^2$ , consider

$$D \equiv \psi_{B2}\psi_{B4} - \Gamma_X(0)\psi_{B1}\psi_{B3} + \Gamma_X(1)\psi_{B1}\psi_{B2} - \Gamma_X(0)\psi_{B1}^2\psi_{B4}.$$

Then,  $D = \sigma_u^2(\psi_{B4} + \Gamma_X(1)\psi_{B1})$ . Dividing both sides by  $\psi_{B4} + \Gamma_X(1)\psi_{B1} \neq 0$  gives

$$\sigma_u^2 = \frac{D}{\psi_{B4} + \Gamma_X(1)\psi_{B1}}.$$

Thus  $\Gamma_x(1) \neq 0$  is sufficient to globally identify  $(\beta \neq 0, \sigma_u^2)$ . Finally, to identify  $\Gamma_\epsilon(0)$ , assume  $\Gamma_x(1) \neq 0$ , and  $\beta \neq 0$ . Consider

$$B \equiv \Gamma_X(0)(\psi_{B3} + \psi_{B1}\psi_{B4}),$$

and note that  $B = A\Gamma_\epsilon(0)$ . Since  $A \neq 0$  under our assumptions,

$$\Gamma_\epsilon(0) = \frac{B}{A} = \Gamma_X(0) \frac{\psi_{B3} + \psi_{B1}\psi_{B4}}{\Gamma_X(1)\psi_{B1}^2 + 2\psi_{B4}\psi_{B1} + \psi_{B3}}.$$

Finally, for  $\Gamma_\epsilon(1)$ , let  $C \equiv -\psi_{B4}^2 + \Gamma_X(1)\psi_{B3}$ , and note that  $C = \Gamma_\epsilon(1)A$ . Under our assumptions,  $A \neq 0$  and  $\Gamma_\epsilon(1)$  is identified as

$$\Gamma_\epsilon(1) = \frac{C}{A} = \frac{-\psi_{B4}^2 + \Gamma_X(1)\psi_{B3}}{\Gamma_X(1)\psi_{B1}^2 + 2\psi_{B4}\psi_{B1} + \psi_{B3}}.$$

**Proof of Lemma 3** The proof is as follows. First, we consider the case when  $\beta = 0$ . Note that

$$1 - \frac{\Gamma_\epsilon(0)\Gamma_y(0)}{\Gamma_y(0)\Gamma_X(0) - \Gamma_{yX}(1,0)^2} = \frac{\Gamma_y(0)\Gamma_x(0) - \Gamma_{yX}(1,0)^2}{\Gamma_y(0)\Gamma_X(0) - \Gamma_{yX}(1,0)^2}$$

and since  $\Gamma_{yX}(1,0) = \Gamma_{yx}(1,0)$  both the numerator and the denominator are determinants of positive definite covariance matrices, thus the above quantity is strictly positive. Thus,  $\psi_2 = 0$  if and only if  $\beta = 0$ . In this case  $\alpha = \psi_1$ , so  $(\alpha, \beta = 0)$  is identified.

Next, consider the case when  $\beta \neq 0$ . There are again two cases to consider:  $\Gamma_{yX}(1,0) = 0$  and  $\Gamma_{yX}(1,0) \neq 0$ . Consider  $\Gamma_{yX}(1,0) = 0$  first. Since  $X_t = x_t + \epsilon_t$  and  $y_t = \sum_{i=0}^{\infty} \alpha^i (\beta x_{t-i} + u_{t-i})$ , we have  $E[X_t y_{t-j}] = \sum_{i=0}^{\infty} \alpha^i (\beta \Gamma_x(j+i))$ . In particular,

$$\Gamma_{yX}(1,0) = \beta \left[ \Gamma_x(1) + \sum_{i=1}^{\infty} \alpha^i \Gamma_x(1+i) \right], \quad (22)$$

so  $\Gamma_{yX}(1,0)$  occurs, for example, whenever  $x$  is white noise. In this case,  $\alpha$  is directly identified off  $\psi_1$ ,  $\alpha = \psi_1$ . As for  $\beta$ , notice that the components  $\psi_{C_2}^-(\theta_C^-), \psi_{C_3}^-(\theta_C^-), \psi_{C_4}^-(\theta_C^-)$  are as in the ADL(0,0) case and identification can proceed as in Lemma 2.

It remains to consider the case  $\beta \neq 0, \Gamma_{yX}(1,0) \neq 0$ . For this, we further expand the elements on the left-hand side of (9). We then obtain that  $\psi_{C^-}(\theta_C^-) = \psi_{C^-}$  has a unique solution in  $\theta_C^-$  if and only if the system:

$$\begin{aligned} \alpha \Gamma_y(0) + \beta \Gamma_{yX}(1,0) &= \pi_1 \\ \alpha \Gamma_{yX}(1,0) + \beta (\Gamma_X(0) - \Gamma_\epsilon(0)) &= \pi_2 \\ \begin{pmatrix} \alpha \Gamma_y(0) + \beta \Gamma_{yX}(1,0) \\ \alpha \Gamma_{yX}(1,0) + \beta (\Gamma_X(0) - \Gamma_\epsilon(0)) \end{pmatrix}' \Gamma_W(0)^{-1} \begin{pmatrix} \alpha \Gamma_y(1) + \beta \Gamma_{yX}(2,0) \\ \alpha \Gamma_{yX}(0,0) + \beta (\Gamma_X(1) - \Gamma_\epsilon(1)) \end{pmatrix} &= \pi_3 \\ \alpha \Gamma_{yX}(2,0) + \beta (\Gamma_X(1) - \Gamma_\epsilon(1)) &= \pi_4 \end{aligned}$$

has a unique solution in  $\theta$ , for given  $\pi = (\pi_1, \dots, \pi_4)$  and  $\psi_{C^-}$ . Of course, the  $\pi$ 's in the above equations are functions of the original  $\psi_{C^-}$ 's: they correspond to the right-hand side of (9). The above equations have a unique solution in  $\theta_C^-$  as a function of  $\pi$ . Replacing  $\pi$ 's with the values on the right-hand side of (9) shows that the solution for  $\alpha$  is:

$$\alpha = \psi_{C_1}^- + \frac{\Gamma_{yX}(1,0)(\psi_{C_3}^- + \psi_{C_2}^- \psi_{C_4}^-)}{(\Gamma_y(0)\Gamma_{yX}(2,0) - \Gamma_y(1)\Gamma_{yX}(1,0))\psi_{C_1}^- - \Gamma_{yX}(1,0)(\Gamma_{yX}(0,0) - \Gamma_{yX}(2,0))\psi_{C_2}^-}.$$

Of course, for the solution to be valid we need to check that the denominator is not zero. For this, write the equality above as:

$$\alpha - \psi_{C_1}^- = \frac{N}{D},$$

with

$$\begin{aligned} N &= \Gamma_{yX}(1,0)(\psi_{C_3}^- + \psi_{C_2}^- \psi_{C_4}^-) \\ D &= (\Gamma_y(0)\Gamma_{yX}(2,0) - \Gamma_y(1)\Gamma_{yX}(1,0))\psi_{C_1}^- - \Gamma_{yX}(1,0)(\Gamma_{yX}(0,0) - \Gamma_{yX}(2,0))\psi_{C_2}^-. \end{aligned}$$

Note that

$$\alpha - \psi_{C_1}^- = -\beta \Gamma_\epsilon(0) \frac{\Gamma_{yX}(1,0)}{\Gamma_y(0)\Gamma_X(0) - \Gamma_{yX}(1,0)^2} \neq 0.$$

Thus,  $D = 0$  if and only if  $\psi_{C3}^- + \psi_{C2}^- \psi_{C4}^- = 0$ . For  $\beta$ , the solution is:

$$\begin{aligned}\beta &= \psi_{C2}^- - \frac{\Gamma_y(0)(\psi_{C3}^- + \psi_{C2}^- \psi_{C4}^-)}{(\Gamma_y(0)\Gamma_{yX}(2,0) - \Gamma_y(1)\Gamma_{yX}(1,0))\psi_{C1}^- - \Gamma_{yX}(1,0)(\Gamma_{yX}(0,0) - \Gamma_{yX}(2,0))\psi_{C2}^-} \\ &= \psi_{C2}^- - \frac{\Gamma_y(0)}{\Gamma_{yX}(1,0)}(\alpha - \psi_{C1}^-).\end{aligned}$$

Thus  $(\alpha, \beta \neq 0)$  are identified from  $\psi_C^-$  if  $\Gamma_x(1) \neq 0$  and  $\psi_{C3}^- + \psi_{C2}^- \psi_{C4}^- \neq 0$ . Finally, for  $\Gamma_\epsilon(0)$  we have:

$$\Gamma_\epsilon(0)\beta = \frac{(\psi_{C3}^- + \psi_{C2}^- \psi_{C4}^-)(\Gamma_{yX}(1,0)^2 - \Gamma_X(0)\Gamma_y(0))}{D},$$

so if in addition  $\beta \neq 0$ ,  $\Gamma_\epsilon(0)$  is identified. Similarly,  $\Gamma_\epsilon(1)$  is then also identified from  $\psi_{C4}^-$ .

**Proof of Lemma 4** We have

$$\Gamma_{\widetilde{W}}(0) = \begin{pmatrix} \Gamma_{\widetilde{y}_{-1}}(0) & \Gamma_{\widetilde{X}\widetilde{y}_{-1}}(0,0) \\ \Gamma_{\widetilde{X}\widetilde{y}_{-1}}(0,0) & \Gamma_{\widetilde{X}}(0) \end{pmatrix},$$

which is assumed to be non-singular. Since  $\Gamma_{\widetilde{W}}(0)$  and  $\Gamma_{\widetilde{W}}(1)$  do not depend on  $\theta_P$ , the system of equations  $\psi_P = \psi_P(\theta_P)$  is equivalent to  $\Psi_P = \Psi_P(\theta_P)$ , where

$$\begin{aligned}\Psi_{P1}(\theta) &= \Gamma_{\widetilde{W}\widetilde{y}}(0,0) = \Gamma_{\widetilde{W}}(0) \begin{pmatrix} \psi_{P1}(\theta) \\ \psi_{P2}(\theta) \end{pmatrix} = \begin{pmatrix} \alpha\Gamma_{\widetilde{y}_{-1}}(0) + \beta\Gamma_{\widetilde{X}\widetilde{y}_{-1}}(0,0) - \sigma_u^2 h_T(\alpha) \\ \alpha\Gamma_{\widetilde{X}\widetilde{y}_{-1}}(0,0) + \beta\Gamma_{\widetilde{X}}(0) - \beta\sigma_\epsilon^2 \end{pmatrix}, \\ \Psi_{P2}(\theta) &= \Gamma_{\widetilde{y}}(0) = \psi_{P3}(\theta) + (\psi_{P1}(\theta) \quad \psi_{P2}(\theta)) \Gamma_{\widetilde{W}}(0) \begin{pmatrix} \psi_{P1}(\theta) \\ \psi_{P2}(\theta) \end{pmatrix} \\ &= \alpha^2\Gamma_{\widetilde{y}_{-1}}(0) + 2\alpha\beta\Gamma_{\widetilde{X}\widetilde{y}_{-1}}(0,0) + \beta^2\Gamma_{\widetilde{X}}(0) + \sigma_u^2 + \beta^2\sigma_\epsilon^2 - 2\sigma_u^2 h_T(\alpha) - 2\beta\sigma_\epsilon^2,\end{aligned}$$

and

$$\begin{aligned}\Psi_{P3}(\theta) &= \Gamma_{\widetilde{W}\widetilde{y}}(0,0)' \Gamma_{\widetilde{W}}(0)^{-1} \Gamma_{\widetilde{W}\widetilde{y}}(1,0) = -\psi_{P4} + (\psi_{P1}(\theta) \quad \psi_{P2}(\theta)) \Gamma_{\widetilde{W}}(1) \begin{pmatrix} \psi_{P1}(\theta) \\ \psi_{P2}(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \alpha\Gamma_{\widetilde{y}_{-1}}(1) + \beta\Gamma_{\widetilde{X}\widetilde{y}_{-1}}(0,1) - \sigma_u^2 [(T+1)h_T(\alpha) - 1] \\ \alpha\Gamma_{\widetilde{X}\widetilde{y}_{-1}}(1,0) + \beta\Gamma_{\widetilde{X}}(1) + \beta\frac{1}{T}\sigma_\epsilon^2 \end{pmatrix}.\end{aligned}$$

Only the right-hand side terms depend on  $\theta_P$ .

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