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Financial Market Breakdown Due to Strategy Constraints and Information Asymmetry

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Abstract: This paper demonstrates the relevance of strategy constraints on market makers to the possibility of financial market breakdown when there is information asymmetry between market makers and investors; both the case of competitive market makers and the case of a monopolistic market marker are included. Specifically, the paper discusses three types of strategy constraints on the market makers and their implications for the equilibria. The results call attention to the need for more precise specifications of institutional environments (beyond information asymmetry and the mode of competition/monopoly) when considering the possibility of financial market breakdown.

JEL classification: G20, G14, D82

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FINANCIAL MARKET BREAKDOWN DUE TO STRATEGY CONSTRAINTS AND INFORMATION ASYMMETRY

1. Introduction

In the ongoing debate about the relative merits of competitive versus monopolistic market making, one important issue is the potential for market breakdown. The concern is that a financial market maker may cease to function if information asymmetry between the market maker and informed traders is too severe. The presence of one or more market makers is purported to maintain the continuity of the operation of an asset market; therefore one who wants to buy or sell the asset can do so at any time and does not have to wait for someone else with a matching order to show up. Being an economic agent, a market maker needs to be motivated to do the job; therefore there is a possibility that she may decide to quit when she cannot be appropriately compensated for making the market, which is called a market breakdown. For example, Glosten (1989) has found that a group of competitive market makers may not be able to break even while making an asset market if they are at a great informational disadvantage relative to the investors. The same paper has also found out that a monopolistic market maker may be resilient enough to make the market and make some profits no matter how severe the information asymmetry between the market maker and the investors is. Given those results, Glosten concluded that a monopolistic market maker is better in the sense that it does not break down.

A problem in conducting this analysis is that merely stating that the market maker is monopolistic or competitive is not a sufficiently detailed description of the trading environment. An exchange that sanctions market makers typically imposes rules on the form of the quotes posted by market makers. Investors may also expect the price schedule quoted by market makers to satisfy certain implicit requirements. Whether a market maker can follow the rules and meet the implicit expectations while breaking even or making some profits will depend on the form of rules and expectations as well as on being competitive or monopolistic.

To exhaust and discuss all the relevant restrictions imposed on market makers by the explicit rules and the implicit expectations is beyond the scope of this paper. Instead I limit my goal to demonstrating the relevance of strategy constraints to the possibility of financial market breakdown. It is shown that the breakdown of a financial market is the result of both the information asymmetry between market makers

and investors and the strategy constraints on market makers. It therefore follows that comparing the resilience of competitive and monopolistic market making against information asymmetry must specify what strategy constraints market makers are facing.

Among many others, two types of strategy constraints on market makers will be discussed. The first is whether a quoted price schedule should cover all the possible trading quantities. The second is whether a quoted price schedule should reflect a gradual spirit in its price change according to the demand/supply change. Regarding the first point, most exchanges only require their market makers to quote a bid and an ask price for a single quantity. But it could also be argued that market makers should be able and willing to quote prices for other reasonable trading quantities in order for the market to be active. Either way, such a constraint should be modeled in the discussion because the existence of an equilibrium depends on it. Regarding the second point, most exchanges require market makers to contribute to "price stability" by preventing big price jumps when demand/supply changes in small amounts. If a quoted price schedule were such that the unit price for buying quantity Q is p but the unit price for buying $Q+\varepsilon$ is $p+\Delta$, where ε is minimal and Δ is a finite number, then it would be against this spirit. It may be tempting to interpret this price jump, if it is allowed in a model, as a price spread. It is especially so when the concerned point is $Q=0$. But such an interpretation is misleading. A price jump in a quoted price schedule is a finite price difference for minimally different trading quantities while a price spread is the price difference between the selling price and buying price for the same trading quantity. In fact, a price spread can exist without any price jumps in a quoted price schedule, and vice versa.

The breakdown of a financial market is represented by the non-existence of equilibrium in the model. When the strategy constraints are such that a price must be quoted for every trading quantity and a price jump may be allowed in the price schedule, the equilibrium with competitive market makers exists if and only if information asymmetry between the market makers and investors is not severe, and the equilibrium with monopolistic market makers always exists no matter how severe the information asymmetry is. This is the case discussed in Glosten (1989) and also listed in the paper. When the strategy constraints are such that a price must be quoted for every trading quantity and no price jump may be allowed in the price schedule, both the equilibrium with competitive market makers and the equilibrium with monopolistic market makers exist if and only if information asymmetry between the market maker(s)

and the investors is not severe; and when they exist, the equilibrium with competitive market makers features better prices for the investors than that with monopolistic market makers. When the strategy constraints are such that the market makers only quote a bid price and an ask price for a single trading quantity, then both the equilibrium with the competitive market makers and the equilibrium with a monopolistic market maker always exist no matter how severe the information asymmetry between market makers and investors is; and the price quoted in the case with competitive market makers is better to the investors than that with a monopolistic market maker.

Before proceeding to the detailed discussion, please note that the idea of competition or monopoly is not always equivalent to the idea of plurality or singularity of market makers. While it is likely that a single market maker acts as a monopolist and quotes a price schedule that maximizes her expected utility, it is plausible that she may not necessarily do so due to, for example, explicit or implicit government interventions. On the other hand, more than one market maker is likely to compete with others by quoting attractive price schedules to entice orders, and the competition may drive the price schedules to such a level that any incoming order, if executed at the quoted price, will make the market maker break even in expected utility; but it is also plausible that market makers may collude with each other and quote a price schedule that is less beneficial to the investors than that quoted under perfect competition. I avoid the empirical issue of how likely a single market maker in a particular market is to quote a monopolistic price schedule or how likely multiple market makers in a comparable market are to quote a competitive price schedule and concentrate on the theoretical implications of competition and monopoly.

This paper is organized as follows. Section 2 sets up the model as used in Glosten (1989). Section 3 presents all types of strategy constraints that will be discussed. Section 4 presents the competitive equilibrium and the monopolistic equilibrium when the strategy constraints are such that every trading quantity should be covered and there may be a price jump. Section 5 presents the equilibria when a price must be quoted for every trading quantity and there may not be any price jump. Section 6 presents the equilibria when a single trading quantity is covered by the price schedule. Section 7 summarizes the results and concludes the paper.

2. The Model

Let me use the model in Glosten (1989) for the discussion. Consider two securities: one is risky and the other is risk free. A market maker is risk neutral and in charge of dealing with the exchange of the two assets. She has the responsibility to quote a price schedule $P(Q)$ and commit herself to it, where Q is the trading demand for the risky asset from an arriving investor and P is the unit price for the order to be executed at, measured in the unit of the risk free asset. There are many investors and each has a utility function in the following form:

$$U(V) = -e^{-\rho V}, \tag{1}$$

where V is his wealth. Denote by W_0 the shares of the risk free asset and by W the shares of the risky asset endowed to an investor. Among the population of the investors, the endowment of the risky asset W follows a normal distribution:

$$W \sim N(0, \frac{1}{\pi_W}). \tag{2}$$

The distribution of the risk free asset W_0 is inconsequential to the discussion due to the technical property of negative exponential utility function; therefore it need not be specified. The true value of the risky asset, denoted by X, is assumed to follow a normal distribution:

$$X \sim N(m, \frac{1}{\pi_X}). \tag{3}$$

Each investor can observe a signal S about the asset value X where

$$S = X + \varepsilon, \tag{4}$$

with ε being independent of X and normally distributed:

(5) $$\varepsilon \sim \mathrm{N}(0, \frac{1}{\pi_S}).$$

Therefore an investor is better informed than a market maker in that he has observed a signal S about the asset value X. This information asymmetry, together with the strategy constraints on the market maker, gives rise to the possibility of market breakdown due to the adverse selection effect.

[*The Problem of an Investor*]

Given the price schedule $P(Q)$ quoted by the market maker, an investor will adjust his portfolio optimally after he receives his asset endowments (W_0, W) and observes the signal S. Such an adjustment is represented by his demand $Q(S, W_0, W)$ for the risky asset which will be submitted to the market maker; therefore he is to find a demand Q to solve the following problem:

(6) $$\underset{Q}{\mathrm{MAX}}\ \mathrm{E}\{\mathrm{U}[(W+Q)X + W_0 - QP(Q)] \mid S, W_0, W\}.$$

Since his utility function is exponential and the information structure comprises only normal distributions, the above problem is equivalent to

(7) $$\underset{Q}{\mathrm{MAX}} \left\{ QP(Q) + (W+Q)\mathrm{E}[X|S] - \rho(W+Q)^2 \frac{\mathrm{VAR}[X|S]}{2} \right\}.$$

Utilizing the relation between the observed signal and the asset value, i.e., $S = X + \varepsilon$, the investor can obtain the following statistics:

(8) $$\mathrm{E}[X|S] = \frac{\pi_X m + \pi_S S}{\pi_X + \pi_S};$$

(9) $$\mathrm{VAR}[X|S] = \frac{1}{\pi_X + \pi_S}.$$

Then the problem of the investor can be transformed to the following form:

$$(10) \qquad \underset{Q}{\text{MAX}}\left\{QP(Q)(\pi_X+\pi_S)+(W+Q)(\pi_X m+\pi_S S)-\frac{\rho(W+Q)^2}{2}\right\}.$$

[*The Problem of a Competitive Market Maker*]

I assume that a market maker under competition is to earn zero expected profit on every trade, and the most favorable price quote among the market makers wins an incoming order. These characteristics are captured by the following conditions:

$$(11) \qquad P(Q)=\begin{cases}\text{MIN}\ \{p(Q)|p(Q)=\text{E}[X|Q]\} & \forall Q\geq 0;\\ \text{MAX}\ \{p(Q)|p(Q)=\text{E}[X|Q]\} & \forall Q<0.\end{cases}$$

[*The Problem of a Monopolistic Market Maker*]

I assume that a monopolistic market maker maximizes her expected profit by choosing a price schedule to quote, which can be represented by the following expression:

$$(12) \qquad P(.)=\underset{P(.)}{ARG\ MAX}\ \ \text{E}\left\{\text{E}[Q(S,W,W)(P(Q(S,W,W))-X)|Q(S,W,W)=\text{Q}]\right\}.$$

[*The Definition of Equilibrium*]

In such a framework, an equilibrium is a pair $\{P(Q),Q(S,W_0,W)\}$, where $P(Q)$ is the price schedule chosen by a market maker following the competitive or monopolistic pricing rules under certain strategy constraints given the demand function $Q(S,W_0,W)$, and $Q(S,W_0,W)$ is the optimal demand function of an investor who faces the price schedule $P(Q)$.

[*Some Mathematical Notation*]

In the discussion that follows, some auxiliary variables will simplify the notation. Let me define them now. The first one is

$$(13) \qquad Z\equiv S-\frac{\rho W}{\pi_S},$$

which, by definition, leads to the relation $Z = X + \varepsilon - \frac{\rho W}{\pi_S}$. It is easy to verify that Z follows a normal distribution with the mean value m and the standard deviation

$$\sigma_Z = \sqrt{\frac{1}{\pi_X} + \frac{1}{\pi_S} + \frac{\rho}{\pi_W \pi_S^2}}. \tag{14}$$

From the relation of $Z = X + \varepsilon - \frac{\rho W}{\pi_S}$, I can compute VAR[$Z$|$X$] in terms of the parameters in the distributions of X, ε, and W. More specifically, let me define $\pi_Z \equiv \frac{1}{\mathrm{VAR}[Z|X]}$; then

$$\pi_Z = \frac{1}{\frac{1}{\pi_S} + \frac{\rho^2}{\pi_S^2 \pi_W}}. \tag{15}$$

Note that $\frac{1}{\sigma_Z^2}$ is not equal to π_Z, since π_Z is the precision of Z conditional on X while $\frac{1}{\sigma_Z^2}$ is the precision of Z. To standardize Z, I introduce the second auxiliary variable:

$$z \equiv \frac{Z - m}{\sigma_Z}; \tag{16}$$

then z follows the standard normal distribution: z~N(0,1). The value of z, if available, will help a market maker to improve the estimation of the asset value X because $Z = X + \varepsilon - \frac{\rho W}{\pi_S}$ and $z \equiv \frac{Z - m}{\sigma_Z}$; specifically, her improved estimation will be

$$\mathrm{E}[X|z] = m + \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} z. \tag{17}$$

Finally, I want to define the following parameters:

$$(18)\qquad \begin{cases} \alpha \equiv \dfrac{\rho^2 \pi_X}{\rho^2 \pi_X + \pi_W \pi_S (\pi_S + \pi_X)}, \\ \beta \equiv \dfrac{\sigma_Z \pi_S}{\rho}. \end{cases}$$

Among the above two parameters, $\alpha \in (0,1)$ has a special interpretation: It is an inverse measure of information asymmetry. The bigger the value of α is, the less severe the information asymmetry is. Intuitively I justify this interpretation by noting that α is decreasing in the value of $\pi_W \pi_S (\frac{\pi_S}{\pi_X} + 1)$: when $\pi_W \pi_S (\frac{\pi_S}{\pi_X} + 1)$ is big, π_W and π_S should be big and π_X should be small, i.e., the information advantage of the investor is big (π_S big and π_X small) and the liquidity motivated trading is small (π_W big), and therefore the adverse selection problem is severe; and when $\pi_W \pi_S (\frac{\pi_S}{\pi_X} + 1)$ is small, the interpretation is just the opposite.

3. Strategy Constraints of a Market Maker: Some Examples

The strategy constraints on a market maker are an important factor affecting the existence of equilibrium. Together with the degree of information asymmetry (indicated by the parameter α) and the fact of whether the market maker is competitive or monopolistic, the strategy constraints determine if the market breaks down. It may not be possible to exhaust all the strategy constraints in the discussion. I instead list three reasonable examples of strategy constraints and discuss the existence and properties of the corresponding equilibria.

Strategy Constraint I (SC-1): *The market maker is responsible for quoting a price for every non-zero trading quantity, i.e., in an equilibrium, P(Q) is well defined for every Q≠0; and a price spread at Q=0 may be allowed, i.e.,* $\text{Lim}_{Q \to 0^+} P(Q) \neq \text{Lim}_{Q \to 0^-} P(Q)$ *may be allowed.*

Strategy Constraint II (SC-2): *The market maker is responsible for quoting a price for every trading quantity, i.e., in an equilibrium, P(Q) is well defined for every Q; and no price spread at Q=0 is allowed, i.e.,* $\text{Lim}_{Q \to 0^+} P(Q) = \text{Lim}_{Q \to 0^-} P(Q)$ *or P(Q) is continuous at Q=0.*

Strategy Constraint III (SC-3): *The market maker is responsible for quoting an ask price and a bid price for a single trading quantity.*

Technical Condition I (TC-1): *P(Q) must be finite and twice differentiable for every Q at which it is defined.*

Technical Condition II (TC-2): $P(Q=0)=m$.

Technical Condition III (TC-3): $\frac{\mathrm{d}^2[Q(P(Q)-m)]}{\mathrm{d}Q^2} \geq \frac{\mathrm{d}[Q(P(Q)-m)]}{Q\,\mathrm{d}Q}$ for $Q \neq 0$.

4. Equilibria under Strategy Constraint SC-1

This is essentially the case considered by Glosten (1989). The proofs of all the propositions in this paper will be given in the Appendix.

Proposition 1 (Competitive equilibrium C1): If the market maker is competitive and her price schedule $P(Q)$ is finite and twice differentiable for every $Q\neq 0$, i.e., subject to the constraints SC-1 and TC-1, then there is an equilibrium if and only if $\alpha>1/2$. Furthermore, the equilibrium is as follows:

$$(19) \qquad \begin{cases} P(Q) = m + \dfrac{\rho(1-\alpha)}{(\pi_X + \pi_S)(2\alpha-1)}Q; \\ Q(S, W_0, W) = Q(z) = (2\alpha-1)\beta z. \end{cases}$$

Remark:

1. Note that the equilibrium exists if and only if $\alpha>1/2$; therefore the market breaks down when $\alpha\leq 1/2$, i.e., when the adverse selection caused by information asymmetry is severe.

2. It is easy to show that the same equilibrium C1 exists if the strategy constraint SC-1 is replaced by SC-2.

Proposition 2 (Monopolistic equilibrium M1): Suppose the market maker is monopolistic and her price

schedule $P(Q)$ is finite and twice differentiable for every $Q\neq 0$, i.e., subject to the constraints SC-1 and TC-1, then there exists an equilibrium as follows:

$$Q(S,W_0,W)=Q(z)=\begin{cases}\beta(\alpha z+\dfrac{F(z)}{f(z)}) & \text{for } z<-z^*;\\ \beta(\alpha z+\dfrac{F(z)-1}{f(z)}) & \text{for } z>z^*;\\ 0 & \text{for } -z^*\le z\le z^*,\end{cases} \tag{20}$$

$$P(Q)=m+\frac{\rho}{\pi_X+\pi_S}\left[-\frac{Q}{2}+\frac{\beta}{Q}\int_0^Q z(Q)\,dQ\right]\pm\frac{\rho\beta}{\pi_X+\pi_S}z^* \quad \begin{cases}\text{for } Q>0;\\ \text{for } Q<0,\end{cases} \tag{21}$$

where $z(Q)$ is the inverse function of $Q(z)$ and z^* is uniquely determined by

$$\alpha z^*+\frac{F(z^*)-1}{f(z^*)}=0. \tag{22}$$

Remark:

1. The equilibrium M1 exists no matter what value α takes. Based on this property, Glosten (1989) argues that monopolistic market making is better than competitive market making since the latter breaks down when $\alpha\le\frac{1}{2}$. His argument holds only when the strategy constraint is SC-1.

2. If the strategy constraint SC-1 is replaced by SC-2, i.e., if the price schedule is limited to those that are continuous and smooth at all Q values (including Q=0), then the equilibrium M1 does not exist.

3. A similar equilibrium can be obtained if the discontinuous point Q=0 is changed to any other point. The key to obtain such a type of equilibrium is not that there is "a price spread at Q=0," but that there is a price jump at some/any point Q.

5. Equilibria under Strategy Constraint SC-2

Proposition 3 (Competitive Equilibrium C2): Suppose the market maker is competitive and her pricing

schedule is continuous and twice differentiable for all Q, i.e., subject to the constraints SC-2 and TC-1. If the price schedule also satisfies the technical constraints TC-2 and TC-3, then there exists an equilibrium if and only if $\alpha>1/2$. Furthermore, the equilibrium is as follows:

$$(23)\qquad \begin{cases} P(Q) = m + \dfrac{\rho(1-\alpha)}{(\pi_X + \pi_S)(2\alpha-1)}Q; \\ Q(S, W_0, W) = Q(z) = (2\alpha-1)\beta z. \end{cases}$$

Remark:

1. This equilibrium features the same strategies as the competitive equilibrium C1.

2. As I have noted in the remark following *Proposition 1*, the same equilibrium can be obtained without the technical conditions TC-2 and TC-3. TC-2 and TC-3 are added here so that the equilibrium C2 can be compared with the monopolistic equilibrium M2 (see next).

Proposition 4 (Monopolistic Equilibrium M2): Suppose the market maker is monopolistic and her pricing schedule is continuous and twice differentiable for all Q, i.e., subject to the constraints SC-2 and TC-1. If the price schedule also satisfies the technical constraints TC-2 and TC-3, then there exists an equilibrium if and only if $\alpha>1/2$. Furthermore, the equilibrium is as follows:

$$(24)\qquad \begin{cases} P(Q) = \mathrm{m} + \dfrac{\rho\,(3-2\alpha)}{(\pi_X + \pi_S)(2\alpha-1)}Q; \\ Q(S, W_0, W) = Q(z) = \dfrac{\beta\,(2\alpha-1)}{2}z. \end{cases}$$

Remark:

1. This is an equilibrium with linear strategies. Its pricing schedule is steeper than that of C2.

2. It breaks down if and only if $\alpha\leq 1/2$, which is the same condition for the competitive equilibrium C2 to breakdown. Therefore the competitive market making is better than monopolistic market making in this case.

6. Equilibria under Strategy Constraint SC-3

Proposition 5 (Competitive equilibrium C3): Suppose a market maker is competitive and is only to quote ask and bid prices for a single quantity $q>0$, i.e., the price schedule is subject to the strategy constraint SC-3. Then for any degree of information asymmetry $\alpha\in(0,1)$, there exists an equilibrium as follows:

$$P(Q)-m=\begin{cases} p & \text{for } Q=q; \\ p & \text{for } Q=-q; \\ +\infty & \text{for } Q>0 \text{ and } Q\neq q; \\ -\infty & \text{for } Q<0 \text{ and } Q\neq -q, \end{cases} \tag{25}$$

$$Q(z)=\begin{cases} q & \text{for } z>z_0; \\ -q & \text{for } z<-z_0; \\ 0 & \text{for } -z_0\le z\le z_0, \end{cases} \tag{26}$$

and p and z_0 are uniquely determined by the following equation set:

(27)

$$\begin{cases} z_0-(1-\alpha)\dfrac{\int_{z_0}^{\infty} z f(z)dz}{\int_{z_0}^{\infty} f(z)dz}=\dfrac{q}{2\beta}, \\ p=\dfrac{\rho\beta}{\pi_X+\pi_S}(z_0-\dfrac{q}{2\beta}), \end{cases}$$

where f(z) is the probability density function of the standard normal distribution.

Remark:

1. Such an equilibrium always exists no matter how severe the information asymmetry is.

2. There are an infinite number of such equilibria, one for each primary quantity $q>0$.

Proposition 6 (Monopolistic equilibrium M3): Suppose a market maker is monopolistic and is only to quote the ask and bid prices for a single quantity $q>0$, i.e., the price schedule is subject to the strategy constraint SC-3. Then for any degree of information asymmetry $\alpha\in(0,1)$, there exists an equilibrium as follows:

$$P(Q)\text{- m} = \begin{cases} p' & \text{for } Q = q; \\ -p' & \text{for } Q = -q; \\ +\infty & \text{for } Q > 0 \text{ and } Q \neq q; \\ -\infty & \text{for } Q < 0 \text{ and } Q \neq -q, \end{cases} \tag{28}$$

$$Q(z) = \begin{cases} q & \text{for } z > z_1; \\ -q & \text{for } z < -z_1; \\ 0 & \text{for } -z_1 \le z \le z_1, \end{cases} \tag{29}$$

and p' and z_1 are uniquely determined by the following equation set:

$$\begin{cases} z_1 = ARG\ \underset{z_1}{MAX} \left[(z_1 - \frac{q}{2\beta}) \int_{z_1}^{\infty} f(z)dz - (1-\alpha) \int_{z_1}^{\infty} z f(z)dz \right]; \\ p' = \frac{\rho\beta}{\pi_X + \pi_S}(z_1 - \frac{q}{2\beta}), \end{cases} \tag{30}$$

where f(z) is the probability density function of the standard normal distribution.

Remark:

1. Such an equilibrium always exists no matter how severe the information asymmetry is.
2. There are an infinite number of such equilibria, one for each primary quantity $q>0$.
3. For the same quantity $q>0$, the equilibrium C3 is better to the investors than M3 because p<p'.

7. Conclusion

The main results of this paper are summarized in Table 1. From the table, it can be seen that the strategy constraints are as important as whether the market maker is competitive or monopolistic in determining the possibility of market breakdown. Information asymmetry, which is represented by the parameter $\alpha \in (0,1)$, also plays a crucial role on that matter. From the viewpoint of an investor, C2 is better than M2, and C3 is better than M3. That is to say, with certain strategy constraints, competitive market makers are better than monopolistic market makers for the welfare of the investors. Glosten (1989) considered only the case in the first row (shaded). Without looking at other possibilities as in the second and the third rows, it would be natural to see an advantage in monopolistic market making since it does not break down. But obviously such a comparison is not complete.

Table 1: Summary of Main Results

Competitive or Monopolistic / Strategy Constraints	Market maker is competitive.	Market maker is monopolistic.
SC-1: Price schedule covers all non-zero quantities and there is no price spread at Q=0; the technical condition TC-1 holds.	For α>1/2, equilibrium C1 exists. For $\alpha\leq$1/2, no equilibrium exists.	Equilibrium M1 exists for all $\alpha\in(0,1)$.
SC-2: Price schedule covers all quantities and there may be a price spread at Q=0; the technical conditions TC-1, TC-2, and TC-3 hold.	For α>1/2, equilibrium C2 exists. For $\alpha\leq$1/2, no equilibrium exists.	For α>1/2, equilibrium M2 exists. For $\alpha\leq$1/2, no equilibrium exists.
SC-3: Price schedule covers only a single quantity (buy and sell).	Equilibrium C3 exists for all $\alpha\in(0,1)$.	Equilibrium M3 exists for all $\alpha\in(0,1)$.

APPENDIX: Mathematical Proofs

The strategy constraints and the technical conditions are listed in the following for easy reference in the proofs.

Strategy Constraint I (SC-1): *The market maker is responsible for quoting a price for every non-zero trading quantity, i.e., in an equilibrium, P(Q) is well defined for every Q≠0; and a price spread at Q=0 may be allowed, i.e.,* $\mathrm{Lim}_{Q\to 0^+} P(Q) \neq \mathrm{Lim}_{Q\to 0^-} P(Q)$ *may be allowed.*

Strategy Constraint II (SC-2): *The market maker is responsible for quoting a price for every trading quantity, i.e., in an equilibrium, P(Q) is well defined for every Q; and no price spread at Q=0 is allowed, i.e.,* $\mathrm{Lim}_{Q\to 0^+} P(Q) = \mathrm{Lim}_{Q\to 0^-} P(Q)$ *or P(Q) is continuous at Q=0.*

Strategy Constraint III (SC-3): *The market maker is responsible for quoting an ask price and a bid price for a single trading quantity.*

Technical Condition I (TC-1): *P(Q) must be finite and twice differentiable for every Q at which it is defined.*

Technical Condition II (TC-2): *P(Q=0)=m.*

Technical Condition III (TC-3): $\frac{\mathrm{d}^2[Q(P(Q)-m)]}{\mathrm{d}Q^2} \geq \frac{\mathrm{d}[Q(P(Q)-m)]}{Q\,\mathrm{d}Q}$ for $Q \neq 0$.

Lemma 1: If $P(Q)$ satisfies the strategy constraint SC-2 and the technical condition TC-1 then the optimal demand of an investor, $Q(S,W_0,W)$ is a function of z, i.e., $Q(S,W_0,W)=Q(z)$; and $Q(z)$ is differentiable and monotonically increasing in z. Furthermore, if $P(Q)$ also satisfies the technical condition TC-2, then $Q(z=0)=0$ and $Q(z)$ has the same sign as z.

Proof:

Since the price schedule $P(Q)$ satisfies the strategy constraint SC-2 and technical condition TC-1, it is well defined for every trading quantity Q and is continuous and twice differentiable everywhere. Then

the first and second order conditions for an investor's optimization problem can be written as

(A1) $$\frac{(\pi_X + \pi_S)[QP'(Q) + P(Q) - m]}{\pi_S} + \frac{\rho Q}{\pi_S} = \sigma_Z z; \qquad (*)$$

(A2) $$-QP''(Q) - 2P'(Q) - \frac{\rho}{\pi_X + \pi_S} < 0. \qquad (**)$$

These conditions guarantee that the optimal demand $Q(S,W_0,W)$ of the investor is a function of z, i.e., $Q(S,W_0,W)=Q(z)$. By (*) I know that $z(Q)$ is differentiable. Differentiating both sides of (*) with respect to Q yields

(A3) $$\frac{(\pi_X + \pi_S)[QP''(Q) + 2P'(Q)]}{\pi_S} + \frac{\rho}{\pi_S} = \sigma_Z z'(Q).$$

By the condition (**), I know that

(A4) $$QP''(Q) + 2P'(Q) + \frac{\rho}{\pi_X + \pi_S} > 0.$$

Therefore $z'(Q)>0$. This leads to the conclusion that $Q'(z)$ exists and is positive. If $P(Q=0)=m$ as given by TC-2, then $z(Q=0)=0$ by (*). But $Q(z)$ is strictly monotonic in z, so $Q(z=0)=0$. It follows that $Q(z)$ has the same sign as z. Q.E.D.

Remark:

1. It can be similarly proven that if the price schedule $P(Q)$ satisfies the strategy constraint SC-1 (instead of SC-2) and the technical condition TC-1, then for any possible trading quantity $Q\neq 0$, the conditions (*) and (**) still hold, the optimal demand of an investor is still a function of z, i.e., $Q(S,W_0,W)=Q(z)$, and $Q(z)$ is differentiable and monotonically increasing in z.

2. When the price schedule $P(Q)$ is subject to the strategy constraint SC-2 and the technical condition TC-1, the market maker is able to figure out the value of z when she receives the demand Q since

the inverse function $z(Q)$ exists and is monotonic. She will be able to do the same except for $Q=0$ when the price schedule is under the strategy constraint SC-1 and the technical condition TC-1. Recall that $\mathrm{E}[X|z] = m + \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} z$, so I have

$$\text{(A5)} \qquad \mathrm{E}[X|Q]=\mathrm{E}[X|z] = m + \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} z, \begin{cases} \text{for all Q if SC-2 and TC-1 hold;} \\ \text{for all } Q \neq 0 \text{ if SC-1 and TC-1 hold.} \end{cases}$$

Lemma 2: If $P(Q)$ satisfies the strategy constraint SC-2 and the technical conditions TC-1, TC-2 and TC-3, then

$$\text{(A6)} \qquad Q'(z) \leq \frac{Q(z)}{z} \qquad \text{for } z \neq 0.$$

Proof:

By *Lemma 1*, I know that $Q'(z)$ exists, $Q'(z)>0$, $Q(0)=0$ and $Q(z)$ has the same sign as z. The condition (*) can be rewritten as

$$\text{(A7)} \qquad \sigma_Z z = \frac{\pi_X + \pi_S}{\pi_S} \frac{\mathrm{d}[(P(Q)-m)Q]}{\mathrm{d}Q} + \frac{\rho Q}{\pi_S}.$$

Therefore

$$\text{(A8)} \qquad \begin{aligned} \sigma_Z \frac{\mathrm{d}z}{\mathrm{d}Q} &= \frac{\pi_X + \pi_S}{\pi_S} \frac{\mathrm{d}^2[(P(Q)-m)Q]}{\mathrm{d}Q^2} + \frac{\rho}{\pi_S} \\ &\geq \frac{\pi_X + \pi_S}{\pi_S} \frac{\mathrm{d}[(P(Q)-m)Q]}{Q\mathrm{d}Q} + \frac{\rho}{\pi_S} \\ &= \frac{\sigma_Z z}{Q(z)}, \end{aligned}$$

where the inequality holds for $Q\neq 0$ or equivalently for $z\neq 0$. Since z has the same sign as $Q(z)$, both sides are positive and the inequality can be reversed when both sides take the operation of inversion.

Q.E.D.

Lemma 3. The solutions to the following differential equation and inequality

$$\text{(A9)} \qquad \begin{cases} (\pi_X + \pi_S)[Qp'(Q) + p(Q) - m] + \rho Q = \dfrac{\pi_S(\pi_Z + \pi_X)(p(Q) - m)}{\pi_Z}; \\ -Qp''(Q) - 2p'(Q) - \dfrac{\rho}{\pi_X + \pi_S} < 0 \quad \forall Q \end{cases}$$

exit when $\alpha > \frac{1}{2}$ and are as follows:

$$\text{(A10)} \qquad p(Q) = m + \frac{\rho(1-\alpha)}{(\pi_X + \pi_S)(2\alpha - 1)} Q + K[\text{Sign}(Q)]|Q|^{\gamma} \quad \text{where } K > 0.$$

Proof:

See Glosten (1989). Or—

The equation can be easily solved with the following trick: Let $p(Q) = m + BQ + h(Q)$ where B is a constant to be determined and $h(Q)$ is a differentiable function to be solved. The result is

$$\text{(A11)} \qquad p(Q) - m = \begin{cases} \dfrac{\rho(1-\alpha)}{(\pi_X + \pi_S)(2\alpha - 1)} Q + K[Sign(Q)]|Q|^{\gamma} & \text{if } \alpha \neq \frac{1}{2}; \\ k_0 Q + k_1 Q \log(|Q|) & \text{if } \alpha = \frac{1}{2}, \end{cases}$$

where K and k_0 are *unrestricted* and $\gamma \equiv \dfrac{\alpha}{1-\alpha}$ and $k_1 \equiv \dfrac{\rho}{\pi_X + \pi_S}$. By studying the inequality at $Q \to 0$ and $Q \to \pm\infty$, it can be shown that only the solutions with $\alpha > 1/2$ and $K > 0$ satisfy the above inequality.

Q.E.D.

Lemma 4: Suppose f(z) is the density function of a standard normal distribution, then

(A12) $$\mathrm{Lim}_{z_0\to\infty}\left[\frac{\int_{z_0}^{\infty} z f(z)\,dz}{\int_{z_0}^{\infty} f(z)\,dz} - z_0\right] = 0.$$

Proof:

By l'Hospital rule.

Proposition 1 (Competitive equilibrium C1): If the market maker is competitive and her price schedule $P(Q)$ is finite and twice differentiable for every $Q\neq 0$, i.e., subject to the constraints SC-1 and TC-1, then there is an equilibrium if and only if $\alpha > 1/2$. Furthermore, the equilibrium is as follows:

(A13) $$\begin{cases} P(Q) = m + \dfrac{\rho(1-\alpha)}{(\pi_X+\pi_S)(2\alpha-1)}Q; \\ Q(S, W_0, W) = Q(z) = (2\alpha-1)\beta z. \end{cases}$$

Proof:

Let me solve for a function $p(Q)$ which satisfies the strategy constraint SC-1, the technical condition TC-1 and the equation $p(Q)=\mathrm{E}[X|Q]$. Since $p(Q)$ satisfies SC-1 and TC-1, I know that for all $Q\neq 0$, it is finite and twice differentiable, and satisfies the conditions (*) and (**):

(A14) $$\frac{(\pi_X+\pi_S)[Qp'(Q)+p(Q)-m]}{\pi_S} + \frac{\rho Q}{\pi_S} = \sigma_z z;$$

(A15) $$-Qp''(Q) - 2p'(Q) - \frac{\rho}{\pi_X+\pi_S} < 0.$$

Recall that the following relation always holds:

(A16) $$\mathrm{E}[X|z] = m + \frac{\sigma_Z \pi_Z}{\pi_Z+\pi_X} z.$$

But SC-1 and TC-1 guarantee that $\mathrm{E}[X|Q]=\mathrm{E}[X|z]$, so

(A17) $$p(Q) = m + \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} z.$$

Now let me eliminate z by combining $\frac{(\pi_X + \pi_S)[Qp'(Q) + p(Q) - m]}{\pi_S} + \frac{\rho Q}{\pi_S} = \sigma_z z$ and $p(Q) = m + \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} z$ together, which yields the following ordinary differential equation for $p(Q)$:

(A18) $$(\pi_X + \pi_S)[Qp'(Q) + p(Q) - m] + \rho Q = \frac{\pi_S(\pi_Z + \pi_X)(p(Q) - m)}{\pi_Z}.$$

The solutions for $p(Q)$ are given by *Lemma 4*, i.e.,

(A19) $$p(Q) = m + \frac{\rho(1-\alpha)}{(\pi_X + \pi_S)(2\alpha - 1)} Q + K[\text{Sign}(Q)]|Q|^\gamma \quad \text{where } \alpha > 1/2 \text{ and } K > 0.$$

Therefore

(A20) $$P(Q) = \begin{cases} \text{MIN} \\ \text{MAX} \end{cases} \left\{ p(Q) \middle| p(Q) = m + \frac{\rho(1-\alpha)}{(\pi_X + \pi_S)(2\alpha - 1)} Q + K[\text{Sign}(Q)]|Q|^\gamma \right\} \begin{cases} \text{for } Q \geq 0; \\ \text{for } Q < 0, \end{cases}$$
$$\text{where } \alpha > 1/2 \text{ and } K > 0$$

which leads to

(A21) $$P(Q) = m + \frac{\rho(1-\alpha)}{(\pi_X + \pi_S)(2\alpha - 1)} Q \text{ with } \alpha > 1/2 \text{ for all } Q,$$

and correspondingly, the optimal demand with respect to this price schedule can be computed from the first order condition (*):

(A22) $$Q(z) = (2\alpha - 1)\beta\, z \text{ for all } z.$$ Q.E.D.

Remark:

It can be similarly proven that the same equilibrium is also obtainable with the strategy constraint SC-2 instead of SC-1. Adding the technical conditions TC-2 and TC-3 will not change the equilibrium, either.

Proposition 2 (Monopolistic equilibrium M1): Suppose the market maker is monopolistic and her price schedule $P(Q)$ is finite and twice differentiable for every $Q\neq 0$, i.e., subject to the constraints SC-1 and TC-1, then there exists an equilibrium as follows:

$$Q(\boldsymbol{S},\boldsymbol{W}_0,\boldsymbol{W}) = Q(z) = \begin{cases} \beta(\alpha z + \dfrac{\mathrm{F}(z)}{\mathrm{f}(z)}) & \text{for } z < -z^*; \\ \beta(\alpha z + \dfrac{\mathrm{F}(z)-1}{\mathrm{f}(z)}) & \text{for } z > z^*; \\ 0 & \text{for } -z^* \le z \le z^*, \end{cases} \tag{A23}$$

$$P(Q) = m + \frac{\rho}{\pi_X + \pi_S}\left[-\frac{Q}{2} + \frac{\beta}{Q}\int_0^Q z(Q)\mathrm{d}Q\right] \pm \frac{\rho\beta}{\pi_X + \pi_S} z^* \qquad \begin{cases} \text{for } Q > 0; \\ \text{for } Q < 0, \end{cases} \tag{A24}$$

where $z(Q)$ is the inverse function of $Q(z)$ and z^* is uniquely determined by

$$\alpha z^* + \frac{\mathrm{F}(z^*)-1}{\mathrm{f}(z^*)} = 0. \tag{A25}$$

Proof:

(i) Given that the price schedule satisfies the strategy constraint TC-1 and the technical condition TC-1, I know, by *Lemma 1*, that $\forall Q\neq 0$, $Q(\boldsymbol{W}_0,\boldsymbol{W},\boldsymbol{S})=Q(z)$, $Q'(z)$ exists and $Q'(z)$ is positive. Let me denote the expected profit of the market maker by $\mathrm{E}\Pi$. Then

(A26)

$$
\begin{aligned}
E\Pi &= E\{Q(z)[P(Q(z)) - E(X|z)]\} \\
&= E\left\{Q(z)\left[P(Q(z)) - m - \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} z\right]\right\} \\
&= \int_{-\infty}^{\infty} dz f(z) Q(z)\left[P(Q(z)) - m - \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} z\right].
\end{aligned}
$$

(ii) Given the price schedule $P(Q)$, an investor decides to buy Q shares of the risky asset only if his expected utility is improved, i.e.

$$
\begin{aligned}
&-P(Q)Q + \frac{(W+Q)(\pi_X m + \pi_S S)}{\pi_X + \pi_S} - \frac{\rho(W+Q)^2}{2(\pi_X + \pi_S)} > \\
&> \frac{W(\pi_X m + \pi_S S)}{\pi_X + \pi_S} - \frac{\rho W^2}{2(\pi_X + \pi_S)},
\end{aligned} \tag{A27}
$$

which is equivalent to

$$
z \begin{Bmatrix} > \\ < \end{Bmatrix} \frac{\pi_X + \pi_S}{\rho\beta}(P(Q) - m) + \frac{1}{2\beta} Q \quad \text{if} \begin{cases} Q > 0; \\ Q < 0. \end{cases} \tag{A28}
$$

Since $P(Q)$ is an increasing function, it holds that

$$
z \begin{cases} > \dfrac{\pi_X + \pi_S}{\rho\beta}(P(0^+) - m) + \dfrac{1}{2\beta} 0^+ & \text{if } Q > 0; \\ < \dfrac{\pi_X + \pi_S}{\rho\beta}(P(0^-) - m) + \dfrac{1}{2\beta} 0^- & \text{if } Q < 0. \end{cases} \tag{A29}
$$

It can be easily verified that

$$
P(0^+) = m + \frac{z^* \rho\beta}{\pi_X + \pi_S} \text{ and } P(0^-) = m - \frac{z^* \rho\beta}{\pi_X + \pi_S}, \tag{A30}
$$

So I have

$$\text{(A31)} \qquad \begin{cases} Q > 0 & \text{only if } z > z^*, \\ Q < 0 & \text{only if } z < -z^*. \end{cases}$$

(iii) So I rewrite the expected profit as follows:

$$\text{(A32)} \qquad \mathrm{E}\Pi = \int_{-\infty}^{z^*} \mathrm{d}z\, \mathrm{F}'(z)A(z) + \int_{-z^*}^{\infty} \mathrm{d}z(\mathrm{F}(z)-1)'A(z),$$

where

$$\text{(A33)} \qquad A(z) \equiv Q(z)\left[P(Q(z)) - m - \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} z\right],$$

and $F(z)$ is the cumulative distribution function of z, which follows the standard normal distribution. The trick of the proof is to let the market maker choose a price schedule that maximizes both

$$\text{(A34)} \qquad \int_{-\infty}^{z^*} \mathrm{d}z\, \mathrm{F}'(z)A(z) \text{ and } \int_{-z^*}^{\infty} \mathrm{d}z(\mathrm{F}(z)-1)'A(z).$$

This is valid because any price schedule $P(Q)$, defined for $Q \in (-\infty,-z^*) \cup (z^*,+\infty)$, that maximizes both the objective functions simultaneously must maximize the sum of the two, although the inverse statement is not necessarily true.

(iv) Now let me try to maximize the first integral over $Q \in (-\infty,-z^*)$. Note that

$$\text{(A35)} \qquad \int_{-\infty}^{-z^*} \mathrm{d}z\, \mathrm{F}'(z)A(z) = \mathrm{F}(-z^*)A(-z^*) - \mathrm{F}(-\infty)A(-\infty) - \int_{-\infty}^{-z^*} \mathrm{d}z\, \mathrm{F}(z)A'(z).$$

But $F(-\infty)=0$ and $Q(-z^*)=0$, therefore

$$\int_{-\infty}^{-z^*} dz\, F'(z)A(z) = -\int_{-\infty}^{-z^*} dz\, F(z)A'(z). \tag{A36}$$

(v) Using the condition (*) to eliminate $P(Q)$ in the expression of $A'(z)$, I get

$$A'(z) = \sigma_Z\left(\frac{\pi_S}{\pi_S+\pi_X} - \frac{\pi_Z}{\pi_Z+\pi_X}\right)zQ'(z) - \frac{\rho}{\pi_S+\pi_X}Q'(z)Q(z) - \frac{\pi_Z\sigma_Z}{\pi_Z+\pi_X}Q(z). \tag{A37}$$

(vi) I am now looking for $Q(.)$ that solves the following optimization problem for $z \le -z^*$:

$$\underset{Q(.)}{MAX} -\int_{-\infty}^{-z^*} dz\, F(z)\left[\sigma_Z\left(\frac{\pi_S}{\pi_S+\pi_X} - \frac{\pi_Z}{\pi_Z+\pi_X}\right)zQ'(z) - \frac{\rho}{\pi_S+\pi_X}Q(z)Q(z) - \frac{\pi_Z\sigma_Z}{\pi_Z+\pi_X}Q(z)\right], \tag{A38}$$

which can be rewritten as

$$\underset{\substack{u(.) \\ \text{s.t. } Q'(z)=u(z) \text{ and } Q(-z^*)=0}}{MAX} -\int_{-\infty}^{-z^*} dz\, F(z)\left[\sigma_Z\left(\frac{\pi_S}{\pi_S+\pi_X} - \frac{\pi_Z}{\pi_Z+\pi_X}\right)zu(z) - \frac{\rho}{\pi_S+\pi_X}u(z)Q(z) - \frac{\pi_Z\sigma_Z}{\pi_Z+\pi_X}Q(z)\right]. \tag{A39}$$

The Hamiltonian of this dynamic programming problem is

(A40) $$H \equiv -\mathrm{F}(z)\left\{\begin{array}{l}\sigma_Z\left(\frac{\pi_S}{\pi_S+\pi_X}-\frac{\pi_Z}{\pi_Z+\pi_X}\right)zu(z)- \\ -\frac{\rho}{\pi_S+\pi_X}u(z)Q(z)-\frac{\pi_Z\sigma_Z}{\pi_Z+\pi_X}Q(z)\end{array}\right\}+\lambda(z)u(z).$$

Its first order conditions are[1]

(A41) $$\begin{cases}\lambda'(z)=-\frac{\partial H}{\partial Q}=-\mathrm{F}(z)\left\{\frac{\rho}{\pi_S+\pi_X}u(z)+\frac{\pi_Z\sigma_Z}{\pi_Z+\pi_X}\right\}; \\ 0=\frac{\partial H}{\partial u}=\mathrm{F}(z)\left\{\sigma_Z\left(\frac{\pi_Z}{\pi_Z+\pi_X}-\frac{\pi_S}{\pi_S+\pi_X}\right)z+\frac{\rho}{\pi_S+\pi_X}Q(z)\right\}+\lambda(z); \\ \lambda(-\infty)=0 \text{ and } Q(-z^*)=0.\end{cases}$$

From $\frac{\partial H}{\partial u}=0$ I can derive the expression of $\lambda'(z)$:

(A42) $$\begin{aligned}\lambda'(z)=-\mathrm{f}(z)&\left\{\sigma_Z\left(\frac{\pi_Z}{\pi_Z+\pi_X}-\frac{\pi_S}{\pi_S+\pi_X}\right)z+\frac{\rho}{\pi_S+\pi_X}Q(z)\right\} \\ -\mathrm{F}(z)&\left\{\sigma_Z\left(\frac{\pi_Z}{\pi_Z+\pi_X}-\frac{\pi_S}{\pi_S+\pi_X}\right)+\frac{\rho}{\pi_S+\pi_X}u(z)\right\}.\end{aligned}$$

Substituting this expression into $\lambda'(z)=-\frac{\partial H}{\partial Q}$, I get

(A43) $$\mathrm{f}(z)\left\{\left(\frac{\pi_Z(\pi_S+\pi_X)}{\pi_S(\pi_Z+\pi_X)}-1\right)z+\frac{\rho}{\sigma_Z\pi_S}Q(z)\right\}-\mathrm{F}(z)=0,$$

which gives the solution:

---

1. See, for example, Kamien and Schwartz (1981, pp.111-120) for the algorithm to solve dynamic programming problems of this type.

(A44) $$Q(z) = \frac{\sigma_Z \pi_S}{\rho}(\alpha z + \frac{F(z)}{f(z)}) \qquad \text{for } z \le -z^*.$$

(vii) Using a similar derivation, I know that

(A45) $$\int_{z^*}^{\infty} dz(F(z)-1)' A(z) = -\int_{z^*}^{\infty} dz(F(z)-1)A'(z).$$

And I can get

(A46) $$Q(z) = \frac{\sigma_Z \pi_S}{\rho}(\alpha z + \frac{F(z)-1}{f(z)}) \qquad \text{for } z \ge z^*.$$

(iix) Summarizing the result, I have

(A47) $$Q(z) = \begin{cases} \beta(\alpha z + \frac{F(z)}{f(z)}) & \text{for } z < -z^*; \\ \beta(\alpha z + \frac{F(z)-1}{f(z)}) & \text{for } z > z^*; \\ 0 & \text{for } -z^* \le z \le z^*, \end{cases}$$

(ix). By the condition (*), I get $P(Q)$. Q.E.D.

Proposition 3 (Competitive Equilibrium C2): Suppose the market maker is competitive and her pricing schedule is continuous and twice differentiable for all Q, i.e., subject to the constraints SC-2 and TC-1. If the price schedule also satisfies the technical constraints TC-2 and TC-3, then there exists an equilibrium if and only if $\alpha > 1/2$. Furthermore, the equilibrium is as follows:

$$\text{(A48)}\qquad \begin{cases} P(Q) = m + \dfrac{\rho(1-\alpha)}{(\pi_X + \pi_S)(2\alpha - 1)} Q; \\ Q(S, W_0, W) = Q(z) = (2\alpha - 1)\beta z. \end{cases}$$

Proof: See *Remark* after the *Proof* of *Proposition 1*.

Proposition 4 (Monopolistic Equilibrium M2): Suppose the market maker is monopolistic and her pricing schedule is continuous and twice differentiable for all Q, i.e., subject to the constraints SC-2 and TC-1. If the price schedule also satisfies the technical constraints TC-2 and TC-3, then there exists an equilibrium if and only if $\alpha > 1/2$. Furthermore, the equilibrium is as follows:

$$\text{(A49)}\qquad \begin{cases} P(Q) = m + \dfrac{\rho(3-2\alpha)}{(\pi_X + \pi_S)(2\alpha - 1)} Q; \\ Q(S, W_0, W) = Q(z) = \dfrac{\beta(2\alpha - 1)}{2.} z. \end{cases}$$

Proof: I know that

$$\text{(A50)}\quad E\Pi = \left(\int_{-\infty}^{0} dz + \int_{0}^{\infty} dz\right) F(z) \left[\begin{array}{l} \sigma_Z \left(-\dfrac{\pi_S}{\pi_S + \pi_X} + \dfrac{2\pi_Z}{\pi_Z + \pi_X}\right) z Q'(z) + \\ + \dfrac{\rho}{\pi_S + \pi_X} Q'(z) Q(z) + \dfrac{\pi_Z \sigma_Z}{\pi_Z + \pi_X} (Q(z) - z Q'(z)) \end{array} \right].$$

Following the approach used in *Proposition 2*, let me maximize $E\Pi$ piecewise. Comparing this with the problem in *Proposition 2*, I notice that there are two additional constraints:

$$\text{(A51)}\qquad \begin{cases} zQ'(z) \le Q(z) \text{ for } z \ge 0, \\ zQ'(z) \ge Q(z) \text{ for } z \le 0. \end{cases}$$

Using the Hamiltonian method, I get

$$\text{(A52)}\qquad Q(z) = \frac{\beta\,(2\alpha - 1)}{2} z.$$

By the condition (*), I get

$$P(Q) = m + \frac{\rho\,(3-2\alpha)}{(\pi_X + \pi_S)(2\alpha-1)}Q + \frac{K'}{Q}, \tag{A53}$$

where K' is a constant. But $P(0)=m$, so $K'=0$. Q.E.D.

Proposition 5 (Competitive equilibrium C3): Suppose a market maker is competitive and is only to quote ask and bid prices for a single quantity $q>0$, i.e., the price schedule is subject to the strategy constraint SC-3. Then for any degree of information asymmetry $\alpha\in(0,1)$, there exists an equilibrium as follows:

$$P(Q) - m = \begin{cases} p & \text{for } Q = q; \\ p & \text{for } Q = -q; \\ +\infty & \text{for } Q > 0 \text{ and } Q \neq q; \\ -\infty & \text{for } Q < 0 \text{ and } Q \neq -q, \end{cases} \tag{A54}$$

$$Q(\boldsymbol{S},\boldsymbol{W}_0,\boldsymbol{W}) = Q(z) = \begin{cases} q & \text{for } z > z_0; \\ -q & \text{for } z < -z_0; \\ 0 & \text{for } -z_0 \le z \le z_0, \end{cases}$$

and p and z_0 are uniquely determined by the following equation set:

(A55)

$$\begin{cases} z_0 - (1-\alpha)\dfrac{\int_{z_0}^{\infty} z f(z)dz}{\int_{z_0}^{\infty} f(z)dz} = \dfrac{q}{2\beta}; \\ p = \dfrac{\rho\beta}{\pi_X + \pi_S}(z_0 - \dfrac{q}{2\beta}), \end{cases}$$

where f(z) is the probability density function of the standard normal distribution.

Proof:

(i). Given the price schedule $P(Q)$, an investor decides to buy q shares of the risky asset if and only if his expected utility is improved, i.e.

$$\begin{aligned} &-(m+p)q + \frac{(W+q)(\pi_X m + \pi_S S)}{\pi_X + \pi_S} - \frac{\rho(W+q)^2}{2(\pi_X + \pi_S)} > \\ &> \frac{W(\pi_X m + \pi_S S)}{\pi_X + \pi_S} - \frac{\rho W^2}{2(\pi_X + \pi_S)}, \end{aligned} \tag{A56}$$

which, by the definitions of z and z_0, is equivalent to

$$z > z_0. \tag{A57}$$

Similarly, an investor decides to sell q shares of the risky asset if and only if

$$z < -z_0. \tag{A58}$$

Otherwise, the investor does not trade, i.e., $Q(z)=0$ for any other value of z.

(ii). Given the trading strategy $Q(z)$, a market maker has the following expectation of the asset value X if she receives a buy order:

$$\mathrm{E}[X|\text{an investor buys } q \text{ shares}] =$$

$$(A59) \qquad = \frac{\int_{z_0}^{\infty} \mathrm{E}[X|z]\mathrm{f}(z)\mathrm{d}z}{\int_{z_0}^{\infty} \mathrm{f}(z)\mathrm{d}z}.$$

Recall that the definition of z gives rise to the following relation:

$$(A60) \qquad \mathrm{E}[X|z] = m + \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} z;$$

therefore I get

$$\mathrm{E}[X|\text{an investor buys } q \text{ shares}] =$$

$$(A61) \qquad = m + \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} \frac{\int_{z_0}^{\infty} z\mathrm{f}(z)\mathrm{d}z}{\int_{z_0}^{\infty} \mathrm{f}(z)\mathrm{d}z}.$$

By the definition of z_0, I know that

$$(A62) \qquad m + p = m + \frac{(\beta z_0 - \frac{q}{2})\rho}{\pi_S + \pi_X}.$$

In equilibrium, the price $m+p$ should be equal to E[X|an investor buys q shares]. Therefore I have

$$\text{(A63)}\qquad \frac{(\beta z_0 - \frac{q}{2})\rho}{\pi_S + \pi_X} = \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} \frac{\int_{z_0}^{\infty} z f(z)dz}{\int_{z_0}^{\infty} f(z)dz}.$$

Or

$$\text{(A64)}\qquad \frac{z_0 - \frac{q}{2\beta}}{1-\alpha} = \frac{\int_{z_0}^{\infty} z f(z)dz}{\int_{z_0}^{\infty} f(z)dz}.$$

Similarly, the equilibrium requirement as an investor sells q shares leads to the same equation. That is to say, if I can find $z_0>0$ to satisfy this equation, then I can find a real number $p>0$ to constitute an equilibrium. By *Lemma 4*, the right-hand side approaches to z_0 as z_0 goes to positive infinity; or equivalently, the right-hand side approaches to the 45-degree straight line asymptotically. Also note that the left-hand side is a positively sloped straight line intersecting the vertical axis in the negative domain. Thus the problem of proving the existence and uniqueness of a value of z_0 to solve the equation is the same as looking for a (different) value of z_0 to solve

$$\text{(A65)}\qquad \frac{z_0}{1-\alpha} > \frac{\int_{z_0}^{\infty} z f(z)dz}{\int_{z_0}^{\infty} f(z)dz}.$$

Since $1>\alpha>0$, the answer is positive and an equilibrium exists. Q.E.D.

Proposition 6 (Monopolistic equilibrium M3): Suppose a market maker is monopolistic and is only to quote the ask and bid prices for a single quantity $q>0$, i.e., the price schedule is subject to the strategy constraint SC-3. Then for any degree of information asymmetry $\alpha\in(0,1)$, there exists an equilibrium as follows:

(A67)
$$P(Q) - \mathrm{m} = \begin{cases} p' & \text{for } Q = q; \\ -p' & \text{for } Q = -q, \\ +\infty & \text{for } Q > 0 \text{ and } Q \neq q; \\ -\infty & \text{for } Q < 0 \text{ and } Q \neq -q, \end{cases}$$

$$Q(S, W_0, W) = Q(z) = \begin{cases} q & \text{for } z > z_1; \\ -q & \text{for } z < -z_1; \\ 0 & \text{for } -z_1 \leq z \leq z_1, \end{cases}$$

and p' and z_1 are uniquely determined by the following equation set:

(A68)
$$\begin{cases} z_1 = \underset{z_1}{ARG\ MAX} \left[(z_1 - \frac{q}{2\beta}) \int_{z_1}^{\infty} f(z)dz - (1-\alpha) \int_{z_1}^{\infty} z f(z)dz \right]; \\ p' = \frac{\rho\beta}{\pi_X + \pi_S}(z_1 - \frac{q}{2\beta}), \end{cases}$$

where f(z) is the probability density function of the standard normal distribution.

Proof:

(i). Given the price schedule $P(Q)$, an investor decides to buy q shares of the risky asset if and only if his expected utility is improved, i.e.

(A69)
$$-(m + p')q + \frac{(W+q)(\pi_X m + \pi_S S)}{\pi_X + \pi_S} - \frac{\rho(W+q)^2}{2(\pi_X + \pi_S)} > \frac{W(\pi_X m + \pi_S S)}{\pi_X + \pi_S} - \frac{\rho W^2}{2(\pi_X + \pi_S)},$$

which, by the definitions of z and z_1, is equivalent to

(A70)
$$z > z_1.$$

Similarly, an investor decides to sell q shares of the risky asset if and only if

(A71) $z<-z_1$.

Otherwise, the investor does not trade, i.e., $Q(z)=0$ for any other value of z.

(ii). Given the trading strategy $Q(z)$, a market maker has the following expectation of the asset value X if she receives a buy order:

$$\text{(A72)} \qquad \mathrm{E}[X|\text{an investor buys } q \text{ shares}] = \frac{\int_{z_1}^{\infty} \mathrm{E}[X|z]\mathrm{f}(z)\mathrm{d}z}{\int_{z_1}^{\infty} \mathrm{f}(z)\mathrm{d}z}.$$

Recall that the definition of z gives rise to the following relation:

$$\text{(A73)} \qquad \mathrm{E}[X|z] = m + \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} z;$$

therefore I get

$$\text{(A74)} \qquad \mathrm{E}[X|\text{an investor buys } q \text{ shares}] = m + \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X} \frac{\int_{z_1}^{\infty} z\mathrm{f}(z)\mathrm{d}z}{\int_{z_1}^{\infty} \mathrm{f}(z)\mathrm{d}z}.$$

By the definition of z_1, I know that

$$\text{(A75)} \qquad m + p' = m + \frac{(\beta z_1 - \frac{q}{2})\rho}{\pi_S + \pi_X}.$$

The expected profit for the market maker from selling q is therefore

$$\text{(A76)} \qquad q\left\{\frac{(\beta\, z_1 - \frac{q}{2})\rho}{\pi_S + \pi_X} - \frac{\sigma_Z \pi_Z}{\pi_Z + \pi_X}\frac{\int_{z_1}^{\infty} z f(z)dz}{\int_{z_1}^{\infty} f(z)dz}\right\}\int_{z_1}^{\infty} f(z)dz,$$

where $\int_{z_1}^{\infty} f(z)dz$ is the probability that an investor decides to buy q, i.e., the probability that $z > z_1$.

Similarly, the expected profit for the market maker from buying q is represented by the same expression. That is to say, the market maker is to find a value of $z_1>0$ to maximize this expression. The problem is equivalent to the following one:

$$\text{(A77)} \qquad \underset{z_1}{\text{MAX}}\left[(z_1 - \frac{q}{2\beta})\int_{z_1}^{\infty} f(z)dz - (1-\alpha)\int_{z_1}^{\infty} z f(z)dz\right].$$

With a tedious procedure it can be proven that the above maximization problem has a solution z_1 which leads to a positive p' and $p'>p$, where p is the price parameter in the equilibriuim C3. Q.E.D.

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