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Empirical Tests of Two State-Variable HJM Models

Robert R. Bliss and Peter Ritchken

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Abstract: Models for pricing interest rate claims, developed under the Heath-Jarrow-Morton paradigm, differ according to the volatility *structure* imposed on forward rates. For most general HJM structures the resultant path dependence creates implementation problems. Ritchken and Sankarasubramanian have recently identified necessary and sufficient conditions on the class of volatility structures of forward rates that enable the term structure dynamics to be captured by a finite set of state variables. The class is quite rich. The instantaneous spot rate volatility may be quite general, but the model curtails the structure of forward rate volatilities relative to this spot rate volatility. This paper provides empirical tests for this class of volatility structures. Unlike other studies, the volatility structure is examined over the a broad section of maturities in the yield curve. Using Treasury data over the period 1982-1994, we find support for this class. Furthermore, unlike other studies, no evidence of a "volatility" hump is identified.

JEL classification: G13

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Empirical Tests of Two State–Variable HJM Models

1 Introduction

In the last decade increased attention has been placed on the volatility structure of both spot and forward interest rates. There are two main reasons for this. First, the volatility structure of forward rates plays a key role in pricing and hedging interest rate contingent claims. Second, the structure of forward rate volatilities determines how shocks to the underlying state variables are transmitted along the yield curve. An empirical characterization of this volatility structure will provide insight into the underlying process that causes yields to change over time.

Heath, Jarrow and Morton (1992) (hereafter HJM) show that, given an initial term structure, arbitrage-free price paths of all bonds can be established once the *structure* of all forward rate volatilities is supplied. The simplest forward rate volatility structure is the constant variance structure, first proposed by Ho and Lee (1986). In this model, all shocks to the term structure are permanent; that is, the absolute magnitude of the change in all forward rates to new information is constant across maturities, and thus independent of both the level of the forward rate and its maturity. Over the last decade, more realistic volatility structures for forward rates have been proposed, in which volatilities fluctuate according to forward rate levels and maturities. Unfortunately, if the structure for these volatilities is left quite general, the evolution of the term structure is usually path dependent and it may not be possible to characterize its dynamics by a finite collection of state variables. In such cases it is difficult to implement the HJM theory, especially for longer term interest rate dependent claims that have early exercise features. As a result, while many proposed forward rate volatility structures have been motivated by the need to incorporate added realism, other structures have been motivated by a desire to eliminate the path dependence problem and to make the resulting models for interest rate claims more tractable.

Recently, Ritchken and Sankarasubramanian (1995) (hereafter RS) identified necessary and sufficient conditions on volatility structures that permit the term structure of HJM models to be characterized by *two* state-variables regardless of the structure for spot interest

rates. Indeed, the volatility of the spot interest rate could depend on its level, or on any set of rates drawn from the present or past term structures. For example, spot rate volatility structures could take on the form of all the models examined by Chan, Karolyi, Longstaff and Sanders (1992), including the constant volatility model of Vasicek (1977), the square root structure of Cox, Ingersoll, and Ross (1985), and the proportional structure of Dothan (1978). Alternatively, spot interest rate volatilities could be modeled as GARCH processes as in Brenner and Kroner (1994). For a two state-variable representation to exist, the relationship of forward rate volatilities relative to the spot rate volatility has to be curtailed in a way that will be made precise later on. The resulting class of forward rate volatility structures include the Ho-Lee and Generalized Vasicek structures as well as others.

The purpose of this article is to establish whether there is any empirical support for these two state-variable HJM models. If volatility structures belong to the RS class, then informational events are transmitted across the yield curve in a constrained fashion. In particular, knowledge of just two points on the term structure at any date, together with the original term structure, is sufficient to uniquely characterize the entire yield curve at that date. Since this implies a relatively simple intertemporal linkage between term structures, pricing and hedging interest rate claims is greatly simplified. On the other hand, rejection of the class will imply that simple two state-variable HJM models are unlikely to describe the linkages between term structures. In this case, HJM models, which are more consistent with empirical evidence, will either have to retain all their path dependence or will require more state variables to capture the intertemporal relationships between term structures.

Most tests of term structure models have assumed the number of state variables that capture the dynamics of the term structure equals the number of sources of uncertainty (i.e. the number of factors). All the tests reported in Chan, Karolyi, Longstaff, and Sanders (1992), for example, assume one state-variable is sufficient to characterize the term structure, in a one factor economy. The single-factor/single state-variable models have had mixed success at best. One approach to extending these models is to increase the number of sources of risk. For example, Longstaff and Schwartz (1992) added stochastic volatility; while Brennan and Schwartz (1979) added a stochastic long rate. An alternative approach is to keep the single factor, or source of risk, but to have that factor effect the term structure through not only

its current value, but also its history. In the HJM paradigm, for example, one can construct one factor models, where the number of state-variables that are necessary to characterize the term structure is infinite. With constraints on the volatilities, models with a finite number of state-variables are possible. For example, the single factor generalized Vasicek model, is a *one* state-variable HJM model developed under a specific deterministic forward rate volatility assumption and the single factor Li, Ritchken and Sankarasubramanian (1995) model is a *two* state-variable HJM model. Indeed, under appropriate volatility restrictions, it is possible to develop k state-variable models where $k > 2$, all developed in a single factor framework. This article will focus on one factor models, but will investigate whether there is support for *any* two state-variable HJM model.

The paper proceeds as follows. In section 2 we review the HJM paradigm, emphasizing the consequences of alternative volatility structures. In section 3 we transform the finite state-variable models into alternative forms that are more useful for empirical analysis. Section 4 develops specific hypotheses, reports the tests, and examines whether the extension from the class of volatility structures that lead to one state-variable HJM models to the larger class of two state-variable models derived from the RS class leads to economically significant differences. Section 5 concludes.

2 Path Dependence, Volatility Structures, and Finite State-Variable HJM Models

Let $f(t,s)$ be the instantaneous forward rate for time s viewed from date t. Forward rates are assumed to follow a diffusion process of the form

$$df(t,T) \;=\; \mu_f(t,T)\,dt + \sigma_f(t,T)\,dw(t) \quad ; \quad f(0,T) \text{ given } \forall T.$$

Here $\mu_f(t,T)$ and $\sigma_f(t,T)$ are the drift and volatility parameters which could depend on the level of the term structure itself, and $dw(t)$ is the Wiener increment.

HJM (1992) show that, given the initial forward rate curve, $f(0,T)$, and a structure for volatilities of all forward rates, $\sigma_f(t,T)$, interest rate claims can be uniquely priced

without explicitly modeling utility dependent parameters such as the market price of risk. In particular, pricing can proceed as if the local expectations hypothesis holds under the following modified process:

$$df(t,T) = \left[\sigma_f(t,T)\int_t^T \sigma_f(t,s)ds\right]dt + \sigma_f(t,T)dw(t)$$

Unfortunately, in general, the evolution of this term structure will not be Markovian with respect to a finite collection of state variables. That is, knowledge of a finite numbers of forward rates at any future date may not be sufficient to characterize other forward rates at that time. Moreover, the dynamics of any interest rate, including the instantaneous spot rate, will usually depend on the entire path taken by forward rates since the initialization date. These issues create difficulties for developing efficient numerical procedures for pricing interest rate claims.

While the general HJM model has these problems, for special subclasses derived using curtailed forward rate volatility structures, the path dependence issues fall away. Caverhill (1994), Hull and White (1993) and Ritchken and Sankarasubramanian (1995) show, for example, that if the volatility structure of forward rates has the form:

$$\sigma_f(t,T) = \sigma k(t,T)$$

where $k(t,T)$ is a deterministic function satisfying the following semi-group property

$$\begin{aligned} k(t,T) &= k(t,u)k(u,T) \qquad t \le u \le T \\ k(u,u) &= 1 \end{aligned}$$

then, conditional on knowing the initial term structure, knowledge of any single point on the term structure at date t is sufficient to characterize the full yield curve at that date. Usually, the single point is taken as the instantaneous spot interest rate, and its dynamics are path independent. If no time varying parameters are used in the volatility structure, then the structure for $k(t,T)$ is necessarily the exponentially dampened structure:

$$k(t,T) = e^{-\kappa(T-t)}. \tag{1}$$

Such volatility structures are referred to as Generalized Vasicek (hereafter GV) structures. This structure implies that the volatility of the spot rate is a constant, independent of its level. Under the GV model, forward rate volatilities are exponentially declining in their maturities, and the future value of the state variable is normally distributed. As a result, it is not surprising that analytical solutions exist for a variety of European claims and that efficient numerical procedures exist for American claims. Empirical tests of the Ho-Lee model, which is a special case of the GV model with $\kappa = 0$ in equation (1), has been conducted by Flesakar (1992) while Amin and Morton (1994) examine the more general structure.[1] No support is found for the Ho-Lee model. However, the GV model was found useful in that it was capable of generating abnormal returns in particular trading strategies. Unfortunately, there appears to be little empirical support for the constant spot rate volatility structure. Chan, Karolyi, Longstaff and Sanders (1992), for example, conclude that the volatility of the spot interest rate should depend on its level, and that models which allow volatilities to be independent of their level will be misspecified.

RS (1995) consider a class of interest rate processes in which volatilities can fluctuate according to their levels. Let $\sigma_r(t,.)$ represent the volatility of the instantaneous spot rate, $r(t)$, at date t. This structure could depend on any term structure information available at time t. As examples, this structure could depend on the level of rates or it could take on a GARCH form. RS then show that if the volatility structure of forward rates is given by:

$$\sigma_f(t,T) = \sigma_r(t,.)\ k(t,T) \tag{2}$$

where $k(t,T)$ again is a deterministic function satisfying the semi-group property, then, given an initial term structure, there exists a *two* state-variable representation of the evolution of future interest rates.

[1]Cohen and Heath (1992) and Abken and Cohen (1994) investigate some volatility structures that do not permit a Markov representation, and Raj, Sim and Thurston (1995) compare some HJM models to the Cox, Ingersoll, and Ross models.

The class of volatility structures admitted by equation (2) is quite large. As an example,

$$\sigma_r(t,.) = \sigma[r(t)]^\gamma.$$

Notice that for $\gamma = 0$ the volatility structure is identical to that of Vasicek (1977) while $\gamma = 0.5$ yields the square root volatility used in Cox, Ingersoll and Ross (1985).[2] For volatility structures that do not contain time-varying parameters, the only feasible $k(t,.)$ function is again the deterministic, exponentially dampened function given by equation (1), and thus

$$\sigma_f(t,T) = \sigma_r(t,.)\ e^{-\kappa(T-t)}.$$

Equivalently, the volatility of the forward rate, normalized by the volatility of the spot interest rate, must be a deterministic, exponentially dampened function of maturity

$$\frac{\sigma_f(t,T)}{\sigma_r(t,.)} = e^{-\kappa(T-t)}. \tag{3}$$

Heath, Jarrow, Morton and Spindel (1992) provide cursory evidence that is inconsistent with this structure. In particular, they conclude that the term structure of volatilities is humped, increasing and then decreasing. Amin and Morton (1994) estimate the exponentially declining volatility structure model. They find point estimates of κ that are negative, implying that volatilities of forward rates are increasing in maturity. This result is implausible for all maturities, and may result from the fact that their analysis is limited to data sets with short maturities. Over this narrow region a negative κ might be consistent with the theory that the term structure of forward rate volatilities is humped. Unfortunately, a humped term structure is inconsistent with any of the path-independent models described in this paper.

Other examples of volatility structures not in the two-state-variable HJM class include some of the forms tested by Amin and Morton (1994). For example, if volatilities are proportional to their levels as:

$$\sigma_f(t,T) = \sigma[f(t,T)]^\gamma,$$

[2]The drift terms for $r(t)$ in these models are different from their single state-variable counterparts.

then a two state-variable representation is not possible.

For volatilities belonging to the two state-variable class, RS show that there is a simple analytical linkage between term structures at dates s and t. Let

$$P[t, t+T] = e^{-\int_t^{t+T} f(t,u)du}$$

represent the time t price of a T-maturity pure discount bond that pays \$1 at date $t+T$. Given an initial term structure, $P(s,.)$ at time s, the price of a bond, at any future date, t, $s \leq t \leq T$, must be defined in terms of its forward price at date s, the short interest rate at date t, and a third variable capturing the history of the path of interest rates from s to t as follows:

$$P[t, t+T] = \left(\frac{P[s, t+T]}{P[s,t]}\right) \exp\left\{-\frac{1}{2}\beta^2(T)\,\phi(t) + \beta(T)\,\psi(t)\right\} \tag{4}$$

where

$$\begin{aligned} \beta(T) &\equiv \tfrac{1}{\kappa}(1 - e^{-\kappa T}) \\ \phi(t) &\equiv \textstyle\int_s^t \sigma_f^2(s,t)ds \\ \psi(t) &\equiv f(s,t) - r(t) \end{aligned}$$

If this is not the case the no-arbitrage assumption will be violated. In this representation, conditional on the time s term structure, the entire term structure at a subsequent time t can be reconstructed once $\phi(t)$ and $\psi(t)$ are given. Neither of these factors depends on maturity T. Thus, this model is a two state-variable model, even though there is still only a single stochastic driver. Equation (4) identifies the two state-variables as the ex post forward premium on the spot interest rate, $\psi(t)$, and the "integrated variance" factor, $\phi(t)$. RS characterize the dynamics of the two state-variables, $\phi(t)$ and $\psi(t)$, in terms of their current values and the forward rate curve at an earlier date s. Specifically, interest rate claims can be priced as if the local expectations hypothesis applied; if the dynamics of the state variables are taken as:

$$\begin{aligned} d\psi(t) &= [\kappa\psi(t) + \phi(t)]\,dt + \sigma_r(t,.)dw(t) \\ d\phi(t) &= [\sigma_r^2(t,.) - 2\kappa\phi(t)]dt \end{aligned}$$

The dynamics of the instantaneous spot interest rate under the RS assumption is given by

$$dr(t) = \left[\kappa\psi(t) + \phi(t) + \frac{d}{dt}f(s,t)\right]dt + \sigma_r(t,.)dw(t).$$

Note that since the dynamics depend only on the values of the two variables ϕ and ψ at time t, the evolution of the spot rate is Markovian. This contrasts with the general HJM models in which the spot interest rate process cannot be described by a Markov process with a finite number of state variables.

Let HJM-RS denote the restricted form of HJM models that follow from the assumption in equation (3) and presented in equation (4).

3 Empirical Implications of the HJM-RS Models

Let $y_s[t, t+T]$ represent the continuously-compounded annualized forward yield over the time period, $[t, t+T]$ measured at date $s \leq t$.[3] That is,

$$y_s[t, t+T] = \frac{1}{T}\int_t^{t+T} f(s,x)dx.$$

Equation (4) can be rewritten in yield form as

$$y_t[t, t+T]\,T = y_s[t, t+T]\,T + \beta(T)\psi(t) - \frac{\beta^2(T)}{2}\phi(t) \tag{5}$$

That is, the yield at date t equals its original forward yield plus a deviation which is fully determined at date t by the two state-variables, $\psi(t)$ and $\phi(t)$. Let

$$\Delta y_t[t, t+T] \equiv y_t[t, t+T] - y_s[t, t+T], \tag{6}$$

[3]Thus the yield on a T-maturity pure discount bond paying \$1 at time $t+T$ and costing $P[t, t+T]$ at date t will be

$$y_s[t, t+T] = -\frac{1}{T}\ln P[t, t+T]$$

and the forward rate observed at time $s < t$ for the period t to $t+T$ is

$$y_s[t, t+T] = -\frac{t+T-s}{T}y_s[s, t+T] + \frac{t-s}{T}y_s[s, t].$$

represent the deviation between the actual yield at date t and the original forward yield at previous observation date s. We shall refer to this deviation simply as the "forecast error" which it would be under the pure expectations hypothesis, though this reference is only for convenience and does not imply a maintained hypothesis that the expectations hypothesis is correct. Substituting we have:

$$\Delta y_t[t, t+T]\ T = \beta(T)\psi(t) - \frac{\beta^2(T)}{2}\phi(t) \tag{7}$$

3.1 Empirical Implementation of HJM-RS Model

Unfortunately, the state variables $\psi(t)$ and $\phi(t)$ are not readily observed. Duffee (1995) shows that using a short maturity bill to proxy for the spot rate $r(t)$ is problematic, and even shorter rates, such as Fed. Funds or overnight repo rates, are subject to shocks specific only to these markets.

Our solution to the problem of unobservable state-variables is to transform the state-variables. Note that by selecting any two distinct maturities τ_1 and τ_2 say, and observing the values for $\Delta y_t[t, t+\tau_1]$ and $\Delta y_t[t, t+\tau_2]$ we can invert the following pair of equations for $\phi(t)$ and $\psi(t)$

$$\begin{aligned}
\Delta y_t[t, t+\tau_1]\ \tau_1 &= \beta(\tau_1)\psi(t) - \frac{\beta^2(\tau_1)}{2}\phi(t) \\
\Delta y_t[t, t+\tau_2]\ \tau_2 &= \beta(\tau_2)\psi(t) - \frac{\beta^2(\tau_2)}{2}\phi(t).
\end{aligned}$$

Substituting back into equation (5) we obtain:

$$\Delta y_t[t, t+T] = \Delta y_t[t, t+\tau_1]H_1(T; \tau_1, \tau_2) + \Delta y_t[t, t+\tau_2]H_2(T; \tau_1, \tau_2) \tag{8}$$

where

$$H_1(T; \tau_1, \tau_2) \equiv \frac{\tau_1\,\beta(T)[\beta(\tau_2) - \beta(T)]}{T\,\beta(\tau_1)[\beta(\tau_2) - \beta(\tau_1)]}$$

$$H_2(T;\tau_1,\tau_2) \equiv \frac{\tau_2\,\beta(T)[\beta(T)-\beta(\tau_1)]}{T\,\beta(\tau_2)[\beta(\tau_2)-\beta(\tau_1)]}$$

Thus, we transform the equation (7), which involves unobservable state-variables, $\phi(t)$ and $\psi(t)$, into equation (8), which involves the observable state-variables, $\Delta y_t[t, t+\tau_1]$ and $\Delta y_t[t, t+\tau_2]$.

In summary, if the volatility structure in equation (3) obtains, and the no-arbitrage condition holds, then the forecast error over the period $[s,t]$, for yields of any arbitrary maturity, T, can be linked to the forecast error of any two benchmark maturities τ_1 and τ_2. Equivalently, if two state-variables are to characterize a term structure in an HJM model with a single stochastic driver, then the relationship in equation (8) must hold. Conversely, if equation (8) does not hold, then the volatility structure is not of the form given in equation (3) and hence is not in the class of HJM-RS models, and more assumptions will be necessary for two or fewer state-variables to characterize the term structure.

Equation (8) does not contain any parameters that characterize the volatility of the spot rate, $\sigma_r(t,.)$. Regardless of its structure, if a two state-variable HJM-RS representation is to obtain, then the linkage between term structures, in terms of two state-variables, only depends on the parameters characterizing $\beta(.)$, namely, κ.[4]

In order to test this relationship, we first recognize that annualized yields are measured with error. We assume that:

$$y_t[t, t+T] = y_t^m[t, t+T] + \epsilon[t, t+T] \tag{9}$$

where $y_t^m[t, t+T]$ is the measured yield on a T-maturity pure discount bond at date t, and $\epsilon[t, t+T]$ is the measurement error term, the set of which are assumed to be independent, identically distributed normal random variables with mean 0 and variance η^2. As a first approximation this is reasonable. Bliss (1994), for example, shows that bonds of longer maturities have larger pricing errors. Measurement errors resulting from stale prices may be reasonably expected to be random. While other sources of error, such as liquidity differences,

[4] Actually, in equation (8), we have assumed the parameters of the volatility structure are not time varying. For a discussion on the use of time varying parameters see RS (1995).

may be systematic, modeling these is problematic at best. Since we use non-parametric hypothesis tests, any misspecification of our error structure is going to reduce the efficacy of our estimates, and bias against finding positive results.

The simple error structure, (9), assumed for raw yield measurements, implies a very specific error structure for the forecast error. Substituting equation (9) into (6) we obtain, after simplification:

$$\Delta y_t^m[t, t+T] = y_t^m[t, t+T] - y_s^m[t, t+T] + \epsilon^*[t, t+T] \tag{10}$$

where

$$\epsilon^*[t, t+T] \equiv \epsilon[t, t+T] - \frac{(t+T)}{T}\epsilon(s, t+T) + \frac{t}{T}\epsilon(s, t)$$

The $\epsilon^*(\cdot)$'s are normally distributed with mean 0 and covariances

$$Cov[\epsilon^*[t, t+T], \epsilon^*(t, t+S)] = \begin{cases} 2\eta^2(1 + (\frac{t}{T}) + (\frac{t}{T})^2) & \text{for } T = S \\ \eta^2(\frac{t^2}{TS}) & \text{for } T \neq S \end{cases}$$

Substituting equation (10) into equation (8) and rearranging leads to:

$$\Delta y_t^m[t, t+T] = \Delta y_t^m[t, t+\tau_1]H_1(T) + \Delta y_t^m[t, t+\tau_2]H_2(T) + \epsilon^{**}[t, t+T] \tag{11}$$

where

$$\epsilon^{**}[t, t+T] \equiv \epsilon^*[t, t+\tau_1]H_1(T) + \epsilon^*[t, t+\tau_2]H_2(T) - \epsilon^*[t, t+T]$$

The $\epsilon^{**}[t, \cdot]$s are normally distributed with mean 0 and covariances given by:

$$\chi(T, S) \equiv Cov\left(\epsilon^{**}[t, t+T], \epsilon^{**}[t, t+S]\right) = \begin{cases} \alpha'(T)\Sigma(T, S)\gamma(S) & \text{for } T \neq S \\ \alpha'(T)\Sigma(T, T)\alpha(T) & \text{for } T = S \end{cases}$$

where

$$\begin{aligned} \alpha'(T) &\equiv (H_1(T), H_2(T), -1, 0) \\ \gamma'(S) &\equiv (H_1(S), H_2(S), 0, -1) \end{aligned}$$

and

$$\Sigma(T,S) \equiv \eta^2 \begin{pmatrix} 2(1+(\frac{t}{\tau_1})+(\frac{t}{\tau_1})^2) & \frac{t^2}{\tau_1\tau_2} & \frac{t^2}{\tau_1 T} & \frac{t^2}{\tau_1 S} \\ \frac{t^2}{\tau_1\tau_2} & 2(1+(\frac{t}{\tau_2})+(\frac{t}{\tau_2})^2) & \frac{t^2}{\tau_2 T} & \frac{t^2}{\tau_2 S} \\ \frac{t^2}{\tau_1 T} & \frac{t^2}{\tau_2 T} & 2(1+(\frac{t}{T})+(\frac{t}{T})^2) & \frac{t^2}{TS} \\ \frac{t^2}{\tau_1 S} & \frac{t^2}{\tau_2 S} & \frac{t^2}{TS} & 2(1+(\frac{t}{S})+(\frac{t}{S})^2) \end{pmatrix}$$

This empirical HJM-RS model has two unknown parameters, κ and η^2 which must be estimated from the data.

3.2 Empirical Implementation of HJM-GV Model

Equation (7) holds regardless of the volatility structure, $\sigma_r(t)$, of the spot interest rate. If, however, the volatility of the spot interest rate is a constant, independent of the level of rates, then the dynamics of the forward rates correspond to the GV model. In this case, $\phi(t)$ becomes

$$\phi(t) = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)$$

which is deterministic, and $\psi(t)$ is the sole state-variable. Only a single benchmark maturity, τ, is needed in this case. The GV analog to equation (11) is:

$$\Delta y_t^m[t, t+T] = h(T)\Delta y_t^m[t, t+\tau] + \sigma^2 k(T) + \epsilon^{**}[t, t+T] \tag{12}$$

where

$$\begin{aligned} h(T) &\equiv \frac{\tau\ \beta(T)}{T\ \beta(\tau)} \\ k(T) &\equiv \frac{1}{4\kappa^2}\beta(T)\beta(\tau)[\beta(T) - \beta(\tau)](1 - e^{-2\kappa t}) \\ \epsilon^{**}[t, t+T] &\equiv h(T)\epsilon^{*}[t, t+\tau] - \epsilon^{*}[t, t+T] \end{aligned}$$

The $\epsilon^{**}[t,\cdot]$s are normally distributed with mean 0 and covariance

$$\chi^*(T,S) = \begin{cases} \alpha^{*'}(T)\Sigma^*(T,S)\gamma^*(S) & \text{for } T \neq S \\ \alpha^{*'}(T)\Sigma^*(T,T)\alpha^*(T) & \text{for } T = S \end{cases}$$

where

$$\begin{aligned} \alpha^{*'}(T) &\cong (h(T), -1, 0) \\ \gamma^{*'}(S) &\equiv (h(S), 0, -1) \end{aligned}$$

and

$$\Sigma^*(T,S) \equiv \eta^2 \begin{pmatrix} 2(1+(\frac{t}{\tau_1})+(\frac{t}{\tau_1})^2) & \frac{t^2}{\tau_1 T} & \frac{t^2}{\tau_1 S} \\ \frac{t^2}{\tau_1 T} & 2(1+(\frac{t}{T})+(\frac{t}{T})^2) & \frac{t^2}{TS} \\ \frac{t^2}{\tau_1 S} & \frac{t^2}{TS} & 2(1+(\frac{t}{S})+(\frac{t}{S})^2) \end{pmatrix}$$

This empirical GV model requires estimating three parameters, κ, η^2, and σ^2.

4 Empirical Tests and Results

4.1 The Data and Methodology

Our raw data set consists of the monthly unsmoothed Fama-Bliss yields, over the period 1982 to 1994.[5] A set of 9 target maturities were chosen to cover the term structure. These were 0.25, 0.5, 1, 2, 3, 5, 7, 10, and 19 years.[6] For each adjacent pair of months we computed the forecast errors $\Delta y_t[t, t+T]$ for each of the target maturities.

Figure 1a shows the box and whiskers plot of the forecast errors for each of the 9 maturities. The plots are based on $13 \times 12 = 156$ data points for each maturity.

[5]See Fama-Bliss (1987) and Bliss (1994) for details of the method used to extract pure discount yields from the set of available coupon bond prices each month.

[6]These maturities correspond to the T's above. To compute the month-before forward prices we also need maturities of $\frac{1}{12}$, $\frac{4}{12}$, $\frac{7}{12}$, $1\frac{1}{12}$, etc.

If forward yields were perfect predictors of future spot yields, then there would be no unexplained variability, and the whiskers in the plot would have zero length. Figure 1a shows that forecast error dispersion varies moderately with maturity and there is evidence of a maturity dependent bias in the forecasts; at short maturities forward rates tend to over state subsequently realised yields on average, but this effect disappears at longer maturities. To see if there was a term structure of forecast errors we subtracted the errors each month by the forecast error of the shortest maturity. The "normalized" results, presented in Figure 1b show a positive relation between the premium of long-maturity forecast errors over the shortest forecast error, and maturity. In addition, normalized forecast errors show dispersion increasing markedly with maturity.

The question is whether our two models, HJM-RS and HJM-GV, can "explain" these forecast errors in terms of two (or one) state-variables as proxied by selected benchmark maturities and in the manner implied by the models.

We divided our 156 months of observations into 13 annual groupings. This was done to minimize reliance in stationarity of the parameters to be estimated, and to provided multiple independent estimates of the parameters and to be able to compare the models' individual and relative performances.

Two maturities were selected as τ_1 and τ_2. The HJM-GV model required a single benchmark to substitute for the single state variable, so one of the pair was used for estimation and the other was simply discarded. This was done so that the set of remaining maturites, over which the models were be tested would be identical in both cases.[7]

The next step was to estimate the model parameters, κ and η for HJM-RS and κ, η and σ for HJM-GV. This was done by maximum likelihood, making use of the variance-covariance structure developed above. The process was repeated for each of the 13 years and τ_1 and τ_2 were varied to investigate the dependence of the results on the maturities selected as benchmarks. The following analyses were based on the residuals from the estimated models, i.e.

$$\hat{\epsilon}[t, t+T] = \Delta y_t^m[t, t+T] - \Delta y_t^m(t, t+\tau_1)\hat{H}_1(T) - \Delta y_t^m(t, t+\tau_2)\hat{H}_2(T)$$

[7]We also compared the results to models which retained the best of two GV models based on separate use of the two candidate state variables. Similar results were obtained.

4.2 HJM-RS Model Results

Using $\tau_1 = 0.25$ and $\tau_2 = 5$ years respectively, Figure 2a shows a box and whiskers plot of the monthly residuals for each of the remaining maturities. As in figure 1a, each box and whiskers is based on $13 \times 12 = 156$ residuals. There is a marked reduction in variance and the bias observed in Figure 1a is no longer apparent. Figure 2b shows an elimination of the heteroskedasticity evident in the raw data

The lack of biases in the above plots suggest that the structural form of the model in equation (11) may have some merit, and indicates that there may be empirical support for the normalized volatility restriction. What remains, therefore, is the task of testing this specification against some plausible alternatives. If the model provides unbiased estimates of yields for all maturities, and if these estimates have smaller mean squared errors than alternative estimators, then the underlying assumptions must be somewhat consistent with actual market behavior. In particular, such evidence would indicate that the normalized volatility constraint, given in equation (3) is not severe and that a two state-variable Markov representation of the term structure is permissible.

Table 1 shows the point estimates and standard errors for the parameters κ and η for each of the 13 years when $\tau_1 = 0.5$ and $\tau_2 = 5$. In addition, in column 2 of the table we report statistics on the total sum of squared forecast errors, TSS, and in column 3 we report the sum of squared residuals, RSS, that remain unexplained after equation (11) has been fit.

Clearly, at any date, knowledge of two points on a *future* term structure must be useful, in that forward yields can now be modified so as to reflect this information and reduce the "unexplained" variability. The question that needs to be addressed is whether our model uses this information to update forecasts in such a way that it explains more of this variability than some other simple specifications which use the same information.

On average the HJM-RS model is explaining about 90% of the forecast errors. While this number might seem large, it is possible that there are superior ways of using the "extra" information provided by these two points, to revise predictions of future spot yields than proposed by equation (11). Of course, in order to perform statistical hypothesis tests, very specific alternatives need to be developed. We shall return to this issue later.

The actual point estimates of the parameters κ range from 0.09 to 0.28 while those for η range from 0.04 to 0.20. In general, the standard errors of κ are quite large, while those of η are small. In all 13 years the point estimates of κ are positive. A non- parametric test of the hypothesis that $\kappa \leq 0$ against the alternative $\kappa > 0$ is strongly rejected at the 0.05% level of significance. This finding is consistent with the results of Flesaker (1992). Our results, however, are not consistent with those of Amin and Morton (1994). As noted above they obtained negative estimates of κ in tests of their GV structure. Flesaker (1992), and Heath, Jarrow, Morton and Spindel (1993) also remark that the volatility structure of forward rates may not be monotone in the maturity of the rates. In the next section we shall return to study this issue more carefully.

Under the normalized volatility restriction, *any* two points on the yield curve can act as the state variables. Moreover, regardless of the two state-variable proxies used, the parameters are the same, and hence the estimates should all be "close" to one another. Further, there should be no systematic relationship between the estimates and the maturities of the state variables. Table 2 provides the point estimates of κ for each year, under a variety of choices of the two state variables. The results show that the point estimates of κ depend on the choice of the state variables. In particular, given τ_1, as τ_2 increases, the point estimates of κ decrease. This implies that the model is not able to capture all elements of volatility shocks.[8]

Notice that under an exponentially dampened structure, with $\kappa > 0$, very long term forward rates will remain somewhat unchanged. The parameter κ controls the speed of adjustment to an informational event across the yield curve. If the volatility of yields at distant maturities is high, then the model will produce point estimates of κ that are lower then would be the case if the volatility were low. If shocks to the yield curve have permanent effects, (i.e. the shocks are parallel) then higher than anticipated volatilities may be encountered on distant maturities, and as τ_2 increases, one would expect the estimates produced by the model to decline. In this regard, Table 2 provides some evidence that the RS constraint may not be fully satisfied and that other specifications could improve upon

[8] An alternative source of the dependence of κ estimate on τ_1 and τ_2 may lie in the assumption that the $\epsilon[t, t+T]$ in equation (9) are homoskedastic and independent.

this postulated structure. For example, a two factor model, with an exponentially dampened structure for the first factor, and a constant structure associated with the second structure might have some advantages. A model with these features, would, in the HJM paradigm, have 3 state-variables.

In summary, the lack of systematic patterns in the residuals and the fact that the signs of the estimates are consistently positive indicates that the model may have some predictive attributes. On the other hand, the fact that the estimates of κ decrease as τ_2 increases, suggests that the model could be missing a factor that captures parallel shifts. While more complex structures may provide added realism, the two state-variable model still may have predictive capability beyond that of well specified alternative models, such as the GV model.

4.3 Hypothesis Tests

There are two reasons for using HJM-GV as a benchmark against which to compare the HJM-RS model. First, the HJM-GV model is the only single state-variable model in the HJM class and has been used in a number of theoretical and empirical studies. Second, Amin and Morton (1994) have shown that this simple structure has predictive power, and that trading schemes based on estimates of the parameters of this structure can generate abnormal risk adjusted returns. Our goal, therefore, is to establish whether the HJM-RS structure has significantly greater predictive capability, beyond that of the GV model. In particular, we are keen to investigate whether the residuals generated by the model (11) are smaller than those generated by the GV model, for all maturities across the spectrum of the term structure.

Towards this goal, for each year we estimate the three parameters of the GV model and the two parameters of the two state-variable model. Then for each month we record the absolute value of the residuals for all the remaining maturities. For each maturity we count the number of times (out of 12) that the RS model had a smaller deviation than the GV model. Table 3 shows these numbers for each maturity and for each year, when $\tau_1 = 0.5$, and $\tau_2 = 5$, for the RS model and $\tau = \tau_2$ for the GV model.

The "winner" for the year was then identified. This experiment was then repeated for all 13 years, and a non-parametric test performed to establish if the RS model out-performed

the GV model. The result is an overwhelming rejection of the hypothesis that the two models are not different. In particular, in each year the two state-variable model out-performed the GV model.

Table 4 reports summary statistics when alternative pairs of maturities are used as state-variables.

In summary, we can conclude that models developed under the normalized forward rate volatility restriction, provide a significant improvement over the GV model. Since the difference between the two models arises solely from the fact that the spot rate volatilities in the GV models are constant, we can conclude that alternative spot rate volatility specifications can lead to further improvements.

4.4 The Volatility Hump

Several studies on the volatility structure of forward rates claim or infer that the volatility structure of forward rates is humped, increasing over short horizons before decaying. If this structure is correct, then the exponentially dampened normalized volatility structure, with no time-varying parameters, equation (3), will be misspecified. We now investigate whether there is a "hump" in the normalized volatility structure. We accomplish this without making assumptions on the spot rate volatility structure.

First, parallelling Amin and Morton's (1994) study, we re-estimate the HJM-RS model using only short maturities. In the presence of a humped term structure of volatilities, an exponentially declining model such as HJM-RS or HJM-GV might come up with positive point estimates of κ when fitted across the *entire* maturity spectrum, but should produce negative estimates when fitted to short maturities only. Our short-horizon maturity targets were $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$ and 3 years. From this set, $\frac{1}{2}$ and 3 years were selected as tau_1 and τ_2. Table 5 reports the results. For all 13 years, the estimates of κ remain positive. This suggests that the normalized volatility structure is a decreasing function of maturity, even near the short end of the curve.

Returning to our original target-maturity set, as a final test to establish whether the estimates of κ are maturity dependent, we estimate κ for each maturity separately, again using $\tau_1 = 0.5$ and $\tau_2 = 5$. If there is no maturity dependence, then plots of these estimates

against maturities should not reveal distinct patterns. Figure 3a presents the cross-maturity κ estimates for each of the 13 years of data. Although κ estimates vary across maturities within each year, there is no obvious pattern to the point estimates. To emphasize this point we plot, in Figure 3a, the κ estimates for each maturity "normalized" by subtracting out the estimate for the shortest maturity.

The only pattern evident here is that the short-maturity κ does not seem to fit in with the others, appearing to be unusually low relative to the others part of the time, and too high at other times. As for the remaining maturities, the only pattern is one of increasing dispersion with maturity.

In neither Figure 3a nor 3b is there any evidence of a "humped" pattern in the κ estimates. This confirms the evidence in Table 5 and is consistent with the success of the HJM-RS model in fitting the forecast errors. Our results are thus strongly supportive of an exponentially declining term structure of normalized forward rate volatilities consistent with the HJM-RS class of models. This conclusion contrasts with the results of Amin and Morton (1994) and Heath, Jarrow, Morton and Spindel (1993)

5 Summary and Conclusions

One of the main drawbacks of the HJM paradigm for pricing interest rate claims concerns the fact that for an arbitrary forward rate volatility structure, a finite collection of state variables, which are sufficient to describe the price process of bonds, may not exist. Implementations of the general HJM model requires keeping track of the entire history of the forward rate process since the initialization date. For short term contracts, Heath, Jarrow, and Morton (1990) discuss feasible lattice based procedures. However, for longer term contracts, the path-dependence causes complications. Lattice models, for example, do not recombine, and actually grow exponentially in the number of periods. However, if the structure for forward rate volatilities belongs to the HJM-RS class, given in equation (2), then the computational problems are simplified. In this case, two state-variables can be identified and the path-dependence is avoided, resulting in computationally efficient HJM models for pricing a wide variety of European and American type interest rate contingent claims.

This article explores whether there is any empirical support for the HJM-RS volatility constraint. The constraint makes no particular assumptions about spot rate volatilities, but rather about forward rate volatilities relative to spot rate volatilities. Our empirical tests allows us to avoid having to postulate a particular spot rate volatility and thus avoid a joint-hypothesis dilema.

Based on the analysis it appears that the HJM-RS model, based on the normalized volatility structure of forward rates being exponentially dampened in maturity, outperforms the GV model. In particular, there is support for a decaying volatility structure, with no evidence of a "hump" existing in the volatility structure.

Given the two state variables, corresponding to forecast errors at maturities τ_1 and τ_2, the estimates of individual κ's vary with maturity, but without pattern. The ability of a cross-maturity fixed-κ to fit the data suggests that a constant κ produces an adequate representation of the data.

The fact that estimates of κ depend on the selection of the state variables implies that not all information is being captured by the exponential structure.

One possible explanation for the results in Table 2 is that a two factor model might be more appropriate. It remains for future research to test such models. While alternative models could do equally well or better than the single-factor/ two state-variable model, with unexplained forecast-error variance already reduced to 10%, it is unlikely that such an approach can produce economically beneficial advantages.

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Figure 1a: Box Plots of Raw Data

$\Delta y_t[t,t+T] = y_t[t,t+T] - y_0[t,t+T]$

(5, 10, 25, 50, 75, 90, and 95th percentiles; dashed line represents the mean)

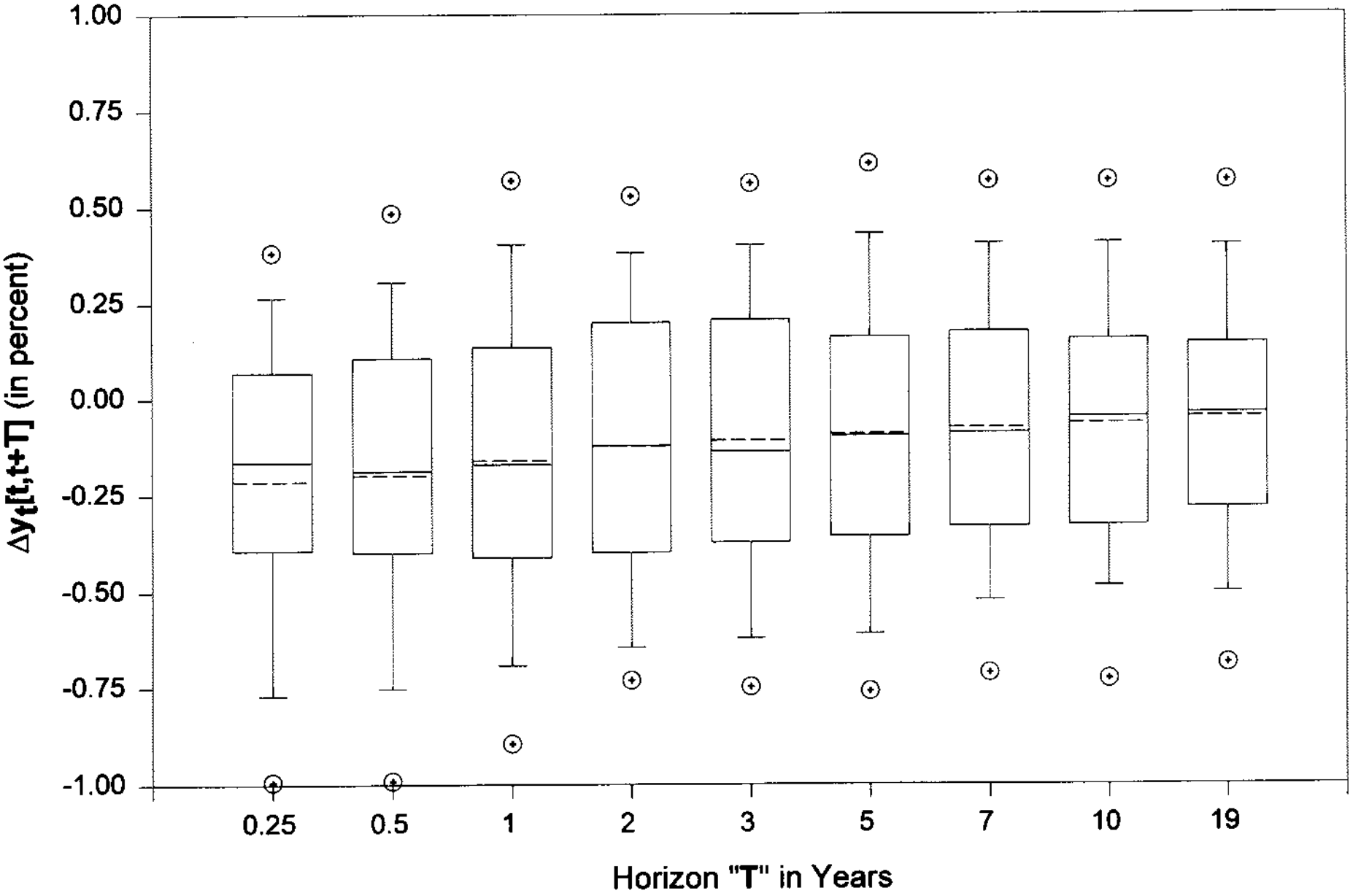

Figure 1b: Box Plots of Data Relative to $\Delta y_t[t,t+0.25]$

$\Delta y_t[t,t+T] = y_t[t,t+T] - y_0[t,t+T]$

(5, 10, 25, 50, 75, 90, and 95th percentiles; dashed line represents the mean)

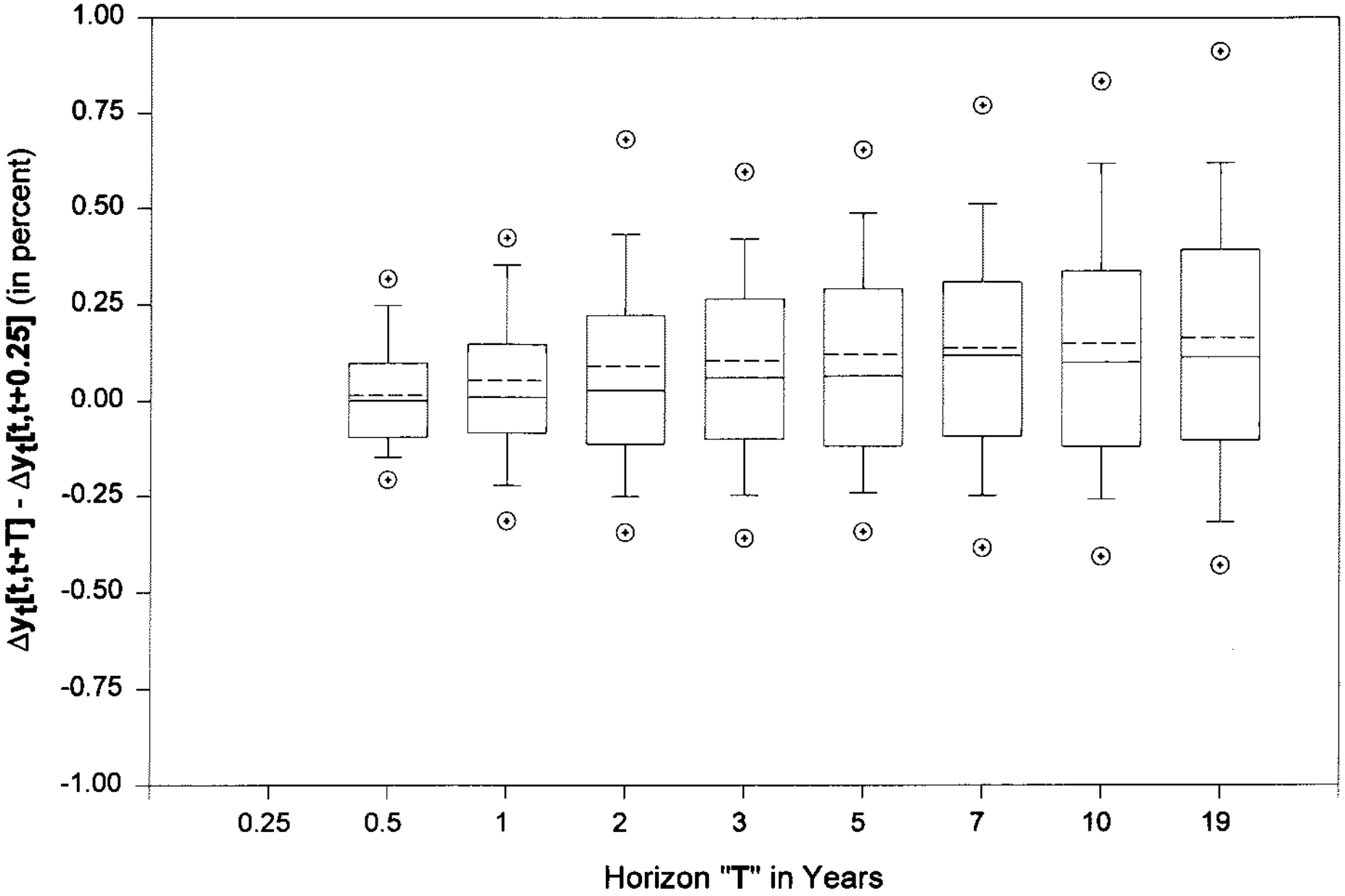

Figure 2a: Box Plots of Raw Errors

$$\varepsilon[t,t+T] = \Delta y_t[t,t+T] - H_1(T;\tau_1, \tau_2)\, \Delta y_t[t,t+\tau_1] - H_2(T;\tau_1, \tau_2)\, \Delta y_t[t,t+\tau_2]$$

(5, 10, 25, 50, 75, 90, and 95th percentiles; dashed line represents the mean)

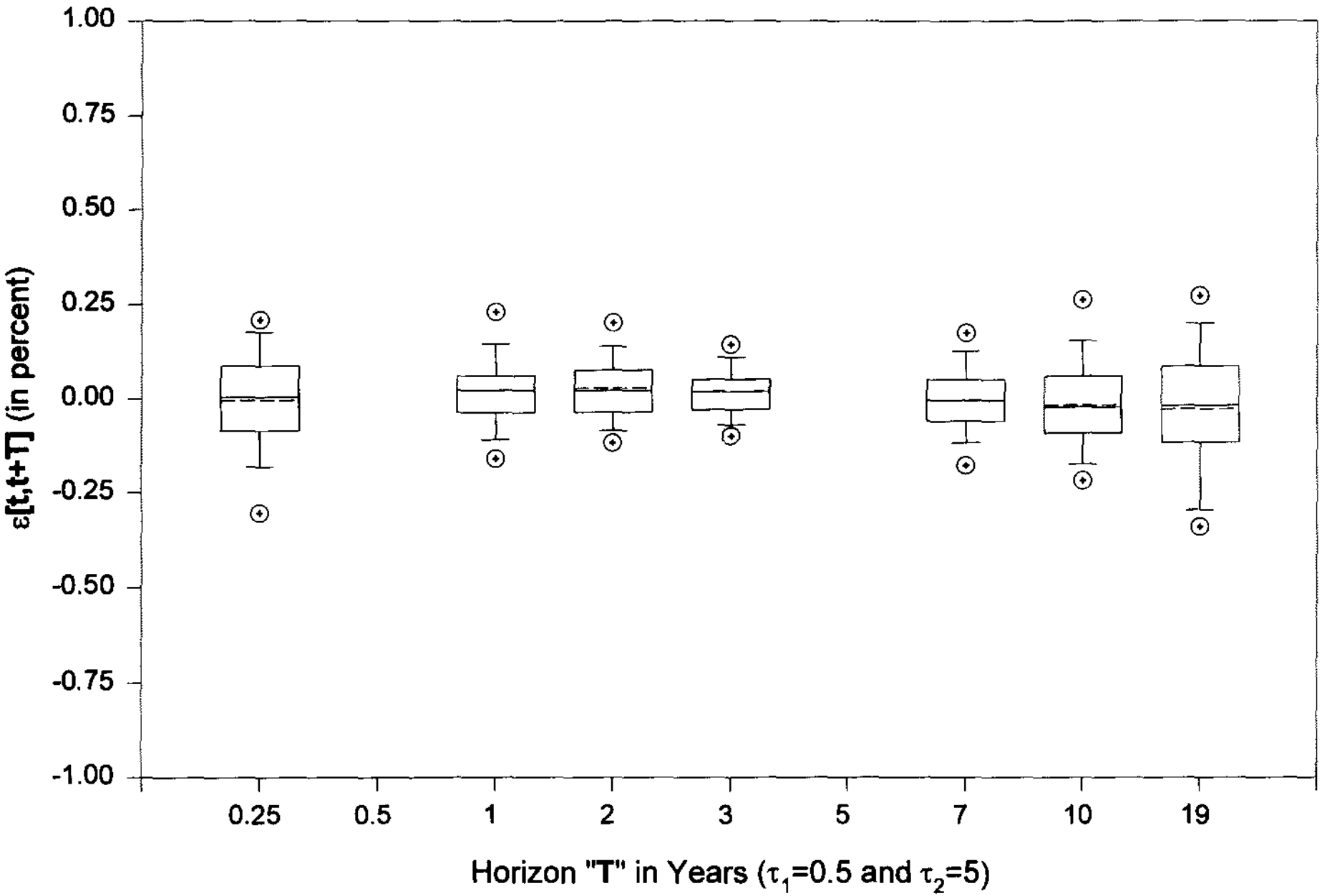

Figure 2b: Box Plots of "Normalized" Errors

$\varepsilon[t,t+T] = \Delta y_t[t,t+T] - H_1(T;\tau_1, \tau_2)\, \Delta y_t[t,t+\tau_1] - H_2(T;\tau_1, \tau_2)\, \Delta y_t[t,t+\tau_2]$

(5, 10, 25, 50, 75, 90, and 95th percentiles; dashed line represents the mean)

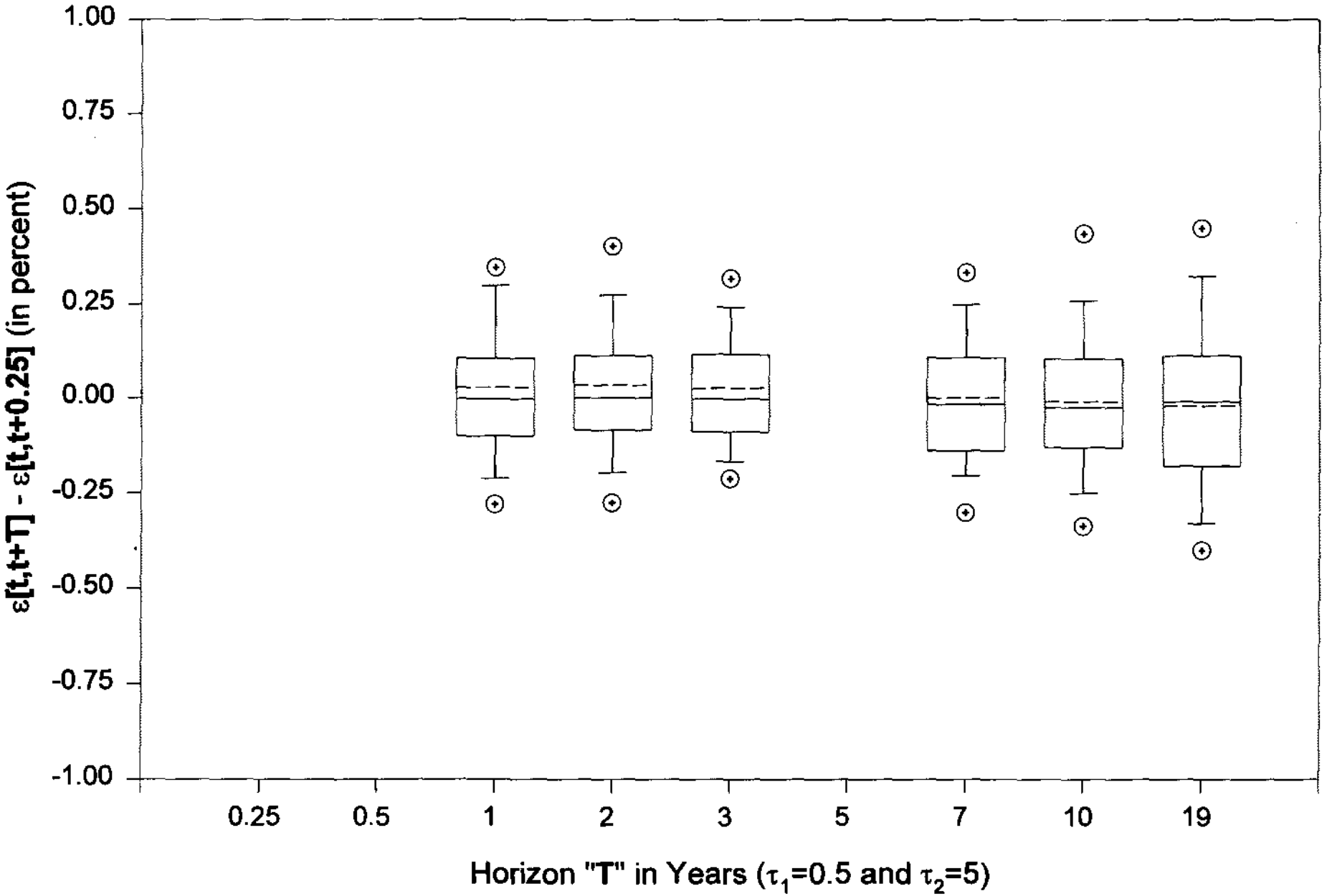

Figure 3a: Plot of Annually Computed K's
$\tau_1 = 0.5, \tau_2 = 5$

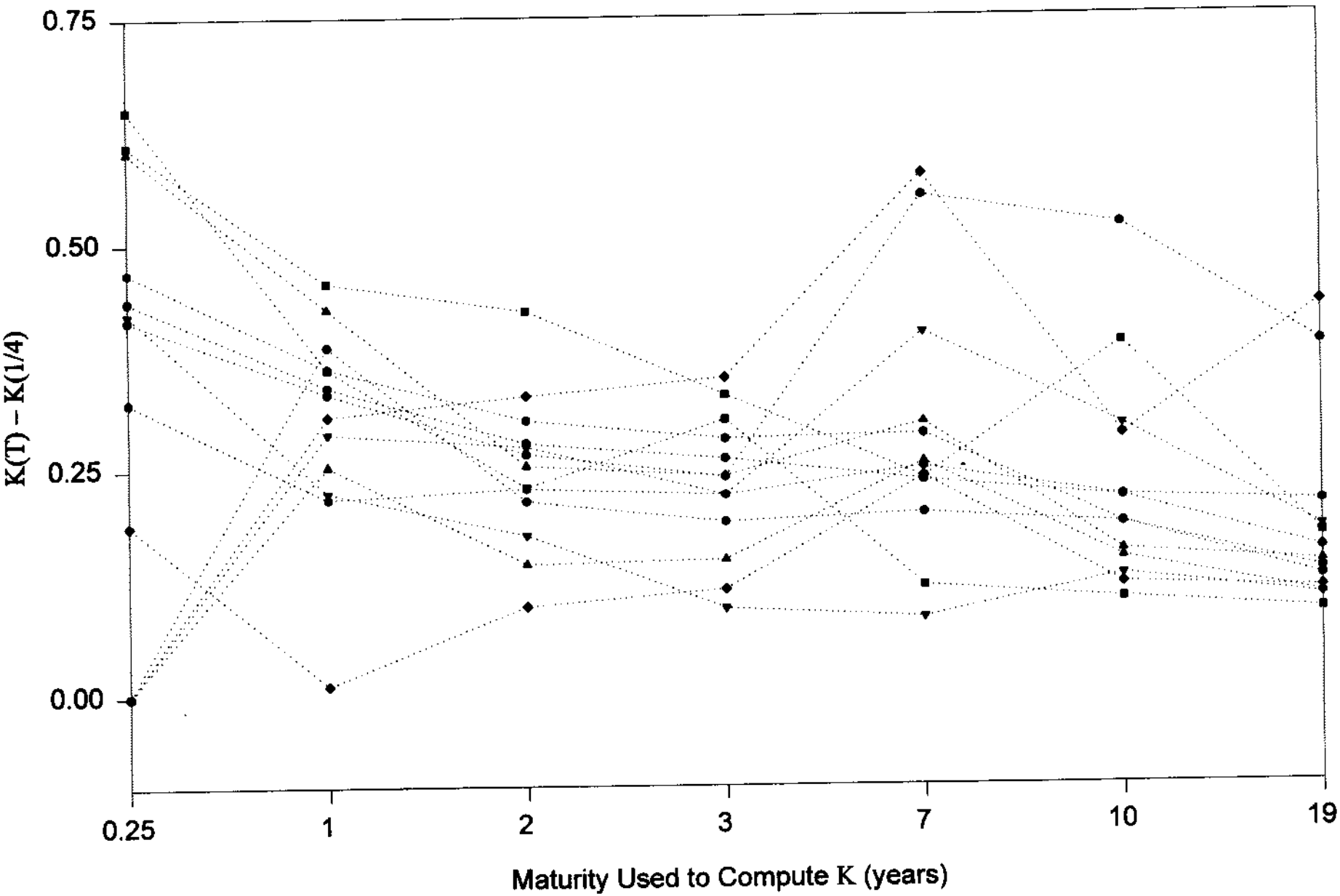

Figure 3b: Plot of Annually Computed K's Relative to K(1/4)
$\tau_1 = 0.5, \tau_2 = 5$

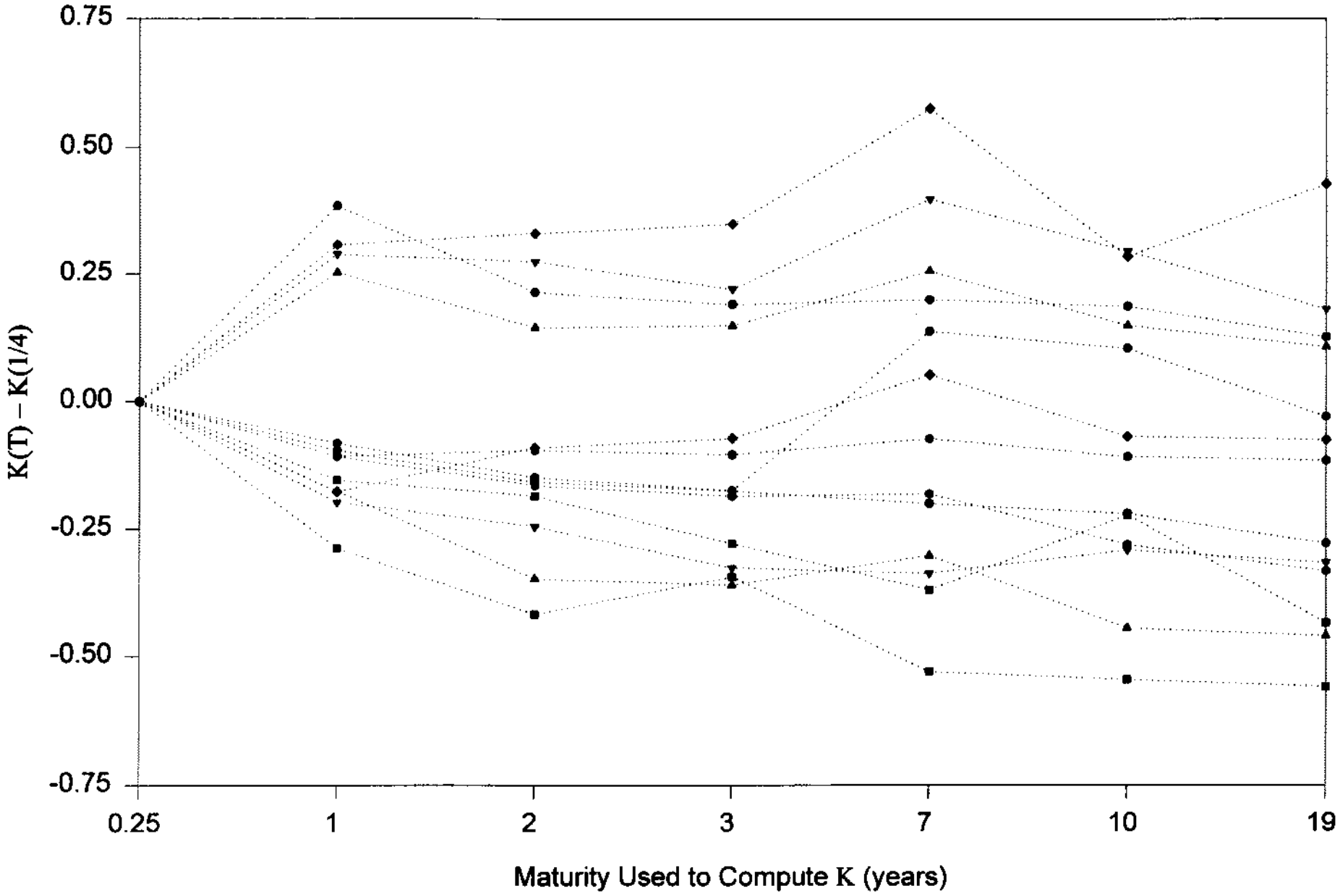

Table 1 — HJM-RS Model Results for $\tau_1 = 0.5$ and $\tau_2 = 5$

Year	Δy TSS	RSS	R^2	κ	S.E.	η	S.E.
1982	62.894	9.525	0.849	0.276	0.079	0.207	0.016
1983	15.172	2.420	0.841	0.081	0.013	0.072	0.006
1984	24.985	1.892	0.924	0.178	0.022	0.084	0.007
1985	21.247	1.213	0.943	0.104	0.014	0.067	0.005
1986	18.229	1.957	0.893	0.094	0.011	0.069	0.005
1987	13.433	1.207	0.910	0.170	0.026	0.071	0.005
1988	9.578	0.775	0.919	0.091	0.017	0.043	0.003
1989	11.694	0.628	0.946	0.349	0.039	0.053	0.004
1990	7.143	0.754	0.894	0.098	0.013	0.049	0.004
1991	8.313	0.652	0.922	0.164	0.034	0.048	0.004
1992	8.299	0.913	0.890	0.287	0.036	0.058	0.004
1993	4.556	0.553	0.879	0.207	0.022	0.046	0.004
1994	6.524	1.221	0.813	0.210	0.038	0.067	0.005

Overall R^2: 89.4%

Table 2 — $\hat{\kappa}$ and Overall Performance of HJM-RS Model for Various τ_1 and τ_2

τ_1	0.25	0.25	0.25	0.25	0.25	0.25
τ_2	2	3	5	7	10	19
Year						
1982	0.537	0.427	0.300	0.222	0.154	0.081
1983	0.088	0.085	0.082	0.078	0.074	0.063
1984	0.196	0.190	0.180	0.172	0.164	0.148
1985	0.111	0.109	0.105	0.101	0.096	0.083
1986	0.098	0.097	0.094	0.092	0.089	0.081
1987	0.195	0.186	0.172	0.161	0.148	0.124
1988	0.102	0.098	0.092	0.087	0.079	0.057
1989	0.388	0.375	0.355	0.339	0.324	0.305
1990	0.105	0.103	0.099	0.096	0.091	0.081
1991	0.209	0.193	0.168	0.149	0.128	0.088
1992	0.325	0.313	0.292	0.276	0.258	0.227
1993	0.223	0.218	0.209	0.202	0.194	0.180
1994	0.265	0.244	0.215	0.195	0.173	0.133
Overall R^2:	70.7%	82.4%	86.7%	85.4%	83.9%	75.8%

τ_1	0.5	0.5	0.5	0.5	0.5	0.5
τ_2	2	3	5	7	10	19
Year						
1982	0.485	0.390	0.276	0.206	0.146	0.079
1983	0.087	0.085	0.081	0.078	0.073	0.062
1984	0.194	0.188	0.178	0.171	0.162	0.146
1985	0.111	0.108	0.104	0.100	0.095	0.083
1986	0.098	0.097	0.094	0.092	0.089	0.081
1987	0.192	0.184	0.170	0.159	0.147	0.122
1988	0.101	0.097	0.091	0.086	0.078	0.056
1989	0.383	0.370	0.349	0.333	0.317	0.296
1990	0.105	0.102	0.098	0.095	0.091	0.080
1991	0.204	0.188	0.164	0.146	0.125	0.086
1992	0.321	0.308	0.287	0.271	0.252	0.218
1993	0.221	0.216	0.207	0.200	0.192	0.179
1994	0.258	0.238	0.210	0.190	0.169	0.130
Overall R^2:	75.1%	85.5%	89.4%	89.2%	88.8%	83.6%

Table 3 — Comparison of HJM-RS and HJM-GV Model Results for $\tau_1 = 0.5$ and $\tau_2 = 5$

	$Delta_y$	HJM-RS Model		HJM-GV Model		RSS	Wins
Year	TSS	RSS	R^2	RSS	R^2	Ratio	Ratio
1982	62.89	9.52	0.85	16.34	0.74	1.72	1.07
1983	15.17	2.42	0.84	3.47	0.77	1.43	1.02
1984	24.98	1.89	0.92	6.66	0.73	3.52	1.19
1985	21.25	1.21	0.94	2.95	0.86	2.43	1.26
1986	18.23	1.96	0.89	4.24	0.77	2.16	1.31
1987	13.43	1.21	0.91	3.12	0.77	2.58	1.21
1988	9.58	0.77	0.92	1.11	0.88	1.44	0.81
1989	11.69	0.63	0.95	1.23	0.89	1.96	1.14
1990	7.14	0.75	0.89	1.95	0.73	2.59	1.14
1991	8.31	0.65	0.92	1.25	0.85	1.91	1.02
1992	8.30	0.91	0.89	1.85	0.78	2.03	1.24
1993	4.56	0.55	0.88	1.88	0.59	3.40	1.33
1994	6.52	1.22	0.81	2.12	0.68	1.73	1.31
Overall R^2:			89.4%		77.2%		

Table 4 — Comparison of HJM-RS and HJM-GV Model Results for Various τ_1 and τ_2

Number of years that HJM-RS Model beats HJM-GV					Ratio of Annual	
Criterion	RSS Ratio		Individual $\lvert\epsilon[t, t+T]\rvert$		Ave(RSS/TSS)	
τ_1	0.25	0.50	0.25	0.50	0.25	0.50
τ_2						
2	3	7	2	5	0.61	0.91
3	7	11	7	8	0.95	1.47
5	11	13	10	12	1.38	2.15
7	11	13	12	13	1.60	2.50
10	11	13	12	13	1.91	3.13
19	11	12	11	13	2.10	3.18

Table 5 — Short-Maturity HJM-RS Model Results for $\tau_1 = 0.5$ and $\tau_2 = 3$

Year	Δy TSS	RSS	R^2	κ	S.E.	η	S.E.
1982	73.841	5.501	0.925	0.718	0.129	0.116	0.010
1983	12.668	0.994	0.922	0.426	0.118	0.053	0.004
1984	25.975	1.111	0.957	0.728	0.070	0.053	0.004
1985	18.144	0.609	0.966	0.446	0.090	0.049	0.004
1986	12.061	0.493	0.959	0.035	0.106	0.040	0.003
1987	12.337	0.549	0.956	0.513	0.069	0.044	0.004
1988	8.781	0.273	0.969	0.349	0.080	0.029	0.002
1989	13.095	0.246	0.981	0.593	0.049	0.031	0.003
1990	5.942	0.452	0.924	0.445	0.096	0.038	0.003
1991	9.348	0.311	0.967	0.325	0.108	0.033	0.003
1992	8.717	0.301	0.965	0.238	0.074	0.034	0.003
1993	3.202	0.189	0.941	0.388	0.056	0.025	0.002
1994	6.022	0.847	0.859	0.628	0.099	0.049	0.004

Overall R^2: 94.6%