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**Estimation of Risk-Neutral and Statistical Densities  
By Hermite Polynomial Approximation:  
With an Application to Eurodollar Futures Options**

Peter A. Abken, Dilip B. Madan,  
and Sailesh Ramamurtie

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**Abstract:** This paper expands and tests the approach of Madan and Milne (1994) for pricing contingent claims as elements of a separable Hilbert space. We specialize the Hilbert space basis to the family of Hermite polynomials and use the model to price options on Eurodollar futures. Restrictions on the prices of Hermite polynomial risk for contingent claims with different times to maturity are derived. These restrictions are rejected by our empirical tests of a four-parameter model. The unrestricted results indicate skewness and excess kurtosis in the implied risk-neutral density. These characteristics of the density are also mirrored in the statistical density estimated from a time series on LIBOR. The out-of-sample performance of the four-parameter model is consistently better than that of a two-parameter version of the model.

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## **Estimation of Risk-Neutral and Statistical Densities by Hermite Polynomial Approximation: With an Application to Eurodollar Futures Options**

This paper expands and tests the approach of Madan and Milne (1994) for pricing contingent claims as elements of a separable Hilbert space. Madan and Milne point out that pricing in terms of a Hilbert space basis is analogous to the use of discount bonds as a basis for pricing fixed income securities or the construction of branches of a binomial tree in pricing options. The application of Madan and Milne's approach used here specializes the Hilbert space basis to the family of Hermite polynomials.

Using this approach we infer the underlying risk-neutral density from traded security prices. There has been considerable interest recently in pricing contingent claims based on implied distributions. Examples include Longstaff (1992, 1995), Shimko (1993), Dupire (1994), Rubinstein (1994), Derman and Kani (1994), Jackwerth and Rubinstein (1995), Ait-Sahalia and Lo (1995), and Dumas, Fleming, and Whaley (1996). Empirical work has been confined to the S&P 100 and S&P 500 index options. Aside from considering a different market—short-term interest rate options—what sets our work apart from these studies is that the Hilbert space basis approach is amenable to standard econometric tests. We also perform extensive out-of-sample tests of the pricing of our model that corroborate the parametric statistical tests of different specifications of our model. Finally, we estimate the actual or statistical probability density and compare its characteristics with those of the risk-neutral density.

This paper presents a new result that extends Madan and Milne (1994). Their original model could only be applied to one option maturity class at a time. This current work derives a restriction on the coefficients weighting the Hermite polynomial basis elements that allows us to use all traded option maturities jointly in estimation. This is analogous to Derman and Kani's construction of an implied binomial tree using all available options.

We apply the Hermite-polynomial-basis approach to the pricing of call and put Eurodollar futures options traded at the Chicago Mercantile Exchange (CME). The Black-Scholes model is a parametric special case of the Madan and Milne model. As in the Black-Scholes model, interest rates are assumed to be constant. Nevertheless, the Black (1976) model is widely used to price Eurodollar futures options, and, in fact, the Pit Committee at the CME uses this model (together with Barone-Adesi/Whaley 1987 early-exercise approximation) to price illiquid options in their determination of settlement prices. Flesaker (1993) and Grinblatt and Jegadeesh (1994) have shown that the difference between forward and futures contracts in the Eurodollar market is negligible for maturities under one year. The effect of a stochastic discount factor on valuation does not appear to be critical for short-dated options; therefore, we restrict our investigation to options maturities of less than nine months.

Another concern is that Eurodollar futures options are American-style options, but our model does not value the early-exercise feature of these options. Ramaswamy and Sundaresan (1985) have shown that the value of the early-exercise feature for futures options is very small. Nevertheless, we check the sensitivity of our results to this omission by excluding in-the-money options. Our ignoring early exercise does not appear to bias our results.

For the 1990–1994 sample period as well as subperiods, we reject the Black model in favor of a four-parameter Hermite polynomial model. This is not surprising since we have three extra degrees of freedom to fit in sample. However, the jointly estimated four-parameter model is rejected when tested against an unrestricted four-parameter model estimated separately by option maturity. The apparent cause of the rejection is the upward slope of the term structure of forward volatility (also reported by Amin and Morton 1994). The model assumes a flat volatility structure. The unrestricted results indicate skewness and excess kurtosis in the implied risk-neutral density. These characteristics of the distribution are also mirrored in the statistical density estimated from a time series on LIBOR.

Out-of-sample tests using the four-parameter model versus a two-parameter model that does not allow for skewness or excess kurtosis were conducted. Although the gain was small from using the four-parameter model, the consistency of better performance was high: 95 percent of the out-of-sample days showed smaller mean absolute and root mean square errors. This result holds equally well when in-the-money options are filtered out of the sample.

The paper is organized as follows: Section 1 reviews the framework of Madan and Milne (1994) and gives an overview of the density measures to be estimated. Section 2 gives an exposition of option pricing using a Hermite polynomial basis and provides an explicit representation of a four-parameter risk-neutral density. Section 3 introduces the corresponding statistical density. Section 4 presents the application of the model to Eurodollar futures options. The estimation approach is reviewed and estimation results for both the risk-neutral and statistical densities are summarized and interpreted. The fit of the model is evaluated both in- and out-of-sample. Section 5 offers concluding remarks.

## 1. A Review of the Hermite Polynomial Basis for Pricing

In this section we describe the model setup and assumptions underlying the Hermite polynomial approximation approach. Madan and Milne (1994) develop a model for valuation of contingent claims and static hedging strategies by identifying a set of “basis” claims and pricing them. The model has the following assumptions. There is an underlying probability space  $(\Omega, \mathcal{F}, P)$  for time  $t \in [0, T]$  with a complete, increasing, and right-continuous filtration  $\{\mathcal{F}_t | 0 \leq t \leq T\}$ , generated by a  $d$ -dimensional Brownian motion  $z(t)$  initialized at zero. There exists a finite set of  $d$  primary securities that can be traded continuously. There also exists a risk-free or money market account that grows in value at an instantaneous rate given by a positive process,  $r(t)$ . The theoretical economy includes a finite set of assets that can be traded continuously. The model shifts to a discrete-time setting for pricing and hedging a wide class of

contingent claims, which are expressed as functions of the primary assets at specific discrete points in time. This is the first step in the discretization of the model that makes static hedging strategies feasible.

The functional dependence of the contingent claims on the primary assets is assumed to be very general and covers a very wide variety of real-life contingent claims. The next major assumption of the model is that the set of all contingent claims is rich enough to form a Hilbert space that is separable and for which an orthonormal basis exists as a consequence. The markets are assumed to be complete and free of arbitrage opportunities. This has the same flavor as valuation in continuous time, in which contingent claims are redundant. However, in discrete time, the set of contingent claims forming the basis for the Hilbert space has a static usefulness akin to that of the set of Arrow-Debreu securities. The set of contingent claims forms a minimally statically spanning collection of claims, which can be used to value and statically hedge any other contingent claim. Consequently, dynamic portfolio rebalancing is unnecessary.

The price one pays for this setup lies in the dimensionality of the orthonormal basis set, which typically can be very large and in general does not give any operational advantage in describing or pricing the basis. However, one can construct a finite basis and use it for pricing and hedging strategies. Once the basis representation of claims and the claim representation of basis elements are obtained, valuation and hedging—static or dynamic—of any other claim are feasible.

The critical assumption for obtaining an operational basis is that only a finite number of basis elements need to be “financed” for the hedging of contingent claims. This financing is accomplished by investing in a suitable set of assets. Use of only a finite number of basis elements introduces a “basis” risk—in the futures market sense of the term—to using this approach. It is important to note that this approximate hedge is technically a Hilbert space approximation in that the difference between the actual claim cash flow and the hedge cash flow is a random variable with an arbitrarily small mean and variance. In other words, there may be instances, albeit with

appropriately small probabilities, when the hedge cash flow differs greatly from the required cash flow. The magnitude of the basis risk depends on the validity of the underlying probability model assumed (i.e., model error) and also on the finite Hilbert space approximation employed (i.e., approximation error).

For practical applications of the approach, the model is restricted to a finite set of embedded discrete-time equivalents of the underlying continuous-time stochastic processes driving the money market and primary assets. Thus, contingent claims are functions of finitely many variables, and the Hilbert space is consequently separable with a finite basis.

The next major step in the application of the model lies in the change of measures. We defined above the existence of the probability space  $(\Omega, F, P)$ . However, an effective basis cannot be constructed without a complete knowledge of  $P$ , which is typically unknown to the practitioner. Madan and Milne rely on an approach developed by Elliot (1993), in which there is a change of measure from  $P$  to a reference measure  $R$ . The latter is assumed to be Gaussian in a discrete context. This change of measure introduces errors or deviations that are assumed to be sufficiently well bounded.

Finally, while we assume the structure of the reference measure  $R$ , what we have in practice, through discrete-time observation of the evolution of the prices (or returns) of both the primary assets and the contingent claims, is a statistical discrete-time model  $(\tilde{\Omega}, \tilde{F}, \tilde{R})$  that is assumed to be a sufficiently close approximation of  $(\Omega, F, R)$ .

With all of these assumptions and approximations in hand, we proceed with the exercise of identifying the “basis” set and valuing the various contingent claims. In the application of the model, we assume that there is only one primary asset and that the Hilbert space may be viewed as the space of functions defined on  $R^M$ , where  $M$  represents the number of sample observations

at discrete times over the interval  $[0, T]$ . The probability measure  $R$  is assumed to be Gaussian with a Hermite polynomial basis (Roazanov 1982).

The statistical density  $P(z)$  or risk-neutral density  $Q(z)$  can be represented as a product of a change of measure density and a reference measure density:

$$\begin{aligned} P(z) &= \nu(z)n(z), \\ Q(z) &= \lambda(z)n(z), \end{aligned} \tag{1}$$

where  $\nu(z)$  and  $\lambda(z)$  are the statistical and risk-neutral change of measure densities, respectively. The reference measure density,  $R(z) = n(z) \equiv (1/\sqrt{2\pi})e^{-z^2/2}$ , is the standard Gaussian density, where  $z$  is a standardized normal random variate. The focus of the analysis of this paper is on the specification and estimation of the change of measure densities. The change of measure densities must be bounded above and below by constants for the Hilbert space to be identical with respect to both measures. The analysis applies in any case to claims that are reference-measure square integrable. Given this assumption and that of Gaussian random process generating uncertainty, a basis for the Gaussian reference space may be constructed using Hermite polynomials. These polynomials are defined in terms of the normal density as

$$\Phi_k(z) = (-1)^k \frac{\partial^k n(z)}{\partial z^k} \frac{1}{n(z)}, \tag{2}$$

and normalized to unit variance

$$\phi_k(z) = \Phi_k(z) / \sqrt{k!}. \tag{3}$$

The Hermite polynomials form an orthonormal system.

Although Madan and Milne show how to price multiple assets in terms of the Hermite polynomial basis, we work exclusively with the simpler case of a single asset (also exposted in

Madan and Milne) and a one-dimensional Gaussian reference measure. We suppose that under the reference measure the asset price evolves as geometric Brownian motion:

$$S_t = S_0 e^{\mu t + \sigma \sqrt{t} z_t - \sigma^2 t / 2} = S_0 e^{\eta t}, \quad (4)$$

with drift rate  $\mu$ , variance rate  $\sigma$ , and  $z_t \sim N(0,1)$ . All parameters are assumed to be constant.<sup>2</sup> The risk-free rate is also assumed to be constant. The exponent in equation (4) is the continuously compounded return process

$$\eta \sim N\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{t}\right). \quad (5)$$

This process will be applied below for the case of LIBOR, where LIBOR is substituted directly for the asset price in (4) and thus LIBOR's dynamics will be assumed to follow geometric Brownian motion. Of course, the interpretation of  $\eta$  as a return process does not apply for an interest rate.

The parameters of the risk-neutral change of density measure  $\lambda(z)$  are estimated from Eurodollar futures options prices. The parameters of the statistical change of density measure  $\nu(z)$  are estimated from the underlying interest-rate process using a time-series on LIBOR.

## 2. Option Pricing Using a Hermite Polynomial Basis

Madan and Milne show that any contingent claim payoff  $g(z)$  can be represented in terms of the Hermite polynomial basis as

$$g(z) = \sum_{k=0}^{\infty} a_k \phi_k(z), \quad (6)$$

where

$$a_k = \int_b^a g(z) \phi_k(z) n(z) dz.$$

The  $a_k$  coefficient is the covariance of the  $k$ -th Hermite polynomial risk with the contingent claim payoff, and may be interpreted as the number of units of the  $k$ -th basis-element contingent claim  $\phi_k(z)$  to hold in a portfolio that replicates the contingent claim payoff. The price of this contingent claim is expressed as

$$V[g(z)] = \sum_{k=0}^{\infty} a_k \pi_k, \quad (7)$$

where  $\pi_k$  is the implicit price of Hermite polynomial risk  $\phi_k(z)$ . The market price of risk  $\pi_k$  may be interpreted as the forward price, for delivery upon the contingent claim's maturity, of basis-element contingent claim  $\phi_k(z)$ .

Equation (7) is the primary focus for our empirical work, and its empirical counterpart is discussed below. Given the structure of the contingent claims and the assumed probability model, the Hermite polynomial coefficients  $a_k$  are well defined and hence the  $\pi_k$  can be inferred from the observed prices.

The payoff function  $g(z)$  is specialized to standard European call and put payoffs in the empirical work below. The call option payoff is denoted by

$$c(z, S_0, x, \mu, \sigma, t) = [S_0 e^{\mu + \sigma \sqrt{t} z - \sigma^2 t/2} - x]^+ \quad (8)$$

and the put option payoff by

$$p(z, S_0, x, \mu, \sigma, t) = [x - S_0 e^{\mu + \sigma \sqrt{t} z - \sigma^2 t/2}]^+. \quad (9)$$

The call option payoff can be expressed in terms of the Hermite polynomial basis using a call option generating function

$$\Phi(u, S_0, x, \mu, \sigma, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(z, S_0, x, \mu, \sigma, t) e^{-(z-u)^2/2} dz, \quad (10)$$

where  $u$  is a dummy variable in the generating function for Hermite polynomials:

$$\frac{e^{-(z-u)^2/2}}{\sqrt{2\pi}} = \sum_{k=0}^{\infty} \phi_k(z) n(z) \frac{u^k}{\sqrt{k!}}. \quad (11)$$

The  $\alpha_k$  coefficients in (6) are defined as follows for the call option payoffs:

$$\alpha_k(S_0, x, \mu, \sigma, t) = \left. \frac{\partial^k \Phi(u, S_0, x, \mu, \sigma, t)}{\partial u^k} \right|_{u=0} \frac{1}{\sqrt{k!}}. \quad (12)$$

Hence  $\Phi$  is a call option generating function.

Upon evaluating the integral in (10), the call option generating function has the following form, as reported in Madan and Milne (1994):

$$\Phi(u, S_0, x, \mu, \sigma, t) = S_0 e^{\mu + \sigma \sqrt{t} u} N(d_1(u)) - x N(d_2(u)), \quad (13)$$

where

$$d_1(u) = \frac{1}{\sigma \sqrt{t}} \ln \frac{S_0}{x} + \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) \sqrt{t} + u,$$

and

$$d_2(u) = d_1(u) - \sigma \sqrt{t}.$$

The corresponding put option generating function is

$$\Psi(u, S_0, x, \mu, \sigma, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(z, S_0, x, \mu, \sigma, t) e^{-(z-u)^2/2} dz. \quad (14)$$

Put options have Hermite polynomial coefficients

$$b_k(S_0, x, \mu, \sigma, t) = \frac{\partial^k \Psi(u, S_0, x, \mu, \sigma, t)}{\partial u^k} \Big|_{u=0} \frac{1}{\sqrt{k!}}.$$

The empirical counterpart of equation (7), the option pricing equation in terms of the Hermite polynomial basis, is estimated below. As shown in Madan and Milne, setting the drift  $\mu$  equal to the risk-free rate specializes the reference measure to the equivalent martingale measure under Black-Scholes, and the Hermite polynomial pricing model collapses to the Black-Scholes model under the parametric restriction yielding  $\lambda(z) \equiv 1$ .

### 2.1 The Risk-Neutral Density

The risk-neutral density is given by

$$Q(z) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} e^{\pi} \pi_k \phi_k(z) e^{-z^2/2} \quad (15)$$

with respect to the standard normal reference measure on  $z$ . In turn, the variate  $z$  may be expressed in terms of the return process  $\eta$  as

$$z = \frac{\eta - (\mu - \sigma^2/2)}{\sigma/\sqrt{t}}. \quad (16)$$

Under the reference measure,  $z$  is normally distributed with the standard moments:

$$z \sim N(0,1); \quad E z = 0, \quad E z^2 = 1, \quad E z^3 = 0, \quad E z^4 = 3, \quad \dots \quad (17)$$

However, the actual risk-neutral distribution of  $z$  may be non-normal; thus, the more general density specification in equation (15). For the empirical work below, the Hermite polynomial expansion of the equivalent martingale measure is truncated at the fourth order.

Using the definitions of the Hermite polynomials, the truncated density is shown explicitly to be:

$$Q(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \left[ \left( \beta_0 - \frac{\beta_2}{\sqrt{2}} + \frac{3\beta_4}{\sqrt{24}} \right) + \left( \beta_1 - \frac{3\beta_3}{\sqrt{6}} \right) z + \left( \frac{\beta_2}{\sqrt{2}} - \frac{6\beta_4}{\sqrt{24}} \right) z^2 + \frac{\beta_3}{\sqrt{6}} z^3 + \frac{\beta_4}{\sqrt{24}} z^4 \right], \quad (18)$$

where  $\beta_k = e^r \pi_k$ , the future value of the  $k$ -th price of risk coefficient. Appendix A derives the central moments of  $Q(z)$  in terms of the  $\beta_k$ . That derivation shows how the change of measure density, given by the bracketed term containing the  $\beta_k$  in (18), alters the central moments of  $z$  and  $\eta$  under the reference measure.

## 2.2 Restrictions on the Market Prices of Risk across Time

The market prices of Hermite polynomial risk are functionally related across time. The restriction has the following form:

$$\pi_k(s) = \pi_k(t) e^{r(t-s)} \frac{s^{k/2}}{t^{k/2}}, \quad (19)$$

where  $s < t$ . Appendix B shows the derivation of this restriction on the hypothesis that the measure change is path independent and volatility is constant. The practical implication of equation (19) is that all traded contingent claims can be used jointly in estimation of the  $\pi_k$ 's. Only the longest available maturity's prices of risk need to be estimated to infer all others.

## 3. The Statistical Density

The statistical density is sometimes referred to as the actual or the true probability density. It is estimated from the underlying price. The basic object for analysis is the asset return process,  $\eta$ , the continuously compounded return.

The fourth-order representation of the statistical density is

$$P(z) = \sum_{k=0}^4 \alpha_k \phi_k \left( \frac{\eta - \mu}{\sigma} \sqrt{t} + \frac{\sigma}{2} \sqrt{t} \right) \frac{1}{\sqrt{2\pi\sigma}} \sqrt{te}^{-\frac{t}{2} \left( \frac{\eta - \mu}{\sigma} + \frac{\sigma}{2} \right)^2} \quad (20)$$

This equation simplifies to

$$P(z) = \frac{1}{\sqrt{2\pi\sigma}} \sqrt{te}^{-\frac{t}{2\sigma^2} \left( \eta - \left( \mu - \frac{\sigma^2}{2} \right) \right)^2}, \quad (21)$$

when  $\alpha_0 = 1$  and  $\alpha_k = 0$ , for  $k = 1 \dots 4$ , i.e., if the reference measure is the actual probability measure.

#### 4. Application to Eurodollar Futures Options

This section estimates the prices of Eurodollar futures options by Hermite polynomial approximation and tests the restrictions on the model. The reference measure parameters and market prices of third- and fourth-order Hermite polynomial risk are estimated by a GMM procedure that uses a nonparametric kernel estimator for the covariance matrix. These estimates are used to construct a daily time series of the risk-neutral density. The out-of-sample performance of the four-parameter model is compared with that of the two-parameter model.

##### 4.1 Data and Institutional Background

The futures and options price data in this study consist of daily closing prices for all three-month Eurodollar futures and options contracts traded at the Chicago Mercantile Exchange. The sample period covered January 1990 through September 1994. LIBOR is the rate of interest paid on three-month time deposits in the London interbank market. The interest is paid in the form of an add-on yield, calculated on a 360-day calendar basis, for a \$1 million deposit, which is also the notional size of both the Eurodollar futures and futures options contracts.

Eurodollar futures contracts mature in a quarterly cycle, with contracts maturing two London business days before the third Wednesday in March, June, September, and December.<sup>4</sup> On any day, Eurodollar futures are traded for these months out to ten years in the future, with

substantial open interest for contract months running out about three years. The price of the Eurodollar futures contract is an index value constructed as 100 minus the add-on yield expressed as a percent. The minimum index movement, the tick size, is one basis point, or \$25.

Quarterly Eurodollar futures options expire simultaneously with their underlying futures contracts and are cash settled. If an option is exercised, the Eurodollar futures call writer (seller) becomes short one Eurodollar futures contract while the call purchaser receives one long Eurodollar futures contract. Eurodollar futures calls gain value as the index rises and LIBOR falls. At expiration, the CME automatically exercises options that are in the money, resulting in an immediate marking to market of the futures position. The tick size for the Eurodollar futures options is also one basis point, implying a minimum price change of \$25 for an option contract. Another important feature of Eurodollar futures options is that they are American-style options, which means that they can be exercised before their expiration date if early exercise is to the advantage of the option holder.

The data set of option prices was filtered in two ways. First, they were filtered by time to expiration. Options were excluded that had less than 30 days to expiration. At such a short time horizon, these options contain little information about implicit distributions. Options with more than one year to expiration were also excluded. In addition to the concern mentioned in the introduction that the assumption of a constant discount factor becomes untenable as an approximation beyond this horizon, the volume of such options diminishes as well. These longer-term options are less likely to represent equilibrium market prices. Second, the options were filtered by volume. The CME data set includes many options for which there was zero trading volume on a given day. The CME Pit Committee imputes an option price using the futures settlement price and the Black futures option model with Barone-Adesi/Whaley (1987) early-exercise correction.

Figure 1 shows the daily aggregate volume across all option maturity classes for puts in the upper panel and calls in the lower during the January 1990 to September 1994 sample. There are a total of 1188 trading days. Figure 2 shows the range of “moneyness” for the options in the sample. Moneyness is defined as the ratio of strike, expressed as an interest rate, to LIBOR. The mean moneyness is just slightly out of the money for both puts and calls.

The statistical density is estimated from a daily time series on LIBOR. The three-month LIBOR series used is the *Financial Times London Interbank Fixing*, which is a deposit rate for \$10 million, computed as the arithmetic average of rates at five banks quoted at 11:00 a.m. GMT. The banks are National Westminster Bank, Bank of Tokyo, Deutsche Bank, Banque National de Paris, and Morgan Guaranty Trust.

#### 4.2 Estimation

We estimate the drift and variance rate parameters  $\mu$  and  $\sigma$  in the reference measure as well as the third- and fourth-order Hermite polynomial risk prices  $\pi_3$  and  $\pi_4$  in the change of measure density.

The empirical counterpart of equation (7) is

$$\begin{aligned} C_i &= \sum_{k=0}^4 \pi_k a_k(\mu, \sigma) + u_i \\ P_j &= \sum_{k=0}^4 \pi_k b_k(\mu, \sigma) + u_j, \end{aligned} \tag{22}$$

where  $C_i$  is the price of the  $i$ -th call option and  $P_j$  is the price of the  $j$ -th put option in a daily panel of options. Appendix A shows that the lower-order  $\pi_k$ 's are restricted in estimation to be:  $\pi_0 = e^{-r}$ ,  $\pi_1 = 0$ , and  $\pi_2 = 0$ . Equation (22) is estimated restricted and unrestricted with regard to the treatment of time to maturity. The restricted version stacked all puts and calls across all maturities and imposed the restriction derived in Appendix B that relates  $\pi_k$  of different maturities. The unrestricted version estimated all four parameters separately by option expiration.

A standard nonlinear Wald test (see Judge et al. 1985, pp. 215-216) is used as a formal test of the time-to-maturity restrictions. The Black-Scholes restriction that  $\mu = r$ ,  $\pi_3 = 0$ , and  $\pi_4 = 0$  is also tested by a nonlinear Wald statistic for the restricted and each of the unrestricted runs by option expiration.

The error terms  $u_i$  and  $u_j$  are assumed to arise for two reasons: (1) the infinite dimension basis representation of the contingent claim price is truncated to a finite dimension and (2) the market prices of traded claims are assumed to be noisy (e.g., some settlement prices may be based on relatively “stale” prices) and not necessarily the true equilibrium prices.

Equation (22) is estimated using a GMM procedure in which a quadratic form,  $u'(\theta)W(\theta)u(\theta)$ , in the error pricing vector  $u$  is minimized with respect to a weighting matrix  $W(\theta)$ , where  $\theta$  is the parameter vector. The standard GMM covariance matrix is

$$\frac{1}{T}[D'(\theta)S^{-1}(\theta)D(\theta)]^{-1},$$

where  $D(\theta)$  is the Jacobian matrix  $\partial u / \partial \theta$  and  $S = E[u(\theta)u'(\theta)]$ .

The usual GMM covariance matrix cannot be used in our application because our time and panel data series on option prices have irregular dimensions. The number of calls and puts available each day was screened by volume and consequently varies over time. Our solution to this problem is to use a kernel estimator for the cross-moment matrix of the errors,  $u(\theta)u'(\theta)$ .<sup>5</sup> The kernel estimator allows us to interpolate values for the cross-moment matrix based on a rolling sample of errors in a window including the current observations and lagged observations on  $u$ . We used a 20-day window. The cross-moment estimator has the following form:

$$\zeta(x, x', t, t') = \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^{N_t} \sum_{j=1}^{N_{t'}} u_{it}(x_i, t_i) u_{jt}(x_j, t_j) w(x_i, x_j, t_i, t_j, x, x', t, t'), \quad (23)$$

where

$$w(x_i, x_j, t_i, t_j, x, x', t, t') = \frac{\exp\left[-\frac{1}{2}(x_i - x, x_j - x', t_i - t, t_j - t')V^{-1}(x_i - x, x_j - x', t_i - t, t_j - t')'\right]}{\sum_j \exp\left[-\frac{1}{2}(x_i - x, x_j - x', t_i - t, t_j - t')V^{-1}(x_i - x, x_j - x', t_i - t, t_j - t')'\right]} \quad (24)$$

is a Gaussian kernel with weighting matrix

$$V = \begin{bmatrix} h_x & & & 0 \\ & h_x & & \\ & & h_t & \\ 0 & & & h_t \end{bmatrix}$$

The  $u$ 's have two arguments: moneyness  $x$  and time to maturity  $t$  of the option to which they correspond. The cross-moment matrix element for an option with moneyness  $x'$  and time to maturity  $t'$  is a function of all current errors and errors in the lag window. (In order to preserve the underlying covariance structure of calls and puts, cross moments involving two calls, two puts, or call and put are kept distinct in applying the kernel.) These errors are weighted by matrix  $w$ , which is a Gaussian weighting matrix commonly used in kernel estimation (see Silverman 1986). The matrix  $V$  determines the degree of smoothing for the kernel, i.e., the "band-width." The number of days in the lag window is  $T$ , and  $N_t$  is the number of options included in the sample on date  $t$ .

In practice, the cross-moment matrix constructed using (23) failed to be positive definite. However, a detailed examination of the errors revealed that the cross correlations were unstable over time and appear to average out to zero. To achieve positive definiteness, a damping matrix was applied to the off-diagonal elements of the cross-moment matrix. This matrix consists of ones on its diagonal and a constant less than one for each off-diagonal element. An off-diagonal value of .3 was the threshold at which the matrix usually became positive definite. The resulting parameter standard errors rose monotonically as the damping value was driven to zero. A

diagonal cross-moment matrix turns out to be a conservative choice. (In fact, the resulting standard errors using the diagonal kernel are quite similar to, though generally slightly smaller than, the standard errors from using a homoscedastic diagonal covariance matrix using the contemporaneous  $u$  vector.) Since the off-diagonal covariances are assumed to be zero, the kernel was applied only to on-diagonal cross moments in the lag window.

As discussed below, the time-to-maturity restrictions of the model are rejected. Consequently, the diagonal covariance matrix is less tenable for the restricted runs. However, the key Wald tests below are evaluated using the covariance matrix estimated from the unrestricted model.

#### *4.3 Estimation Results*

##### Restricted Model

The output of the daily nonlinear Hermite polynomial regression runs for the restricted model is summarized in Figures 3 and 4. The daily time series for the point estimates of each the four parameters,  $\mu$ ,  $\sigma$ ,  $\pi_3$ , and  $\pi_4$ , are shown, with two standard error bounds around them.

The basic findings are: (1) the risk-neutral drift  $\mu$  is usually insignificantly different from zero; (2)  $\sigma$  is very tightly estimated and peaks around the time of the breakdown of the European Exchange Rate Mechanism; (3) the market price of skewness  $\pi_3$  was insignificantly different from zero in 1990–1991, often negative during 1992, usually negative in early 1993, and then positive through the rest of the sample; and (4) the sample uniformly shows significant excess kurtosis, with a positive value of  $\pi_4$ . (Appendix A shows that skewness and kurtosis are directly proportional to  $\pi_3$  and  $\pi_4$ , respectively.)

The nonlinear Wald test of the Black-Scholes restrictions is usually strongly rejected, except in isolated, short periods of the sample. [To conserve space, the Wald test results are not reported.<sup>6</sup>] Acceptance of the Black-Scholes null usually coincides with a relatively flat forward

volatility curve, which often occurs at times of high volatility. The forward volatility curve for LIBOR usually slopes upward out to a one-year maturity.

The residuals from regression (22), i.e., the pricing errors, were tested for any remnant of the volatility smile. Linear regressions were run, separately for puts and calls, of each day's pricing errors from the restricted four-parameter model on moneyness, moneyness squared, time to maturity, and time to maturity squared. If the four-parameter model "flattens" the volatility smile, the  $R^2$ 's should be zero. In fact, they are never close to zero. This is largely a reflection of a key result in the next section on the unrestricted model: the restricted model is rejected, hands down. The time to maturity regressors are also usually significant in the daily regressions.

However, the regression results indicate nothing about the magnitude of the errors. This is assessed in Figure 5, which shows the daily mean absolute error (MAE) and root mean square error (RMSE), where the unit of measurement is a basis point. Both measures are usually much less than one basis point, the minimum tick size for price movements. The MAE and RMSE are also usually close together, indicating that the residual smile is not that pronounced, even though it is statistically significant in the regressions. However, there are frequent, large spikes in the RMSE relative to the MAE.

#### Unrestricted Model

The results for the unrestricted model are presented in a similar format to those from the restricted model. To summarize those results, the time series of parameter estimates were stratified into two subsamples that derived from options with minimum and maximum maturities, respectively. The minimum and maximum maturity sample results are presented in separate panels. For the 1990–1994 sample, the minimum maturity options had a mean time to maturity of 75 days with standard deviation of 27 days; the maximum maturity options had a mean time to maturity of 225 days with standard deviation of 28 days. A check on the robustness of the results was done by filtering out all of the in-the-money options and rerunning the Hermite polynomial

regressions. Because the results for the full and filtered samples turned out to be so similar, only the latter appear in the following figures.

Figures 6–9 show respectively the daily time series for the point estimates of each the four parameters,  $\mu$ ,  $\sigma$ ,  $\pi_3$ , and  $\pi_4$ , with two standard error bounds around them. Especially for the filtered sample, the kernel estimator produces apparently excessively wide confidence bounds episodically. Qualitatively, the results for the minimum and maximum maturity series are similar to those reported for the restricted model. In Figure 8, there is significant skewness for the minimum maturities in early 1993 and somewhat later in 1993 for the maximum maturities. Excess kurtosis is more evident in Figure 9 for the minimum maturity series than for the maximum maturity series. Recall that the confidence bounds were computed using the diagonal kernel estimator described in the previous section. The full kernel (with damped off-diagonal cross-moment elements) would produce much tighter confidence bounds.

Figure 10 shows the  $p$ -values from the daily nonlinear Wald tests of the time-to-maturity restrictions. The panels of this figure summarize the tests of the restricted value of the parameters for a given maturity option versus their unrestricted values. The restrictions are rejected much more frequently for the shorter-term options: rejection at the 5 percent level occurs for 92 percent of the sample days for the minimum maturity options as compared with 65 percent for the maximum maturity options.

Figure 11 reports the  $p$ -values from the Wald tests of the Black-Scholes restrictions applied to a given maturity class. The Black-Scholes specification is also rejected more frequently, in 94 percent of the sample days, for the minimum maturity options as compared with 86 percent for the maximum maturity options.

The diagnostic regressions for the volatility smile in the pricing errors again indicates that even for the unrestricted model, the  $R^2$ 's tend to be high for both calls and puts. However, Figure

12 shows that the MAE's and RMSE's are small, and, in contrast to Figure 5 for the restricted model, the gap between the two measures is smaller, with fewer large outlier spikes.

Using the parameter estimates and equation (18) for  $Q(z)$ , Figure 13 shows the risk-neutral density, contrasting the 1990–1992 subperiod with the 1993–1994 subperiod. The upper plot is derived from options with the longest maturities in the sample; the lower, from options with the shortest maturities. For both maturity extremes, skewness reverses sign and kurtosis decreases going from the 1990–1992 to the 1993–1994 subperiod. The statistical significance of these shifts is clear from Table 1, which gives the sample means for  $\pi_3$  and  $\pi_4$  for the two subperiods and two maturity groups. While statistically significant shifts in  $\pi_3$  and  $\pi_4$  occurred, their economic significance is open to question. The latter is investigated using out-of-sample performance measurements of the four-parameter Hermite polynomial model.

#### Out-of-Sample Tests

The out-of-sample performance of the four-parameter model is compared with a separately estimated two-parameter model. Both the full-moneyness sample and the filtered (out-of-the-money and at-the-money) sample were used in estimation. The two-parameter model includes only  $\mu$  and  $\sigma$ . All parameters are estimated on a given day and then plugged into the models on the next trading day. The daily pricing errors are analyzed using the MAE and RMSE measures.

Figure 14 shows the daily time series for the in-sample MAE and RMSE for each model for the filtered sample. The results for the full-moneyness sample were very similar and are not graphed. Figure 15 shows the four-parameter model's in-sample versus out-of-sample comparisons for the MAE and RMSE. It is clear that the four-parameter model consistently beats the two-parameter model—particularly from mid-1993 to the end of the sample, when skewness is significant. Although there are a number of large spikes, the out-of-sample MAE's and RMSE's are usually less than the tick size.

Table 2 summarizes the out-of-sample results. The times series averages of the daily MAE's and RMSE's were computed along with their standard errors. For the overall sample, the mean MAE is about .25 basis points, regardless of whether in-the-money options are included in estimation. The similarity of results gives some assurance that the use of a European pricing model instead of an American one is inconsequential in this application. Most strikingly, the four-parameter model had lower MAE and RMSE in about 95 percent of the sample days.

The key finding is for the 1993–1994 subsample. Here mean MAE and RMSE are about .4 basis points, and the superior performance of the four-parameter model is realized in 97 to 98 percent of the sample days, again for both full-moneyness and filtered samples. The gain to accounting for skewness and kurtosis is slight—less than the tick size—but it is extremely consistent and essentially costless to realize. The additional computational effort is negligible. We would like to note that dollar value of the gain becomes more economically significant for positions involving multiple contracts.

### Interpretation

Our reference measure is one under which the underlying primary asset value process (in our case LIBOR) is geometric Brownian motion with drift  $\mu$  and variance rate  $\sigma$ . Initially, we assume that  $\mu$  and  $\sigma$  are *constant* for the entire sample time horizon such that there is a well-specified relationship between the basis prices  $\pi_k$  for different times to maturity. This is equation (19) above, repeated here:

$$\pi_k(s) = \pi_k(t) e^{r(t-s)} \frac{S^{k/2}}{t^{k/2}},$$

for  $s < t$  and where  $k$  is the  $k$ -th basis element.

The restrictions have been derived in a preference-free setting commonly employed in option pricing models. Equivalently, as shown in Appendix C, this relationship is consistent with an economy in which investors maximize their terminal consumption or wealth and their

preferences do not depend on the price path of the underlying asset. The appendix gives the technical conditions on the risk-neutral and reference measures in order for equation (19) to hold.

The intuition of those conditions is that with constant parameters  $\mu$  and  $\sigma$ , the preferences are such that investors care only about terminal consumption. In an asset pricing context, the investors' marginal utilities depend only on the final asset price, not the path followed by that price. In other words, empirical rejection of (19) could imply that investors' utilities may depend on intermediate consumption. This implies that the relationship between basis prices at different dates is more complex than that derived in Appendix B.

An alternative reason for the rejection of the deterministic restrictions of equation (19) could be that  $\mu$  and  $\sigma$  do vary over time as functions of some other state variables. In fact, Madan and Milne's general model does specify that  $\mu$  and  $\sigma$  could be functions of another Markov process  $\xi(t)$ .<sup>7</sup> With time-varying  $\mu$  and  $\sigma$ , we would have to explicitly specify the  $\xi(t)$  process and the functional dependence of the parameters on that process in order to derive restrictions similar to (19), which could then be empirically verified. Under this assumption we make no claims about investor preferences.

In our empirical tests, we retained the constant parameter assumption and also estimated an unrestricted model, i.e., one without restriction (19). A justification for the unrestricted model is that contingent claims can be arranged in different maturity classes and that each maturity class is driven by a *separate*, but possibly correlated, Wiener process, with similar Hermite polynomial representations. As discussed above, the unrestricted model performed well.

We recognize the need for a more general setup that incorporates time-varying parameters, with corresponding relationships between the parameters for different maturity classes. The unrestricted model effectively implies separate Wiener processes for different option maturities. However, all of the options are claims on the same underlying asset; therefore, the treatment of the unrestricted model appears to be ad hoc.

However, while there may be only one underlying Wiener process driving the primary asset value *directly*, there may be several other Wiener processes (through  $\xi(t)$ ) that affect the asset values through their impact on  $\mu$  and  $\sigma$ . Our unrestricted Hermite polynomial regression results may be giving indirect evidence on the impact of the state variable or stochastic process  $\xi(t)$ , with its own embedded discrete-time representation.

Thus, estimation of the Hermite polynomial regression parameters separately for different maturities is defensible. The fact that the  $\mu$ 's and  $\sigma$ 's, especially the latter, are different for different maturities makes the unrestricted approach more reasonable and more valuable in practical applications than the full restricted model.

#### Statistical Density

The statistical density given by equation (20) was estimated by maximum likelihood using the *Financial Times London Interbank Fixing* for three-month LIBOR. The full-sample standard deviation,  $\sigma$ , of 25 percent reported in Table 3 for LIBOR exceeds the 1990–1994 average 20 percent value of the daily  $\sigma$  estimates from the risk-neutral density estimation. This discrepancy is largely the result of the volatility spike in late 1992. For the entire sample, the  $\alpha_3$  coefficient for skewness is insignificantly different from zero and that for kurtosis is .537, somewhat higher than the corresponding average value of  $\beta_3$  for minimum sample maturities of .461 and more than double the  $\beta_3$  for maximum sample maturities of .214. However, skewness for the statistical density is significantly negative during the 1990–1992 subperiod, as it is for the risk-neutral density (see Table 1).

The January 1993 to September 1994 subperiod shows no statistically significant skewness, unlike the subsample used in estimation of the risk-neutral  $\pi_3$ . The kurtosis implied by  $\alpha_4$  is also relatively high. However, this result is sensitive to the inclusion of the month of January 1993, during which LIBOR hardly moved, driving volatility down and kurtosis up sharply. Once January is excluded from the subperiod,  $\alpha_3$  is about .2, significant at the 5 percent level, and the

degree of significance increases as more early months are dropped from 1993.<sup>8</sup> The  $\alpha_3$  value compares with a corresponding  $\beta_3$  of approximately three-quarters this size. Positive statistical-density skewness is highly significant in 1994, mirroring that for the risk-neutral density.

The three-month Treasury bill yield was also used to estimate equation (20). As expected, the results, reported in Table 3, are very similar to those from the LIBOR series. In contrast to the LIBOR series, the T-bill yield series shows significant positive skewness for the 1993–1994 subperiod, including January 1993.

Figure 16 displays graphs of the risk-neutral and statistical densities for the minimum and maximum maturity samples and the two subperiods. The risk-neutral density parameters used in generating these graphs are those from the filtered samples (in-the-money options excluded). The 1993–1994 subperiod used for the graphs excludes January 1993. The most striking feature of these graphs is the close match of the two densities for the minimum maturity sample. (The corresponding graphs, not shown, for the option sample including the in-the-money options exhibits an even closer match for the minimum maturity sample.) The greater kurtosis of the statistical density relative to the risk-neutral is especially evident in the maximum maturity sample.

## 5. Conclusion

This paper expands and tests the approach of Madan and Milne (1994) for pricing contingent claims as elements of a separable Hilbert space. We specialize the Hilbert space basis to the family of Hermite polynomials and use the model to price options on Eurodollar futures. Restrictions on the prices of Hermite polynomial risk for contingent claims with different times to maturity are derived. This allows all traded options to be used in estimation of the market prices of risk. However, these restrictions are rejected by our empirical tests of the four-parameter model.

We drop the market price of risk restrictions and estimate the model separately by option maturity. This can be justified as an indirect way to accommodate the likely time variation of the underlying parameters of the asset price dynamics, which the current model does not explicitly include. The out-of-sample performance of the unrestricted four-parameter model is consistently better than that of an unrestricted two-parameter version of the model, although the gain is small.

The price of risk estimates for the restricted as well as the unrestricted models indicate skewness and excess kurtosis in the implied risk-neutral density. For both estimated risk-neutral and statistical densities, the sign of skewness changed from negative to positive after 1992. The statistical density has greater excess kurtosis than the risk-neutral density, especially when derived from longer-term option prices. In our sample, the risk-neutral density implied by shorter-term options closely matches the shape of the statistical density.

Future work can test the static hedging properties of the model. In particular the Hermite polynomial coefficients have the interpretation of being basis-mimicking portfolios. If the Hermite polynomial coefficients for a contingent claim can be derived under this model, arbitrage-free pricing of these claims is feasible by Hermite polynomial approximation. Many contracts, such as path-dependent options, exotic options, swaps, etc., are candidates for this approach.

## Appendix A: Central Moments of $z$ and $\eta$ under the Risk-Neutral Measure

This appendix derives the moments of the return process  $\eta$  under the risk-neutral density  $Q(z)$ . Under the reference measure, the first four central moments of  $\eta$  are

$$\begin{aligned}E[\eta] &= \mu - \sigma^2 / 2; \\E[(\eta - E\eta)^2] &= \sigma^2 / t; \\E[(\eta - E\eta)^3] &= 0; \\E[(\eta - E\eta)^4] &= 3\sigma^4 / t^2.\end{aligned}$$

For  $Q(z)$  to be a density,

$$\int_{-\infty}^{\infty} Q(z) dz = 1$$

This is easily verified by substituting for  $Q(z)$  using equation (11) and using the moments of  $z$  from (12):

$$\begin{aligned}\beta_0 - \frac{\beta_2}{\sqrt{2}} + \frac{3\beta_4}{\sqrt{24}} + \frac{\beta_2}{\sqrt{2}} - \frac{6\beta_4}{\sqrt{24}} + \frac{3\beta_4}{\sqrt{24}} &= 1; \\ \beta_0 &= 1.\end{aligned}$$

*Mean:*

The mean of  $z$  under  $Q(z)$  is given by

$$E^Q[z] = \int_{-\infty}^{\infty} z Q(z) dz = \beta_1 - \frac{3}{\sqrt{6}}\beta_3 + \frac{3}{\sqrt{6}}\beta_3 = \beta_1.$$

Since

$$\eta = \mu - \frac{\sigma^2}{2} + \frac{\sigma}{\sqrt{t}}z,$$

the mean of  $\eta$  under  $Q(z)$  is given by

$$E^Q[\eta] = \mu - \frac{\sigma^2}{2} + \frac{\sigma}{\sqrt{t}}\beta_1.$$

Thus,  $\beta_1$  gives the mean shift of  $\eta$  relative to the reference measure.

*Variance:*

The variance of  $z$  and  $\eta$  under  $Q(z)$  are given by

$$E^Q\left[(z - E^Q z)^2\right] = 1 + \sqrt{2}\beta_2 - \beta_1^2;$$
$$E^Q\left[(\eta - E^Q \eta)^2\right] = \frac{\sigma^2}{t}\left(1 + \sqrt{2}\beta_2 - \beta_1^2\right),$$

where the restriction that  $\beta_0 = 1$  is imposed (and is imposed for the remaining moments).

*Third Central Moment:*

$$E^Q\left[(z - E^Q z)^3\right] = \sqrt{6}\beta_3 - 3\sqrt{2}\beta_1\beta_2 - 2\beta_1^3;$$
$$E^Q\left[(\eta - E^Q \eta)^3\right] = \frac{\sigma^3}{t^{3/2}}\left(\sqrt{6}\beta_3 - 3\sqrt{2}\beta_1\beta_2 - 2\beta_1^3\right).$$

*Fourth Central Moment:*

$$\begin{aligned} E^Q\left[(z - E^Q z)^4\right] &= 3 - 6\beta_1^2 - 3\beta_1^4 + \frac{12}{\sqrt{2}}\beta_2 + 6\sqrt{2}\beta_1^2\beta_2 - 4\sqrt{6}\beta_1\beta_3 + \sqrt{24}\beta_4; \\ E^Q\left[(\eta - E^Q \eta)^4\right] &= \frac{\sigma^4}{t^2} \left( 3 - 6\beta_1^2 - 3\beta_1^4 + \frac{12}{\sqrt{2}}\beta_2 + 6\sqrt{2}\beta_1^2\beta_2 - 4\sqrt{6}\beta_1\beta_3 + \sqrt{24}\beta_4 \right). \end{aligned}$$

*Parameter Restrictions*

The reference measure parameters  $\mu$  and  $\sigma$  can be specified arbitrarily. In this case, the estimated  $\beta_k$  parameters fit the density  $Q(z)$  to the risk-neutral return process implicit in options prices. As shown above,  $\beta_1$  gives the mean shift for the mean of  $\eta$  under  $Q(z)$  relative to the reference measure, and similarly  $\beta_1$  and  $\beta_2$  give the variance shift. On the other hand, if  $\mu$  and  $\sigma$  are estimated from options prices, they are estimated under the risk-neutral distribution  $Q(z)$  and no parameter shift is necessary. If we impose the condition that the first two moments of  $\eta$  under the transformed measure are equal to the true moments under the reference measure,  $\beta_1$  and  $\beta_2$  must each equal zero. Therefore, the following restrictions can be imposed:  $\beta_0 = 1$ ,  $\beta_1 = 0$ , and  $\beta_2 = 0$ . These restrictions were used in the estimation of equations (22) for Eurodollar futures options.

With the foregoing restrictions, the third and fourth central moments simplify to the following:

*Third Central Moment:*

$$\mathbf{E}^{\varrho}\left[(z - \mathbf{E}^{\varrho} z)^3\right] = \sqrt{6}\beta_3;$$

$$\mathbf{E}^{\varrho}\left[(\eta - \mathbf{E}^{\varrho} \eta)^3\right] = \frac{\sigma^3}{t^{3/2}} \sqrt{6}\beta_3.$$

*Fourth Central Moment:*

$$\mathbf{E}^{\varrho}\left[(z - \mathbf{E}^{\varrho} z)^4\right] = 3 + \sqrt{24}\beta_4;$$

$$\mathbf{E}^{\varrho}\left[(\eta - \mathbf{E}^{\varrho} \eta)^4\right] = \frac{\sigma^4}{t^2} (3 + \sqrt{24}\beta_4).$$

## Appendix B: Restrictions on Basis Prices for Different Maturities

This appendix derives the relationship between basis prices at different maturities. Let  $S_t$  denote the asset price at time  $t$ , and let  $r$  represent a constant interest rate paid by a money market account. We shall distinguish three stochastic processes that describe the behavior of  $S_t$  under three supposedly equivalent measures.

Let  $P$  be the statistical measure on  $F$  that defines the probability or relative frequency of the occurrence of events. Hence for each set  $A \in F$  the magnitude  $P(A)$  is the asymptotic relative frequency of occurrence of an even  $\omega \in A$ , where we are capable of replaying history over the time interval  $[0, T]$  infinitely often.

Let  $Q$  be the risk-neutral or equivalent martingale measure also defined on  $(\Omega, F)$ . Asymptotic relative frequency  $Q(A)$ , for  $A \in F$ , defines the time-zero price of trading in the futures market for  $T$  delivery of the claim that pays unit face value if  $\omega \in A$  and zero otherwise.

Let  $R$  be the reference measure, also defined on  $F$ . For any set  $A \in F$ ,  $R(A)$  is the probability that  $\omega \in A$  under measure  $R$ . The asset price process will have a simple description under the measure  $R$ . Suppose that under the measure  $R$  we may write that

$$S_t = S_0 e^{\mu t + \sigma z'_t - \sigma^2 t / 2} = S_0 e^{\eta t} \tag{B1}$$

for a standard Brownian motion  $z'_t$ . Alternatively, we have that under  $R$  the continuously compounded return

$$z'_t = \ln(S_t / S_0) / \sigma - (\mu / \sigma - \sigma / 2)t \tag{B2}$$

is a standard Brownian motion. In particular, the density of the continuously compounded return under the reference measure is normal with mean zero and variance  $t$ . Equivalently, the density of the standardized return

$$z_t = z_t' / \sqrt{t} = \eta / (\sigma / \sqrt{t}) - (\mu / \sigma - \sigma / 2) \sqrt{t}$$

is normal with mean zero and unit variance. (In accordance with the discrete-time setting of their model, Madan and Milne 1994 used the normalized variate  $z_t$  in their derivations.)

Consider the class of densities given by the collection of square integrable functions with respect to the standard normal density. Any such density may be written in the form of equation (15) in the text as

$$Q(z_t, t) = \sum_{k=0}^{\infty} e^{r_k t} \pi_k(t) \phi_k(z_t) n(z_t), \quad (\text{B3})$$

where  $n(z)$  is the standard normal density and  $\pi_k(t)$  is a square summable sequence that has the interpretation of the forward price at time zero for delivery of the contingent claim  $\phi_k(z_t)$ , when  $Q$  defines the risk-neutral density.

Our working hypothesis is that both the risk-neutral and statistical densities for all maturities lie in this class. Hence the statistical density for  $z_t$  is

$$P(z_t, t) = v(z_t, t) n(z_t), \quad (\text{B4})$$

where

$$v(z_t, t) = \sum_{k=0}^{\infty} \alpha_k(t) \phi_k(z_t),$$

and the risk-neutral density for  $z_t$  is

$$Q(z_t, t) = \lambda(z_t, t) n(z_t), \quad (\text{B5})$$

where

$$\lambda(z_t, t) = \sum_{k=0}^{\infty} e^{r_k t} \pi_k(t) \phi_k(z_t).$$

Note that under the reference measure the process  $z'_t = z_t \sqrt{t}$  is a standard Brownian motion with statistical and risk-neutral densities given on applying the change of variable formula to equations (B4) and (B5) respectively by

$$P'(z'_t, t) = \frac{1}{\sqrt{t}} v(z'_t/\sqrt{t}) n(z'_t/\sqrt{t}) \quad (\text{B6})$$

and

$$Q'(z'_t, t) = \frac{1}{\sqrt{t}} \lambda(z'_t/\sqrt{t}) n(z'_t/\sqrt{t}). \quad (\text{B7})$$

We may also define the adapted processes for the Radon-Nikodym derivatives. First let  $\Lambda_t$  be defined by

$$\Lambda_t = E^R \left[ \frac{dQ}{dR} \middle| F_t \right],$$

as the change of measure density process from the reference measure  $R$  to the risk-neutral measure  $Q$ . By construction the process  $\Lambda_t$  is an  $R$  martingale.

Similarly, define the  $R$ -martingale process  $N_t$  by

$$N_t = E^R \left[ \frac{dP}{dR} \middle| F_t \right].$$

Consider now the process

$$\Phi_t = E^P \left[ \frac{dQ}{dP} \middle| F_t \right]$$

that defines the density of  $Q$  with respect to  $P$ . This is by construction a  $P$  martingale, and hence  $\Phi_t N_t$  is an  $R$  martingale. It follows that

$$\Phi_t = \frac{1}{N_t} E^R \left[ N_t \frac{dQ}{dP} \middle| F_t \right] = \frac{1}{N_t} E^R \left[ N_t \frac{dQ}{dR} \frac{dR}{dP} \middle| F_t \right] = \frac{\Lambda_t}{N_t}.$$

The coefficient restriction on the market prices of risk  $\pi_k$  derived below is predicated on the hypothesis that

$$\Lambda_t = \lambda(z'_t / \sqrt{t}, t) \quad (\text{B8})$$

for  $z'_t$ , a standard Brownian motion.

The following derivation shows the relation between the market price of risk  $\pi_k$  at an arbitrary time  $t$  to the market price of risk  $\pi_k$  at an earlier time  $s$ .

Conditional on  $F_s$ ,

$$z'_t \sim N(z'_s, t - s).$$

Equations (B7) and (B8),  $z'_t$ , a standard Brownian motion, and  $\Lambda_t$ , an  $R$  martingale, imply

$$\int_{-\infty}^{\infty} \lambda(z'_t / \sqrt{t}, t) \frac{e^{-\frac{(z'_t - z'_s)^2}{2(t-s)}} \sqrt{t}}{\sqrt{2\pi} \sqrt{t-s}} dz'_t = \lambda(z'_s / \sqrt{s}, s) = \lambda(z_s, s).$$

Now make the change of variable in the integration of  $z_t = z'_t / \sqrt{t}$ . This gives

$$\int_{-\infty}^{\infty} \lambda(z_t, t) \frac{e^{-\frac{(z_t \sqrt{t} - z_s \sqrt{s})^2}{2(t-s)}} \sqrt{t}}{\sqrt{2\pi} \sqrt{t-s}} dz_t = \lambda(z_s, s).$$

This may be rewritten as

$$\int_{-\infty}^{\infty} \lambda(z_t, t) \frac{e^{-\frac{(z_t - z_s \sqrt{s/\sqrt{t}})^2}{2(t-s)/t}}}{\sqrt{2\pi} \sqrt{(t-s)/t}} dz_t = \lambda(z_s, s).$$

Now let us write the left-hand side as an inner product with respect to  $\frac{e^{-\frac{z_t^2}{2}}}{\sqrt{2\pi}}$ , giving

$$\int_{-\infty}^{\infty} \lambda(z_t, t) \frac{e^{-\frac{(z_t - z_s \sqrt{s/\sqrt{t}})^2}{2(t-s)/t}}}{\sqrt{(t-s)/t}} \frac{e^{-\frac{z_t^2}{2}}}{\sqrt{2\pi}} dz_t = \lambda(z_s, s).$$

If we write

$$\frac{e^{\frac{(z_t - z_s \sqrt{s}/\sqrt{t})^2 + z_t^2}{2(t-s)t}}}{\sqrt{(t-s)/t}} = \sum_{k=0}^{\infty} \theta_k \phi_k(z_t), \quad (\text{B9})$$

then the desired inner product is

$$\sum_{k=0}^{\infty} \pi_k(t) e^{rt} \theta_k = \sum_{k=0}^{\infty} \pi_k(s) e^{rs} \phi_k(z_s) = \lambda(z_s, s). \quad (\text{B10})$$

Multiplying (B9) by  $n(z_t)$  gives

$$\frac{e^{\frac{(z_t - z_s \sqrt{s}/\sqrt{t})^2}{2(t-s)t}}}{\sqrt{(t-s)/t} \sqrt{2\pi}} = \sum_{k=0}^{\infty} \theta_k \phi_k(z_t) \frac{e^{-\frac{z_t^2}{2t}}}{\sqrt{2\pi}}.$$

From Hilbert space representation theory, we have that

$$\theta_k = \int_{-\infty}^{\infty} \frac{e^{\frac{(z_t - z_s \sqrt{s}/\sqrt{t})^2 + z_t^2}{2(t-s)t}}}{\sqrt{(t-s)/t}} \phi_k(z_t) \frac{e^{-\frac{z_t^2}{2t}}}{\sqrt{2\pi}} dz_t.$$

Hence,

$$\sum_{k=0}^{\infty} \sqrt{k!} \theta_k \frac{u^k}{k!} = \int_{-\infty}^{\infty} \frac{e^{\frac{(z_t - z_s \sqrt{s}/\sqrt{t})^2 + z_t^2}{2(t-s)t}}}{\sqrt{(t-s)/t}} \sum_{k=0}^{\infty} \phi_k(z_t) \frac{e^{-\frac{z_t^2}{2t}}}{\sqrt{2\pi}} \frac{u^k}{\sqrt{k!}} dz_t,$$

and from the definition of the Hermite polynomial generating function,

$$\frac{e^{-(z-u)^2/2}}{\sqrt{2\pi}} = \sum_{k=0}^{\infty} \phi_k(z) n(z) \frac{u^k}{\sqrt{k!}},$$

we have that

$$\sum_{k=0}^{\infty} \sqrt{k!} \theta_k \frac{u^k}{k!} = \int_{-\infty}^{\infty} \frac{e^{\frac{(z_t - z_s \sqrt{s}/\sqrt{t})^2 + z_t^2}{2(t-s)t}}}{\sqrt{(t-s)/t}} \frac{e^{-(z_t - u)^2/2}}{\sqrt{2\pi}} dz_t \equiv \Phi(u, z_s, s, t). \quad (\text{B11})$$

The desired coefficients are then

$$\theta_k = \left. \frac{\partial \Phi(u, z_s, s, t)}{\partial u^k} \right|_{u=0} \frac{1}{\sqrt{k!}}. \quad (\text{B12})$$

The exponent in (B11) can be rewritten as

$$\frac{\left(z_t - \left(\frac{z_s/\sqrt{s}}{\sqrt{t}} + \frac{u(t-s)}{t}\right)\right)^2}{2(t-s)/t} - \frac{u^2}{2} + \frac{u^2(t-s)}{2t} + \frac{u\sqrt{s}z_s}{\sqrt{t}}.$$

Substituting, we have

$$\Phi(u) = e^{\frac{u\sqrt{s}z_s}{\sqrt{t}} - \frac{u^2s}{2t}} \int_{-\infty}^{\infty} \frac{e^{-\frac{\left(z_t - \left(\frac{z_s/\sqrt{s}}{\sqrt{t}} + \frac{u(t-s)}{t}\right)\right)^2}{2(t-s)/t}}}{\sqrt{2\pi}\sqrt{(t-s)/t}} dz_t. \quad (\text{B13})$$

The integral in (B13) is that of  $N\left(\frac{z_s/\sqrt{s}}{\sqrt{t}} + \frac{u(t-s)}{t}, \frac{t-s}{t}\right)$  over the whole range, and this is unity;

thus,

$$\begin{aligned} \Phi(u) &= e^{\frac{u\sqrt{s}z_s}{\sqrt{t}} - \frac{u^2s}{2t}} \\ &= e^{-\frac{1}{2}\left(\frac{u\sqrt{s}z_s}{\sqrt{t}} - z_t\right)^2 + \frac{z_t^2}{2}}. \end{aligned}$$

Using the generating function of the Hermite polynomials, this may be rewritten as

$$\begin{aligned} \Phi(u) &= e^{\frac{z_t^2}{2}} \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{\phi_k(z_s)}{\sqrt{2\pi}} \frac{e^{-\frac{z_t^2}{2}} u^k s^{k/2}}{t^{k/2} \sqrt{k!}} \\ &= \sum_{k=0}^{\infty} \frac{\phi_k(z_s) s^{k/2} \sqrt{k!}}{t^{k/2}} \frac{u^k}{k!}, \end{aligned}$$

and it follows from (B12) that

$$\theta_k = \frac{\phi_k(z_s) s^{k/2}}{t^{k/2}}.$$

From (B10), the desired inner product is

$$\sum_{k=0}^{\infty} \pi_k(t) e^{\pi} \frac{\phi_k(z_s) s^{k/2}}{t^{k/2}} = \lambda(z_s, s).$$

But we also have

$$\lambda(z_s, s) = \sum_{k=0}^{\infty} \pi_k(s) e^{rs} \phi_k(z_s)$$

Hence,

$$\pi_k(t) e^{\pi} \frac{s^{k/2}}{t^{k/2}} = \pi_k(s) e^{rs},$$

or

$$\pi_k(s) = \pi_k(t) e^{r(t-s)} \frac{s^{k/2}}{t^{k/2}}.$$

## Appendix C: An Interpretation of the Basis Price Restrictions

The restrictions on the market prices of risk derived in Appendix B are predicated on the hypothesis that

$$\Lambda_t = \lambda(z'_t/\sqrt{t}, t), \quad (\text{C1})$$

which is equation (B8) in Appendix B. This hypothesis is in fact strongly rejected by the data. One could ask which economic assumptions ensure the validity of equation (C1). In general, the process  $\Lambda = \{\Lambda_t, 0 \leq t \leq T\}$  is adapted to the filtration  $F = \{F_t | 0 \leq t \leq T\}$  and  $\Lambda_t$  is  $F_t$  measurable. The  $\Lambda$  process does not reduce to a function of  $z'_t$  as stated in equation (C1).

Suppose that  $\Lambda_T$  were a function of just  $z'_T$ , say

$$\Lambda_T = L(z'_T). \quad (\text{C2})$$

This is a very special hypothesis in the context of asset pricing that asserts that marginal utilities depend on just the final asset price, independent of the price path taken. If preferences were for final wealth with no regard for intermediate consumption, then one would expect such to be the case.

Now since under  $R$  the process  $z'_t$  is Markov, the expectation of  $\Lambda_T$  conditional on  $F_t$  is a function of just  $z'_t$ , say  $g(z'_t)$ . This gives us the validity of the expression

$$\Lambda_t = g(z'_t). \quad (\text{C3})$$

For any bounded Borel test function  $h(z'_t)$ , we have that

$$\begin{aligned} \mathbb{E}^Q[h(z'_t)] &= \mathbb{E}^R[\Lambda_T h(z'_t)] \\ &= \mathbb{E}^R[\mathbb{E}^R[\Lambda_T h(z'_t) | F_t]] \\ &= \mathbb{E}^R[\Lambda_t h(z'_t)] \\ &= \mathbb{E}^R[g(z'_t)h(z'_t)]. \end{aligned} \quad (\text{C4})$$

But we also have by the definition of  $\lambda(z'_i/\sqrt{t}, t)$  that

$$E^{\varrho}[h(z'_i)] = E^{\rho}[\lambda(z'_i/\sqrt{t}, t)h(z'_i)]. \quad (C5)$$

It follows from (C3), (C4), and (C5) that  $\Lambda_t = \lambda(z'_i/\sqrt{t}, t)$ . Hence, the hypothesis tested and rejected in this paper is given by equation (C2).

## Endnotes

1. The Hermite polynomials through the fourth order are

$$\phi_0(z) = 1;$$

$$\phi_1(z) = z;$$

$$\phi_2(z) = \frac{1}{\sqrt{2}}(z^2 - 1);$$

$$\phi_3(z) = \frac{1}{\sqrt{6}}(z^3 - 3z);$$

$$\phi_4(z) = \frac{1}{\sqrt{24}}(z^4 - 6z^2 + 3).$$

2. Time-varying, deterministic drifts and volatilities can be used in the reference measure. See Madan and Milne, p. 236, for technical conditions that must be satisfied.

3. From Madan and Milne (1994), pp. 237-238, the  $b_k$  coefficients can be obtained from the put-call parity condition:

$$\Phi(u, S_0, x, \mu, \sigma, t) - \Psi(u, S_0, x, \mu, \sigma, t) + x = S_0 e^{\mu + \sigma \sqrt{t}};$$

$$b_0(S_0, x, \mu, \sigma, t) = a_0(S_0, x, \mu, \sigma, t) - S_0 e^{\mu};$$

$$b_k(S_0, x, \mu, \sigma, t) = a_k(S_0, x, \mu, \sigma, t) - S_0 e^{\mu} \frac{(\sigma \sqrt{t})^k}{\sqrt{k!}}, \quad k > 0.$$

4. The CME determines the final settlement price of the Eurodollar contract based on LIBOR prevailing in the cash market by the following procedure. On the last day of trading of a maturing Eurodollar futures contract, the CME polls sixteen banks that are active in the London Eurodollar market. These banks are randomly selected from a group of no less than 20 banks. In the final 90 minutes of trading, the CME asks these banks for three-month LIBOR quotes at a random time during this period and again at the close of trading. The CME specifically asks each bank for "its perception of the rate at which three-month Eurodollar Time Deposit funds are currently offered by the market to prime banks" (CME, 1994, Chapter 39, p. 3). The four highest and four lowest quotes at both the random and closing-time polls are eliminated and the remaining quotes are averaged together and rounded to the nearest basis point to give the LIBOR value for the determination of the final settlement price.

5. See Silverman (1986) for an introduction to kernel estimation. Ait-Sahalia and Lo (1995) use a kernel estimator for the option pricing function.

6. The results are available from the authors.

7. See Madan and Milne (1994), Assumption 3.1.

8. The low volatility estimate of .149 for 1993-1994 also increases to .22 when January is excluded.

## References

- Ait-Sahalia, Yacine, and Andrew W. Lo. "Nonparametric Estimation of State-Price Densities Implicit in Financial Asset Prices." NBER Working Paper 5351, November 1995.
- Amin, Kaushik I., and Andrew J. Morton. "Implied Volatility Functions in Arbitrage-Free Term Structure Models." *Journal of Financial Economics* 35 (1994): 141-180.
- Barone-Adesi, Giovanni, and Robert E. Whaley. "Efficient Analytic Approximation of American Option Values." *Journal of Finance* 42 (1987): 301-320.
- Black, Fischer. "The Pricing of Commodity Contracts." *Journal of Financial Economics* 3 (1976): 167-179.
- Chicago Mercantile Exchange. *Rules of the CME*, November 1994.
- Derman, Emanuel, and Iraj Kani. "Riding on a Smile." *Risk* 7 (February 1994): 32-39.
- Dumas, Bernard, Jeff Fleming, and Robert E. Whaley. "Implied Volatility Functions: Empirical Tests." Unpublished manuscript, February 1996.
- Dupire, Bruno. "Pricing with a Smile." *Risk* 7 (January 1994): 18-20.
- Elliot, Robert J. "A General Recursive Discrete Time Filter." *Journal of Applied Probability* 30 (1993): 575-588.
- Flesaker, Bjorn. "Arbitrage-Free Pricing of Interest Rate Futures and Forward Contracts." *Journal of Futures Markets* 13 (1993): 77-91.
- Grinblatt, Mark, and Narasimhan Jegadeesh. "The Relative Pricing of Eurodollar Futures and Forward Contracts." Anderson Graduate School of Management, UCLA, Working Paper #5-94, March 1994.
- Jackwerth, Jens Carsten, and Mark Rubinstein. "Implied Probability Distributions: Empirical Analysis." Institute of Business and Economic Research Finance Working Paper #250, University of California at Berkeley, June 1995.
- Judge, George G., W.E. Griffiths, R. Carter Hill, Helmut Lütkepohl, and Tsoung-Chao Lee. *The Theory and Practice of Econometrics*. 2nd edition. New York: John Wiley and Sons, 1985.
- Longstaff, Francis A. "An Empirical Examination of the Risk-Neutral Valuation Model." Working Paper, College of Business, Ohio State University, and the Anderson Graduate School of Management, UCLA, 1992.
- \_\_\_\_\_. "Option Pricing and the Martingale Restriction." *Review of Financial Studies* 8 (Winter 1995): 1091-1124.
- Madan, Dilip B., and Frank Milne. "Contingent Claims Valued and Hedged by Pricing and Investing in a Basis." *Mathematical Finance* 4 (July 1994): 223-245.
- Ramaswamy, Krishna, and Suresh M. Sundaresan. "The Valuation of Options on Futures Contracts." *Journal of Finance* 40 (1985): 1319-1340.
- Rozanov, Y. A. *Markov Random Fields*. New York: Springer Verlag, 1982.

Rubinstein, Mark. "Implied Binomial Trees." *Journal of Finance* 49 (July 1994): 771-818.

Shimko, David. "Bounds of Probability." *Risk* 6 (April 1993): 33-37.

Silverman, B.W. *Density Estimation for Statistics and Data Analysis*. New York: Chapman and Hall, 1986.

Whaley, Robert E. "Valuation of American Futures Options: Theory and Empirical Tests." *Journal of Finance* 41 (1986): 127-150.

**Table 1**  
**Mean Subperiod Parameter Values for**  
**the Market Prices of Skewness and Kurtosis Risk**

*Maximum Maturity Sample*

<i>Subperiod</i>	$\pi_3$	$\pi_4$
<i>1990-1992</i>	-.040 (.002)	.247 (.006)
<i>1993-1994</i>	.118 (.003)	.139 (.004)

*Minimum Maturity Sample*

<i>Subperiod</i>	$\pi_3$	$\pi_4$
<i>1990-1992</i>	-.056 (.003)	.477 (.007)
<i>1993-1994</i>	.157 (.004)	.422 (.010)

*Maximum Maturity Sample*  
*In-the-money Options Excluded*

<i>Subperiod</i>	$\pi_3$	$\pi_4$
<i>1990-1992</i>	-.031 (.002)	.237 (.005)
<i>1993-1994</i>	.108 (.003)	.144 (.004)

*Minimum Maturity Sample*  
*In-the-money Options Excluded*

<i>Subperiod</i>	$\pi_3$	$\pi_4$
<i>1990-1992</i>	-.038 (.003)	.457 (.007)
<i>1993-1994</i>	.131 (.003)	.339 (.007)

Standard errors of the sample means are given in parentheses. The means are taken over the daily estimates of each of the parameters in each subperiod.  $\pi_3$  is the market price of skewness risk and  $\pi_4$  is the price of kurtosis risk.

**Table 2**  
**Out-of-Sample Results‡**

*Four-Parameter Model vs. Two-Parameter Model*

1990-1994

	<i>All Strikes</i>	<i>In-the-Money Excluded</i>
<i>Mean difference in MAE*</i>	.252	.249
<i>Standard Error</i>	.006	.005
<i>Mean difference in RMSE</i>	.261	.277
<i>Standard Error</i>	.007	.007
<i>% 4-param smaller MAE†</i>	95.3	94.9
<i>% 4-param smaller RMSE</i>	94.7	94.8
<i>Sample Size</i>	1087	1004

1990-1992

	<i>All Strikes</i>	<i>In-the-Money Excluded</i>
<i>Mean difference in MAE</i>	.163	.208
<i>Standard Error</i>	.004	.006
<i>Mean difference in RMSE</i>	.164	.213
<i>Standard Error</i>	.006	.008
<i>% 4-param smaller MAE</i>	93.4	93.8
<i>% 4-param smaller RMSE</i>	92.4	92.8
<i>Sample Size</i>	668	601

1993-1994

	<i>All Strikes</i>	<i>In-the-Money Excluded</i>
<i>Mean difference in MAE</i>	.394	.313
<i>Standard Error</i>	.010	.008
<i>Mean difference in RMSE</i>	.414	.371
<i>Standard Error</i>	.011	.009
<i>% 4-param smaller MAE</i>	98.3	96.5
<i>% 4-param smaller RMSE</i>	98.3	97.8
<i>Sample Size</i>	419	403

‡The parameters estimated based on one day's options were used in pricing options the next trading day. Daily mean absolute errors and root mean square errors across all calls and puts were computed.

\*Results are expressed in basis points.

†Percentage of the daily MAEs or RMSEs in out-of-sample results for the four-parameter model that were smaller than the corresponding two-parameter values.

**Table 3**  
**Estimation of the Statistical Density\***

*Parameter:  $\mu$*

<i>Interest Rate Series</i>	<i>1990-1994</i>		<i>1990-1992</i>		<i>1993-1994</i>		<i>1994</i>	
	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>
<i>3-month LIBOR</i>	-.132	.151	-.469	.182	.236	.142	.766	.298
<i>3-month T-bill</i>	-.009	.143	-.545	.190	.506	.175	.865	.262

*Parameter:  $\sigma$*

<i>Interest Rate Series</i>	<i>1990-1994</i>		<i>1990-1992</i>		<i>1993-1994</i>		<i>1994</i>	
	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>
<i>3-month LIBOR</i>	.254	.003	.270	.009	.149	.008	.225	.014
<i>3-month T-bill</i>	.213	.002	.224	.008	.205	.009	.229	.012

*Parameter:  $\alpha_3$*

<i>Interest Rate Series</i>	<i>1990-1994</i>		<i>1990-1992</i>		<i>1993-1994</i>		<i>1994</i>	
	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>
<i>3-month LIBOR</i>	-.012	.058	-.132	.067	.039	.100	.258	.088
<i>3-month T-bill</i>	.086	.067	-.138	.094	.219	.071	.213	.088

*Parameter:  $\alpha_4$*

<i>Interest Rate Series</i>	<i>1990-1994</i>		<i>1990-1992</i>		<i>1993-1994</i>		<i>1994</i>	
	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>	<i>Estimate</i>	<i>Std. Error</i>
<i>3-month LIBOR</i>	.537	.031	.554	.047	.651	.050	.403	.094
<i>3-month T-bill</i>	.564	.032	.685	.052	.393	.063	.422	.078

\*Maximum likelihood estimation of equation (20) The three-month LIBOR series consists of daily observations from the *Financial Times London Interbank Fixing*, and the three-month Treasury bill series consists of daily observations on constant-maturity three-month yields from the Federal Reserve Board.

Figure 1

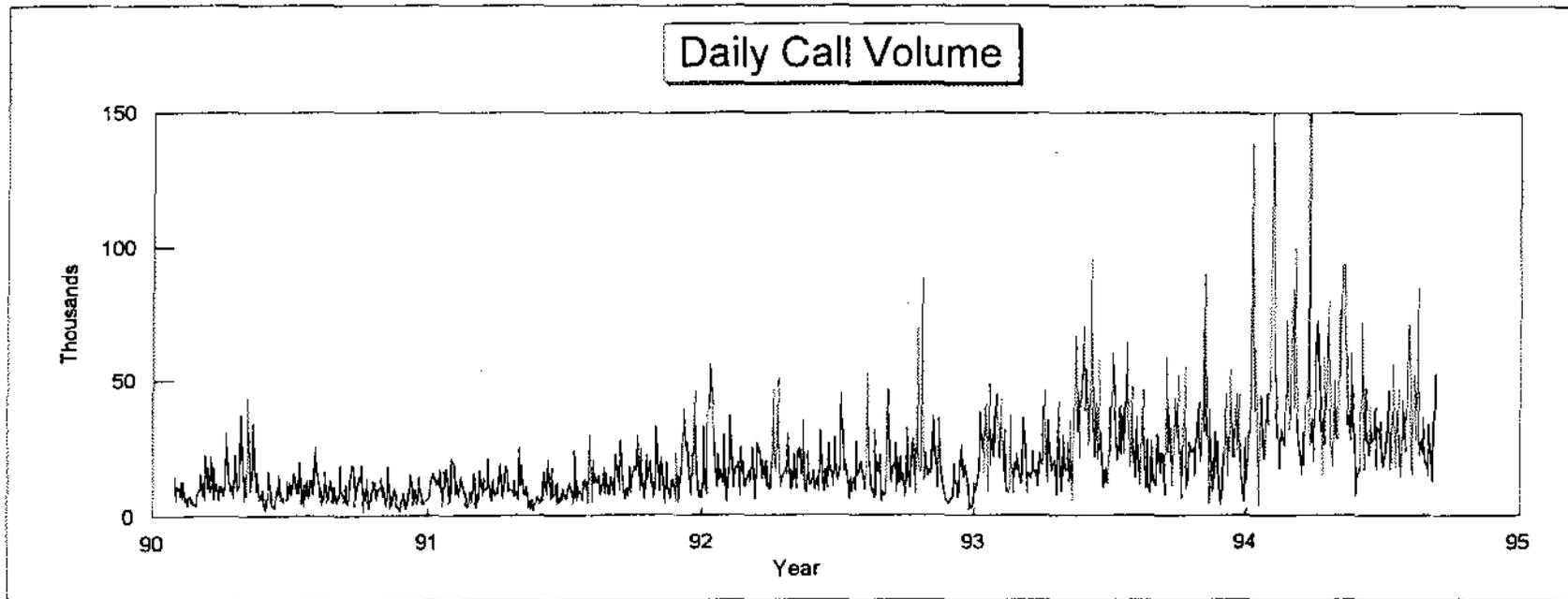
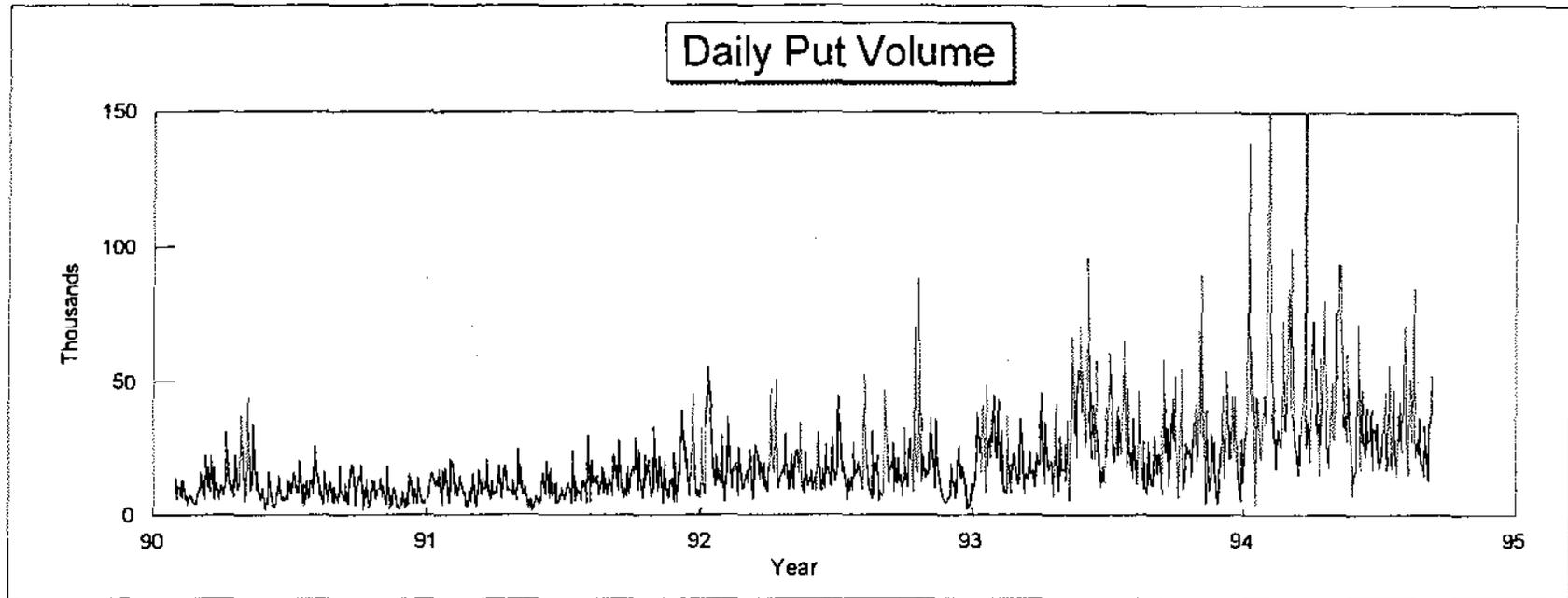


Figure 2

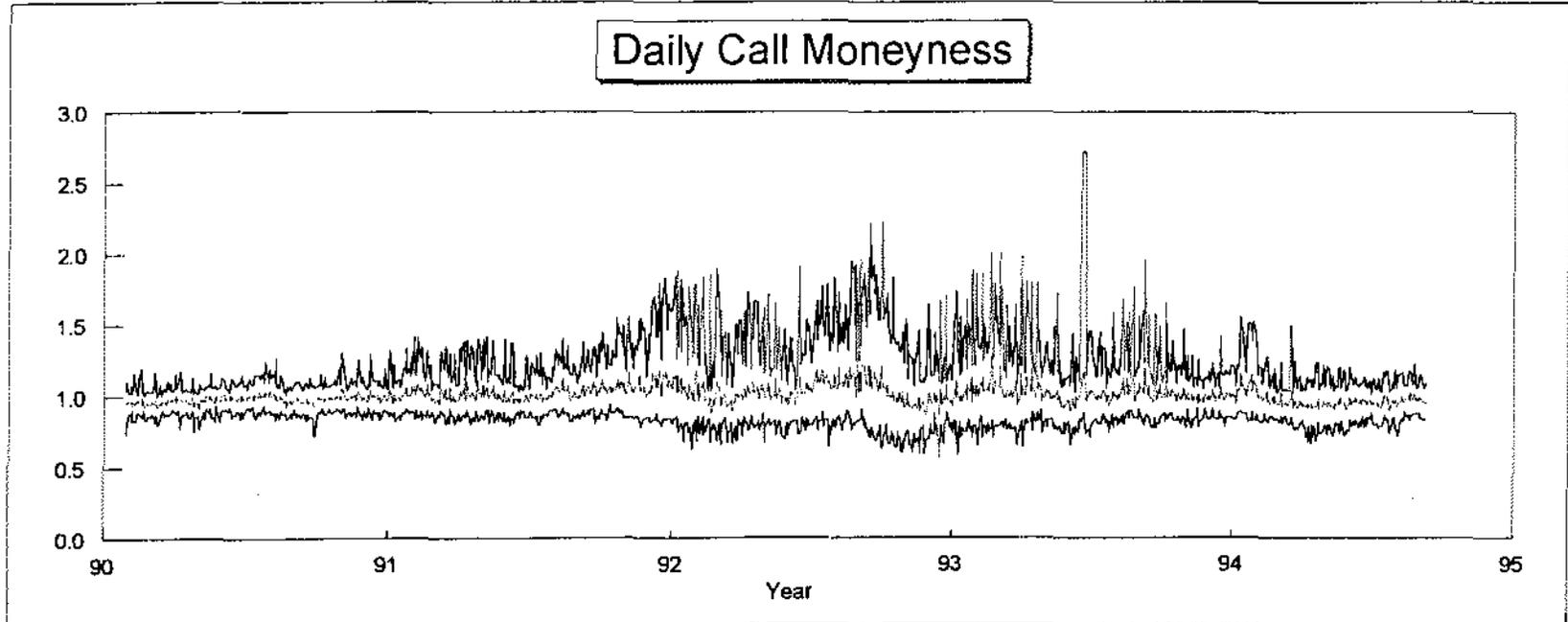
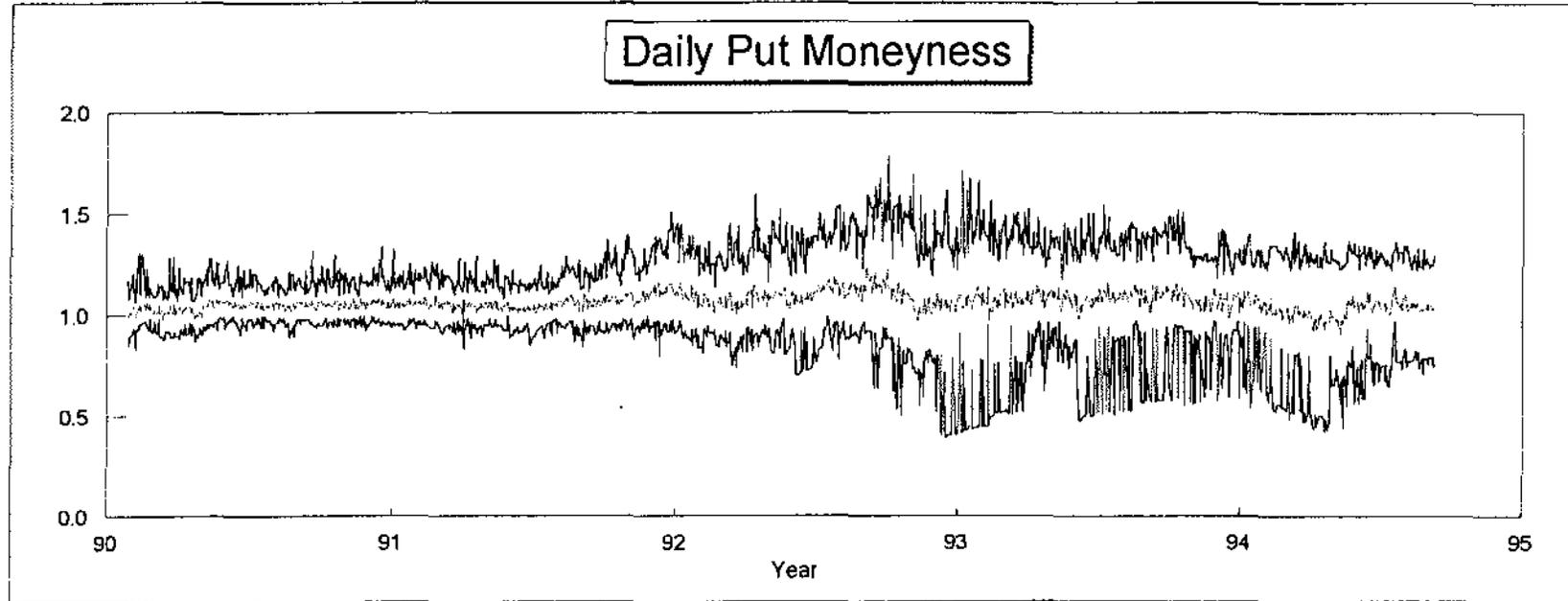


Figure 3 - Restricted Model

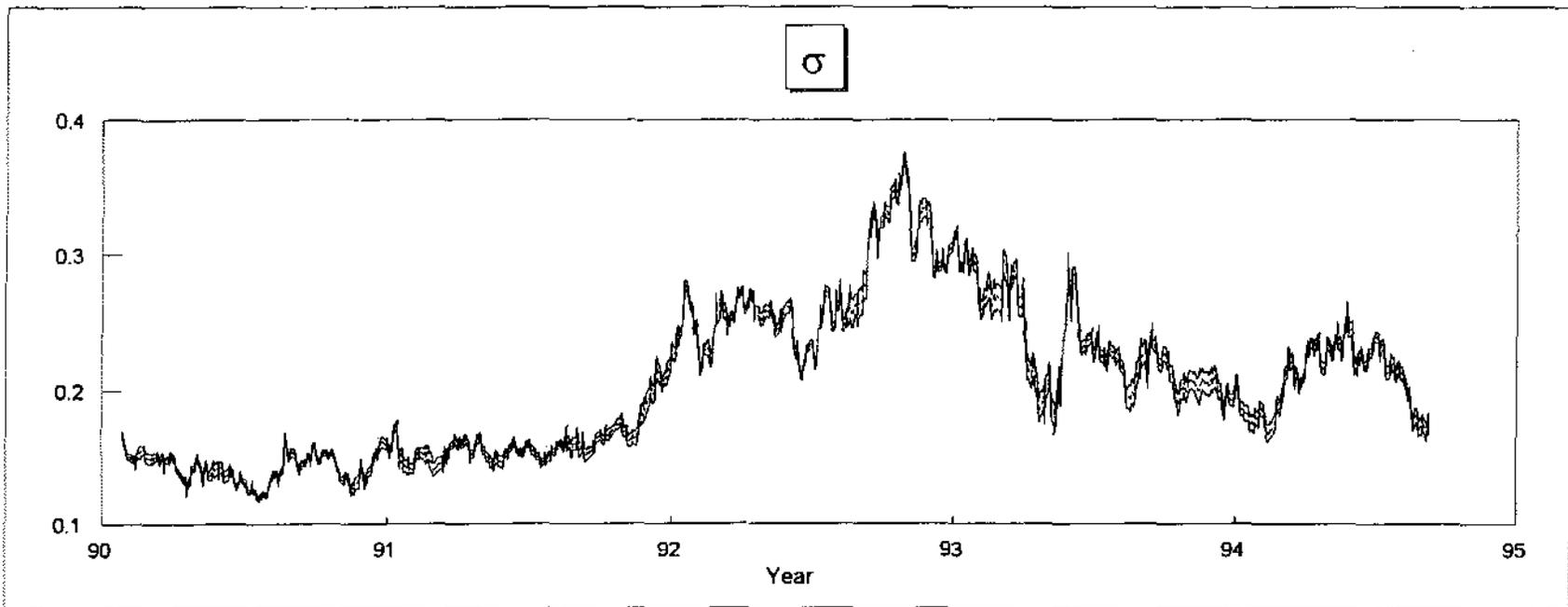
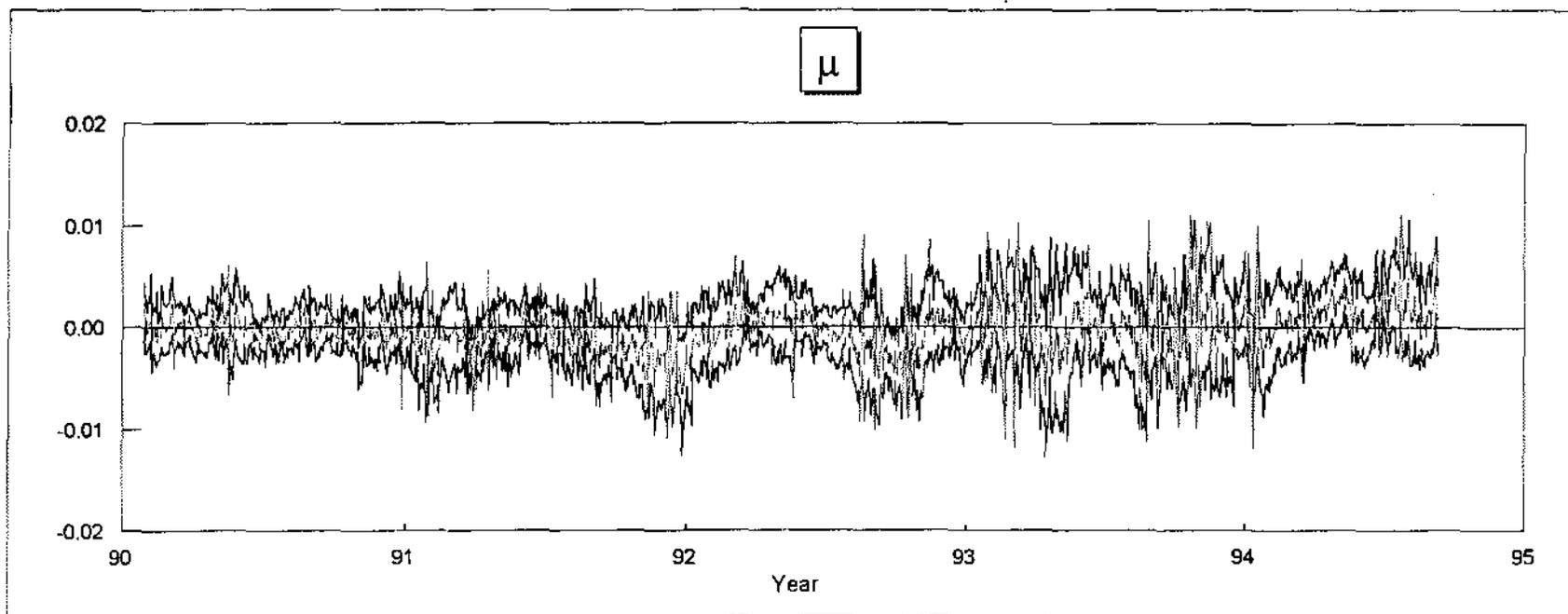


Figure 4 - Restricted Model

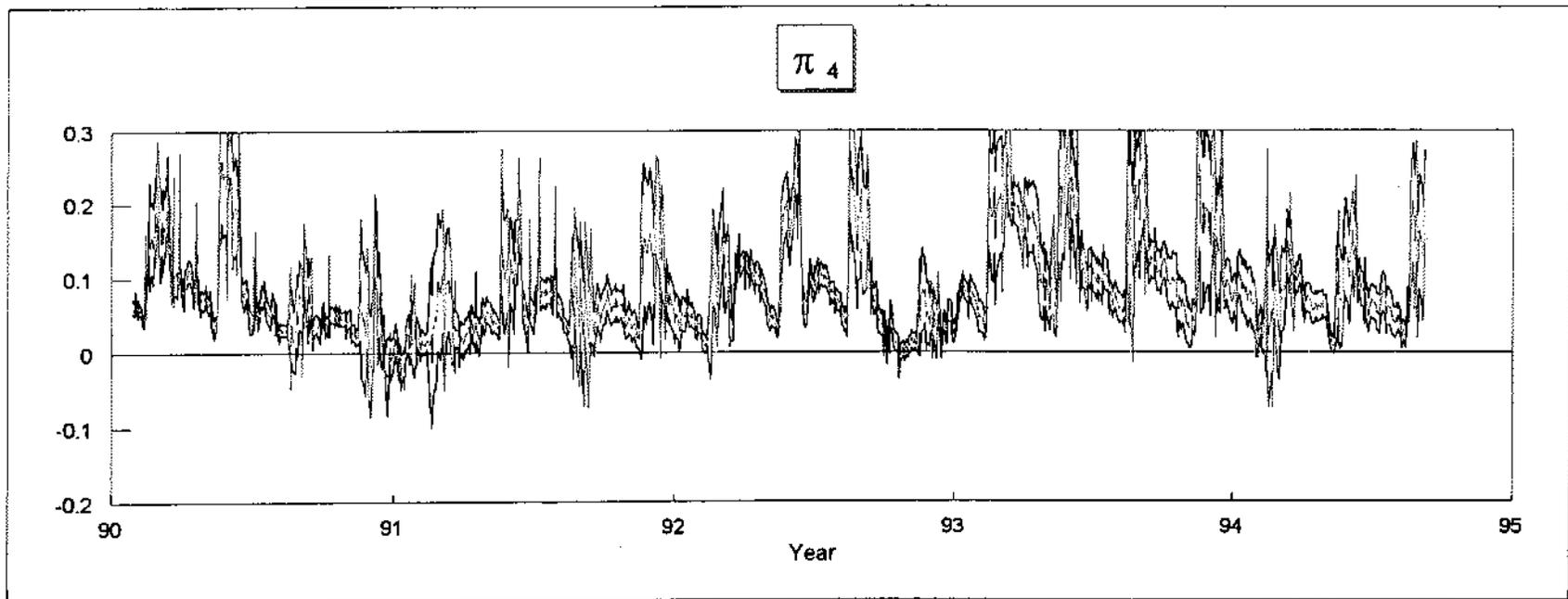
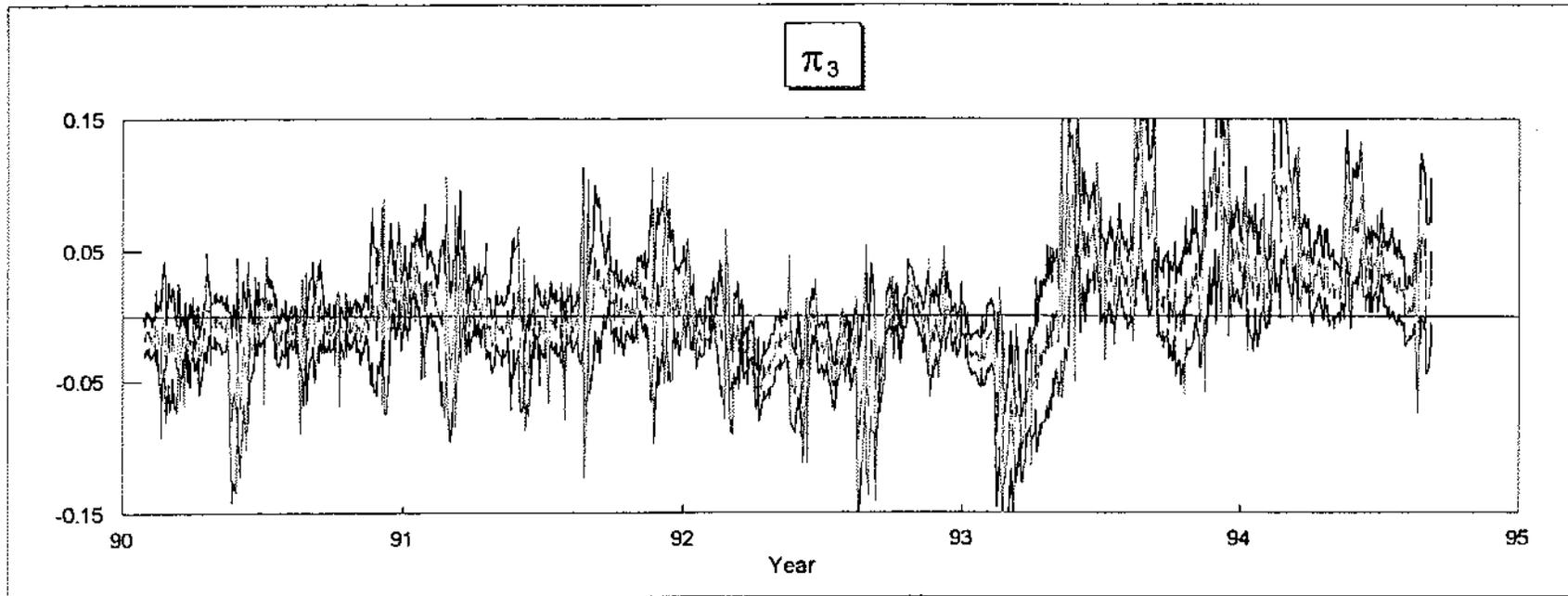


Figure 5 - Restricted Model

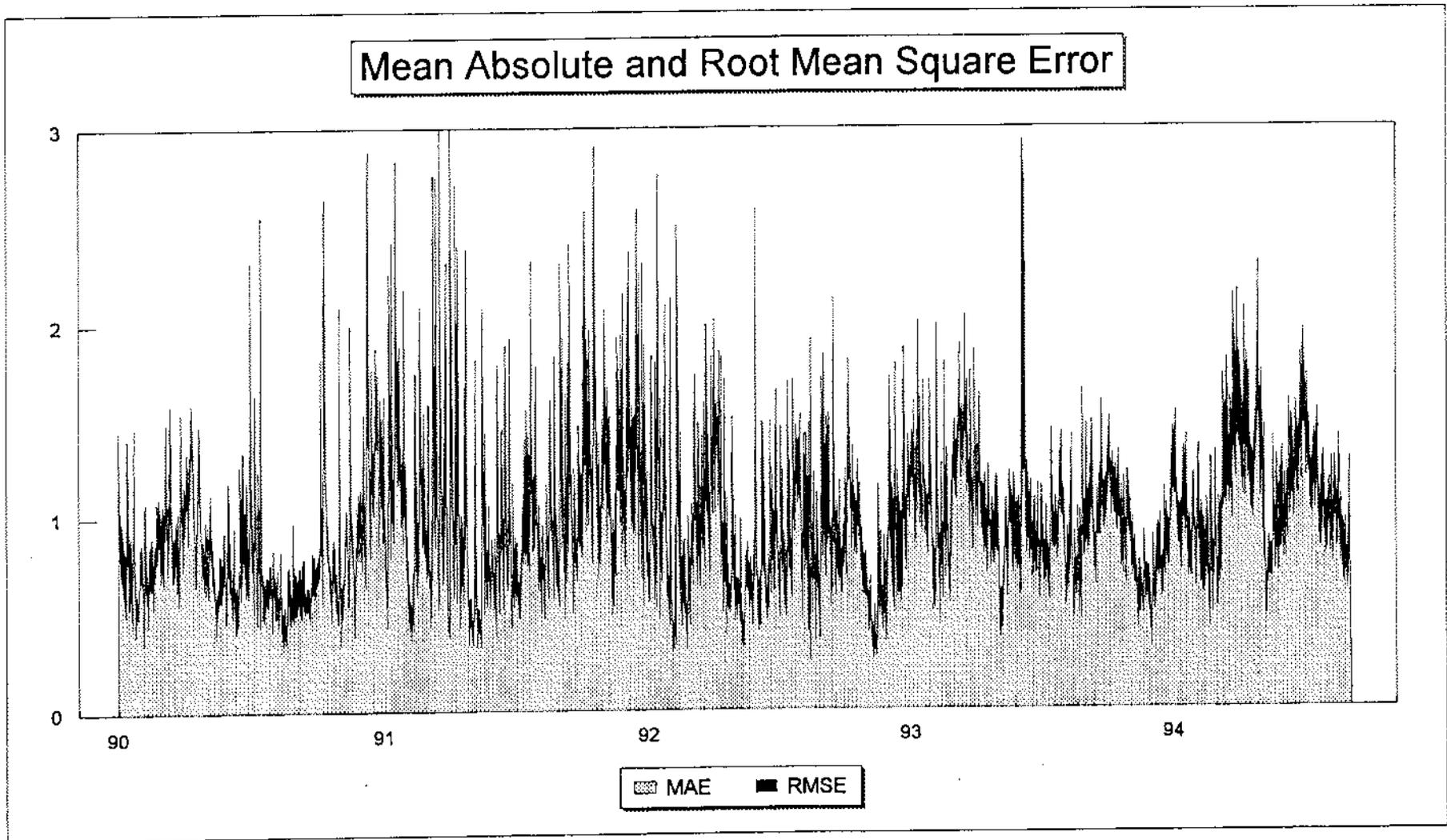


Figure 6 - Unrestricted Filtered Sample

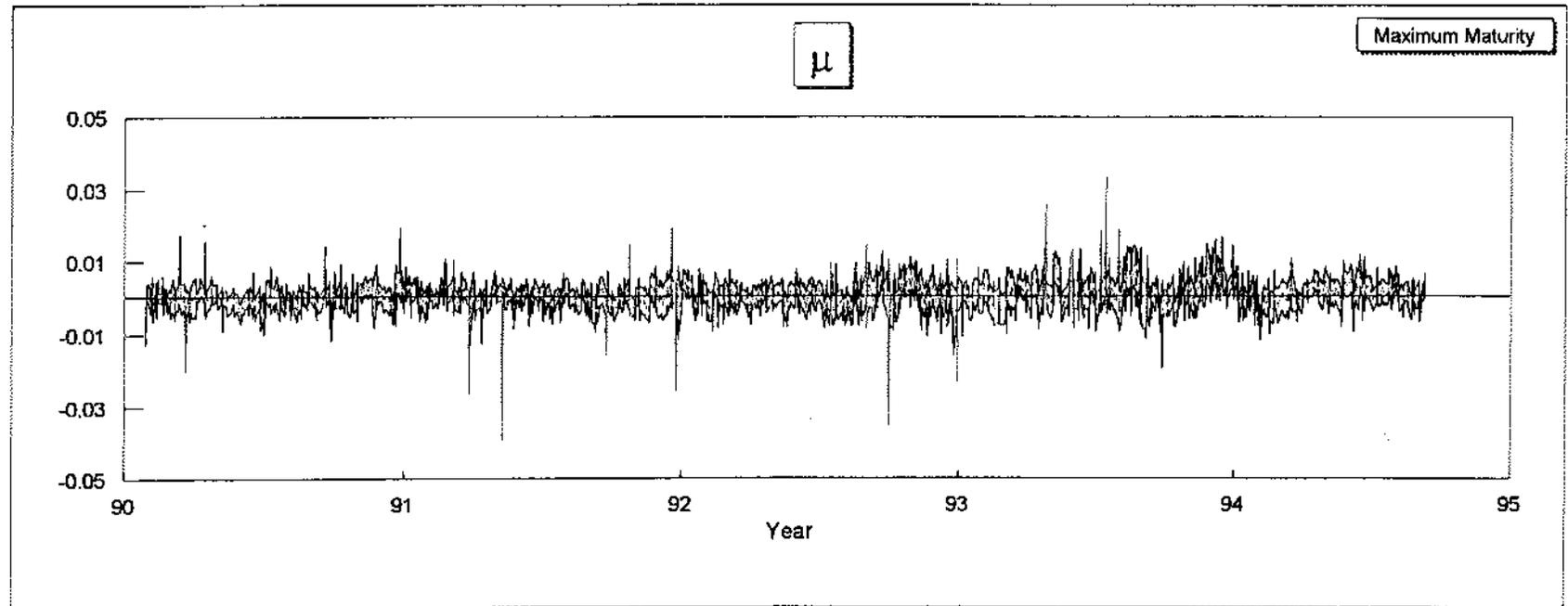
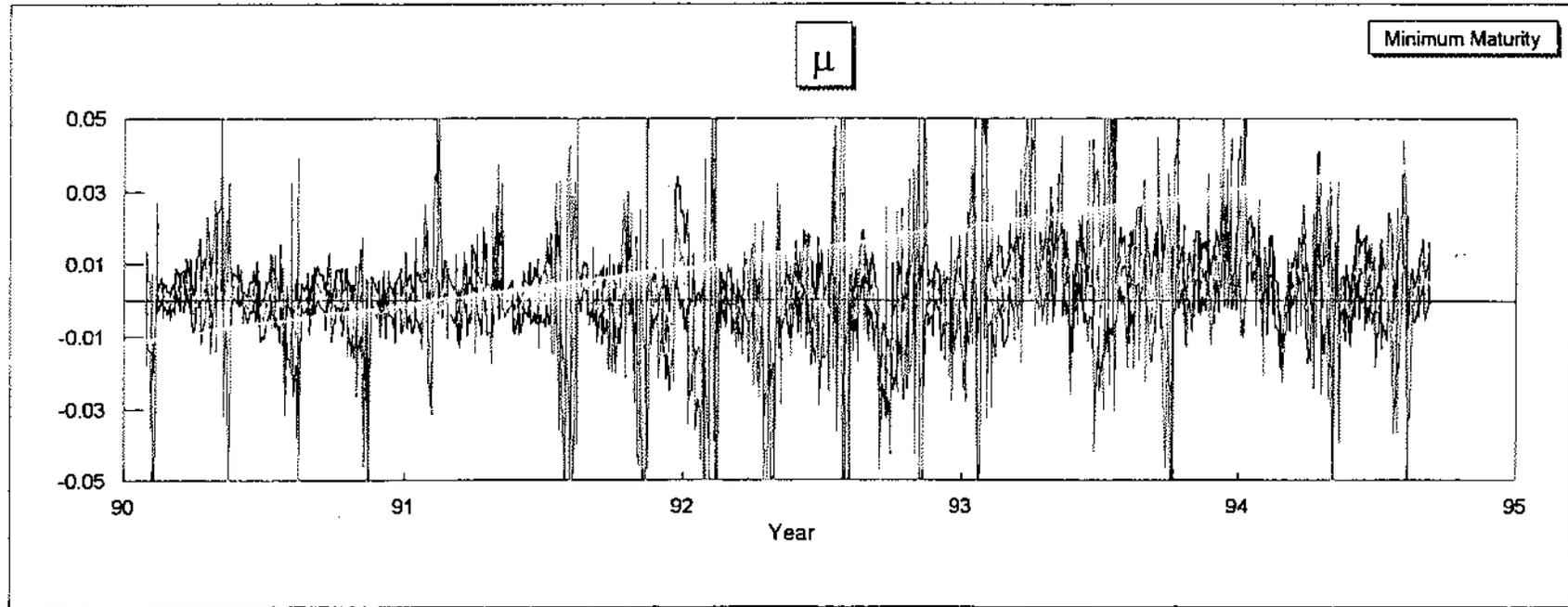


Figure 7 - Unrestricted Filtered Sample

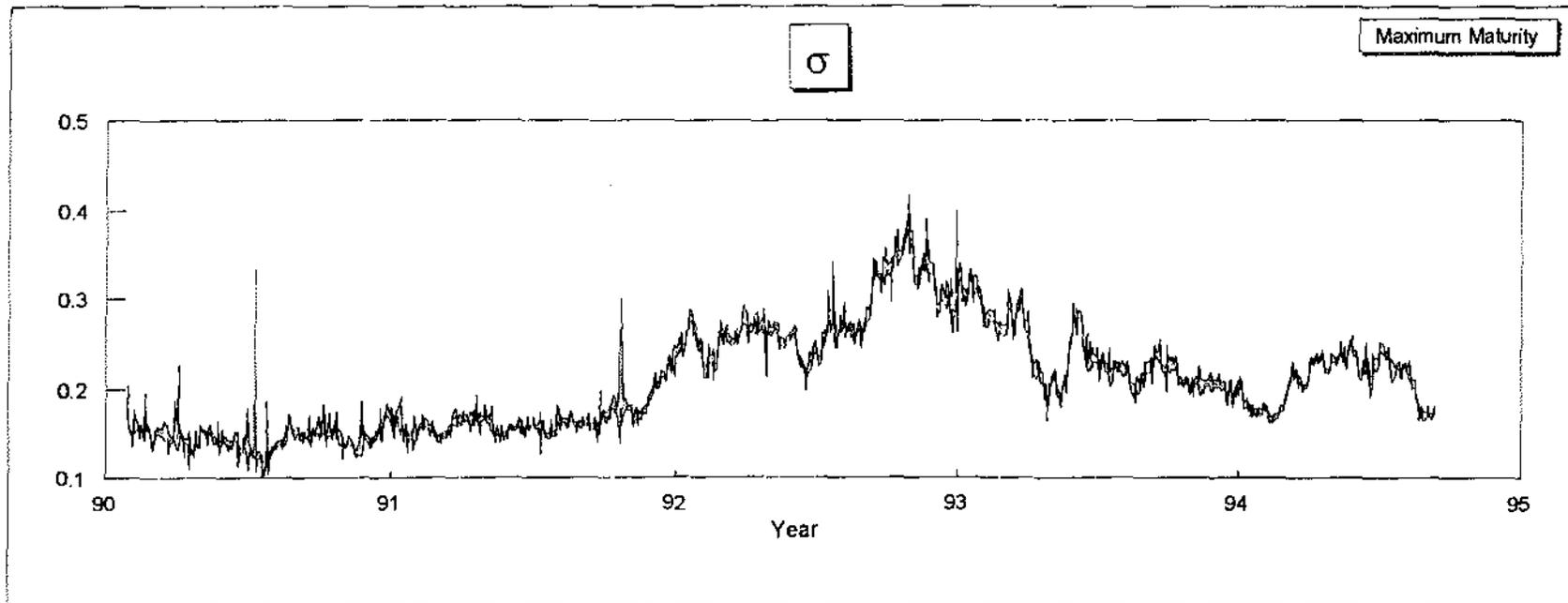
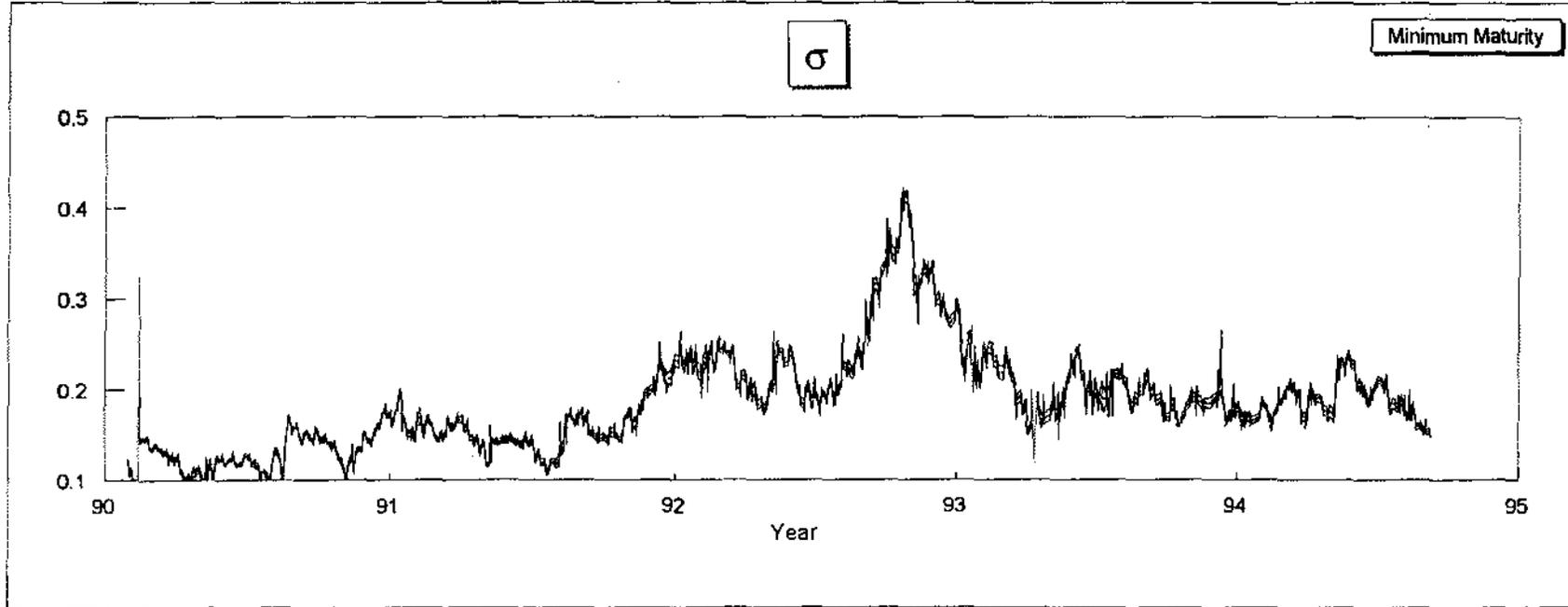


Figure 8 - Unrestricted Filtered Sample

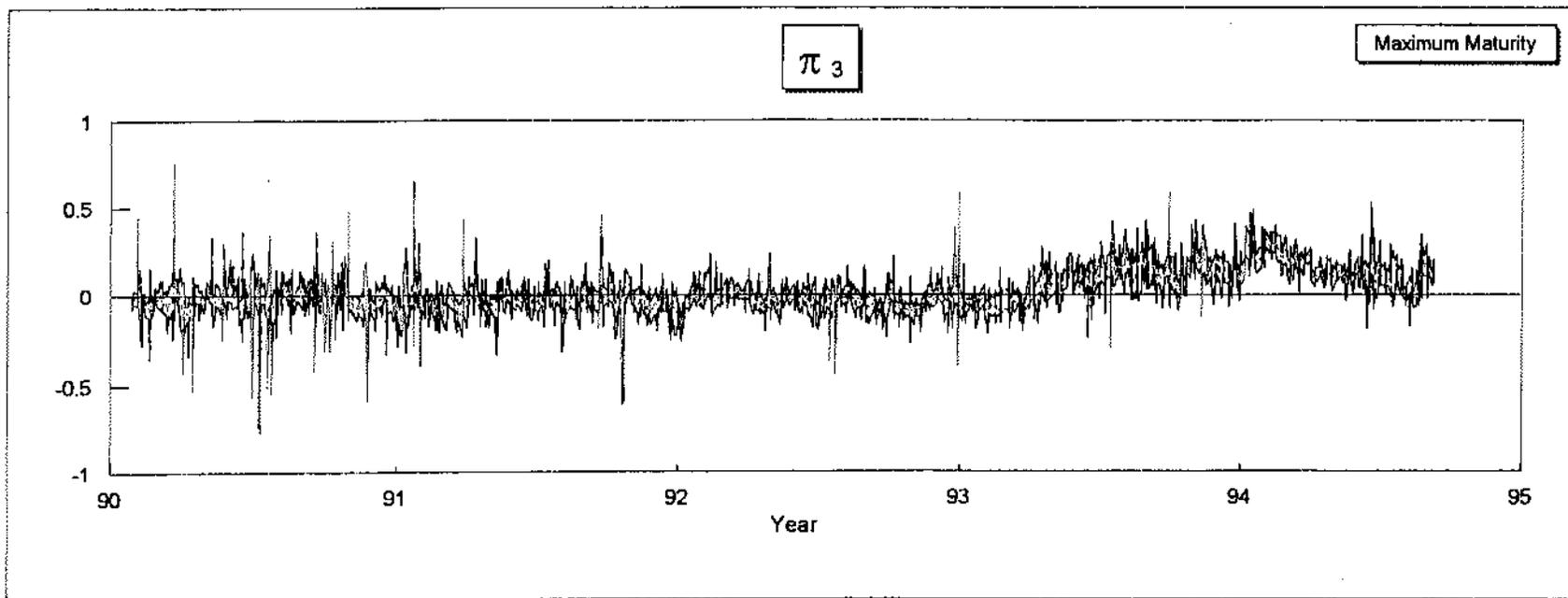
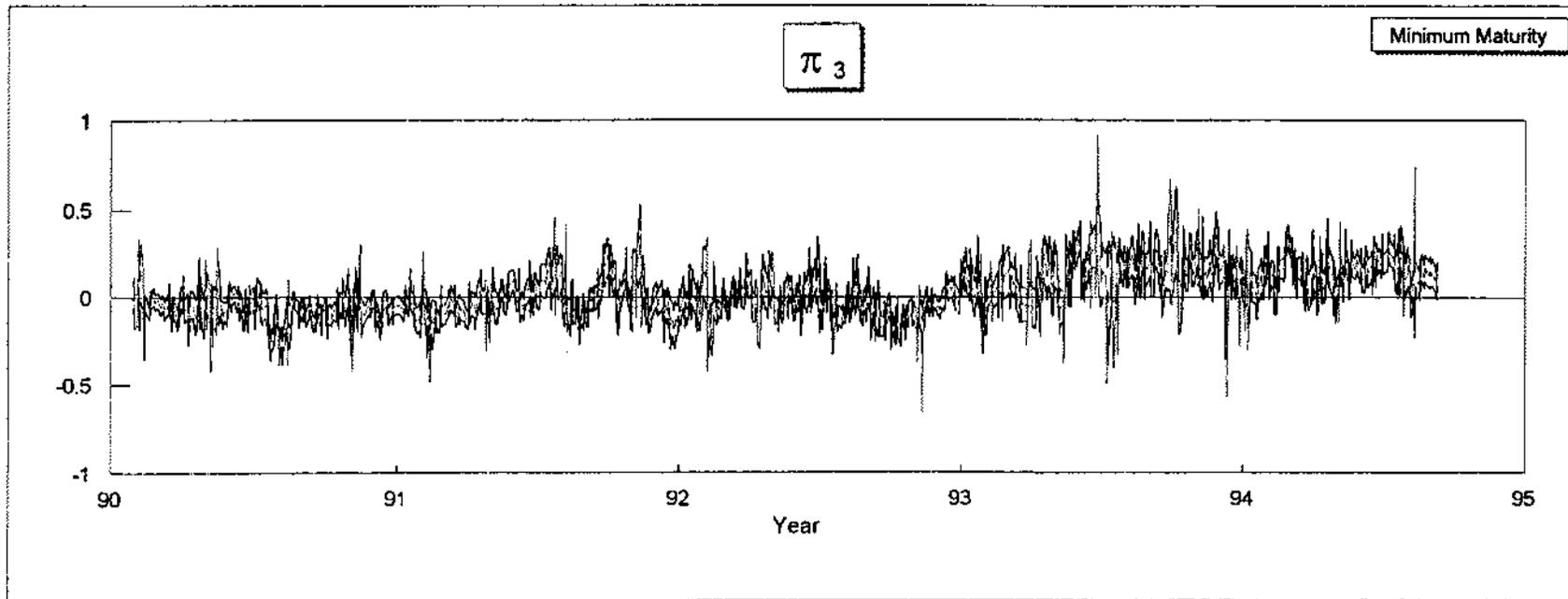


Figure 9 - Unrestricted Filtered Sample

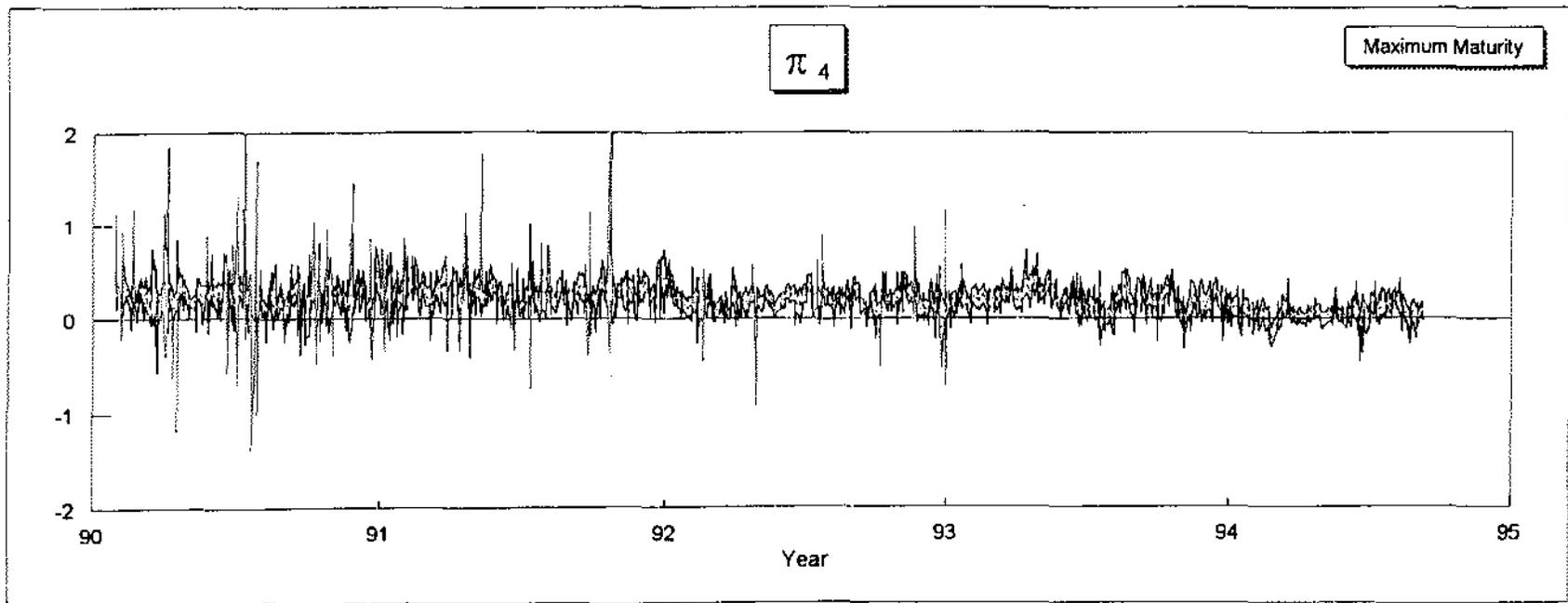
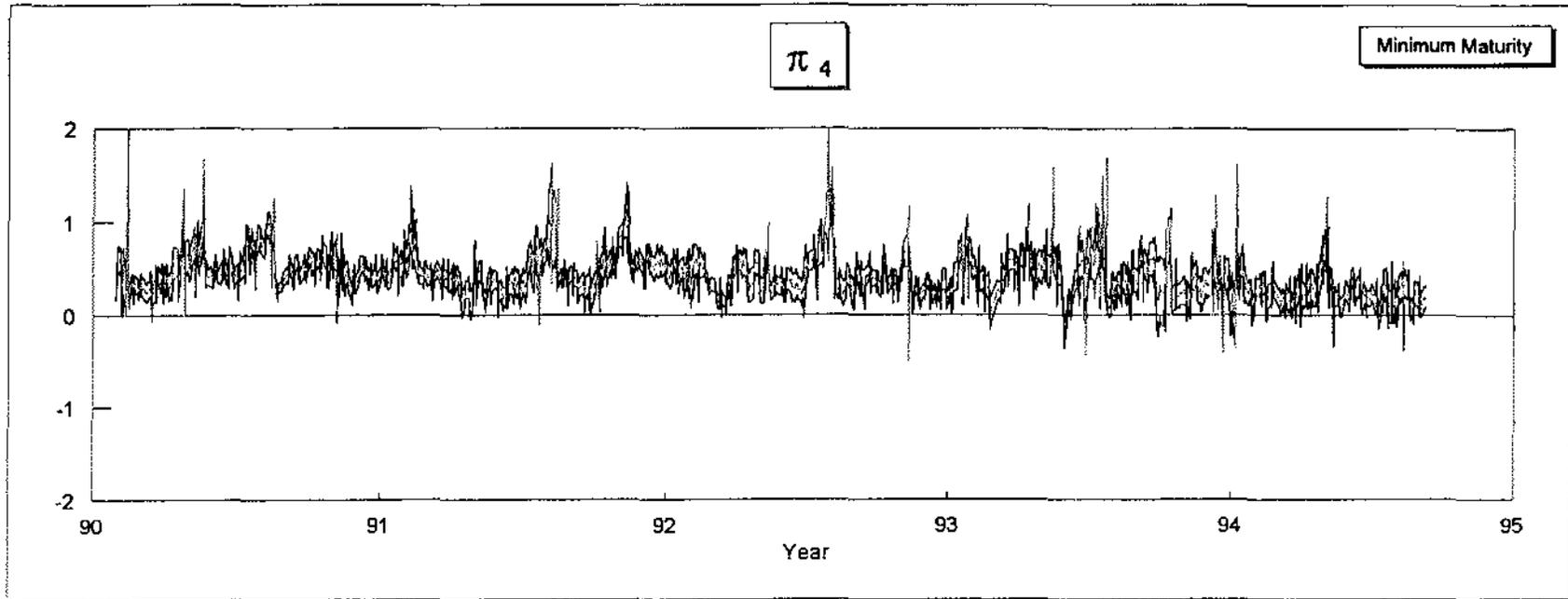


Figure 10 - Unrestricted Filtered Sample

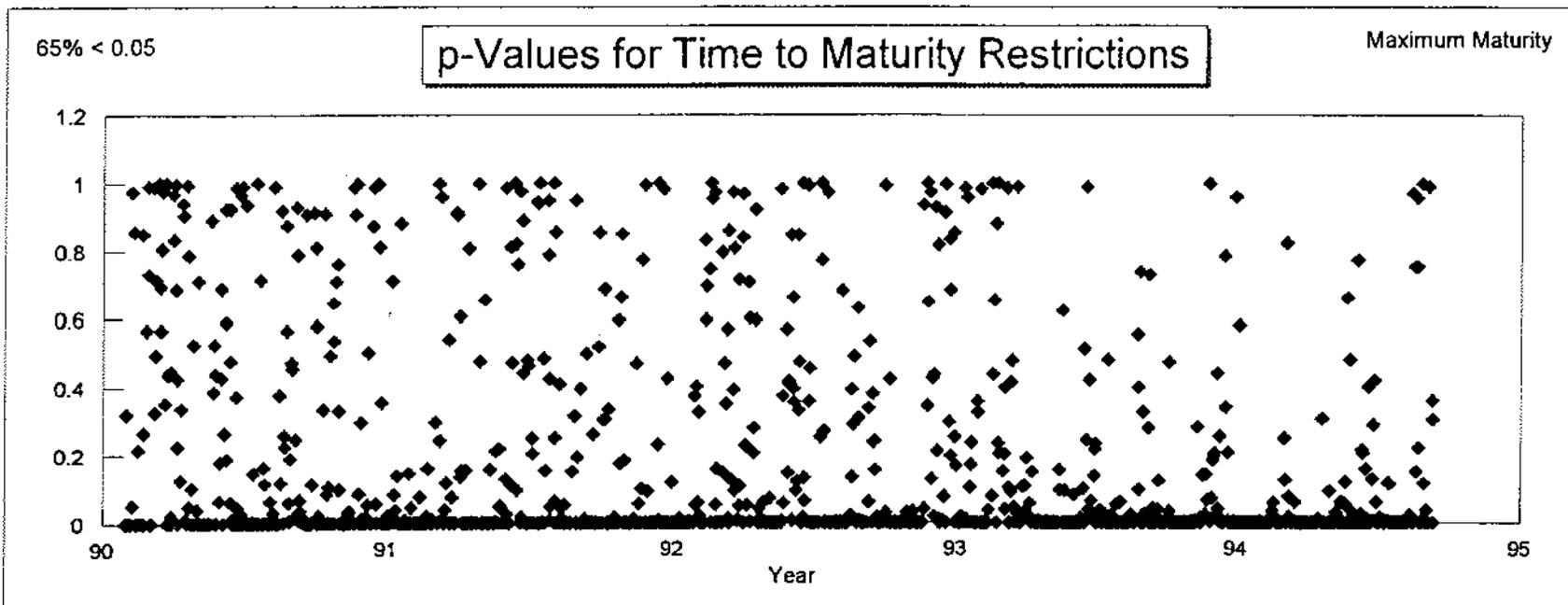
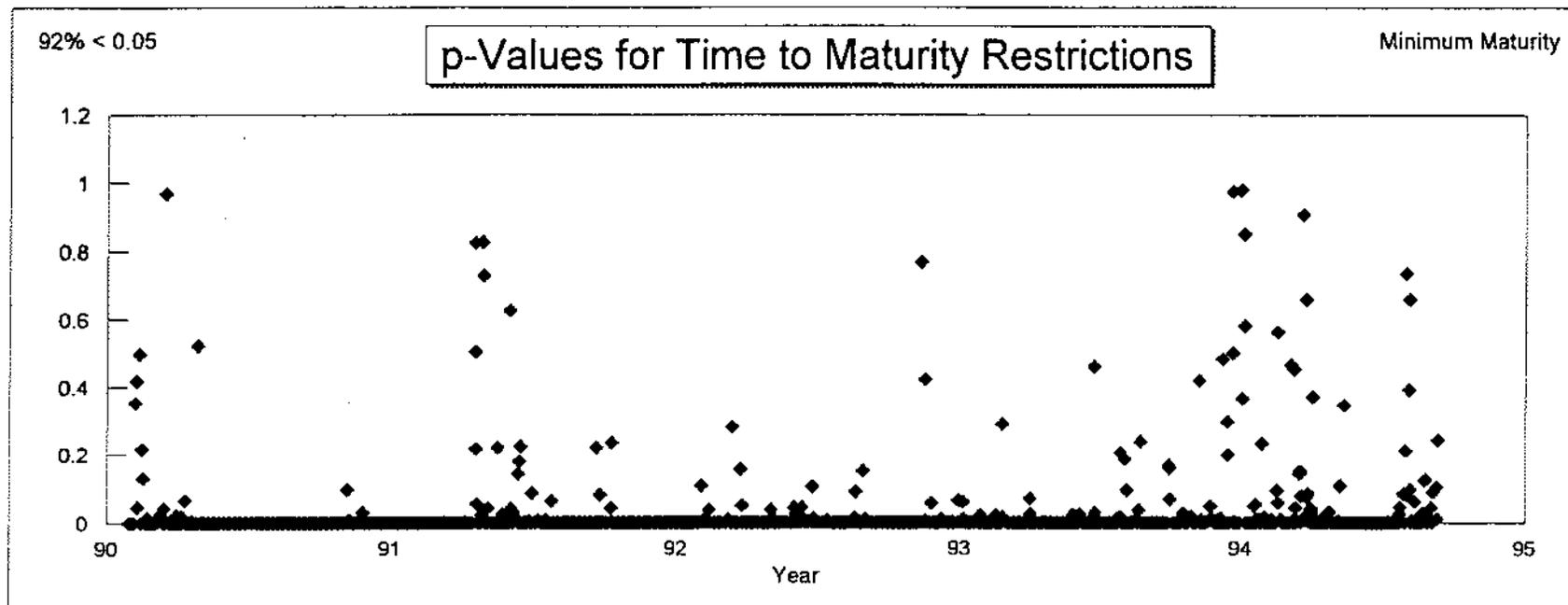


Figure 11 - Unrestricted Filtered Sample

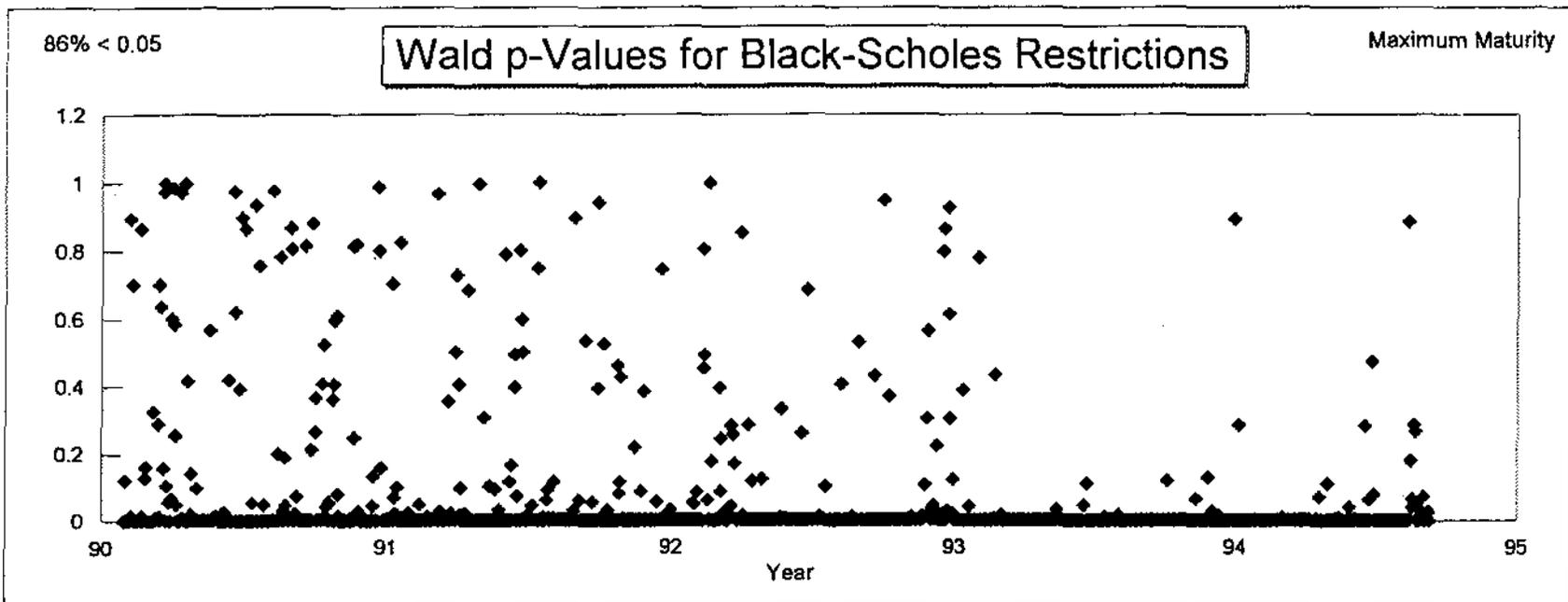
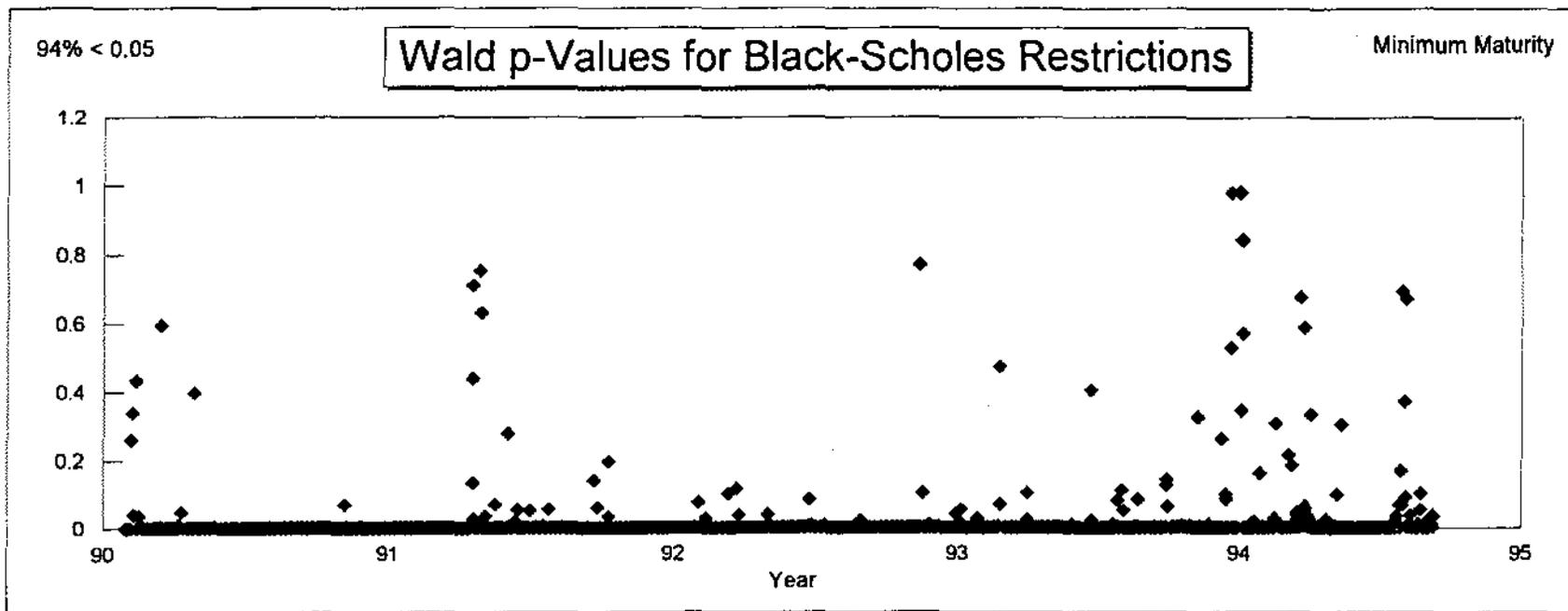


Figure 12 - Unrestricted Filtered Sample

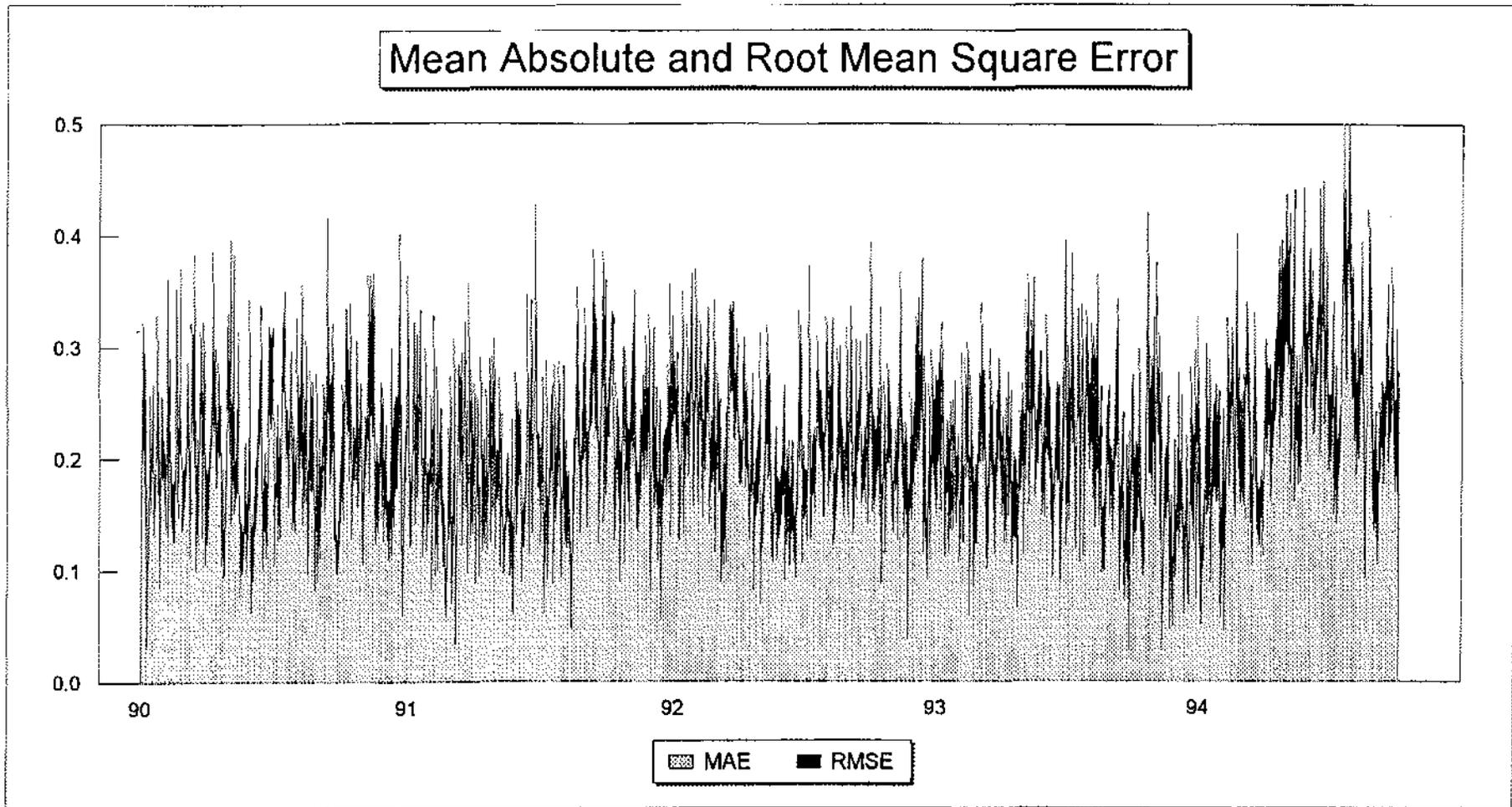
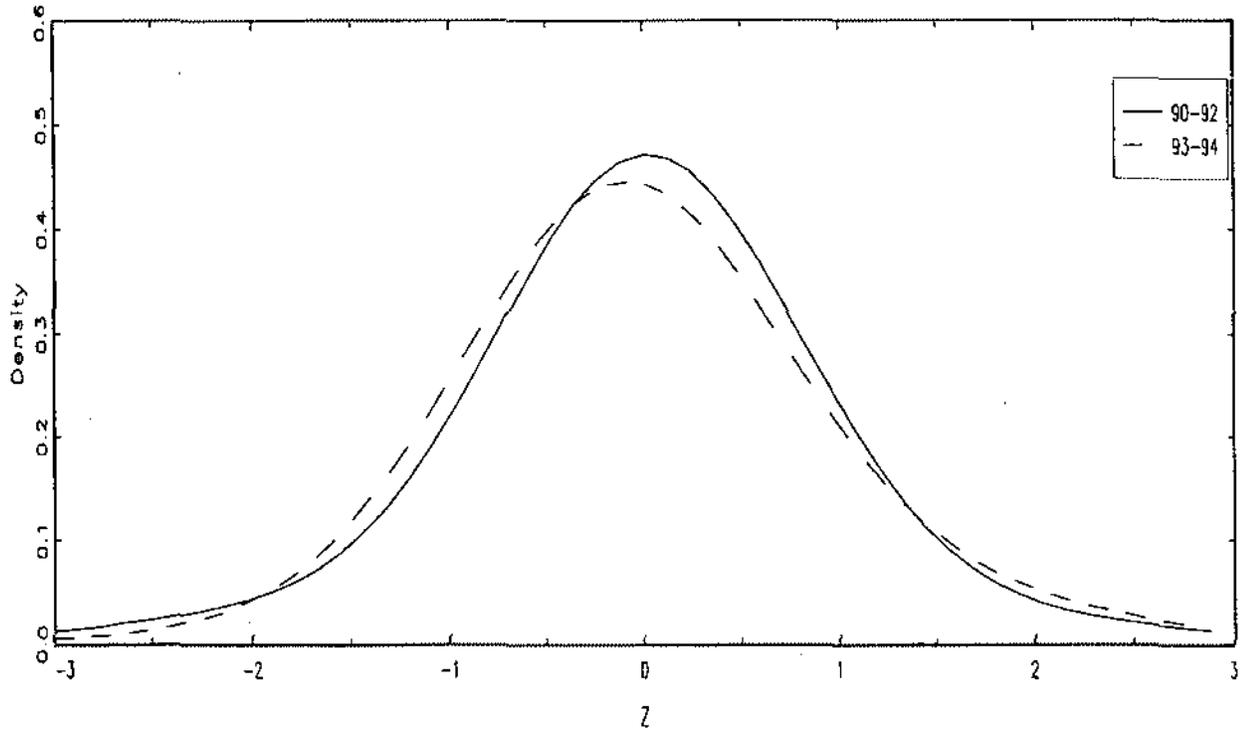


Figure 13  
Risk-Neutral Densities: 1990 - 1992 vs 1993 - 1994  
Eurodollar Futures Options: Maximum Sample Maturities



Risk-Neutral Densities: 1990 - 1992 vs 1993 - 1994  
Eurodollar Futures Options: Minimum Sample Maturities

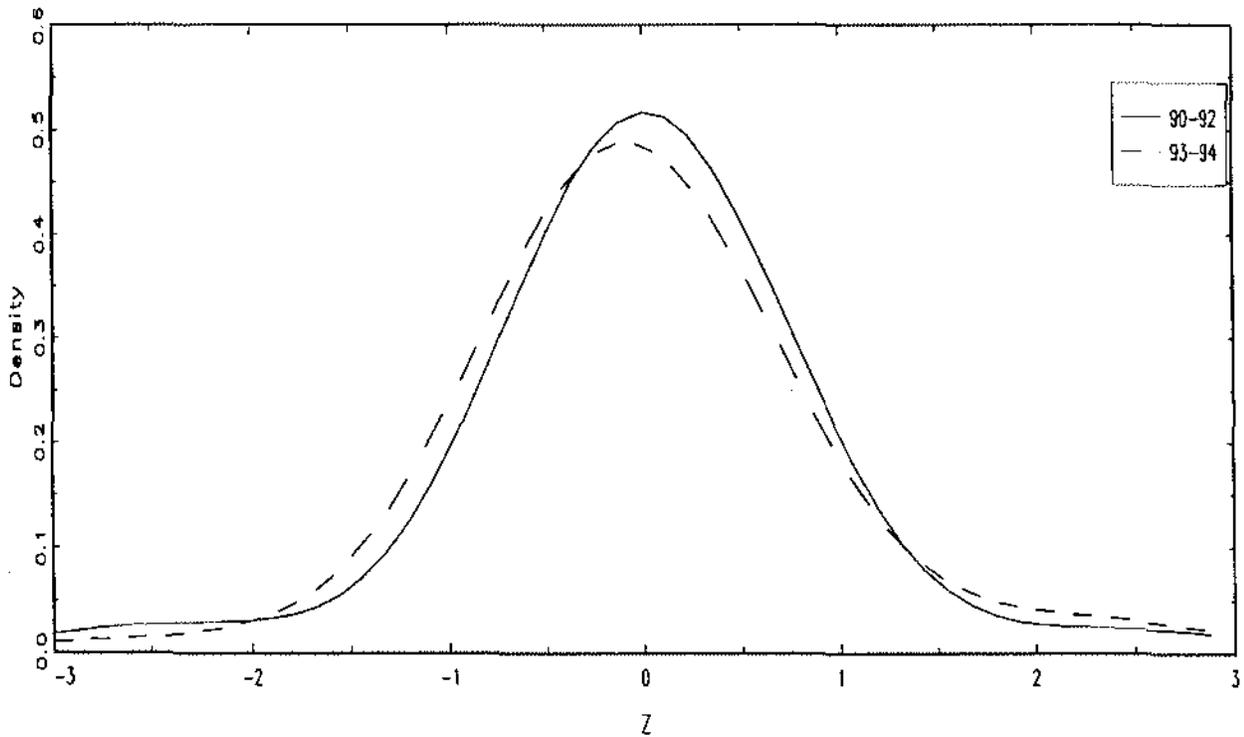


Figure 14 - Unrestricted Filtered Sample

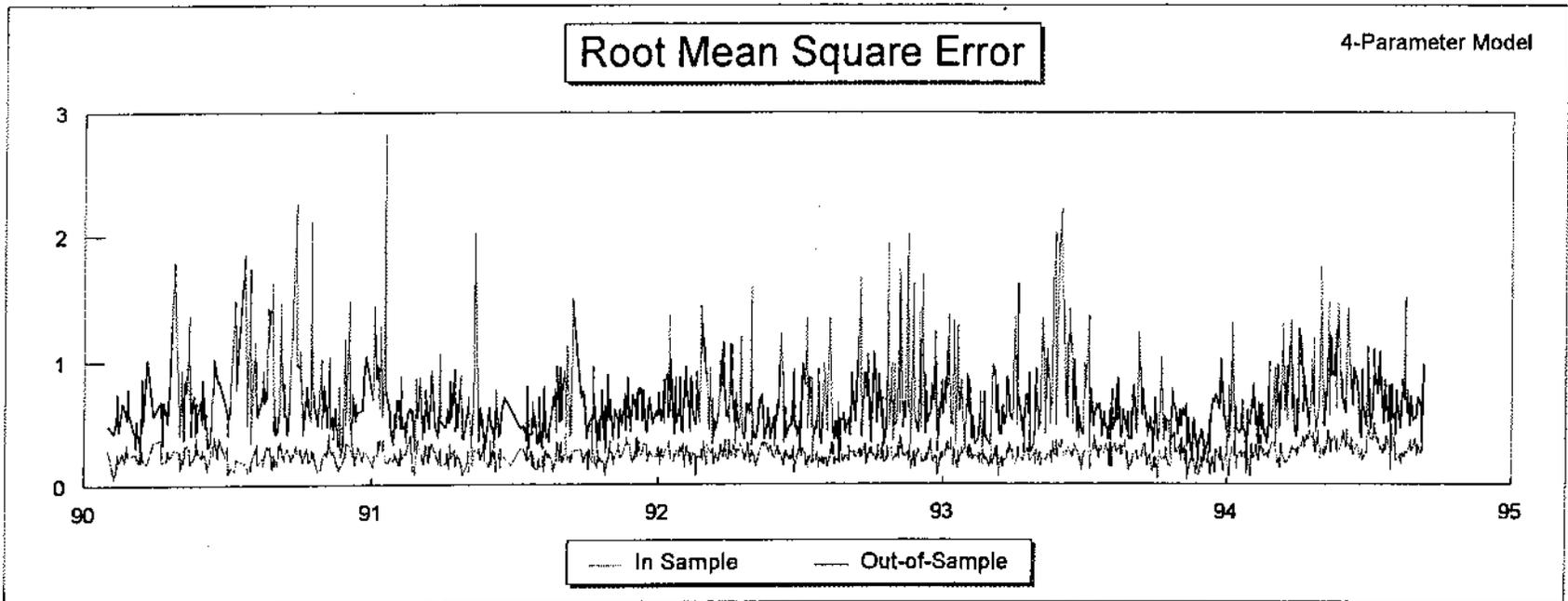
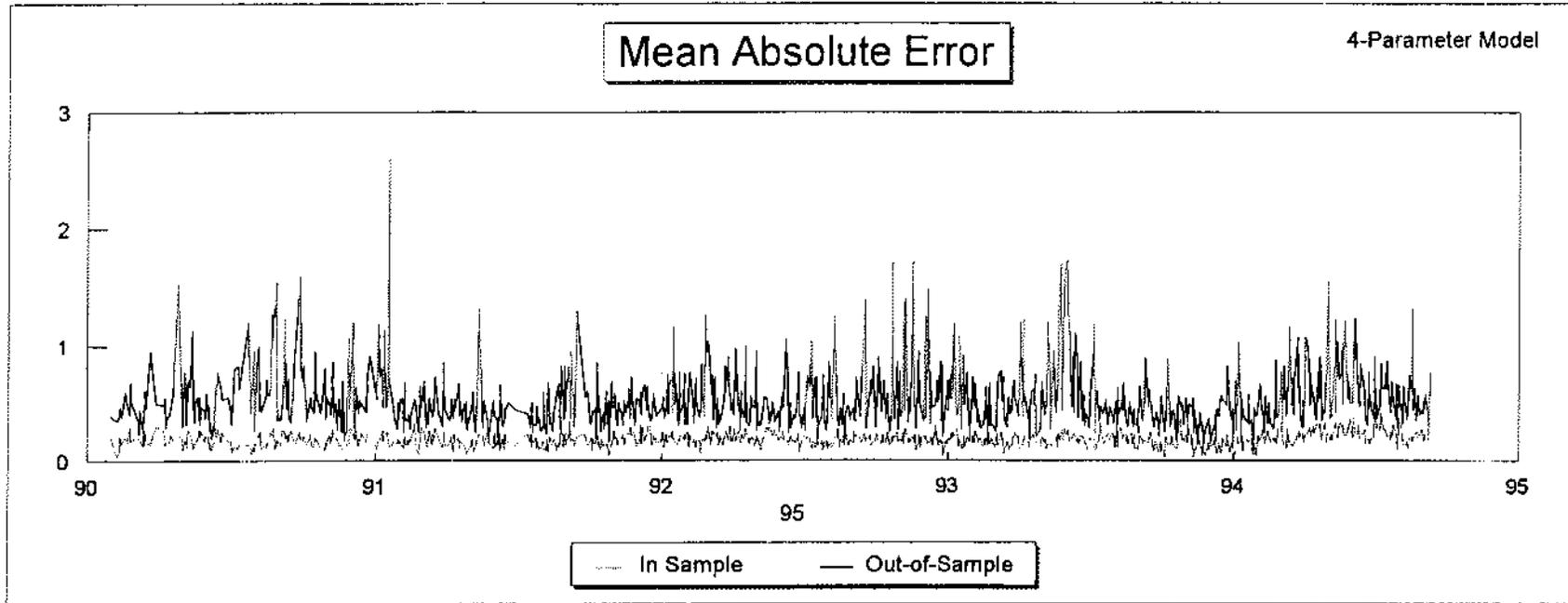


Figure 15 - Unrestricted Filtered Sample

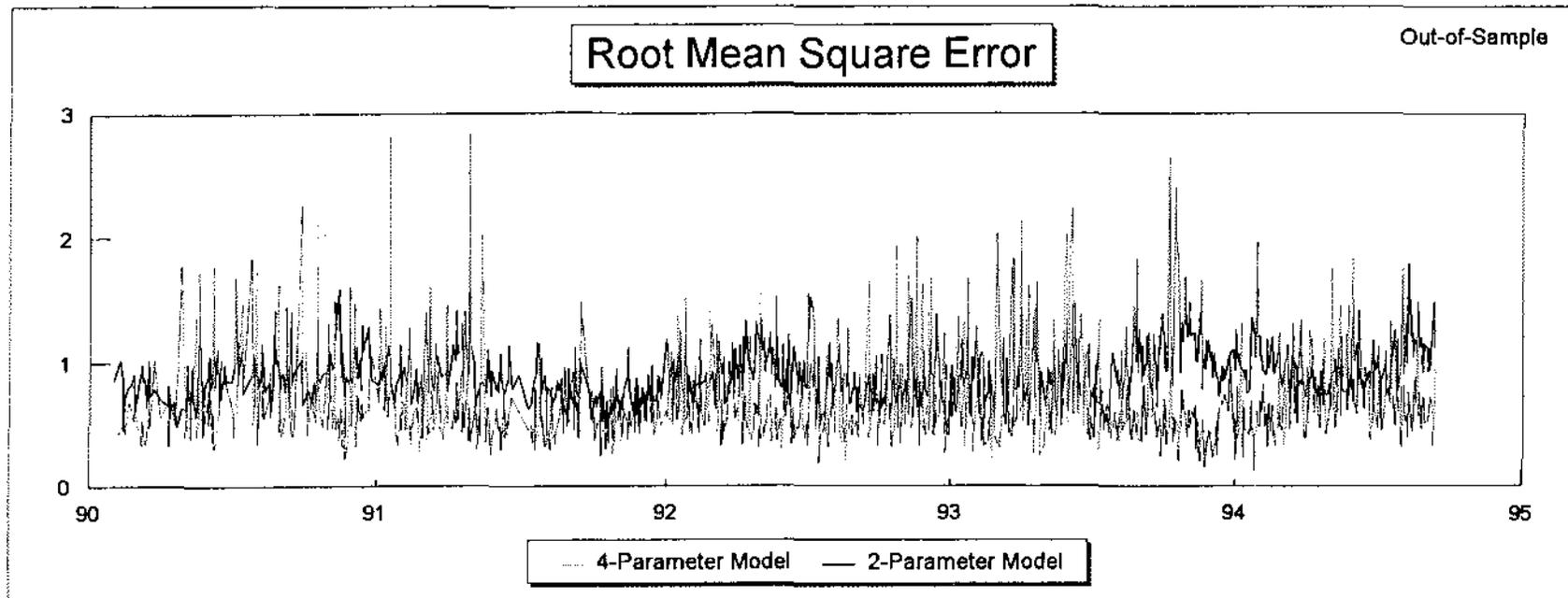
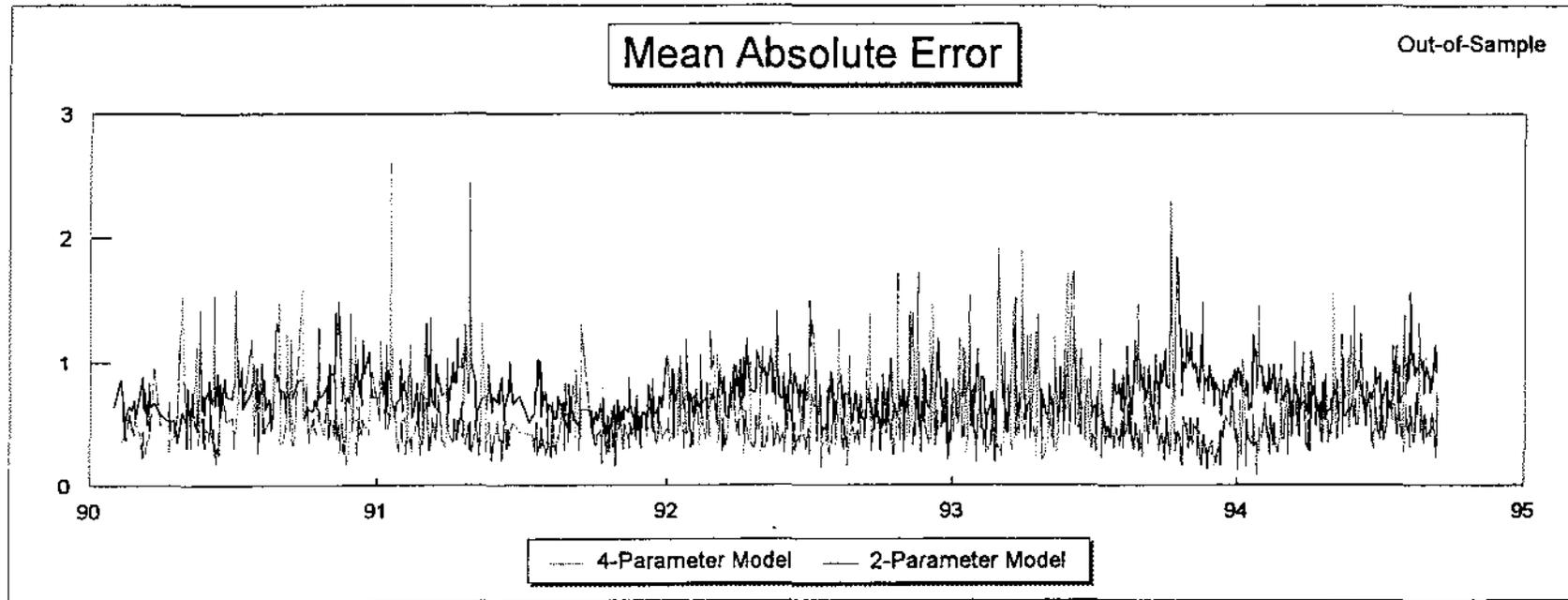


Figure 15 - Unrestricted Filtered Sample

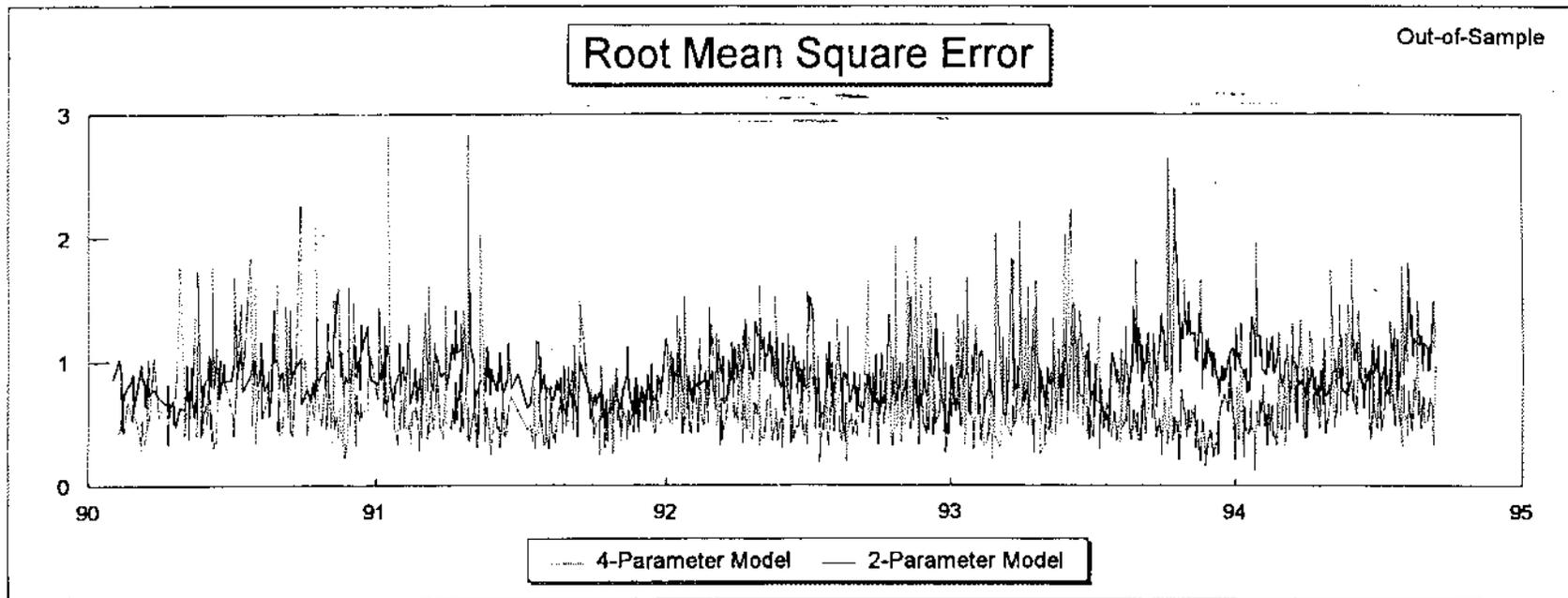
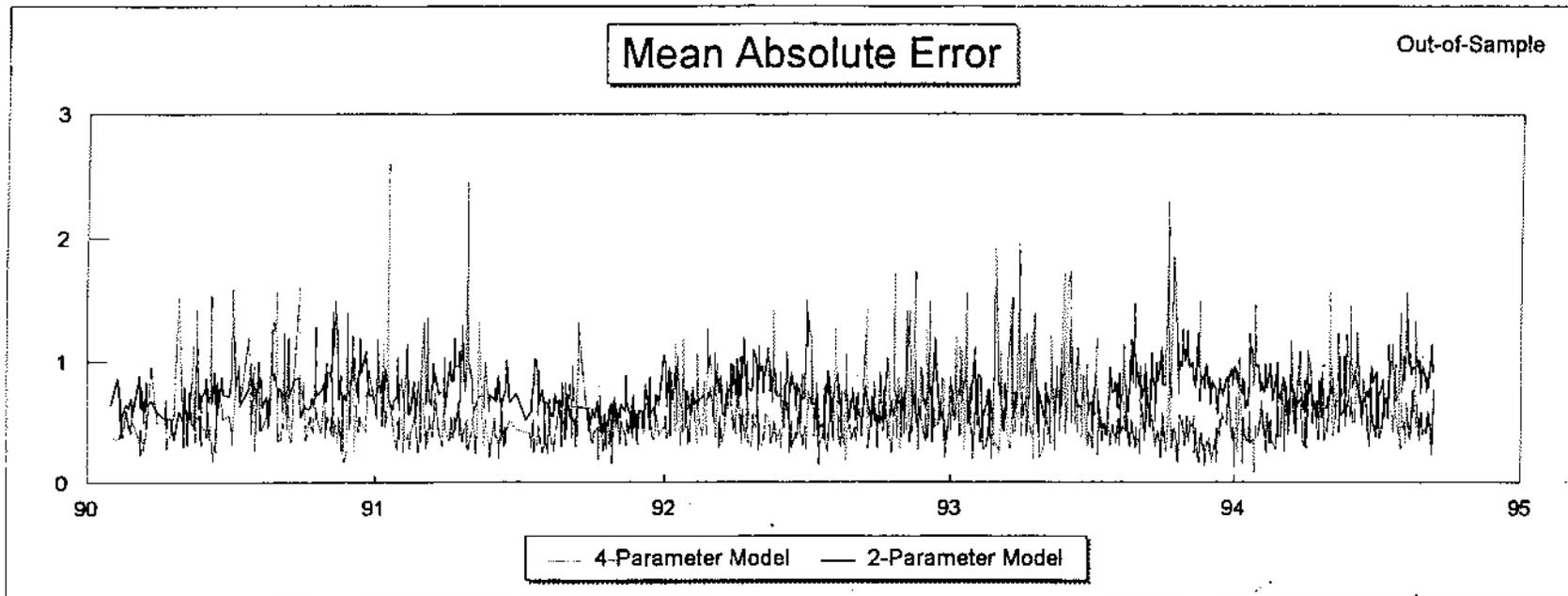
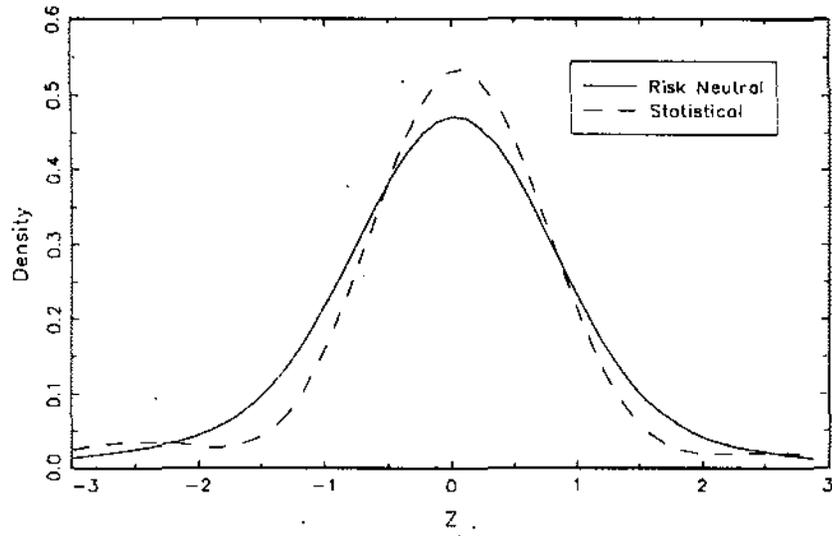
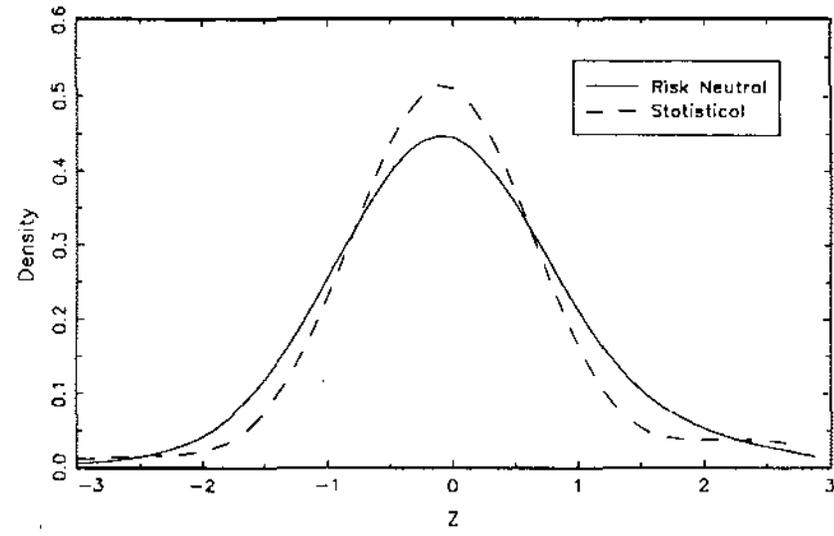


Figure 16

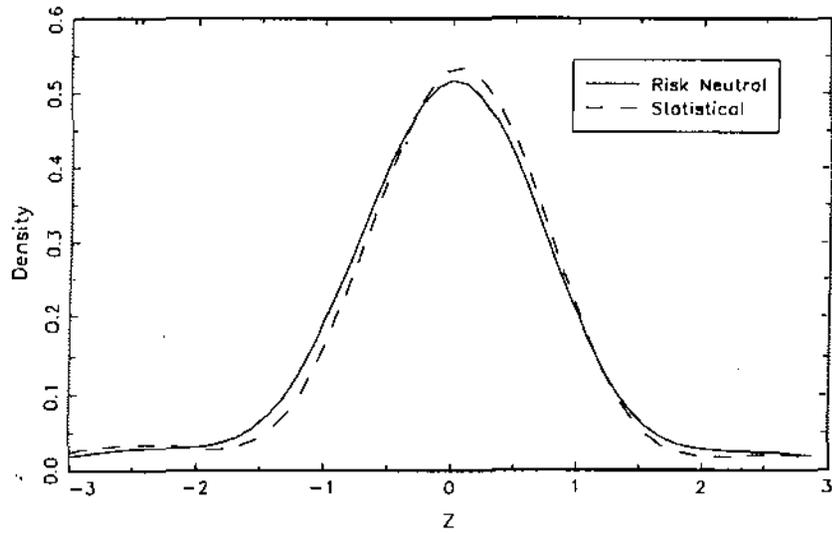
Risk-Neutral and Statistical Densities: 1990 - 1992  
Eurodollar Futures Options: Maximum Sample Maturities



Risk-Neutral and Statistical Densities: 1993 - 1994  
Eurodollar Futures Options: Maximum Sample Maturities



Risk-Neutral and Statistical Densities: 1990 - 1992  
Eurodollar Futures Options: Minimum Sample Maturities



Risk-Neutral and Statistical Densities: 1993 - 1994  
Eurodollar Futures Options: Minimum Sample Maturities

