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Endogenous Capital- and Labor-Augmenting Technical Change in the Neoclassical Growth Model

- Preliminary and Incomplete -

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Abstract: The determinants of the direction of technical change and the implications for economic growth are studied in the one-sector neoclassical growth model of Ramsey (1928), Cass (1965), and Koopmans (1965) extended to allow for endogenous capital- and labor-augmenting technical change. For this purpose, we develop a novel micro-foundation for the competitive production sector. It rests upon the idea that the fabrication of the final good requires tasks to be performed by capital and labor. Firms may engage in innovation investments that increase the productivity of capital and labor in the performance of their respective tasks. These investments are associated with new technological knowledge that accumulates over time. We analyze a version of the model with only labor-augmenting and one with capital- and labor-augmenting technical change. When only labor-augmenting technical change is allowed for we find that steady-state growth depends on the efficient capital intensity and, thus, on household preferences. When it is included, capital-augmenting technical change must vanish in the steady state. Moreover, the mere feasibility of capital-augmenting technical change drastically changes the comparative-static properties of the steady state, e. g., household preferences loose their effect on steady-state growth.

Keywords: Endogenous Technical Change, Induced Innovation, Neoclassical Production.

JEL-Classification: O31, O33, O41,

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1 Introduction

This paper studies the determinants of the direction of technical change and highlights their implications for economic growth. To accomplish this, we integrate endogenous capital- and labor-augmenting technical change into the one-sector neoclassical growth model of Ramsey (1928), Cass (1965), and Koopmans (1965) and develop a novel micro-foundation for the competitive production sector. It rests upon the idea that the fabrication of the final good requires tasks to be performed. Some tasks will be carried out by capital, others by labor. Firms may engage in innovation investments that increase the productivity of capital and labor in the performance of their respective tasks. These investments are associated with new technological knowledge that accumulates over time.

Our main findings may be summarized as follows. First, a key determinant of the direction of technical change is the relative scarcity of capital with respect to labor. This ratio turns out to affect the relative profitability of innovation investments. More precisely, if a factor of production becomes scarcer, then it also becomes more expensive. Accordingly, an investment enhancing the productivity of this factor becomes more advantageous and the direction of technical change shifts towards this factor.

Second, along the transition towards the steady state, the growth rate of the economy reflects both capital- and labor-augmenting technical progress. However, in steady state capital-augmenting technical progress vanishes. Hence, in the long run, the growth rate of per-capita variables is fully determined by labor-augmenting technical change.[1] The reason for this finding is closely related to the generalization of Uzawa's steady-state growth theorem established in (Irmen (2013a)): under full employment of both factors of production the net output function of the economy under scrutiny here is shown to exhibit constant returns to scale in capital and labor. Therefore, there cannot be capital-augmenting technical progress in steady state.

The third set of results relates to the comparative-static properties of the steady state. The steady-state growth rate is predicted to increase in a parameter reflecting the positive effect of institutions, technical infrastructure, or geography on the efficiency of the production process. However, other parameters that often bring about growth effects such as the discount factor of the representative household or the population growth rate have no impact on the steady-state growth rate. The mere feasibility of capital-augmenting technical change is shown to be the reason for this. Due to its presence, the steady-state growth rates of capital- and labor-augmenting technical change are determined by the properties of the production sector alone.

Fourth, we show that the model with endogenous capital- and labor-augmenting technical change nests several important variants of the one-sector neoclassical growth model.

[1] See, e. g., Klump, McAdam, and Willman (2007) for an empirical study of the US economy that confirms this pattern of technical change for the period 1953 to 1998.

For instance, the economy can readily be simplified to one that allows only for endogenous labor-augmenting technical change. We emphasize this framework in our analysis since it helps to elicit the role of capital-augmenting technical change. We establish that absent of the latter, the population growth rate has a negative effect on the steady-state growth rate of the economy. Finally, we observe that the model variant with endogenous labor-augmenting technical change can be turned into the standard neoclassical growth model with or without exogenous technical change.

Fifth, we study the local stability properties of the steady state and establish saddle-path stability.

Finally, we link our analysis to the so-called "induced innovations" literature of the 1960s.[2] This literature stipulates a somewhat arbitrary innovation possibility frontier and firms choose their technology to maximize the current rate of cost reduction.[3] We show that our framework give rise to an innovation possibility frontier that emerges endogenously. This frontier is time-invariant and states a functional relationship between the equilibrium growth rates of capital- and labor-augmenting technical progress.

This paper is organized as follows. Section 2 relates the paper to the existing literature. Section 3 presents the details of the model and establishes the existence of an endogenous innovation possibility frontier. Section 4 defines the general equilibrium in the economy under scrutiny. In section 5 we set up the dynamical system of the economy. The main results are contained in Sections 5.1 and 5.2. Section 6 concludes. All proofs are contained in Section 7, the Appendix.

2 Related Literature

The present paper builds on and contributes to several strands of the literature. First, it is a natural extension of the standard neoclassical growth framework. We show that the dynamical system of the economy under scrutiny with endogenous capital- and labor-augmenting technical change nests the previously known specifications with and without exogenous technical change and several, thus far, unknown specifications. They include the economy with endogenous labor-augmenting technical change and the economy with capital- as well as labor-augmenting technical change.

Moreover, the paper relates to the theory of directed technical change. Interest in explaining the direction of technical change rests upon the observation that technical progress requires resource expenditure and is subject to economic market incentives, or, in other words, that technical progress is endogenous. If labor and capital are independent production factors, there might be times when it is reasonable to invest resources

[2] See, **?**, Kennedy (1964), Samuelson (1965), or Drandakis and Phelps (1966).

[3] This approach has been criticized by, e. g., Nordhaus (1973), Funk (2002), or Acemoglu (2003a).

in increasing the efficiency of one factor rather than the other. It is then natural to ask when it is relatively more profitable to invest resources in increasing the efficiency of labor rather than capital and vice versa. In other words, what determines whether technical change is labor-augmenting or capital-augmenting.[4]

In this respect, our paper contributes to both the modern theory of directed technical change as well as to the 'old' induced innovations literature. It complements the modern theory developed by Acemoglu (2003b), Acemoglu (2009), and Irmen (2013a). Acemoglu's model of directed technical change is an extended version of the expanding varieties model of endogenous growth in which the final good is produced using two types of intermediate goods. One type is produced with labor, while the other is produced using only capital. Each of the two types of intermediate goods is produced and supplied by monopolists. An increase in the number of the two types of intermediate goods represents, respectively, capital- or labor-augmenting technical change. Irmen introduces capital- and labor-augmenting technical change in an overlapping generations economy and studies the effects of population aging on the direction of technical change.

With regard to the induced innovations literature of the 1960s, our paper overcomes the lack of microfoundations in the latter. The idea of an IPF was introduced by von Weizsäcker (2010) and Kennedy (1964) who argue that the objective of technical improvements is a reduction in unit costs of production: any technical improvement reduces the amount of labor and the amount of capital required to produce a unit of product in a certain proportion. They formalize this idea suggesting an IPF which is assumed to be a strictly concave, downward sloping locus describing all technical improvements available to the firm at any time. Samuelson (1965), Drandakis and Phelps (1966), and Nordhaus (1967) embed this idea into a neoclassical growth framework. In their models technical change is factor-augmenting and the IPF depicts the fixed menu from which a firm can choose the desired labor and capital augmentation. One of the main concerns with the induced innovation literature was the exogenous nature of the IPF. It was independent of the state of the economy and simply "falls from the sky every period"[5]. The firm is assumed to choose an innovation along the IPF but it remains unexplained how the firm finds the IPF. Moreover, it can choose from a fixed menu of alternative technologies with varying degrees of labor- and capital-augmentation without any expenses for R&D or innovation adoption.[6]

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[4]The first answer to this question was suggested by Hicks (1932) who argued that technical change is directed toward saving the factor that becomes relatively more expensive, or, in other words, that technical change is induced by changes in relative factor prices. See also Fellner (1961) for this line of thought.

[5]Nordhaus (1967), p. 65.

[6]See also Elster (1983).

3 The Model

We consider a competitive closed economy in discrete time, i. e., $t = 0, 1, \ldots$, with a household sector and a production sector. In each period there is a single final good that can be consumed or invested. If invested, it is either accumulated as capital or it serves as innovation investments that raise the productivity of capital, labor, or both. Firms are owned by the households. In each period produced output serves as numeraire.

3.1 Household Sector

The economy is populated by a continuum of identical, infinitely-lived households of measure one. Under this assumption and those that follow it is admissible to study the household sector through the lens of a single representative household. It cares about the utility its members derive from consumption. More precisely, denote

$$u(\hat{c}_t) = \ln \hat{c}_t \tag{3.1}$$

the utility a household member derives if it consumes $\hat{c}_t$ units of the final good at t. At time t the representatvie household has $L_t = (1+\lambda)^t$ members, where $\lambda > -1$ is the population growth rate and the population size at $t = 0$ is normalized to unity. At all t, household members are endowed with one unit of labor which is inelastically supplied. Hence, L_t is also equal to the aggregate labor supply at t. Let $\beta \in (0,1)$ denote the discount factor. Then, the representative household evaluates sequences of consumption $\{c_t\}_{t=0}^{\infty}$ according to

$$U_0 = \sum_{t=0}^{\infty} [\beta(1+\lambda)]^t \ln \hat{c}_t, \tag{3.2}$$

where $\beta(1+\lambda) < 1$.

The representative household owns all firms and the capital stock. Since profits, i.e., dividends, vanish in equilibrium, we shall not explicitly account for the profit distribution. Capital is the only asset in the economy. Capital at t is installed at $t-1$ and firms pay a real rental rate, R_t, per unit of capital they use. For simplicity, and without loss of generality, we assume that the capital stock fully depreciates after one period. Then the household's flow budget constraint at time t in per capita terms may be written as

$$(1+\lambda)\hat{k}_{t+1} = R_t \hat{k}_t + w_t - \hat{c}_t, \tag{3.3}$$

where $\hat{k}_t \equiv K_t / L_t$ denotes capital holdings per capita and w_t is the real wage.

Given $\hat{k}_t > 0$, the representative household maximizes utility U_0 of equation (3.2) subject to (3.3), $\hat{c}_t \geq 0$, $\hat{k}_{t+1} \geq 0$ for all t, and an appropriate No-Ponzi Game condition by

choosing a sequence $\{\hat{c}_t\}_{t=0}^{\infty}$. The solution to this problem satisfies the flow budget constraint (3.3), the Euler condition,

$$\frac{\hat{c}_{t+1}}{\hat{c}_t} = \beta R_{t+1}, \tag{3.4}$$

and the transversality condition

$$\lim_{t\to\infty} [\beta(1+\lambda)]^t \hat{k}_{t+1} = 0. \tag{3.5}$$

3.2 The Production Sector

3.2.1 Technology

The production sector of the economy is represented by a continuum of identical, competitive firms of measure one. Without loss of generality, the analysis proceeds through the lens of a competitive representative firm. To produce output two types of tasks need to be performed. Denote by $m \in \mathbb{R}_+$ a task performed by capital, and let $n \in \mathbb{R}_+$ be a task performed by labor. Further, denote by m_t and n_t the measure of all tasks of the respective type performed at time t such that $m \in [0, m_t]$ and $n \in [0, n_t]$. Tasks of the respective type are identical. Therefore, total output hinges only on m_t and n_t. More precisely, the representative firm has access to the production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ which assigns the maximum output, Y_t, to each pair $(m_t, n_t) \in \mathbb{R}_+^2$, i. e.

$$Y_t = F(m_t, n_t), \tag{3.6}$$

where F has constant returns to scale in its arguments and is $\mathcal{C}^2$ with $F_1 > 0 > F_{11}$ and $F_2 > 0 > F_{22}$. Let κ_t denote the period-t task intensity,

$$\kappa_t = \frac{m_t}{n_t}. \tag{3.7}$$

Then, output in intensive form is $F(\kappa_t, 1) \equiv f(\kappa_t)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$, with $f'(\kappa_t) > 0 > f''(\kappa_t)$ for all $\kappa_t > 0$.

At t a task m requires $k_t(m) = 1/b_t(m)$ units of capital whereas a task n needs $l_t(n) = 1/a_t(n)$ units of labor. Hence, $b_t(m)$ and $a_t(n)$ denote the productivity of capital and labor in the respective task. These productivities are given by

$$b_t(m) = B_{t-1}(1-\delta)(1+q_t^B(m)) \quad \text{and} \quad a_t(n) = A_{t-1}(1-\delta)(1+q_t^A(n)), \tag{3.8}$$

where $\delta \in (0,1)$ represents the rate of depreciation of technological knowledge. Think of $B_{t-1}(1-\delta)$ and $A_{t-1}(1-\delta)$ as the level of technological knowledge inherited from the past. Then, $(q_t^B(m), q_t^A(n)) \in \mathbb{R}_+^2$ are indicators of productivity growth associated with task m and n, respectively.

To achieve positive productivity growth, i. e., $q^j > 0$, $j = A, B$, the firm must engage in an innovation investment. More precisely, at t it must invest $i(q_t^B(m)) > 0$ units of the final good to achieve $q_t^B(m) > 0$ and, similarly, $i(q_t^A(n)) > 0$ units of the final good to obtain $q_t^A(n) > 0$.

The function $i : \mathbb{R}_+ \to \mathbb{R}_+$ is the same for all tasks, time invariant, $\mathcal{C}^2$ on $\mathbb{R}_{++}$, strictly increasing and strictly convex in q. Moreover, denoting $i'(q^j) = di(q^j)/dq^j$ for $j = A, B$, it satisfies the following regularity conditions:

$$i(0) = 0, \quad \lim_{q^j \to 0} i'(q_t^j) = 0, \quad \lim_{q \to \infty} i'(q_t^j) = \lim_{q^j \to \infty} i(q_t^j) = \infty. \tag{3.9}$$

Notice that (3.9) implies that satisfies an Inada-type condition since the first marginal unit of q^j is costless.

Any new peace of technological knowledge is proprietory knowledge of a particular firm only in the period when it occurs. Subsequently, it becomes public and embodied in aggregate economy-wide productivity indicators $(A_t, B_t), (A_{t+1}, B_{t+1}), ...$ (to be specified below). If at t the firm makes no investment in a productivity enhancing technology, it has access to the economy-wide technology given by $A_{t-1}(1-\delta)$ and $B_{t-1}(1-\delta)$ such that its task-specific productivity of labor and capital is given by $a_t(n) = A_{t-1}(1-\delta)$ and $b_t(m) = B_{t-1}(1-\delta)$.

3.2.2 Firm's Optimization

The representative firm takes the sequence $\{R_t, w_t, A_{t-1}, B_{t-1}\}_{t=0}^{\infty}$ of real wages, real rental rates of capital, and aggregate productivity indicators as given. Its choice involves a production plan comprising a sequence

$$\left\{m_t, n_t, k_t(m), l_t(n), q_t^B(m), q_t^A(n)\right\}_{t=0}^{\infty}$$

for $m \in [0, m_t]$ and $n \in [0, n_t]$, respectively. Because an innovation investment generates private knowledge only in the period when it is made, the intertemporal profit maximization problem of the firm boils down to the maximization of per-period profits given by

$$F(m_t, n_t) - C_t, \tag{3.10}$$

where C_t is the firm's total cost, comprising factor cost and investment outlays for all performed tasks. In other words,

$$C_t = \int_0^{m_t} \left[R_t k_t(m) + i(q_t^B(m))\right] dm + \int_0^{n_t} \left[w_t l_t(n) + i(q_t^A(n))\right] dn,$$

where,

$$k_t(m) = \frac{1}{B_{t-1}(1-\delta)(1+q_t^B(m))} \quad \text{and} \quad l_t(n) = \frac{1}{A_{t-1}(1-\delta)(1+q_t^A(n))}.$$

The maximization of (3.10) can be split up into two parts. First, for each $m \in [0, m_t]$ and $n \in [0, n_t]$ a choice of $q_t^B(m)$ and $q_t^A(n)$ must minimize C_t. This leads to the first-order sufficient conditions

$$q_t^B(m): \quad \frac{-R_t}{B_{t-1}(1-\delta)(1+q_t^B(m))^2} + i'(q_t^B(m)) = 0, \quad \forall m \in [0, m_t], \tag{3.11}$$

$$q_t^A(n): \quad \frac{-w_t}{A_{t-1}(1-\delta)(1+q_t^A(n))^2} + i'(q_t^A(n)) = 0, \quad \forall n \in [0, n_t], \tag{3.12}$$

In light of (3.9), and assuming $w_t > 0$ and $R_t > 0$, the conditions (3.11) and (3.12) determine a unique $q_t^A(n) = q_t^A > 0$ and $q_t^B(m) = q_t^B > 0$. Accordingly, $a_t(n) = a_t$ and $b_t(m) = b_t$.

In a second step the firm decides how many tasks of either type to perform. From the cost minimum we know that the cost of all tasks of each type will be the same. Hence, the respective first-order sufficient conditions are given by

$$f'(\kappa_t) - \left(\frac{R_t}{B_{t-1}(1-\delta)(1+q_t^B)} + i(q_t^B) \right) = 0, \tag{3.13}$$

$$f(\kappa_t) - \kappa_t f'(\kappa_t) - \left(\frac{w_t}{A_{t-1}(1-\delta)(1+q_t^A)} + i(q_t^A) \right) = 0. \tag{3.14}$$

In economic terms, the firm produces tasks up to the point at which the marginal value product of the last produced task of the respective type is equal to its (marginal) cost.

With $q^B(m) = q^B > 0$ and $q^A(n) = q^A > 0$, equations (3.13) and (3.14) guarantee that the number of tasks will be chosen optimally. Notice that Π_t has CRS in (m_t, n_t) at $q^B(m) = q^B$ and $q^A(n) = q^A$. Therefore, equations (3.13) and (3.14) will only pin down the task intensity $\kappa_t = m_t/n_t$. The number of tasks performed in equilibrium will be determined by market clearing conditions. We summarize the main result arising from the firm's optimality conditions in the following proposition.

Proposition 1 *Suppose* (3.11)-(3.14) *are satisfied. Then, the following holds:*

1. *There are maps* $g^A : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ *and* $g^B : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ *such that for all* $\kappa_t > 0$:

$$q_t^B = g^B(\kappa_t), \quad \textit{with} \quad g_\kappa^B < 0, \tag{3.15}$$

$$q_t^A = g^A(\kappa_t), \quad \textit{with} \quad g_\kappa^A > 0. \tag{3.16}$$

2. *There are maps* $w : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ *and* $R : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ *such that the real wage and the real rental rate of capital satisfy*

$$R_t = R(\kappa_t, B_{t-1}) \quad \textit{with} \quad R_\kappa < 0, \quad R_B > 0. \tag{3.17}$$

$$w_t = w(\kappa_t, A_{t-1}) \quad \textit{with} \quad w_\kappa > 0, \quad w_A > 0, \tag{3.18}$$

Claim 1 highlights the role of the task intensity for productivity growth rates of capital and labor in their respective tasks. While capital-augmenting technical change decreases in the task intensity, labor-augmenting technical change increases in it. Claim 2 reveals that the task intensity has a similar effect on factor prices as well. The real rental rate of capital decreases in the task intensity and the real wage increases. In addition, the real rental rate of capital and the real wage depend positively on previous period's aggregate level of technology.

To gain intuition for these findings consider two things: the role the task intensity plays for the equilibrium incentives to engage in innovation investments and its effect on equilibrium factor prices. In equilibrium, the first-order conditions (3.11) and (3.13) require the marginal product of the last task performed by capital to be equal to the minimum cost of performing that task. Now, suppose the task intensity, κ_t, increases. Then, because of diminishing returns, the marginal value product of tasks performed by capital decreases and requires a lower rental rate of capital, R_t. However, a decrease in R_t implies a smaller marginal benefit of investing in productivity growth of capital and therefore leads to a lower level of q_t^B.

The following corollary shows that Claim 1 of Proposition 1 implies a time-invariant innovation possibility frontier, whereas Claim 2 implies a time-varying factor price frontier.

Corollary 1 *(Innovation Possibility Frontier and Factor Price Frontier)*

Suppose (3.11)-(3.14) *are satisfied. Then, the following holds:*

1. *There is a time-invariant innovation possibility frontier, i. e., there is a function* $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ *such that*

$$q^B = g(q^A), \quad \textit{with} \quad g'(q^A) < 0. \tag{3.19}$$

2. *There is a time-varying factor price frontier, i. e., there is a function* $h : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ *such that*

$$R_t = h(w_t, A_{t-1}, B_{t-1}) \quad \textit{with} \quad h_w(w_t, A_{t-1}, B_{t-1}) < 0. \tag{3.20}$$

Statement 1 of the corollary establishes the existence of an innovation possibility frontier (IPF). It describes the relationship between productivity growth rates of capital and labor in performing their respective tasks. More precisely, the IPF relates a unique productivity growth rate of capital to any attainable productivity growth rate of labor.

Statement 2 of the corollary establishes the existence of a factor price frontier. The factor price frontier describes the relationship between the marginal products - and under

perfect competition, prices - of the factors of production. More precisely, it depicts the price of one of the factors for any given level of the other. The position of the factor price frontier at t depends on inherited aggregate technological knowledge, A_{t-1} and B_{t-1}. Hence, unlike the standard neoclassical model, here the factor price frontier moves over time.

3.3 The Evolution of Technological Knowledge

Let A_t and B_t be the highest level of labor productivity and capital productivity attained across all tasks of the respective type at t, i. e.,

$$A_t = \max\left\{a_t(n) = A_{t-1}(1-\delta)(1+q_t^A(n)) \mid n \in [0, n_t]\right\},$$

$$B_t = \max\left\{b_t(m) = B_{t-1}(1-\delta)(1+q_t^B(m)) \mid m \in [0, m_t]\right\}.$$

Firm's optimization implies $q_t^B(m) = q_t^B$ and $q_t^A(n) = q_t^A$, as well as $a_t(n) = a_t$ and $b_t(m) = b_t$ so that

$$\begin{aligned} A_t = a_t &= A_{t-1}(1-\delta)(1+q_t^A), \\ B_t = b_t &= B_{t-1}(1-\delta)(1+q_t^B), \end{aligned} \tag{3.21}$$

for all $t = 0, 1, 2, \cdots$ with $A_{-1} > 0$ and $B_{-1} > 0$ given.

4 Dynamic Competitive Equilibrium

Definition 1 *Given $L_t = (1+\lambda)^t$, initial values of the physical capital stock, $K_0 > 0$, and of technological knowledge, $A_{-1} > 0$ and $B_{-1} > 0$, a dynamic competitive equilibrium is a sequence*

$$\left\{m_t, n_t, q_t^A(n), q_t^B(m), a_t(n), b_t(n), l_t(n), k_t(m), w_t, R_t, \hat{c}_t, \hat{k}_t, Y_t\right\}_{t=0}^{\infty},$$

for all $m \in [0, m_t]$ and $n \in [0, n_t]$, such that

(E1) the behavior of the representative household is described by (3.3), (3.4), (3.5), and $\hat{k}_0 > 0$.

(E2) the production sector satisfies Proposition 1,

(E3) for all t both factors are fully employed, i. e.,

$$\int_0^{m_t} k_t(m)dm = K_t \quad \text{and} \quad \int_0^{n_t} l_t(n)dn = L_t,$$

(E4) the productivity indicators evolve according to equation (3.21).

Condition $(E1)$ ensures household optimization while $(E2)$ requires optimal behavior of firms and zero profits. Full employment of the factors of production, $(E3)$, implies, for any symmetric configuration, that the total number of each task type is given by the amount of the respective *production factor in efficiency units*, i. e.,

$$m_t = b_t K_t \quad \text{and} \quad n_t = a_t L_t. \tag{4.1}$$

Moreover, observe that $(E2)$ together with the evolution of technological knowledge, $(E4)$, imply that in equilibrium the task intensity, as given by eq. (3.7), may be rewritten as

$$\kappa_t = \frac{b_t K_t}{a_t L_t} = \frac{B_{t-1}(1 + g^B(\kappa_t))K_t}{A_{t-1}(1 + g^A(\kappa_t))L_t}. \tag{4.2}$$

Thus, in equilibrium the task intensity corresponds to efficient capital per unit of efficient labor, or, 'efficient capital intensity'. The following proposition establishes a unique value $\kappa_t > 0$ that satisfies (4.2).

Proposition 2 *There is a unique equilibrium task intensity $\kappa_t > 0$, i. e.,*

$$\kappa_t = \kappa\left(\frac{B_{t-1}K_t}{A_{t-1}L_t}\right),$$

that satisfies eq. (4.2).

5 The Dynamical System

The evolution of the economy is fully captured by three variables: efficient capital intensity, κ_t, consumption per efficient unit of labor, $c_t \equiv \frac{\hat{c}_t}{A_t}$, and the economy-wide stock of capital-augmenting knowledge, B_t. As we show in the following proposition, this leads to a system of three non-linear difference equations, in which κ_t and B_t are the two state variables, and c_t is the control variable. Before we state the dynamical system we introduce the notion of net output.

We define net output as the difference between final good production and total innovation investment. Consider a symmetric technology choice in the sense that the firm invests the same amount into all tasks of the same type, i. e., $q_t^B(m) = q_t^B$, and $q_t^A(n) = q_t^A$. Hence, net output is given by

$$V(m_t, n_t) \equiv F(m_t, n_t) - n_t i(q_t^A) - m_t i(q_t^B). \tag{5.1}$$

Evaluating the latter *at firms' optimum* and *at full employment of factors of production*, i. e., using (E2) and (E3), respectively, delivers *net equilibrium output*:

$$V(B_tK_t, A_tL_t, \kappa_t) \equiv F(B_tK_t, A_tL_t) - B_tK_t i(g_t^B(\kappa_t)) - A_tL_t i(g_t^A(\kappa_t)). \tag{5.2}$$

Notice that $V(B_tK_t, A_tL_t, \kappa_t)$ has constant returns to scale in (B_tK_t, A_tL_t), a consequence of $F(m_t, n_t)$ having constant returns to scale in m_t, n_t. Thus, dividing both sides of (5.2) by A_tL_t we may express *net equilibrium output per unit of efficient labor* as

$$v(\kappa_t) = f(\kappa_t) - \kappa_t i(g_t^B)(\kappa_t) - i(g_t^A(\kappa_t)). \tag{5.3}$$

The dynamical system may now be stated as follows.

Proposition 3 *Given initial conditions* $\Big(A_{-1}, B_{-1}, K_0, L_0\Big) > 0$ *there is a unique equilibrium sequence* $\{\kappa_t, c_t, B_t\}_{t=0}^{\infty}$ *determined by*

$$(1+\lambda)\frac{(1+g^A(\kappa_{t+1}))}{(1+g^B(\kappa_{t+1}))}\kappa_{t+1} = B_t\Big(v(\kappa_t) - c_t\Big), \tag{5.4}$$

$$\frac{(1+g^A(\kappa_{t+1}))}{(1+g^B(\kappa_{t+1}))}c_{t+1} = \beta B_t[f'(\kappa_{t+1}) - i(g^B(\kappa_{t+1}))]c_t, \tag{5.5}$$

$$B_t = B_{t-1}(1-\delta)(1+g^B(\kappa_t)), \tag{5.6}$$

the transversality condition

$$[\beta(1+\lambda)] < 1,$$

and for $t = 0$, κ_0 *satisfies*

$$\kappa_0 = \frac{B_{-1}(1+g^B(\kappa_0))K_0}{A_{-1}(1+g^A(\kappa_0))L_0}. \tag{5.7}$$

Equation (5.4) corresponds to the economy's resource constraint and describes the evolution of efficient capital per unit of efficient labor. To obtain it substitute (3.13) and (3.14), using Claim 1 of Proposition 1, into the representative houshold's budget constraint, (3.3). The Euler equation, (5.5), results from (3.4) after Claim 1 of Proposition 1 and (3.13) have been used. Together with the transversality condition the resource constraint (5.4), the Euler equation (5.5), and the evolution of technological knowledge 5.6 form a three-dimensional system of first-order, non-linear difference equations. Starting from an initial value κ_0 they characterize a unique sequence for (κ_t, c_t, B_t).

Given the initial conditions, κ_0 is determined by equation (5.7). Using this in (5.6) gives a unique $B_t > 0$. The resource constraint describes a relation between c_t and κ_{t+1} for a given pair (κ_t, B_t). Then, for any given pair $(\kappa_t, B_t) \in \mathbb{R}^2_{++}$ the transversality

condition pins down the initial choice of consumption c_t and equation (5.4) gives a unique $\kappa_{t+1} > 0$. The Euler equation then determines a unique c_{t+1}.

Observe that the dynamical system of Proposition 3 is a natural extension of the standard neoclassical growth model (the Ramsey-Cass-Koopmans model). It is easily reducible, e. g., to the neoclassical growth model with exogenous technical change. In the latter, there is no capital-augmenting technical change nor is the labor-augmenting technical change endogenous. Hence, first of all this requires us to fix $q_t^B(m) = 0$, and $B_t = B = 1$. Moreover, assuming that the growth rate of labor productivity is exogenously given at some rate $q_t^A(n) > -1$ and costless, i. e., $i(q^A) = 0$, reduces the dynamical system to the one of the standard neoclassical growth model:

$$\begin{aligned}(1+\lambda)k_{t+1} &= f(k_t) - c_t,\\ c_{t+1} &= \beta f'(k_{t+1})(1+q^A)c_t,\end{aligned}$$

where k_t denotes capital per unit of efficient labor and $k_0 = K_0/A_{-1}(1-\delta)(1+q^A)L_0$. The above two first-order difference equations, together with the initial conditions and the tranversality condition, govern the path of c_t and k_t. There is a unique steady-state equilibrium in this model in which per-capita variables grow at the exogenous rate of labor-augmenting technological knowledge. Indeed, households' preferences may affect the effective capital-labor ratio κ^*, and, thus, the growth rate of the economy along the transition to steady state, but in steady state the growth rate is exogenous. In fact, if $q^A = \frac{\delta}{1-\delta}$, then $A_t = A_{t-1}$ for all t and we are in the Ramsey-Cass-Koopmans economy without technical change and without per capita growth.

We now turn to the effects of introducing endogenous technical change into the standard neoclassical growth model. To illuminate the role of endogenous technical change for the steady-state growth rate and, consequently, for the predictions of the (extended) neoclassical model, we introduce, in turn, endogenous labor-augmenting technical change and capital-augmenting technical change.

5.1 Neoclassical Growth Model with Endogenous Labor-Augmenting Technical Change

Consider first an economy in which technical change is labor-augmenting and endogenous. There is no capital-augmenting technical change. The productivity growth rate of labor in performing tasks, q_t^A, at time t is the result of firms' optimization. Moreover, it is costly and requires innovation investments per task of $i(q_t^A) > 0$. For clarity, in this version of the model we define $\kappa_t \equiv \frac{K_t}{A_t L_t}$ to denote the capital-labor ratio in efficiency units. The evolution of the key variables in this economy may then be described as follows:

Proposition 4 *Given initial conditions* $\left(A_{-1}, K_0, L_0\right) > 0$, *there is a unique equilibrium sequence* $\{\kappa_t, c_t\}_{t=0}^{\infty}$. *determined by*

$$(1+\lambda)(1-\delta)(1+g^A(\kappa_{t+1}))\kappa_{t+1} = v(\kappa_t) - c_t, \tag{5.8}$$

$$(1-\delta)(1+g^A(\kappa_{t+1}))c_{t+1} = \beta f'(\kappa_{t+1})c_t, \tag{5.9}$$

the transversality condition

$$[\beta(1+\lambda)] < 1,$$

and for $t = 0$, κ_0 *is determined by*

$$\kappa_0 = \frac{K_0}{A_{-1}(1-\delta)(1+g^A(\kappa_0))L_0}. \tag{5.10}$$

Equation (5.8) represents the economy's resource constraint and obtains from the representative household's budget constraint using equilibrium factor prices (3.13) and (3.14) resulting from firm optimization. The Euler equation, (5.9), is obtained upon substitution of the firm's first-order condition (3.13) into (3.4).[7]

Given intitial conditions, κ_0 is determined by equation (5.10). For a given κ_t, the resource constraint describes the relationship between κ_{t+1} and c_t. Then, for any starting value $\kappa > 0$ the transversality condition picks the optimal choice of consumption. Equation (5.8) then delivers next period's capital-labor ratio in efficiency units. With next period's capital at hand, the Euler equation determines the optimal consumption in next period.

We now focus on the implications of endogenous labor-augmenting technical change for the rate of economic growth in steady state. A steady state is defined as the stationary solution to a dynamical system. For the dynamical system of Proposition 4 this implies a stationary solution to the two difference equations (5.8) and (5.9). In other words, we require, simultaneously, $\kappa_t = \kappa_{t+1} = \kappa^*$, and $c_t = c_{t+1} = c^*$.

Thus, a steady state is a solution to

$$(1+\lambda)\Big(1+g^A(\kappa^*)\Big)\kappa^* = v(\kappa^*) - c^*, \tag{5.11}$$

$$\Big(1+g^A(\kappa^*)\Big)c^* = \beta f'(\kappa^*)c^*. \tag{5.12}$$

The stationary solution of the system gives rise to a balanced growth path, i. e., a path $\left\{\hat{k}_t, \hat{c}_t, K_t, C_t, V_t\right\}_{t=0}^{\infty}$ along which per capita and aggregate variables are positive and grow at constant, possibly different and not necessarily positive, rates.

[7]Notice that in the model without capital-augmenting technical change the marginal product of tasks performed by capital is given by $R_t = f'(\kappa_t)$.

Proposition 5 *(Steady State when Labor-Augmenting Technical Change is Endogenous)*

1. *There is a unique* $\kappa^* \in (0, \infty)$*, if and only if*

$$\lim_{\kappa \to 0} f'(\kappa) > \frac{i'(\delta/(1-\delta))}{1-\delta} + i(\delta/(1-\delta)) > \lim_{\kappa \to \infty} f'(\kappa). \tag{5.13}$$

   If, in addition, the transversality condition

$$[\beta(1+\lambda)] < 1$$

   holds, there is a unique steady state satisfying $\kappa^* \in (0, \infty)$, $c^* \in (0, \infty)$.

2. *In steady state, per capita variables grow at rate*

$$g^* \equiv \frac{A_{t+1}}{A_t} = (1-\delta)(1 + g^A(k^*)) - 1.$$

   Aggregate variables grow at approximately $g^* + \lambda$.

3. *Moreover, the steady-state growth rate may be written as* $g^*(\beta, \delta)$*, where* β, δ *are the underlying parameters. Then*

$$\frac{\partial g^*(\beta,\delta)}{\partial \beta} > 0, \quad \textit{and} \quad \frac{\partial g^*(\beta,\delta)}{\partial \delta} > 0.$$

4. *The steady-state equilibrium is locally saddle-path stable.*

Statement 1 of Proposition 5 states that there is a finite profit-maximizing choice of κ if and only if, at the equilibrium allocation, a small (large) stock of efficient capital per unit of efficient labor has a sufficiently high (low) marginal value product. If, moreover, the transversality condition is satisfied, there is a unique steady state.

Statement 2 establishes that the steady-state growth rate of all per capita variables, g^*, is equal to the growth rate of the stock of labor-augmenting technological knowledge.

Statement 3 contains the main result of Proposition 5. The steady-state growth rate of the economy, g^*, is endogenous and depends on household preferences, as well as on the depreciation rate of labor-augmenting technological knowledge. The reason for this is that the productivity growth rate of labor, q_t^A, depends on the efficient capital-labor ratio in the steady state, which, in turn, depends on β and δ (and the functional form of the production function).[8]

According to Statement 4 of Proposition 5, the steady state is asymptotically locally saddle-path stable.

[8]Notice that with a CES utility function the growth rate would also depend on the intertemporal elasticity of substitution.

5.2 Neoclassical Growth Model with Capital- and Labor-Augmenting Technical Change

We now focus on the effects of adding endogenous capital-augmenting technical change to the model of the previous section. Allowing for endogenous captal-augmenting technical change in addition to endogenous labor-augmenting technical change significantly alters the structure of the model and consequently the steady state and its comparative static properties. As stated in Section 5 the model with both capital- and labor- augmenting technical change is fully described by the dynamical system of Proposition 3.

A steady state in this system implies a stationary solution to the three difference equations of which it consists. In other words, we require, simultaneously, $\kappa_t = \kappa_{t+1} = \kappa^*$, $c_t = c_{t+1} = c^*$, and $B_t = B_{t+1} = B^*$. Thus, a steady state is a solution to

$$(1+\lambda)(1-\delta)(1+g^A(\kappa^*))\kappa^* = B^*\Big(v(\kappa^*) - c^*\Big), \tag{5.14}$$

$$(1-\delta)(1+g^A(\kappa^*)) = \beta B^*[f'(\kappa^*) - i(g^B(\kappa^*))], \tag{5.15}$$

$$g^B(\kappa^*) = \frac{\delta}{1-\delta}. \tag{5.16}$$

The stationary solution of the system gives rise to a trajectory $\left\{\hat{k}_t, \hat{c}_t, K_t, C_t, V_t\right\}_{t=0}^{\infty}$ along which per capita and aggregate variables are positive and grow at constant, possibly different, rates.

To study the local stability properties of the steady state we denote the elasticity of the respective productivity growth factor with respect to the efficient capital intensity by

$$\epsilon_\kappa^B(\kappa_t) \equiv \frac{-g_\kappa^B(\kappa_t)\kappa_t}{1+g^B(\kappa_t)} > 0, \quad \epsilon_\kappa^A(\kappa_t) \equiv \frac{g_\kappa^A(\kappa_t)\kappa_t}{1+g^A(\kappa_t)} > 0.$$

Proposition 6 *(Steady State)*

1. *There is a $\kappa^* \in (0,\infty)$ if and only if*

$$\lim_{\kappa\to 0} f'(\kappa) > \frac{i'(\delta/(1-\delta))}{1-\delta} + i(\delta/(1-\delta)) > \lim_{\kappa\to\infty} f'(\kappa). \tag{5.17}$$

Moreover, there is a unique steady state satisfying $\kappa^* \in (0,\infty)$, $c^* \in (0,\infty)$ and $B^* \in (0,\infty)$ if, in addition,

$$\beta(1+\lambda) < 1. \tag{5.18}$$

2. *The steady-state growth rate of the economy is*

$$g^* = (1-\delta)(1+g^A(\kappa^*)) - 1.$$

More precisely, in steady state, we have

a) $\frac{\hat{c}_{t+1}}{\hat{c}_t} = \frac{\hat{k}_{t+1}}{\hat{k}_t} = \frac{w_{t+1}}{w_t} = 1 + g^*$

b) $\frac{V_{t+1}}{V_t} = \frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = (1+g^*)(1+\lambda)$

c) $B^* = \frac{(1+\lambda)(1+g^*)}{\beta[f'(\kappa^*) - i(g^B(\kappa^*))]}, \quad R^* = \frac{(1+\lambda)(1+g^*)}{\beta}, \quad g^B(\kappa^*) = \frac{\delta}{1-\delta}$

3. *The steady-state equilibrium is asymptotically locally stable in the state space if*

$$\frac{v'(\kappa^*)}{[f'(\kappa^*) - i(g^B(\kappa^*))]} < \beta(\epsilon_\kappa^A + \epsilon_\kappa^B + 1) \tag{5.19}$$

Statement 1 of Proposition 6 states that there is a finite profit-maximizing choice of κ if and only if, at the equilibrium allocation, a small (large) stock of efficient capital per unit of efficient labor has a sufficiently high (low) marginal value product. If, moreover, the transversality condition is satisfied, there is a unique steady state.

Statement 2 contains the main result of Proposition 6. It establishes that the steady-state growth rate of all per capita variables, g^*, is equal to the growth rate of the stock of labor-augmenting technological knowledge. Aggregate variables grow approximately at the rate $g^* + \lambda$. The rental rate of capital is constant.

The most important conclusion is that g^* is solely determined by the production side of the economy and is not anymore dependent on parameters of the houshold sector. The growth rate of the economy is determined by the fact that the level of capital-augmenting technological knowledge must remain constant in the steady state. Equation (5.6) then pins down the level of the efficient capital intensity in steady state, κ^*. This result is driven by the neoclassical structure of the model and the specific conditions it imposes for the existence of a balanced growth path. The intuition for this finding stems from the Generalized Growth Theorem of Irmen (2013b). This states that only labor-augmenting technical change can occur in the steady state of a neoclassical economy that uses some of its current output to generate technical progress. Asymptotically, capital-augmenting technical change cannot take place. In the current model this requires that the stock of capital-augmenting technical change remains constant over time. Thus, in the steady state there must be just enough capital-augmenting technical change to offset the depreciation of this stock. Having established κ^* it is the task of the Euler equation (5.5) to pin down the stock of capital-augmenting technological knowledge which will make sure that the household embarks on a consumption path with a constant growth rate equal to g^*. The resource constraint, (5.4), determines the steady-state level of consumption in efficiency units, c^*.

Under the condition in Assertion 3 the steady state is locally asymptotically stable in the state space.

6 Conclusion

TO BE WRITTEN

7 Appendix: Proofs

7.1 Proof of Proposition 1

Proof 1 *Claim 1.) Combine the first-order conditions* (3.11) *and* (3.13), *and* (3.12) *and* (3.14), *to obtain, respectively,*

$$\begin{aligned} f_\kappa(\kappa) =& i(q^B) + (1+q^B)i'(q^B), \\ f(\kappa) - \kappa f_\kappa(\kappa) =& i(q^A) + (1+q^A)i'(q^A). \end{aligned}$$

The properties of $f(\kappa)$ and $i(q^j)$ ensure the existence of some function $g^j : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $j = A, B$ such that (3.15) and (3.16) hold.

Claim 2.) Solving (3.11) and (3.12) for R_t and w_t, respectively, and using (3.15) and (3.16) delivers for all $\kappa_t > 0$:

$$\begin{aligned} R_t &= B_{t-1}(1-\delta)(1+g^B(\kappa_t))i'(q^B(\kappa_t)) \quad \equiv R(\kappa_t, B_{t-1}) >, \\ w_t &= A_{t-1}(1-\delta)(1+g^A(\kappa_t))i'(q^A(\kappa_t)) \quad \equiv w(\kappa_t, A_{t-1}) > 0, \end{aligned}$$

where $w : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ and $R : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$. The derivatives indicated in (3.18) and (3.17) follow immediately from the properties of $i(\cdot)$ and Claim 1. ■

7.2 Proof of Corollary 1

Proof 2 *Claim 1: The claim follows from equations* (3.15) *and* (3.16). *Without loss of generality, we suppress the time argument. Since g^A is increasing on its domain it is invertible. Let $G^A : \mathbb{R}_{++} \to \mathbb{R}_{++}$ denote the inverse of the function g^A. Then, from* (3.16), *$\kappa = G^A(q^A)$. Hence, with* (3.15), *we may write*

$$q^B = g^B(G^A(q^A)) \equiv g(q^A).$$

The slope of the function $g(q^A)$ is given by

$$g'(q^A) \equiv \frac{dq^B}{dq^A} = \frac{dg^B(\kappa)}{d\kappa}\frac{dG^A(q^A)}{dq^A} = \frac{g^B_\kappa(\kappa)}{g^A_\kappa(\kappa)} < 0. \tag{7.1}$$

Claim 2: The claim follows from equations (3.17) and (3.18). In equation (3.18), the function w is strictly increasing in κ_t on its domain. Holding technology A_{t-1} constant and viewing w only as a function of κ_t, it is invertible. Let $W : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ denote the inverse of the function w with respect to κ_t. Then, $\kappa_t = W(w_t : A_{t-1})$. Hence, with (3.17), we may write

$$R_t = R(W(w_t : A_{t-1}), B_{t-1}) \equiv h(w_t, A_{t-1}, B_{t-1}).$$

The partial derivative of $h(w_t, A_{t-1}, B_{t-1})$ with respect to w is given by

$$h_w(w_t, A_{t-1}, B_{t-1}) \equiv \frac{dR_t}{dw_t} = \frac{dR(\kappa_t, B_{t-1})}{d\kappa_t}\frac{dW(w_t : A_{t-1})}{dw_t} = \frac{R_\kappa(\kappa_t, B_{t-1})}{w_\kappa(\kappa_t, A_{t-1})} < 0. \tag{7.2}$$

■

7.3 Proof of Proposition 2

Proof 3 *The proof follows from the properties of $g^A(\kappa_t), g^B(\kappa_t)$ which we established in Proposition* 1. ■

7.4 Proof of Proposition 3

Proof 4 *To show uniqueness of the equilibrium sequence $\{\kappa_t, B_t, c_t, A_t\}_{t=0}^{\infty}$ we first need to derive equations (5.4) and (5.5).*

One may derive (5.4) using Claim 1 *of Proposition (1) in (3.13) and (3.14) to substitute for R_t and w_t in the household's budget constraint. Upon this substitution we may write the latter as*

$$(1+\lambda)\hat{k}_{t+1} = B_t\Big(f'(\kappa_t) - i(g^B(\kappa_t))\Big)\hat{k}_t + A_t\Big(f(\kappa_t) - \kappa_t f'(\kappa_t) - i(g^A(\kappa_t))\Big) - \hat{c}_t.$$

By definition, we have $\hat{k}_t = (A_t\kappa_t)/B_t$ and $\hat{c}_t = A_t c_t$, so that above equation may be written in terms of efficiency units as

$$(1+\lambda)\frac{A_{t+1}}{B_{t+1}}\kappa_{t+1} = B_t\Big(f'(\kappa_t) - i(g^B(\kappa_t))\Big)\frac{A_t}{B_t}\kappa_t + A_t\Big(f(\kappa_t) - \kappa_t f'(\kappa_t) - i(g^A(\kappa_t))\Big) - A_t c_t,$$

$$(1+\lambda)\frac{A_{t+1}}{B_{t+1}}\kappa_{t+1} = A_t\Big(f'(\kappa_t) - i(g^B(\kappa_t))\Big)\kappa_t + A_t\Big(f(\kappa_t) - \kappa_t f'(\kappa_t) - i(g^A(\kappa_t))\Big) - A_t c_t,$$

$$(1+\lambda)\frac{A_{t+1}}{A_t B_{t+1}}\kappa_{t+1} = v(\kappa_t) - c_t,$$

where the last line uses equation (5.3).

It is straightforward to see that employing equations (3.15) and (3.13) in equation (3.4) delivers the Euler equation, (5.5).

Now, we may write the system describing the evolution of the economy as

$$(1+\lambda)\frac{A_{t+1}}{A_t B_{t+1}}\kappa_{t+1} = v(\kappa_t) - c_t, \tag{7.3}$$

$$\frac{A_{t+1}}{A_t}c_{t+1} = \beta B_{t+1}[f'(\kappa_{t+1}) - i(g^B(\kappa_{t+1}))c_t, \tag{7.4}$$

$$A_t = A_{t-1}(1-\delta)(1+g^A(\kappa_t)), \tag{7.5}$$

$$B_t = B_{t-1}(1-\delta)(1+g^B(\kappa_t)). \tag{7.6}$$

Observe that this system of four first-order, non-linear difference equations may be reduced to a system of three equations. Forwarding equations (7.5) and (7.6) and substituting in (7.3) *and* (7.4) *we obtain the three-dimensional system of Proposition 3.*

To prove the uniqueness of the sequence $\{\kappa_t, B_t, c_t, A_t\}_{t=0}^{\infty}$*, notice first that for given* κ_t *equations (5.6) and (Systemeq3) determine a unique* B_t *and* A_t*. Next, consider equation* (5.4) *and define*

$$\Omega^{\kappa}(\kappa_{t+1}) \equiv (1+\lambda)\frac{\Big(1+g^{A}(\kappa_{t+1})\Big)}{\Big(1+g^{B}(\kappa_{t+1})\Big)}\kappa_{t+1}. \tag{7.7}$$

Then (5.4) *may be rewritten as*

$$\Omega^{\kappa}(\kappa_{t+1}) = B_t\Big(v(\kappa_t) - c_t\Big). \tag{7.8}$$

The transversality condition ensures that c_t *is chosen optimally and the agent never consumes all available resources. Hence, the right-hand side of* (7.8) *is strictly positive for all* $\kappa_t > 0, B_t > 0, c_t > 0$*. Therefore, there will be a unique value of* $\kappa_{t+1} > 0$ *satisfying eq.* (7.8) *if* $\Omega^{\kappa}(\kappa_{t+1})$ *is strictly positive, continuous and monotone in* $\kappa_{t+1} > 0$ *and may take any value in* $\mathbb{R}_{++}$*.*

To see that $\Omega^{\kappa}(\kappa_{t+1}) > 0$ *for all* $\kappa_{t+1} > 0$ *recall the properties of functions* g^A *and* g^B*, as established in Proposition* 1*, which guarantee that* $\Omega^{\kappa}_{\kappa}(\kappa_{t+1}) > 0$ *for all* $\kappa > 0$*.*

It remains to be shown that $\lim_{\kappa\to 0}\Omega^{\kappa}(\kappa_{t+1}) = 0$ *and* $\lim_{\kappa\to\infty}\Omega^{\kappa}(\kappa_{t+1}) = \infty$*. To show this, consider the right-hand side of* (7.7)*. Recall from Proposition* 1 *that* $g^B(\kappa)$ *is decreasing on* $\mathbb{R}_{++}$ *and bounded below by zero. Hence,* $\lim_{\kappa\to\infty} g^B(\kappa)$ *is finite, while* $\lim_{\kappa\to 0} g^B(\kappa)$ *is either finite or infinite. Moreover, Proposition* (1) *implies that* $\lim_{\kappa\to 0} g^A(\kappa)$ *is finite while* $\lim_{\kappa\to\infty} g^A(\kappa)$ *is finite or infinite, since* g^A *is increasing on* $\mathbb{R}_{++}$ *and bounded below by zero. Consequently, as* κ *tends to zero we have* $\lim_{\kappa\to 0}\Omega^{\kappa}(\kappa_{t+1}) = 0$ *and as* κ *tends to infinity we have* $\lim_{\kappa\to\infty}\Omega^{\kappa}(\kappa_{t+1}) = \infty$*.*

It follows that the right-hand side of (7.7) *is increasing in* $\kappa_{t+1} > 0$*, approaches zero as* $\kappa \to 0$ *and approaches infinity as* $\kappa \to \infty$*. Therefore, there is a unique* $\kappa_{t+1} > 0$ *that satisfies eq.* (5.4) *for given* $(\kappa_t, B_t, c_t) \in \mathbb{R}_{++}$ *and* $v(\kappa_t) - c_t > 0$*.*

Given a unique $\kappa_{t+1} > 0$ *equation* (5.5) *delivers a unique* $c_{t+1} > 0$*.* ■

7.5 Proof of Proposition 4

Proof 5 *Given* $\Big(A_{-1}, K_0, L_0\Big) > 0$*, eq.* (5.10) *admits a unique solution* $\kappa_0 > 0$*. To show uniqueness of the equilibrium sequence* $\{c_t, \kappa_t\}_{t=0}^{\infty}$ *consider first equation* (5.8) *and define*

$$\Psi^{\kappa}(\kappa_{t+1}) \equiv (1+\lambda)(1-\delta)\Big(1+g^{A}(\kappa_{t+1})\Big)\kappa_{t+1} \tag{7.9}$$

The transversality condition ensures that c_t *is chosen optimally and the agent never consumes all available resources. Hence, the right-hand side of* (5.8) *is strictly positive for all* $\kappa_t > 0$ *and an appropriate initial choice* $c_t > 0$*. Therefore, there will be a unique value* $\kappa_{t+1} > 0$ *satisfying eq.* (5.8) *if* $\Psi^{\kappa}(\kappa_{t+1})$ *is strictly positive, continuous and monotone in* $\kappa_{t+1} > 0$ *and may take any value in* $\mathbb{R}_{++}$*.*

To see that $\Psi(\kappa_{t+1}) > 0$ *for all* $\kappa_{t+1} > 0$ *recall the properties of the function* g^A*, as established in Proposition* 1*, which guarantee that* $\Psi^{\kappa}_{\kappa}(\kappa_{t+1}) > 0$ *for all* $\kappa > 0$*.*

It remains to be shown that $\lim_{\kappa\to 0} \Psi^{\kappa}(\kappa_{t+1}) = 0$ *and* $\lim_{\kappa\to\infty} \Psi^{\kappa}(\kappa_{t+1}) = \infty$*. The only point of concern is the function* g^A *since the other factor in* $\Psi^{\kappa}(\kappa_{t+1})$ *is* κ_{t+1} *itself. Then, since* g^A *is bounded below by zero and may be finite or infinite for* $\kappa \to \infty$ *we have that* $\lim_{\kappa\to 0} \Psi^{\kappa}(\kappa_{t+1}) = 0$ *and* $\lim_{\kappa\to\infty} \Psi^{\kappa}(\kappa_{t+1}) = \infty$*.*

It follows that the left hand side of (5.8) *is increasing in* $\kappa_{t+1} > 0$*, approaches zero as* $\kappa \to 0$ *and approaches infinity as* $\kappa \to \infty$*. Therefore, there is a unique* $\kappa_{t+1} > 0$ *that satisfies eq.* (5.8) *for given* $\kappa_t \in \mathbb{R}_{++}$ *and appropriate initial choice* $c_t > 0$ *under the assumption that* $v(\kappa_t) - c_t > 0$*.*

Given a unique $\kappa_{t+1} > 0$ *and* $c_t > 0$*,* (5.9) *delivers a unique* $c_{t+1} > 0$*.* ■

7.6 Proof of Proposition 5

Proof 6 *1. The question here is whether at the steady-state rate of productivity growth* $g^A = \delta/(1-\delta)$ *it makes economic sense for the representative firm to produce, i.e., to accumulate a finite* κ *or whether it becomes optimal not to produce at all,* $\kappa = 0$*, or to accumulate an infinite amount of* κ*.*

If (5.13) *holds, then equations* (3.14) *and* (3.12) *imply a* $\kappa^{ss} \in (0, \infty)$ *since* $f''(\kappa) < 0$*. It is then straightforward to see that the steady state value of c may be obtained from* (5.8) *and satisfies* $c^* \in (0, \infty)$*.*

Now, suppose $k^* \in (0, \infty)$ *but*

$$\lim_{\kappa\to 0} f'(\kappa) < \frac{i'(\delta/(1-\delta))}{1-\delta} + i(\delta/(1-\delta)) < \lim_{\kappa\to\infty} f'(\kappa).$$

First, consider $\lim_{\kappa\to 0} f'(\kappa) < \frac{i'(\delta/(1-\delta))}{1-\delta} + i(\delta/(1-\delta))$*. This inequality means that at an infinite number of tasks performed by labor the marginal product of tasks* m_t *is smaller than marginal cost incurred to produce those tasks. It follows that the optimal choice for the firm is to produce only with tasks performed by labor, hence* $\kappa^* = 0$*. Equation* (5.8) *reveals that* $\kappa^* = 0$ *implies* $c^* = 0$*.*

Second, consider $\lim_{\kappa\to\infty} f'(\kappa) > \frac{i'(\delta/(1-\delta))}{1-\delta} + i(\delta/(1-\delta))$*. Since this inequality means that the marginal product of tasks* m_t *is greater than the associated cost incurred to produce that task, it would be profit maximizing for the firm to extend the number of tasks* m_t *ad infinitum. Hence, this implies* $\kappa^* = \infty$*. With* $\kappa^* = \infty$ (**??**) *implies* $c^* = \infty$*.*

3. Consider the system given by equations (5.8) *and* (5.9)*. Given* (κ_t, c_t)*, eq.* (5.8) *determines* κ_{t+1}*. Having* κ_{t+1} *then allows to determine* c_{t+1} *from eq.* (5.9)*.*

Notice that eqs. (5.8) *and* (5.9) *each define a continuously differentiable function* $\Phi^\kappa : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ *and* $\Phi^c : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ *such that the above system may be written as*

$$\kappa_{t+1} = \Phi^\kappa(\kappa_t, c_t), \tag{7.10}$$

$$c_{t+1} = \Phi^c(\kappa_t, c_t). \tag{7.11}$$

Approximate this non-linear system locally around a steady-state equilibrium using first order Taylor expansion to obtain

$$\begin{bmatrix} \kappa_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} \Phi^\kappa_\kappa(\kappa^{ss}, c^{ss}) & \Phi^\kappa_c(\kappa^{ss}, c^{ss}) \\ \Phi^c_\kappa(\kappa^{ss}, c^{ss}) & \Phi^c_c(\kappa^{ss}, c^{ss}) \end{bmatrix} \begin{bmatrix} \kappa_t \\ c_t \end{bmatrix} + \begin{bmatrix} \Phi^\kappa(\kappa^{ss}, c^{ss}) - \Phi^\kappa_\kappa(\kappa^{ss}, c^{ss}) - \Phi^\kappa_c(\kappa^{ss}, c^{ss}) \\ \Phi^c(\kappa^{ss}, c^{ss}) - \Phi^c_\kappa(\kappa^{ss}, c^{ss}) - \Phi^c_c(\kappa^{ss}, c^{ss}) \end{bmatrix}.$$

The eigenvalues of the Jacobian

$$J = \begin{bmatrix} \Phi^\kappa_\kappa(\kappa^{ss}, c^{ss}) & \Phi^\kappa_c(\kappa^{ss}, c^{ss}) \\ \Phi^c_\kappa(\kappa^{ss}, c^{ss}) & \Phi^c_c(\kappa^{ss}, c^{ss}) \end{bmatrix} \tag{7.12}$$

determine the local behavior of the non-linear system in the neighborhood of the steady-state equilibrium. The eigenvalues of the Jacobian are obtained as a solution to

$$\mid J - \lambda I \mid,$$

where $\mid J - \lambda I \mid$ *is the determinant of the matrix* $[J - \lambda I]$ *and* I *is the identity matrix.*

In order to say something about the stability properties, we first need to determine $\Phi^\kappa_\kappa, \Phi^\kappa_c, \Phi^c_\kappa, \Phi^c_\kappa$. *To do so, obtain*

$$(1+\lambda)(1-\delta)\Big(1 + g^A(\Phi^\kappa(\kappa_t, c_t))\Big)\Phi^\kappa(\kappa_t, c_t) = v(\kappa_t) - c_t \tag{7.13}$$

$$(1-\delta)\Big(1 + g^A(\Phi^\kappa(\kappa_t, c_t))\Big)\Phi^c(\kappa_t, c_t) = \beta\Big(f'(\Phi^\kappa(\kappa_t, c_t)\Big)c_t \tag{7.14}$$

by substituting (7.10), (7.11) *in* (5.8) *and* (5.9). *Implicit differentiation of* (7.13) *and* (7.14) *and evaluation at steady-state yields:*

$$\Phi_c^c(\kappa^{ss}, c^{ss}) = 1 + \frac{(c^{ss}/\theta) f''(\kappa^{ss})}{v'(\kappa^{ss})} \Phi_c^\kappa(\kappa^{ss}, c^{ss}) - \frac{c^{ss} g_\kappa^A(\kappa^{ss})}{1 + g^A(\kappa^{ss})} \Phi_c^\kappa(\kappa^{ss}, c^{ss}) \tag{7.15}$$

$$\Phi_\kappa^c(\kappa^{ss}, c^{ss}) = \frac{(c^{ss}/\theta) f''(\kappa^{ss})}{v'(\kappa^{ss})} \Phi_\kappa^\kappa(\kappa^{ss}, c^{ss}) - \frac{c^{ss} g_\kappa^A(\kappa^{ss})}{1 + g^A(\kappa^{ss})} \Phi_\kappa^\kappa(\kappa^{ss}, c_{ss}) \tag{7.16}$$

$$\Phi_\kappa^\kappa(\kappa^{ss}, c^{ss}) = \frac{v'(\kappa^{ss})}{(1+\lambda)(1-\delta)(1 + \kappa^{ss} g_\kappa^A(\kappa^{ss}) + g^A(\kappa^{ss}))} \tag{7.17}$$

$$\Phi_c^\kappa(\kappa^{ss}, c^{ss}) = -\frac{1}{(1+\lambda)(1-\delta)(1 + \kappa^{ss} g_\kappa^A(\kappa^{ss}) + g^A(\kappa^{ss}))} \tag{7.18}$$

The implied characteristic equation is

$$c(\lambda) \equiv \lambda^2 - trJ\lambda + detJ = 0, \tag{7.19}$$

where

$$trJ = \Phi_\kappa^\kappa + \Phi_c^c$$

and $detJ$ reduces to

$$detJ = \Phi_\kappa^\kappa.$$

We now show that if both eigenvalues of the system are real and distinct, i.e., if

$$(trJ)^2 > 4detJ,$$

the steady-state equilibrium is a saddle. This follows since

$$c(1) < 0 \quad \textit{and} \quad c(-1) > 0.$$

To see that $c(1) < 0$ observe that

$$\begin{aligned}
c(1) &= 1 - (\Phi_\kappa^\kappa + \Phi_c^c) + \Phi_\kappa^\kappa \\
&= 1 - \Phi_c^c \\
&= 1 - 1 - \frac{(c^{ss}/\theta) f''(\kappa^{ss})}{v'(\kappa^{ss})} \Phi_c^\kappa(\kappa^{ss}, c^{ss}) + \frac{c^{ss} g_\kappa^A(\kappa^{ss})}{1 + g^A(\kappa^{ss})} \Phi_c^\kappa(\kappa^{ss}, c^{ss}) \\
&= -\frac{(c^{ss}/\theta) f''(\kappa^{ss})}{v'(\kappa^{ss})} \Phi_c^\kappa(\kappa^{ss}, c^{ss}) + \frac{c^{ss} g_\kappa^A(\kappa^{ss})}{1 + g^A(\kappa^{ss})} \Phi_c^\kappa(\kappa^{ss}, c^{ss}) < 0,
\end{aligned}$$

where the inequality follows from the fact that $f(\kappa)$ has diminishing returns and $\Phi_c^\kappa < 0$.

To see that $c(-1) > 0$ observe that

$$\begin{aligned} c(-1) &= 1 + (\Phi_\kappa^\kappa + \Phi_c^c) + \Phi_\kappa^\kappa \\ &= 1 + \Phi_\kappa^\kappa + 1 + \frac{(c^{ss}/\theta) f''(\kappa^{ss})}{v'(\kappa^{ss})} \Phi_c^\kappa(\kappa^{ss}, c^{ss}) - \frac{c^{ss} g_\kappa^A(\kappa^{ss})}{1 + g^A(\kappa^{ss})} \Phi_c^\kappa(\kappa^{ss}, c^{ss}) + \Phi_\kappa^\kappa \\ &= 2(1 + \Phi_\kappa^\kappa) + \frac{(c^{ss}/\theta) f''(\kappa^{ss})}{v'(\kappa^{ss})} \Phi_c^\kappa(\kappa^{ss}, c^{ss}) - \frac{c^{ss} g_\kappa^A(\kappa^{ss})}{1 + g^A(\kappa^{ss})} \Phi_c^\kappa(\kappa^{ss}, c^{ss}) > 0, \end{aligned}$$

where the inequality follows from the fact that $\Phi_\kappa^\kappa > 0$, $\Phi_c^\kappa < 0$ and $f''(\kappa) < 0$.

Hence, the steady-state equilibrium is a saddle, with one eigenvalue being explosive, $(\lambda_1 > 1)$, and the other one being stable, $|\lambda_2| < 1$. ■

7.7 Proof of Proposition 6

Proof 7 *1. If* (5.17) *holds, then equations* (3.13) *and* (3.11) *imply a $\kappa^* \in (0, \infty)$ since $f'(\kappa) > 0$. It is then straightforward to see that the steady state value of B may be obtained from eq.* (5.15) *and satisfies $B^* \in (0, \infty)$. Having κ^* and B^* equation* (5.14) *pins down the steady state value of consumption in efficiency units.*

Now, suppose $B^* \in (0, \infty)$ and $c^* \in (0, \infty)$ but $\lim_{\kappa \to 0} f'(\kappa) < \frac{i'(\delta/(1-\delta))}{1-\delta} + i(\delta/(1-\delta)) < \lim_{\kappa \to \infty} f'(\kappa)$. If $\lim_{\kappa \to 0} f'(\kappa) < \frac{i'(\delta/(1-\delta))}{1-\delta} + i(\delta/(1-\delta))$, then the optimal choice is $\kappa^* = 0$. However, with $\kappa^* = 0$ equation (5.14) *implies $c^* = 0$ which is a contradiction. Similarly, if $\frac{i'(\delta/(1-\delta))}{1-\delta} + i(\delta/(1-\delta)) < \lim_{\kappa \to \infty} f'(\kappa)$, the optimal task intensity would be $\kappa = \infty$. In this case, equation* (5.14) *would imply $c^* = \infty$ which is a contradiction.*

Moreover, notice that equation (5.16) *implicitly determines the steady-state value of capital in efficiency units. It is then the Euler equation* (5.15) *which yields the steady-state value of B^* as*

$$B^* = \frac{1}{\beta[f'(\kappa^*) - i(g^B(\kappa^*))]} \frac{(1 + g^A(\kappa^*))}{(1 + g^B(\kappa^*))}. \tag{7.20}$$

Finally, substituting out B^* from equation 5.14 *gives the steady-state value of consumption in efficiency units as*

$$c^* = v(\kappa^*) - (1 + \lambda)\beta[f'(\kappa^*) - i(g^B(\kappa^*))]\kappa^* > 0. \tag{7.21}$$

2. Notice that equations (5.4) and (5.5) each define a continuously differentiable function, $\Phi^i : \mathbb{R}^3_{++} \to \mathbb{R}_{++}$, where $i = \kappa, c$, such that $\kappa_{t+1} = \Phi^\kappa(\kappa_t, c_t, B_t)$, and $c_{t+1} = \Phi^c(\kappa_t, c_t, B_t)$. Consider (5.6) in $t + 1$ and substitute $\kappa_{t+1} = \Phi^\kappa(\kappa_t, c_t, B_t)$ to obtain

$$B_{t+1} = B_t(1 - \delta)(1 + g^B(\Phi^\kappa(\kappa_t, c_t, B_t))) \equiv \Phi^B(\kappa_t, c_t, B_t),$$

where Φ^B is also a continuously differentiable function

Thus, the dynamic system may be written as

$$(1+\lambda)\frac{(1+g^A(\Phi^\kappa(\kappa_t,c_t,B_t)))}{(1+g^B(\Phi^\kappa(\kappa_t,c_t,B_t)))}\kappa_{t+1} = B_t\Big(v(\kappa_t)-c_t\Big), \tag{7.22}$$

$$\frac{(1+g^A(\Phi^\kappa(\kappa_t,c_t,B_t)))}{(1+g^B(\Phi^\kappa(\kappa_t,c_t,B_t)))}\frac{\Phi^c(\kappa_t,c_t,B_t)}{c_t} = \beta[f'(\Phi^\kappa(\kappa_t,c_t,B_t)) - i(g^B(\Phi^\kappa(\kappa_t,c_t,B_t)))], \tag{7.23}$$

$$\Phi^B(\kappa_t,c_t,B_t) = B_t(1-\delta)(1+g^B(\Phi^\kappa(\kappa_t,c_t,B_t))). \tag{7.24}$$

Perform a first order Taylor expansion to approximate this non-linear system locally around the steady-state equilibrium to obtain

$$\begin{bmatrix}\kappa_{t+1}\\ c_{t+1}\\ B_{t+1}\end{bmatrix} = \begin{bmatrix}\Phi^\kappa_\kappa(\kappa^*,c^*,B^*) & \Phi^\kappa_c(\kappa^*,c^*,B^*) & \Phi^\kappa_B(\kappa^*,c^*,B^*)\\ \Phi^c_\kappa(\kappa^*,c^*,B^*) & \Phi^c_c(\kappa^*,c^*,B^*) & \Phi^c_B(\kappa^*,c^*,B^*)\\ \Phi^B_\kappa(\kappa^*,c^*,B^*) & \Phi^B_c(\kappa^*,c^*,B^*) & \Phi^B_B(\kappa^*,c^*,B^*)\end{bmatrix}\begin{bmatrix}\kappa_t\\ c_t\\ B_t\end{bmatrix}$$
$$+\begin{bmatrix}\Phi^\kappa(\kappa^*,c^*,B^*) - \Phi^\kappa_\kappa(\kappa^*,c^*,B^*) - \Phi^\kappa_c(\kappa^*,c^*,B^*) - \Phi^\kappa_B(\kappa^*,c^*,B^*)\\ \Phi^c(\kappa^*,c^*,B^*) - \Phi^c_\kappa(\kappa^*,c^*,B^*) - \Phi^c_c(\kappa^*,c^*,B^*) - \Phi^c_B(\kappa^*,c^*,B^*)\\ \Phi^B(\kappa^*,c^*,B^*) - \Phi^B_\kappa(\kappa^*,c^*,B^*) - \Phi^B_c(\kappa^*,c^*,B^*) - \Phi^B_B(\kappa^*,c^*,B^*)\end{bmatrix}.$$

The eigenvalues of the Jacobian

$$J = \begin{bmatrix}\Phi^\kappa_\kappa(\kappa^*,c^*,B^*) & \Phi^\kappa_c(\kappa^*,c^*,B^*) & \Phi^\kappa_B(\kappa^*,c^*,B^*)\\ \Phi^c_\kappa(\kappa^*,c^*,B^*) & \Phi^c_c(\kappa^*,c^*,B^*) & \Phi^c_B(\kappa^*,c^*,B^*)\\ \Phi^B_\kappa(\kappa^*,c^*,B^*) & \Phi^B_c(\kappa^*,c^*,B^*) & \Phi^B_B(\kappa^*,c^*,B^*)\end{bmatrix} \tag{7.25}$$

determine the local behavior of the non-linear system in the neighborhood of the steady-state equilibrium. To find the eigenvalues obtain the solution to

$$\det(J-\lambda I) = 0,$$

which gives rise to the following characteristic polynomial:

$$c(\lambda) \equiv \lambda^3 - tr(J)\lambda^2 + \sum M_2(J)\lambda - \det(J), \tag{7.26}$$

where $tr(J)$ denotes the trace, $\sum M_2(J)$ denotes the sum of principal minors of order two and $\det(J)$ the determinant of the Jacobian. One can show that

$$\begin{aligned} tr(J) &= \Phi^\kappa_\kappa + \Phi^c_c + \Phi^B_B\\ \sum M_2(J) &= 2\Phi^\kappa_\kappa + \Phi^c_c\Phi^B_B - \Phi^c_B\Phi^B_c\\ det(J) &= \Phi^\kappa_\kappa \end{aligned}$$

By Descartes' rule of signs we know that if the terms of a polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Moreover, the number of negative roots is at most equal to the number of continuations in the signs of the coefficients. Inspection of equation (7.26) *reveals that it has*

(α) *either three positive roots,*

(β) *or one positive root and one pair of complex conjugate roots.*

The stability properties of the three-dimensional system are fully determined by the magnitudes of the eigenvalues λ_1, λ_2 and λ_3. Given that the dynamic system contains two predetermined variables (the two state variables κ_t and B_t), for a unique saddle path to exist there must be two stable eigenvalues: either two real eigenvalues with absolute value smaller than 1 or a pair of complex conjugate eigenvalues λ_1, λ_2 with $\mid \lambda_1\lambda_2 \mid < 1$.

Next, we turn to the study of the determinant of the Jacobian, $det(J)$. We are interested in the absolute value of the determinant as it might help establish the stability properties of the system. Differentiation of equation (7.22) with respect to κ yields

$$\Phi^{\kappa}_{\kappa} = \frac{B^* v'(\kappa^*)}{(1+\lambda)\left[\Gamma\kappa^* + \frac{1+g^A(\kappa^*)}{1+g^B(\kappa^*)}\right][f'(\kappa^*) - i(g^B(\kappa^*))]},$$

where

$$\Gamma = \frac{g^A_{\kappa}(1+g^B(\kappa^*)) - (1+g^A(\kappa^*))g^B_{\kappa^*}}{(1+g^B(\kappa^*))^2}.$$

Thus, using equation 7.20 we may write

$$\begin{aligned} det(J) &= \Phi^{\kappa}_{\kappa} \\ &= \frac{B^* v'(\kappa^*)}{(1+\lambda)\left[\Gamma\kappa^* + \frac{1+g^A(\kappa^*)}{1+g^B(\kappa^*)}\right]} \\ &= \frac{\frac{(1+g^A(\kappa^*))}{(1+g^B(\kappa^*))} v'(\kappa^*)}{\beta\left[\Gamma\kappa^* + \frac{1+g^A(\kappa^*)}{1+g^B(\kappa^*)}\right][f'(\kappa^*) - i(g^B(\kappa^*))]}. \end{aligned}$$

Observe that $det(J) < 1$ iff

$$\begin{aligned} \frac{(1+g^A(\kappa^*))}{(1+g^B(\kappa^*))} v'(\kappa^*) &< \beta\left[\Gamma\kappa^* + \frac{1+g^A(\kappa^*)}{1+g^B(\kappa^*)}\right][f'(\kappa^*) - i(g^B(\kappa^*))] \\ \frac{v'(\kappa^*)}{[f'(\kappa^*) - i(g^B(\kappa^*))]} &< \beta\left[\Gamma\kappa^* \frac{(1+g^B(\kappa^*))}{(1+g^A(\kappa^*))} + 1\right] \\ \frac{v'(\kappa^*)}{[f'(\kappa^*) - i(g^B(\kappa^*))]} &< \beta\Big(\frac{g^A_{\kappa}(\kappa^*)\kappa^*}{1+g^A(\kappa^*)} + \frac{-g^B_{\kappa}(\kappa^*)\kappa^*}{1+g^B(\kappa^*)} + 1\Big) \\ \frac{v'(\kappa^*)}{[f'(\kappa^*) - i(g^B(\kappa^*))]} &< \beta(\epsilon^A_{\kappa} + \epsilon^B_{\kappa} + 1), \end{aligned} \tag{7.27}$$

where ϵ^i_{κ} denotes the elasticity of the respective productivity growth factor with respect to the efficient capital intensity.

Since $det(J)$ is the product of the eigenvalues of the system, we have that if (7.27) holds,

$$det(J) = \lambda_1\lambda_2\lambda_3 \quad \in (0,1),$$

implying the following:

(α) *If all eigenvalues are real, this implies that at least one of them is smaller than one. Without loss of generality, let* $\lambda_1 \in (0,1)$. *We can determine the magnitude of the remaining eigenvalues by checking whether or not they fall on the same side of a given constant along the real line. This can be done by evaluating*

$$c(a) = (a - \lambda_1)(a - \lambda_2)(a - \lambda_3).$$

Now let $a = 1$ *in order to check whether* $\lambda_i, i = 2, 3$, *lie inside or outside the unit circle. Some agebra shows that* $c(1) = (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3) = -\Phi_c^B \frac{c^*}{B^*} < 0$. *But since* $\lambda_1 \in (0,1)$ *the only way for* $c(1)$ *to be negative is by having one of the remaining eigenvalue inside the unit circle and the other outside. Therefore, we may conclude that if all eigenvalues are real and positive, then the system is asymptotically locally stable in the state space. (to be completed)*

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