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Decomposing Risk in Dynamic Stochastic General Equilibrium*

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Abstract

We analyze the theoretical moments of a nonlinear approximation to real business cycle model with stochastic volatility and recursive preferences. We find that the conditional heteroskedasticity of stochastic volatility operationalizes a time-varying risk adjustment channel that induces variability in conditional asset pricing measures and assigns a substantial portion of the variance of macroeconomic variables to variations in precautionary behavior, both while leaving its ability to match key macroeconomic and asset pricing facts untouched. We calculate the theoretical moments directly and decomposes these moments into contributions from shifts in the distribution of future shocks (i.e., risk) and from realized shocks and differing orders of approximation, enabling us to identify the common channel through which stochastic volatility in isolation operates and through which conditional asset pricing measures vary over time. Under frictional investment and varying capital utilization, output drops in response to an increase in risk, but the contributions to the variance of macroeconomic variables from risk becomes negligible.

JEL classification: C63, E32, G12

Keywords: Recursive preferences; stochastic volatility; asset pricing; DSGE; moment calculation

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1 Introduction

Assessing the statistical and structural implications of nonlinear DSGE models with recursive preferences and stochastic volatility for asset pricing and business cycle dynamics is an unfinished task in macroeconomics. We derive the theoretical moments of nonlinear moving average approximations to the model and decompose these moments into contributions from the individual orders of nonlinearity in realized shocks (amplification effects) and from the moments of future shocks (risk adjustment effects). With this decomposition, we find that stochastic volatility activates a time-varying risk adjustment channel in macroeconomic variables accounting for a substantial amount of total variation. We identify this conditional heteroskedastic mechanism as the sole driving force of the conditional asset pricing measures under study. This enables us to tell the story of a varying pattern of risk in the economy eliciting changes in households' precautionary responses as priced by measures such as the conditional market price of risk. We find, however, that stochastic volatility contributes to the model's ability to match asset pricing facts only by increasing the overall volatility of macro variables—taken as given exogenously in endowment settings¹ that reach the opposite conclusion—and that frictional investment and variable capital utilization allow the model to predict a drop in output in response to an increase in risk (positive volatility shock) at the cost of making the importance of this risk channel to the variability of macro variables moot.

While there is growing interest in stochastic volatility and Epstein and Zin's (1989) recursive preferences² in recent literature, there is little work that studies the joint effect of these two elements for both asset pricing and business cycle dynamics.³ Andreasen (2012), focusing on the

¹See, e.g., Bansal and Yaron (2004)

²See also Kreps and Porteus (1978) and Weil (1990). Backus et al. (2005) offers a recent review of these and related preferences.

³Bloom (2009) studies the impact of stochastic volatility at the firm level and documents a short drop followed by an overshooting in aggregate economic activity following a volatility shock. Justiniano and Primiceri (2008) add stochastic volatility to a linearized New Keynesian model to study the documented reduction in volatility of U.S. economy since the early 1980's (See Blanchard and Simon (2001) and Stock and Watson (2003), as well as Sims and Zha (2006) for a review.). Fernández-Villaverde et al. (2011a) and Born and Pfeifer (2013) use New Keynesian models to study the effect of changes in the volatility of policy variables on the aggregate economy. Tallarini (2000) among

different specifications of the conditional heteroskedasticity and the consequential difference in the quantitative performance of a New Keynesian model, takes a brief look at the implications of the model on both sides. Bidder and Smith (2012), taking a model uncertainty perspective à la Hansen and Sargent (2007), study fluctuations in the worst-case distribution as sources for business cycles in a model with stochastic volatility and recursive preferences. We differ from both their works in our aim to analyze the propagation mechanism of stochastic volatility implemented as a volatility shock in a production model, and we examine the role of stochastic volatility in attaining the Hansen-Jagannathan bounds (See Hansen and Jagannathan (1991)) without compromising the fit vis-à-vis the macroeconomy to complement the empirical evaluation of the model regarding replicating asset pricing regularities.

We solve the model using the nonlinear moving average perturbation derived in Lan and Meyer-Gohde (Forthcoming), which takes the infinite sequence of realized shocks, past to present, as its state variable basis and adjusts the deterministic policy function for the effect of future shocks by scaling their distribution with the perturbation parameter. Following Caldara et al.'s (2012) assessment of the accuracy of third order perturbations in a business cycle model with recursive preferences and stochastic volatility and as it is the minimum order needed to capture the time-varying shifts in risk premium as noted in Andreasen (2012, p. 300) and van Binsbergen et al. (2012, p. 638), we approximate the policy function to third order. The nonlinear moving average policy function and its third order approximation can be decomposed straightforwardly into the order of the amplification effects (the impact of the realized shocks) and risk adjustment (the anticipation effect of future shocks). We find, in the analysis of the impulse responses of both macroeconomic and asset pricing variables, a volatility shock by itself propagates solely through the time-varying risk adjustment channel. For conditional asset pricing measures such as the expected risk premium,

others, note recursive preferences can contribute to resolving the longstanding asset pricing puzzles (equity premium and risk free rate) documented in Mehra and Prescott (1985) and Weil (1989) without compromising the model's ability of replicating macroeconomic dynamics; and Rudebusch and Swanson (2012) and van Binsbergen et al. (2012) use a model with recursive preferences to study the dynamics of the yield curve.

volatility shocks and productivity growth shocks propagate individually through the time-varying risk adjustment channel only. Moreover, the effect of stochastic volatility shocks on the expected risk premium is several orders of magnitude larger than that of productivity growth shocks, highlighting again the importance of this time variation in the dispersion of probability measures used to form expectations for conditional asset pricing.

Using the third order nonlinear moving approximation, we derive theoretical moments that are in general not available in the nonlinear DSGE models.⁴ In a similar vein to our nonlinear moving average, Andreasen et al. (2013) compute theoretical moments using a pruned state space perturbation,⁶ since after pruning, the unknown higher moments are nonlinear functions of the known moments of lower order approximations. However, we are able to further derive a decomposition of the theoretical variance that neatly dissects the individual contributions of amplification and risk adjustment effects to the total variance of the model. With this variance decomposition, we find that adding stochastic volatility changes the composition of the variance of the macroeconomic variables. In the presence of stochastic volatility, more variation is generated in the time-varying risk adjustment channel. As for macroeconomic variables, movements in the risk adjustment channel can be explained by the household's precautionary motive. This finding implies households aware of shifts in the distributions of future shocks will adjust their precautionary behavior commensurately.

The paper is organized as follows. The competitive real business cycle model with recursive preferences and stochastic volatility is derived in section 3. In section 4, we present the nonlinear moving average perturbation solution to the model. The calibrations are introduced in section 5. We then derive the theoretical moments in section 6 and apply our method to analyze the model in

⁴The nonlinear moving average approximation, as its policy function directly maps exogenous shocks into the endogenous variables, only needs the moments of the exogenous shocks when computing the theoretical moments. We implement our approach numerically by providing an add-on for the popular Dynare package.⁵ A state space perturbation policy function, by contrast, maps the endogenous variables into themselves and resulting in an infinite regression in theoretical moments requiring higher moments than moments being computed.

⁶See Lan and Meyer-Gohde (2013a) for an overview of pruning and its relation to our nonlinear moving average.

section 7. In section 8, we extend the baseline model to frictional investment and variable capital utilization. Section 9 concludes.

2 Stochastic Volatility

2.1 Related Literature

As documented in Blanchard and Simon (2001), Stock and Watson (2003), Sims and Zha (2006), Fernández-Villaverde and Rubio-Ramírez (2010) and many others, the volatility of employment growth, consumption growth and output of the U.S. economy from 1984 to 2007 has evidently declined by one third comparing to their values during the 1970s and early 1980s. Nominal volatilities also have declined by more than half. This period of volatility reduction in aggregate time series, often labeled as the Great Moderation, motivates the study of its causes. The literature thus far offers three main ways of modeling, and therefore analyzing this volatility shift: i) stochastic volatility, i.e., model the volatility of the exogenous processes under investigation as an autoregressive process, or ii) a GARCH process, or iii) the volatility switches between two (or more) states, i.e., Markov regime switching models. As pointed out by Fernández-Villaverde and Rubio-Ramírez (2010, p. 10), stochastic volatility can capture many important features of the empirical volatility shift and differentiates the special effect of volatility from others, this approach has been adopted in many studies.

By incorporating stochastic volatility, Fernández-Villaverde and Rubio-Ramírez (2007) show variations in the volatility of investment-specific technological shock and preferences shock explain most of variations in the volatility of in output and hours worked. Justiniano and Primiceri (2008) estimate a DSGE model with stochastic volatility and conclude the decline in the volatility of output, hours worked and consumption is largely owing to a change in the variance of the investment-specific technological shock, e.g., a change in the variance of the investment shock explains on average 30% of variability in output growth since mid-1980s. On the other hand, Bidder

and Smith (2013) investigate the implication of stochastic volatility on asset pricing and find that, their endowment model economy generates a much higher unconditional market price of risk with stochastic volatility in the exogenous consumption growth process than without. Meanwhile, the presence of stochastic volatility does not lead to a noticeable change in risk free rate, and thus improves the model's ability in attaining the Hansen-Jagannathan bound. Bansal and Yaron (2004) also find, in the presence of stochastic volatility in the exogenous consumption growth, the maximal Sharpe ratio, i.e., the lower bound of the market price of risk, is about three times larger than its value in the absence of stochastic volatility.

All these findings motivates the study of whether stochastic volatility is a driver of business cycle fluctuations. As noted in Born and Pfeifer (2013), changes in aggregate uncertainty can potentially induce changes in economic activities through i) the precautionary motive as household tends to save more to ensure itself against the increased future risk, ii) the (inverse) Hartman-Abel effect which is, in essence, firm's precautionary reaction in response to the increased future risk, and iii) real option effect at work. These three transmission mechanisms of volatility change are however, partial equilibrium effects. In a general equilibrium model where prices can adjust to accommodate changes in uncertainty, the effect of volatility change on economic activities could differ from that in a partial equilibrium model, both qualitatively and quantitatively.

Using a partial equilibrium model, Bloom (2009) show a positive volatility shock causes a drop in output and employment, both by about 1%. Bloom et al. (2012) report a drop in output by just over 3% when general equilibrium effects are shut off. On the other hand, the effect of stochastic volatility appears, but not without exception, smaller in size when general equilibrium effects are taken into account. Fernández-Villaverde et al. (2011b) show a volatility shock to the real interest rate leads to a drop in output, consumption, investment and hours by 0.2%, 0.5%, 2% and 0.001% respectively.⁷ Fernández-Villaverde et al. (2011a) find the previous four aggregates

⁷This is the case for Argentina, see Fernández-Villaverde et al. (2011b, p. 2550).

fall by about 0.15%, 0.02%, 0.6% and 0.15% respectively in response to an increase in the uncertainty of fiscal policy. Born and Pfeifer (2013) also study the uncertainty of fiscal policy and its effect. They also find, with the baseline parameterization of their model, an increase in uncertainty causes a contraction on economic activities—the four aggregates fall by 0.045%, 0.03%, 0.1% and 0.04% respectively. Basu and Bundick (2012) investigate the effect of stochastic volatility in both technology and preference shock process. A volatility shock to either of the two processes leads to a drop in output, consumption, investment and hours, but at different quantitative level. The four aggregates fall by about 0.04%, 0.06%, 0.01% and 0.06% respectively in response to a volatility shock to technology, and by about 0.17%, 0.16%, 0.2% and 0.21% in response to a volatility shock to preference. While the first order impact effect of aggregate uncertainty is not pictured and explicitly reported, Bachmann and Bayer (Forthcoming) show the contribution from aggregate uncertainty to the volatility of output, consumption, investment and hours is negligible. Still in a general equilibrium framework, Bloom et al. (2012) and Bidder and Smith (2013) both find however, stochastic volatility can have large effect on economic activities. A volatility shock causes a fall in output, consumption, invest and hours by about 3%, 1%, 20% and 7% in Bloom et al. (2012), and by about 2%, 1.5%, 2.5% and 1% in the worse case model of Bidder and Smith (2013).

To summarize, the studies cited above tend to agree an increase in volatility leads to a recession, yet differ in their views on the size of such a recession. Moreover, Bloom (2009) and Bloom et al. (2012) find the recession due to the increase in uncertainty lasts for only 6 and 12 months respectively, as opposed to a more prolonged recession reported by the other studies. Additionally, there is few studies investigating the potential effect of stochastic volatility on the market price of risk in a production economy. It worth noting that the contribution from stochastic volatility to asset pricing in an endowment economy does not necessarily carry over to a production economy, as such contribution is not entirely independent from the reduced form, empirical specification of

consumption growth.

2.2 General Operation within DSGE Models

One general way⁸ to introduce stochastic volatility is to replace a homoskedastic shock ω_t with $e^{\zeta_t} \omega_t$, where ζ_t is a mean zero stochastic variable. The exponential function ensure that e^{ζ_t} is always positive, enabling the interpretation of the product of e^{ζ_t} and homoskedastic standard deviation of ω_t as the shock's conditional standard deviation. As we will be concerned with local approximations in this study, it will be useful to have a Taylor series of a conditionally heteroskedastic shock $e^{\zeta_t} \omega_t$ as we approximate our equilibrium system to a given order.

Lemma 2.1. *The Taylor expansion of*

$$(1) \quad e^{\zeta_t} \omega_t$$

around the point $\zeta_t = \omega_t = 0$ is given by

$$(2) \quad e^{\zeta_t} \omega_t = \left(\sum_{i=0}^{\infty} \frac{1}{i!} \zeta_t^i \right) \omega_t$$

Proof. See the appendices. □

To assess the general equilibrium effects of stochastic volatility, consider the following general model

$$(3) \quad 0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) + H e^{\zeta_t} \omega_t]$$

where y_t is the vector of the endogenous variables, and ε_t the vector of normally distributed exogenous shocks apart from the stochastic volatility shock under consideration, H a constant vector, ω_t is a normally distributed exogenous shock subjected to a stochastic volatility process ζ_t ,⁹ itself given by

$$(4) \quad \zeta_t = \rho \zeta_{t-1} + \tau \eta_t$$

⁸See, e.g., Fernández-Villaverde et al. (2011b)

⁹Our shock subject to stochastic volatility enters the model linearly through $H e^{\zeta_t} \omega_t$. The vector H could, for example, contain zeros everywhere but for the row associated with, say, an autoregressive process for technology; in this row, the entry would be the homoskedastic standard deviation of technology shocks.

where $|\rho| < 1$ is the autocorrelation of the process, η_t its standard normal innovation with $\tau > 0$ scaling these innovations to enable non-unity standard deviations.

The solution to (3) is a time-invariant function y , taking as its state variable basis, the shocks and states induced by stochastic volatility $z_t^\omega \equiv [\zeta_{t-1} \quad \varepsilon_t^{\omega'}]'$, where $\varepsilon_t^\omega \equiv [\omega_t \quad \eta_t]'$, as well as the remaining shocks and states, $z_t \equiv [y'_{t-1} \quad \varepsilon'_t]'$. Solutions are indexed by the perturbation parameter $\sigma \in [0, 1]$ scaling the distribution of future shocks

$$(5) \quad y_t = g(\sigma, z_t, z_t^\omega)$$

$$(6) \quad y_{t+1} = g(\sigma, [y'_t \quad \sigma \varepsilon'_{t+1}]', [\zeta_t \quad \sigma \varepsilon_{t+1}^{\omega'}]')$$

The role of the parameter $\sigma \in [0, 1]$ can be seen in the policy function at $t + 1$, where it premultiplies shocks dated $t + 1$. That is, from the time t perspective of the conditional expectations operator in (3), shocks dated $t + 1$ are unknown and are the source of risk in the model which is scaled by σ . When $\sigma = 0$, the model is deterministic and the deterministic steady state is a fix point of the mappings in (5) when all shock realizations are equal to their mean (i.e., zero) values.

We are now in a position to provide the third order Taylor approximation of (5) about the deterministic steady state.

Proposition 2.2. *The recursive solution of (3) expanded out to third order at the deterministic steady state is given by*

$$(7) \quad \begin{aligned} y_t \approx & \frac{1}{2} g_{\sigma^2} + \frac{1}{2} g_{\sigma^2 z} z_t + \frac{1}{2} g_{\sigma^2 z^\omega} z_t^\omega \\ & + (g_{\zeta \omega} \zeta_{t-1} + g_{\eta \omega} \eta_t) \omega_t \\ & + \frac{1}{2} (g_{\zeta \omega z} \zeta_{t-1} + g_{\eta \omega z} \eta_t) \omega_t z_t \\ & + \frac{1}{2} (g_{\zeta^2 \omega} \zeta_{t-1}^2 + 2g_{\zeta \eta \omega} \zeta_{t-1} \eta_t + g_{\eta^2 \omega} \eta_t^2) \omega_t \\ & + \text{Terms independent of stochastic volatility} \end{aligned}$$

Proof. See the appendices. □

Notice, importantly, that only derivatives with respect to the risk scaling parameter σ provide a

channel for stochastic volatility to interact with variables outside of the stochastic volatility and the shock it impacts. The remaining terms, i.e., those that do not involve derivatives with respect to the risk scaling parameter σ , capture the direct effect of a change in volatility on shocks drawn from the distribution subjected to the change. We split the third order approximation of the recursive solution of (3) in proposition 2.2 into these two components.

First, the component that captures the effect of changes in the dispersion of the distribution of shocks on the magnitude of shocks realized from this distribution.

Definition 2.3. *At third order, the amplification component of y_t is*

$$(8) \quad \begin{aligned} y_t^{amplification} &= (g_{\zeta\omega}\zeta_{t-1} + g_{\eta\omega}\eta_t) \omega_t \\ &\quad + \frac{1}{2} (g_{\zeta\omega z}\zeta_{t-1} + g_{\eta\omega z}\eta_t) \omega_t z_t \\ &\quad + \frac{1}{2} (g_{\zeta^2\omega}\zeta_{t-1}^2 + 2g_{\zeta\eta\omega}\zeta_{t-1}\eta_t + g_{\eta^2\omega}\eta_t^2) \omega_t \end{aligned}$$

In essence, an increase in the dispersion of this distribution serves to magnify the realizations of shocks from the undispersed distribution, hence our label, “amplification.”

Second, the component that captures the effect of changes in the dispersion of the distribution of shocks on the evaluation of expectations

Definition 2.4. *At third order, the risk component of y_t is*

$$(9) \quad y_t^{risk} = \frac{1}{2}g_{\sigma^2} + \frac{1}{2}g_{\sigma^2 z}z_t + \frac{1}{2}g_{\sigma^2 z^w}z_t^w$$

In essence, an increase in the dispersion of this distribution increase the risk or measurable uncertainty regarding future stochastic variables, hence our label, “risk.”

The volatility shock η_t can only affect the conditional distribution of future shocks only if the stochastic process ζ_t is persistent, as we summarize in the following

Corollary 2.5. *At third order, the risk component of y_t , (9), is nonzero if and only if ζ_t is persistent, that is if $\rho \neq 0$.*

Proof. See the appendices. □

We have defined the volatility process in (1) such that changes in the volatility occur simultaneous with realizations of shocks, that is both ζ_t and ω_t enter (1) with a subscript t . Volatility shocks will only affect risk or measurable uncertainty regarding future shocks if ζ_t is serially correlated, such that an innovation to the volatility process also affects future volatility.

3 The Model

In this section, we derive a stochastic neoclassical growth model with the recursive preferences and stochastic volatility. We will follow Tallarini (2000) closely so that our model coincides with his in the special case of constant (i.e., non stochastic) volatility. Preferences are recursive in an exponential certainty equivalent with the period utility function logarithmic in consumption and leisure. Production is neoclassic using time-to-build capital and labor, whose productivity grows as a random walk with drift and innovations subject to stochastically varying volatility.

The economy is populated by an infinitely lived household seeking to maximize its expected discounted lifetime utility given by the recursive preferences

$$(10) \quad V_t = \ln C_t + \psi \ln(1 - N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} V_{t+1} \right) \right] \right)$$

where C_t is consumption, N_t labor, $\beta \in (0, 1)$ the discount factor and

$$(11) \quad \gamma \equiv 2 \frac{(1 - \beta)(1 - \chi)}{1 + \psi}$$

indexes the deviation with respect to the expected utility. χ denotes the coefficient of relative risk aversion (CRRA) for atemporal wealth gambles¹⁰ and $\psi > 0$ controls labor supply. With χ equal to the elasticity of intertemporal substitution (EIS) which is equal to one here, (10) collapses to the expected utility. The household optimizes over consumption and labor supply subject to

$$(12) \quad C_t + K_t = W_t N_t + r_t^K K_{t-1} + (1 - \delta) K_{t-1}$$

¹⁰See also Swanson (2013).

where K_t is capital stock accumulated today for productive purpose tomorrow, W_t real wage, r_t^K the capital rental rate and $\delta \in [0, 1]$ the depreciation rate. Investment is the difference between the current capital stock and the capital stock in the previous period after depreciation

$$(13) \quad I_t = K_t - (1 - \delta)K_{t-1}$$

We assume a perfectly competitive production side of the economy, where output is produced using the labor augmented Cobb-Douglas technology $Y_t = K_{t-1}^\alpha (e^{Z_t} N_t)^{1-\alpha}$. Z_t is a stochastic productivity process and $\alpha \in [0, 1]$ the capital share. Productivity is assumed to be a random walk with drift, incorporating long-run risk into the model¹¹

$$(14) \quad a_t \equiv Z_t - Z_{t-1} = \bar{a} + \bar{\sigma}_z e^{\sigma_{z,t}} \varepsilon_{z,t}, \quad \varepsilon_{z,t} \sim \mathcal{N}(0, 1)$$

with $\varepsilon_{z,t}$ the innovation to Z_t . $\bar{\sigma}_z e^{\sigma_{z,t}}$ can be interpreted as the standard deviation of the productivity growth with $\bar{\sigma}_z$ the homoskedastic component. Following, e.g., Fernández-Villaverde et al. (2011b) and Caldara et al. (2012), we specify the heteroskedastic component, $\sigma_{z,t}$, as

$$(15) \quad \sigma_{z,t} = \rho_\sigma \sigma_{z,t-1} + \tau \varepsilon_{\sigma_{z,t}}, \quad \varepsilon_{\sigma_{z,t}} \sim \mathcal{N}(0, 1)$$

where $|\rho_\sigma| < 1$ and τ is the standard deviation of $\varepsilon_{\sigma_{z,t}}$. The model is closed by the market clearing condition

$$(16) \quad Y_t = C_t + I_t$$

setting consumption and investment equal to output in each period.

The solution is characterized by the intratemporal labor supply/productivity condition equalizing the utility cost of marginally increasing labor supply to the utility value of the additional consumption obtained therewith

$$(17) \quad \frac{\Psi}{1 - N_t} = \frac{1}{C_t} (1 - \alpha) \frac{Y_t}{N_t}$$

¹¹As noted by Bansal and Yaron (2004, p. 1502), in an endowment economy with recursive preferences and stochastic volatility, better long-run growth prospects leads to a rise in the wealth-consumption and the price-dividend ratios. Rudebusch and Swanson (2012, p. 108) incorporate both real and nominal long-run risk in a production economy with recursive preference, and find long-run nominal risk improves the model's ability to fit the data.

and the intertemporal Euler equation

$$(18) \quad E_t [m_{t+1} (1 + r_{t+1})] = 1$$

where the real risky rate r_t comes from combining firms' profit and households' utility maximization

$$(19) \quad 1 + r_t = \alpha K_{t-1}^{\alpha-1} (e^{z_t} N_t)^{1-\alpha} + 1 - \delta = r_t^K + 1 - \delta$$

and where m_{t+1} the stochastic discount factor or pricing kernel is given by

$$(20) \quad m_{t+1} \equiv \frac{\partial V_t / \partial C_{t+1}}{\partial V_t / \partial C_t} = \beta \frac{C_t}{C_{t+1}} \frac{\exp\left(\frac{\gamma}{2} V_{t+1}\right)}{E_t \left[\exp\left(\frac{\gamma}{2} V_{t+1}\right) \right]}$$

with V_t implicitly evaluated at the maximum.

As the economy is nonstationary, growing at the rate a_t , we detrend output, consumption, investment, capital stock and value function to stationarize the model. This is achieved by dividing all nonstationary variables but the value function, which must be detrended differently, by the contemporaneous level of productivity e^{z_t} .¹² Labor supply N_t and leisure $1 - N_t$ as well as the returns r_t and r_t^K are stationary and therefore do not need to be transformed. Stationary variables will be denoted by lower case letters.

Reexpressing the pricing kernel in terms of stationary variables, the stochastic trend or long-run risk can be seen directly in the pricing kernel

$$(21) \quad m_{t+1} = \beta \frac{c_t}{c_{t+1}} e^{-(\bar{a} + \bar{\sigma}_z e^{\sigma_{z,t+1}} \varepsilon_{z,t+1})} \frac{\exp\left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \bar{\sigma}_z e^{\sigma_{z,t+1}} \varepsilon_{z,t+1}) \right]\right)}{E_t \left[\exp\left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \bar{\sigma}_z e^{\sigma_{z,t+1}} \varepsilon_{z,t+1}) \right]\right) \right]}$$

where the stochastic trend, $\bar{\sigma}_z e^{\sigma_{z,t+1}} \varepsilon_{z,t+1}$, enters the kernel through the twisted continuation utility as well as through the stochastic discount factor that would obtain under expected utility, $\beta \frac{c_t}{c_{t+1}}$, and thereby explicitly relates the volatility process $\sigma_{z,t+1}$ to the pricing kernel as in, e.g., Bansal et al. (Forthcoming).

To analyze asset prices, we append the model with the following additional asset pricing variables: the real risk-free rate $1 + r_t^f \equiv E_t(m_{t+1})^{-1}$, the squared conditional market price of risk—

¹²See the appendix for details.

the ratio of the conditional variance of the pricing kernel to square of its conditional mean¹³—
 $cmpr_t \equiv \frac{(E_t[(m_{t+1} - E_t m_{t+1})^2])^{\frac{1}{2}}}{E_t m_{t+1}}$ that measures the excess return the household demands for bearing
an additional unit of risk, the expected (ex ante) risk premium $erp_t \equiv E_t(r_{t+1} - r_t^f)$, and the (ex
post) risk premium $rp_t = r_t - r_{t-1}^f$ as the difference between the risky and risk-free rate.

4 Perturbation Solution and Risk Adjustment Channel

We solve the model in the foregoing section with a third order perturbation. As shown by Caldara et al. (2012), low order local approximations via perturbation methods can solve models such as ours quickly with a degree of accuracy comparable to global methods. Moreover, as at least a third order approximation is necessary for the analysis of time-varying shifts in risk premia and related measures at the heart of our analysis, we solve the model to third order. We use the nonlinear moving average perturbation derived in Lan and Meyer-Gohde (Forthcoming) as it delivers stable impulse responses and simulations and, as we shall show, enables analytical calculation and risk decomposition of moments.

For the implementation of the nonlinear moving average perturbation, we collect the (stationarized) equilibrium conditions into a vector of functions

$$(22) \quad 0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t)]$$

where y_t is the vector of the endogenous variables, and ε_t the vector of the exogenous shocks, assuming the function f in (22) is sufficiently smooth and all the moments of ε_t exist and finite¹⁴.

The solution to (22) is a time-invariant function y , taking as its state variable basis the infinite sequence of realized shocks, past and present, and indexed by the perturbation parameter $\sigma \in [0, 1]$

¹³We square the market price of risk to bestow it with the differentiability at the deterministic steady state necessary for our perturbation approach

¹⁴See for example, Judd (1998, ch. 13) and Jin and Judd (2002) for a complete characterization of these assumptions. While the normal distribution for shocks we choose is at odds with Jin and Judd's (2002) assumption of bounded support, Kim et al. (2008) dispute the essentiality of this assumption, lending support to our distribution choice

scaling the distribution of future shocks

$$(23) \quad y_t = y(\sigma, \varepsilon_t, \varepsilon_{t-1}, \dots)$$

Assuming normality of all the shocks and setting $\sigma = 1$ as we are interested in the stochastic model, the third order approximation—a Volterra expansion, see Lan and Meyer-Gohde (Forthcoming)—of (23), takes the form

$$(24) \quad y_t^{(3)} = \bar{y} + \frac{1}{2} y_{\sigma^2} + \frac{1}{2} \sum_{i=0}^{\infty} (y_i + y_{\sigma^2, i}) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j, i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \\ + \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{k, j, i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i})$$

where \bar{y} denotes the deterministic steady state of the model, at which all the partial derivatives $y_{\sigma^2}, y_{\sigma^2, i}, y_i, y_{j, i}$ and $y_{k, j, i}$ are evaluated. (24) is naturally decomposed into order of nonlinearity and risk adjustment— $y_i, y_{j, i}$ and $y_{k, j, i}$ capture the amplification effects of the realized shocks $(\varepsilon_t, \varepsilon_{t-1}, \dots)$ in the policy function (23) at first, second and third order respectively. The two partial derivatives with respect to σ, y_{σ^2} and $y_{\sigma^2, i}$ adjust the approximation for future risk.¹⁵ While y_{σ^2} is a constant adjustment for risk and a linear function of the variance of future shocks¹⁶, $y_{\sigma^2, i}$ varies over time, interacting the linear response to realized shocks with the variance of future shocks essentially adjusting the model for time variation in the conditional volatility of future risk.

5 Calibration

We select three calibrations for the numerical analysis of the model. For the baseline calibration, most of the parameter values are taken from Tallarini (2000) and are listed below.

[Table 1 about here.]

The discount factor $\beta = 0.9926$ generates an annual interest rate of about 3 percent. The capital share $\alpha = 0.339$ matches the ratio of labor share to national income. The depreciation

¹⁵More generally, a constant term, y_{σ^3} , at third order adjusts (24) for the skewness of the shocks. See Andreasen (2012). As we assume all the shocks are normally distributed, y_{σ^3} is zero and not included in (24) and the rest of our analysis.

¹⁶See, Lan and Meyer-Gohde (Forthcoming, p. 13) for the derivation of this term.

rate $\delta = 0.021$ matches the ratio of investment to output. The parameter χ is set to 100, translating to a relative risk aversion parameter with respect to consumption of about 25 following Tallarini (2000).¹⁷ The labor supply parameter ψ is chosen such that labor in the deterministic steady state, \bar{N} , is 0.2305 to align with the mean level of hours in data following Tallarini (2000). With β , χ , and ψ as above, $\gamma = -0.3676$ in line with Tallarini (2000).

For the parameters of the volatility shock, the literature varies in the range of the persistence— ρ_σ , from 0.9, Caldara et al. (2012) and Bidder and Smith (2012), to 0.95, Fernández-Villaverde and Rubio-Ramírez (2010), and to 0.99 or 1, Andreasen (2012) and Justiniano and Primiceri (2008)—and in the range of its instantaneous standard deviation— τ , from 0.01, Andreasen (2012) and Justiniano and Primiceri (2008), to 0.1, Fernández-Villaverde and Rubio-Ramírez (2010), and to 0.15, Bidder and Smith (2012). We follow the parameterization of Bidder and Smith (2012), implying a cumulative variance comparable to the value in Fernández-Villaverde and Rubio-Ramírez (2010, p. 20), that “generates changes in volatility similar to the ones observed in the [post-war] U.S.” Following Tallarini (2000), we adjust the homoskedastic component of the standard deviation of productivity growth to match the standard deviation of (log) consumption growth.

[Table 2 about here.]

While still allowing preferences to be recursive, the constant volatility calibration shuts down stochastic volatility by setting $\rho_\sigma = \tau = 0$, this enables direct comparison with Tallarini’s (2000) results. In addition, by comparing with the results from the baseline calibration, this exercise helps identify the contribution of the stochastic volatility, by itself and/or in interaction with recursive preferences, to the model. The expected utility calibration shuts stochastic volatility down and is implemented by setting $\chi = 1$ (equivalently, $\gamma = 0$).

¹⁷Tallarini (2000) gives $\frac{\psi+\chi}{1+\psi}$ for this measure of risk aversion. Swanson (2013) incorporating the active labor margin as in Swanson (2012), gives $\frac{\chi}{1+\psi}$ for this same measure of risk aversion. At our calibration, these measures correspond to 25.831 and 25.08 respectively.

We will use all the three calibrations to analyze the contributions of recursive preferences and stochastic volatility to the model's performance in matching empirical macroeconomic and asset pricing statistics. !!!Explain with macro moments and why!!! !!!Explain Hansen-Jagnannathan bounds!!!

6 Theoretical Moments

In this section, we derive the theoretical moments of the third order approximation (24). The nonlinear moving average policy function (23) and its third order approximation (24) both map exogenous shocks directly into endogenous variables. The moments of endogenous variables can therefore be computed directly as they are functions of the known moments of exogenous shocks. We further decompose the theoretical variance, disentangling the individual contributions of the risk adjustment and amplification channels to the total variance. Note that throughout the derivation of theoretical moments, we assume normality of the exogenous shocks and all the approximated variables are covariance stationary.¹⁸

By contrast, the state space perturbation policy function and its nonlinear approximations map the endogenous variables into themselves. Computing the m -th theoretical moment of such a nonlinear approximations of n -th order, for example, requires the knowledge of higher (than m -th) moments of endogenous variables that are in general nonlinear functions of the approximations up to and including n -th order. To this end, the calculation results in an infinite regression in the moments of endogenous variables. While theoretical moments of nonlinear state space perturbation approximations are in general not available, there are attempts in recent literature. Andreasen et al. (2013) calculate theoretical moments by pruning the nonlinear approximations, such that the higher (than m -th) moments are functions of approximations lower than the current order of

¹⁸While removing normality does not disable the calculation of theoretical moments, the derivation will be more complicated as additional terms involving skewness and higher (up to fifth) moments of the shocks emerge. See Lan and Meyer-Gohde (Forthcoming) for proof of the covariance stationarity.

approximation, and therefore computable given the results from all lower orders.¹⁹

6.1 Mean

The mean (first moment) of the third order approximation (24) is straightforward to calculate.

Applying the expectations operator to (24) yields

$$(25) \quad E \left[y_t^{(3)} \right] = \bar{y} + \frac{1}{2} y \sigma^2 + \frac{1}{2} \sum_{j=0}^{\infty} y_{j,j} E [\boldsymbol{\varepsilon}_t \otimes \boldsymbol{\varepsilon}_t]$$

The last term in (24) vanishes as the triple Kronecker product in expectation is the columnwise vectorization of the third moment of the exogenous shocks, equal to zero under normality. Likewise, the Kronecker product in expectation is the columnwise vectorization of the second moment of the exogenous shocks. Only the contemporaneous variance appears because the shock vector is assumed serially uncorrelated. The other two terms containing $\boldsymbol{\varepsilon}_{t-i}$ in (24) also disappear as the shock is mean zero. From a different perspective, the deterministic steady state is the mean of the zeroth order approximation where all shocks, past, present and future are zero. It remains the mean in a first order approximation, as the exogenous shocks are mean zero (first moment is zero). At second order, the second moments of the shocks are included—both past and present (in the term $\sum_{j=0}^{\infty} y_{j,j} E [\boldsymbol{\varepsilon}_t \otimes \boldsymbol{\varepsilon}_t]$) as well as future shocks (in the term y_{σ^2})—which are assumed nonzero, generating an adjustment from the deterministic steady state. When the approximation moves to the third order, the calculation of the mean of (24) would be accordingly adjusted for the first three moments of all the realized and future shocks, but the mean zero and normality assumptions render the first and third moments of the shocks zero, thus leaving the first moment at third order identical to its value from a second order approximation.

¹⁹ As shown in Lan and Meyer-Gohde (2013a), nonlinear moving average perturbations at the third order differ from their pruned state space counterparts in that they are centered around and correct the first derivative of the policy function for the stochastic steady state implied by the order of approximation, leading to accuracy gains in a mean squared sense.

6.2 Variance and Autocovariances

While we could conceivably compute the second moments (variance and autocovariances) of (24) using the Volterra expansion directly, it would be a rather complicated operation on the products of multi-layered infinite summation of coefficients. As an alternative, we use the recursive expression of (24) derived in Lan and Meyer-Gohde (2013a) to compute the second moments.

Computing the second moments using the recursive expression of (24), we need to proceed sequentially through the orders of approximation and exploit the linearly recursive (in order) structure of the solution. That is, the second moments of the approximation at any order can always be expressed as the sum of the second moments of the approximation of the previous order and the second moments of all the previous order increments (the difference between two approximations of adjacent order, subtracting the constant risk adjustment of the higher order). In other words, the embedded decomposition into order of approximation in the nonlinear approximations of the policy function (23) is preserved in its second moments.

The first order approximation of (23) takes the form of a linear moving average, $y_t^{(1)} = \bar{y} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i}$, and can be expressed recursively as²⁰

$$(26) \quad y_t^{(1)} - \bar{y} = \alpha \left(y_t^{(1)state} - \bar{y}^{state} \right) + \beta_0 \varepsilon_t$$

where the difference $y_t^{(1)} - \bar{y}$ is the deviation of the first order approximation with respect to the deterministic steady state, and identical to the first order increment

$$(27) \quad dy_t^{(1)} \equiv y_t^{(1)} - \bar{y}$$

which captures the addition to the approximation contributed by the time varying terms of the current, here first, order of approximation, as \bar{y} is the zeroth order approximation²¹ and the constant

²⁰See Lan and Meyer-Gohde (2013a). This is, of course, an standard result for linear models. Compare, e.g., the state space representations of Uhlig (1999) with the infinite moving average representations of Taylor (1986).

²¹This is the terminology in Anderson et al. (2006, p. 17) and Borovicka and Hansen (2012, p. 22).

risk adjustment of first order, y_σ , is zero. In addition

$$(28) \quad E \left[dy_{t-1}^{(1)} \boldsymbol{\varepsilon}'_t \right] = 0$$

as the current shock is not correlated with the endogenous variables in the past. Under the orthogonality condition (28), the sequence of autocovariances of endogenous variables or, at this order equivalently, of the first order increment $\Gamma_j^{y^{(1)}} = \Gamma_j^{(1)} = E \left[dy_t^{(1)} dy_{t-j}^{(1)'} \right]$, solves the following Lyapunov equation

$$(29) \quad \Gamma_j^{y^{(1)}} = \alpha \Gamma_j^{y^{(1)}} \alpha' + \beta_0 E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_{t-j}'] \beta_0'$$

The second order approximation of the policy function (23) captures the amplification effects of the realized shocks up to second order, and the constant risk adjustment for future shocks

$$(30) \quad y_t^{(2)} = \bar{y} + \frac{1}{2} y_\sigma^2 + \sum_{i=0}^{\infty} y_i \boldsymbol{\varepsilon}_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\boldsymbol{\varepsilon}_{t-j} \otimes \boldsymbol{\varepsilon}_{t-i})$$

Defining the second order increment

$$(31) \quad dy_t^{(2)} \equiv y_t^{(2)} - y_t^{(1)} - \frac{1}{2} y_\sigma^2$$

which more clearly illustrates the notion of increment we use here: the addition the approximation contributed by time varying components of current order (or the difference between the current and previous order of approximation, here $y_t^{(2)} - y_t^{(1)}$, less the additional constant contributed by the current order, here $\frac{1}{2} y_\sigma^2$). With this notion, the second order approximation (30) can be considered as the sum of first order approximation, the constant risk correction term and second order increment

$$(32) \quad y_t^{(2)} = y_t^{(1)} + \frac{1}{2} y_\sigma^2 + dy_t^{(2)}$$

The above decomposition of second order approximation naturally passes on to its moments — Starting with the mean, taking expectation of (32) yields

$$(33) \quad E y_t^{(2)} = E y_t^{(1)} + \frac{1}{2} y_\sigma^2 + E dy_t^{(2)}$$

Therefore the mean of second order approximation is a sum of the mean of first order approximation, i.e., the deterministic steady state, the constant risk correction term, and the mean of the

second order increment. Likewise, the second moments of the second order approximation can be expressed as the sum of the second moments of the first order approximation and those of the order increment. We summarize the results for a second order approximation in the following proposition

Proposition 6.1. *Assuming the exogenous shocks are normally distributed, the j 'th autocovariance of the second order approximation (30) is of the form*

$$(34) \quad \Gamma_j^{y^{(2)}} = \Gamma_j^{y^{(1)}} + \Gamma_j^{(2)}$$

where

$$(35) \quad \Gamma_j^{y^{(2)}} = E \left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_{t-j}^{(2)} \right)' \right]$$

$$(36) \quad \Gamma_j^{y^{(1)}} = \Gamma_j^{(1)} = E \left(dy_t^{(1)} dy_{t-j}^{(1)'} \right)$$

$$(37) \quad \Gamma_j^{(2)} = E \left[\left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_{t-j}^{(2)} \right)' \right]$$

Proof. See the appendices. □

The second order increment $dy_t^{(2)}$ can also be expressed recursively.²² With that recursive expression in hand, the unknown $E dy_t^{(2)}$ in (33) and $\Gamma_j^{(2)}$ in (34) can be obtained by solving some standard linear matrix equations and an appropriate Lyapunov equation. The details are relegated to the appendices.

Similarly, to compute the second moments of endogenous variables using the third order approximation (24), we define the third order increment

$$(38) \quad dy_t^{(3)} \equiv y_t^{(3)} - y_t^{(2)}$$

which is merely the difference between the third and second order approximations, as the third order approximation adds no additional constant terms under normality. We summarize the resulting second moment calculations at third order in the following proposition

²²See, again, Lan and Meyer-Gohde (2013a).

Proposition 6.2. *Assuming the exogenous shocks are normally distributed, the j 'th autocovariance of the third order approximation (24) takes the form*

$$(39) \quad \Gamma_j^{y^{(3)}} = \Gamma_j^{y^{(2)}} + \Gamma_j^{(3)} + \Gamma_j^{(1),(3)} + \left(\Gamma_j^{(1),(3)}\right)'$$

where

$$(40) \quad \Gamma_j^{y^{(3)}} = E \left[\left(y_t^{(3)} - E y_t^{(3)} \right) \left(y_{t-j}^{(3)} - E y_{t-j}^{(3)} \right)' \right]$$

$$(41) \quad \Gamma_j^{(3)} = E \left(dy_t^{(3)} dy_{t-j}^{(3)'} \right)$$

$$(42) \quad \Gamma_j^{(1),(3)} = E \left(dy_t^{(1)} dy_{t-j}^{(3)'} \right)$$

and $\Gamma_j^{y^{(2)}}$ is as defined in Proposition 6.1.

Proof. See the appendices. □

$\Gamma_j^{y^{(3)}}$ is the j 'th autocovariance of endogenous variables computed using the third order approximation (24), $\Gamma_j^{(3)}$ the j 'th autocovariance of the third order increment $dy_t^{(3)}$, and $\Gamma_j^{(1),(3)}$ the j 'th autocovariance between the first and the third order increments $dy_t^{(1)}$ and $dy_t^{(3)}$. Analogous to (34) in Proposition 6.1, (39) decomposes the second moments into order of approximation: When the approximation moves to the third order, the second moments of endogenous variables are those computed using second order approximation (30), adjusted by the second moments of $dy_t^{(3)}$ itself and the interaction with the first order increment $dy_t^{(1)}$.

With the recursive form of the third order increment $dy_t^{(3)}$,²³ the two unknown quantities, $\Gamma_j^{(3)}$ and $\Gamma_j^{(1),(3)}$, in (39) for calculating the covariance matrices of the third order approximation can be computed by formulating appropriate Lyapunov equations. The details are in the appendices.

6.3 A Variance Decomposition

The third order approximation, (24), decomposes naturally into orders of nonlinearity and risk adjustment. This dissects the individual contributions of the sequence of realized shocks and future

²³See, again, Lan and Meyer-Gohde (2013a).

shocks and a variance decomposition can be accordingly derived to analyze the composition of the volatility of endogenous variables.

Let $y_t^{(3)risk} \equiv \frac{1}{2}y_{\sigma^2} + \frac{1}{2}\sum_{i=0}^{\infty}y_{\sigma^2,i}\varepsilon_{t-i}$ denote the risk adjustment channel, with a constant risk adjustment at second order ($\frac{1}{2}y_{\sigma^2}$) and a time-varying risk adjustment channel at third order ($\frac{1}{2}\sum_{i=0}^{\infty}y_{\sigma^2,i}\varepsilon_{t-i}$) and $y_t^{(3)amp}$ collect all the other terms in the third order approximation (24) capturing the amplification effects, we can rewrite (24) as

$$(43) \quad y_t^{(3)} \equiv y_t^{(3)risk} + y_t^{(3)amp}$$

Centering the previous equation around its mean,²⁴ multiplying the resulting expression with its transposition and applying the expectations operator yields the following variance decomposition

Proposition 6.3. *Assuming the exogenous shocks are normally distributed, the covariance of the third order approximation (24) takes the form*

$$(44) \quad \Gamma_0^{y^{(3)}} = \Gamma_0^{y^{(3)risk}} + \Gamma_0^{y^{(3)risk,amp}} + \Gamma_0^{y^{(3)amp}}$$

where

$$(45) \quad \Gamma_0^{y^{(3)risk}} = E \left[\left(y_t^{(3)risk} - E y_t^{(3)risk} \right) \left(y_t^{(3)risk} - E y_t^{(3)risk} \right)' \right]$$

$$(46) \quad \Gamma_0^{y^{(3)amp}} = E \left[\left(y_t^{(3)amp} - E y_t^{(3)amp} \right) \left(y_t^{(3)amp} - E y_t^{(3)amp} \right)' \right]$$

$$(47) \quad \Gamma_0^{y^{(3)risk,amp}} = E \left[\left(y_t^{(3)amp} - E y_t^{(3)amp} \right) y_t^{(3)risk'} \right] + \left(E \left[\left(y_t^{(3)amp} - E y_t^{(3)amp} \right) y_t^{(3)risk'} \right] \right)'$$

Proof. See the appendices. □

The variance of the endogenous variables, $\Gamma_0^{y^{(3)}}$, can thus be expressed as the sum of $\Gamma_0^{y^{(3)risk}}$ that stores the variations come from the time-varying risk adjustment channel alone, $\Gamma_0^{y^{(3)amp}}$ that stores the variations come from the amplification channels of all three orders and $\Gamma_0^{y^{(3)risk,amp}}$ that stores the variations come from the interaction between the two types of channels.

Both $y_t^{(3)risk}$ and $y_t^{(3)amp}$ can be expressed recursively. With those recursive expressions, $\Gamma_0^{y^{(3)risk}}$ and $\Gamma_0^{y^{(3)amp}}$ can be computed by formulating appropriate Lyapunov equations (See the appendices

²⁴Note $E y_t^{(3)risk} = \frac{1}{2}y_{\sigma^2}$ and $E y_t^{(3)amp} = \bar{y} + \frac{1}{2}\sum_{j=0}^{\infty}y_{j,j}E[\varepsilon_t \otimes \varepsilon_t]$.

for details). As $\Gamma_0^{y(3)}$ is already known from Proposition 6.2, $\Gamma_0^{y(3)risk,amp}$ can be computed by subtracting $\Gamma_0^{y(3)risk}$ and $\Gamma_0^{y(3)amp}$ from $\Gamma_0^{y(3)}$.

6.4 Simulated Moments

As an alternative to the theoretical moments, we can simulate the third order approximation (24) and compute the moments of the simulated series to analyze the statistical implications of the model. Lan and Meyer-Gohde (Forthcoming) show that nonlinear approximation of the policy function (23) preserve the stability of the linear approximation or first order approximation and, hence, does not generate explosive time paths in simulations.

Simulation methods for moment calculations are, however, not always feasible for state space perturbations. Aruoba et al. (2006), Fernández-Villaverde and Rubio-Ramírez (2006) and Kim et al. (2008) note that higher order Taylor approximations to state space perturbation policy function can be potentially explosive in simulations. Truncation of the distribution from which exogenous shocks are drawn or the application of pruning schemes, like proposed by Kim et al. (2008) for a second order approximation,²⁵ can prevent such behavior. While this imposes stability on simulations of higher order approximations, pruning is an ad hoc procedure as noted by Lombardo (2010) and potentially distortive even when the simulation is not on an explosive path (See, Den Haan and De Wind (2012)). Though this might give rise to reasonable doubts regarding the accuracy and validity of moments calculated using perturbations, we will show that this is not the case with our nonlinear moving average.

As (24) generates stable time paths, moments computed by simulating (24) should asymptotically converge to their theoretical counterparts.

[Figure 1 about here.]

²⁵See Lan and Meyer-Gohde (2013a) for an overview and comparison of pruning algorithms at second and third order and their relation to our nonlinear moving average.

Figure 1 is an example of this check. It depicts the evolution path of the density of the simulated variance of the pricing kernel in the model described in Section 3 under the baseline calibration. Densities of the simulated variance of the pricing kernel are calculated using a kernel density estimation and 100 simulations at the indicated length. The theoretical variance, denoted by the red dashed line, is 0.0666 and all densities are in general centered around this value. The distributions of simulated variance are more dispersed in short-run simulations, tightening up to the theoretical value as the length increases consistent with asymptotic convergence of the simulated moments to their theoretical counterparts we calculated above.

7 Analysis of the Baseline Model

In this section, we report the performance under different calibrations of the model approximated to third order. We present impulse responses using the method of Lan and Meyer-Gohde (Forthcoming) to shocks in productivity growth and its volatility for both macroeconomic and asset pricing variables. We then proceed to the moments and the results of the variance decomposition introduced in section 6.3 to identify and quantify the individual contribution from the time-varying risk adjustment channel to the total variation. Finally, we cast doubt on the efficacy of stochastic volatility in aiding the model ability in attaining the Hansen-Jagannathan bounds.

7.1 Impulse Responses and Simulations

We analyze the impulse responses to shocks in productivity growth and shock in its volatility for macroeconomic and asset pricing variables. We also simulate the conditional market price of risk under stochastic volatility and with growth shocks of constant variance to observe the change in the variations of this variable under conditional heteroskedasticity.

[Figure 2 about here.]

Figure 2 depicts the impulse response and its contributing components for capital to a positive, one standard deviation shock in volatility, i.e., in $\varepsilon_{\sigma_z,t}$. The upper panel displays the impulse responses at first, second and third order as deviations from their respective (non)stochastic steady states (themselves in the middle right panel). In the the middle left panel and the middle column of panels in the lower half of the figure, the contributions to the total impulse responses from the first, second and third order amplification channels, that is, y_i , $y_{i,i}$ and $y_{i,i,i}$ in the third order approximation (24), are displayed. Notice that there is no response in these amplification channels. All responses to this volatility shock come from the lower left panel of the figure where the time-varying risk adjustment channel $y_{\sigma^2,i}$ is displayed. In other words, for capital, a volatility shock by itself propagates solely through the time-varying risk adjustment channel.

Capital responds positively to a positive volatility shock. This captures the household's precautionary reaction to the widening of the distribution of future shocks.²⁶ Our risk-averse household accumulates a buffer stock in capital to insure itself against the increased future risk of productivity growth shocks from a more dispersed distribution.

[Figure 3 about here.]

Figure 3 displays the responses of macroeconomic variables as deviations from their risk corrected steady states to a positive, one standard deviation volatility shock. The household accumulates a buffer stock of capital by increasing current investment on impact of the shock. As the allocation has not changed, the household finances this investment through a decrease in current consumption, resulting in an increase in the marginal utility of consumption. The intratemporal labor supply equation (17) implies this increased marginal utility of consumption leads to an increase in the marginal utility of leisure, and therefore a decrease in time spend on leisure. The increased labor effort, with the capital stock being fixed on impact as it is a state variable and with the pro-

²⁶See also Fernández-Villaverde and Rubio-Ramírez (2010) and van Binsbergen et al. (2012) for precautionary savings behavior in DSGE perturbation.

ductivity having not changed,²⁷ translates into an increase in current output partially offsetting the costs borne by consumption of the increased investment for the buffer stock of capital. Thus, this model predicts a boom in economic activity following an increase in risk, as firms produce and households work to accumulate the necessary buffer stock. A richer model of frictional investment that, for example, includes variable capacity utilization, capital adjustment cost and consumption habit formation can overturn this result, as discussed in section 8. While the impulse responses for the macroeconomic variables are not pictured with their contributing components, responses of these variables to a volatility shock come solely from the time-varying risk adjustment channel. The volatility shock is persistent but not permanent. As the shock dies out and productivity shocks fail to materialize from their widened distribution, the household winds down its buffer stock of capital by increasing consumption and leisure, leading to a fall in output and investment.

[Figure 4 about here.]

Figure 4 depicts the impulse responses and their contributing components for the expected risk premium to positive, one standard deviation shocks in productivity and its volatility, i.e., in $\epsilon_{z,t}$ and $\epsilon_{\sigma_{z,t}}$ (Figure 4a and 4b respectively). Firstly, note that both the volatility and productivity growth shock propagate solely through $y_{\sigma^2,i}$, the time-varying risk adjustment channel of this variable and there are no responses in the amplification channels of any of the three orders. Second, the response of the expected risk premium to the volatility shock is almost two orders magnitude larger than that to the productivity growth shock, implying the overall majority of variations in this variable is driven solely by volatility shocks with the contribution from the productivity growth shock to the total variation negligible. Moreover, the positive response of the expected risk premium to an increase in volatility highlights the role of long run risk in asset pricing as noted by Bansal and Yaron (2004), Bansal (2008) and Bansal et al.'s (Forthcoming) — risks and volatility in asset prices

²⁷Note that, it is the distribution governing future productivity shocks that is being shocked here, not the level of productivity itself.

are driven by those in economic fundamentals. An increase of volatility in long run productivity growth therefore drives up risks in asset prices and makes holding asset riskier. Household thereby demands a higher compensation for doing so. I.e., an increase in volatility of long run risk carries a positive risk premium.

[Figure 5 about here.]

Figure 5 depicts the simulated time paths of the squared conditional market price of risk²⁸ under the constant volatility and the baseline calibration of the model (Figure 5a and 5b respectively). When there is no volatility shock, the conditional market price exhibits minimal fluctuations along the simulation path. Adding stochastic volatility, however, induces a substantial amount of variations in this variable. This is consistent with the interpretation that volatility shocks are a source of conditional heteroskedasticity. The displayed time variation in the conditional market price of risk is roughly consistent with the empirical variations in the (lower bound of) market price of risk as measured over different periods of time the past 130 odd years (See, Cogley and Sargent (2008, p. 466)).

7.2 Moments Comparison

We compare the mean and standard deviations of the third order approximation (24) to those reported in Tallarini (2000) for his model and post-war U.S. data. The results of the variance decomposition in Section 6.3 are reported, allowing us to pin down the contribution from the time-varying risk adjustment channel to the total variance of the endogenous variables.

[Table 3 about here.]

The third and fourth column of Table 3 report the theoretical means under the baseline and constant volatility calibration of the model. The last column displays the means of the constant

²⁸We square this variable to eliminate the kink at the deterministic steady state and make perturbation applicable.

volatility calibration as reported by Tallarini (2000) with his iterative modified linear quadratic approximation based on Hansen and Sargent (1995). By comparing the last two columns we observe, for both macroeconomic and asset pricing variables, our theoretical means are in line with those of Tallarini (2000).

In the presence of risks induced by the long run productivity growth shocks (with or without stochastic volatility), the means of macroeconomic quantities (reported in the first five rows of the third and fourth column) are uniformly larger than their corresponding deterministic steady state value (reported in the first five rows of the first column), reinforcing the interpretation of household's precautionary reaction to future shocks.²⁹ In contrast, the mean of risky and risk free rates (reported in the last two rows of the third and fourth column) are uniformly lower than their deterministic counterparts (reported in the last two rows of the first column). This follows directly from the increase in the mean of capital which reduces the average return on equity (risky rate) and consequentially the risk free rate as noted by Tallarini (2000).

[Table 4 about here.]

The second and third column of Table 4 report the theoretical standard deviations of the third order approximation (24) under the baseline and constant volatility calibration of the model. Comparing to the standard deviations reported in the last two columns, our theoretical standard deviations are in line with those reported in Tallarini (2000), both model based and empirical.

[Table 5 about here.]

Table 5 reports the results of the variance decomposition under the baseline (stochastic volatility) and the constant volatility calibration. For each calibration, the table reports the percentage contributions of the first order amplification channel $y_t^{(1)}$ and the time-varying risk adjustment

²⁹That the mean of higher order approximation of macroeconomic quantities captures precautionary reactions and hence are different from their deterministic steady state counterparts are also noted by Tallarini (2000), Michel (2011) and Coeurdacier et al. (2011) in state space context.

channel $y_t^{(3)risk}$ to the total variance of the endogenous variables as the overall majority of variations come from these two channels. The second and third column report the decomposition results in absence of volatility shock and the last two columns in presence of volatility shock. For the conditional market price of risk and the expected risk premium, all variation comes from the time-varying risk adjustment channel regardless of whether there is volatility shock. This is consistent with the impulse responses for the expected risk premium (Figure 4), where we observed that both the productivity growth and volatility shock propagate solely through the time-varying risk adjustment channel.

For the risk premium and macroeconomic variables, adding the volatility shock alters the composition of variance. In the absence of the volatility shock, the contribution of the time-varying risk adjustment channel is negligible and almost all variation comes from the first order amplification channel. Adding stochastic volatility, however, operationalizes the time-varying risk channel, as a large portion of variance now comes through changes in risk as measured by conditional heteroskedasticity. Since, for macroeconomic variables, actions in the time-varying risk adjustment channel can be explained by the risk-averse household's precautionary motives, this variance decomposition result implies that such motives account for a larger portion of variance in the presence of stochastic volatility than in the absence thereof. An presence of changing risk induces the pattern of precautionary behavior here—with investment, output, and labor driven substantially by risk adjustments—as the capital margin cannot be freely adjusted contemporaneously in response to shifts in the distribution of technology shocks, pushing the adjustment onto production, the other factor of production, labor, and the component of expenditure, investment, over which the household does not have a direct smoothing motive.

From a methodological point of view, in the absence of volatility shock, a first order linear approximation would thus appear sufficient for computing the theoretical variance of macroeconomic variables. However, theoretical variances need to be computed using a third order approximation in

the presence of stochastic volatility and for conditional asset pricing measures, as otherwise a large portion or all of the variance will be missed through the neglect of time varying risk adjustment and higher order amplification effects.³⁰

7.3 Stochastic Volatility and Hansen-Jagannathan Bounds

We evaluate the model's ability in attaining the Hansen-Jagannathan bounds under the three different calibrations. The bounds present an important empirical measure that depend on the first two moments of the pricing kernel for a model's ability of replicating asset pricing regularities. Contrary to studies in endowments models where the variance of the log consumption growth process is fixed exogenously, the variance of log consumption growth here is endogenous, deriving eventually from the productivity process. While adding stochastic volatility does *ceteris paribus* move the model closer to the Hansen-Jagannathan bounds, it does so at the cost of increasing the variance of the consumption process. Adjusting the homoskedastic component of productivity to hold the variance of log consumption growth constant, the move towards the Hansen-Jagannathan bounds is negated.

[Figure 6 about here.]

Figure 6 depicts the unconditional mean standard deviation pairs of the pricing kernel generated by the model under the three different calibrations. Under the baseline (stochastic volatility) and the constant volatility calibrations, the preferences are in recursive form. While the volatility of the kernel increases with the coefficient of relative risk aversion for atemporal wealth gambles (here from χ equals one to five, ten, twenty, thirty, forty, fifty, and one hundred), the unconditional mean of the kernel is left (essentially) unchanged as the elasticity of intertemporal substitution (EIS) is parameterized independent of risk aversion in recursive preferences, and the model approaches

³⁰This provides insight, and a proviso in the presence of stochastic volatility, into the practice of computing macro variables using first and conditional asset pricing measures with third order approximations as in Rudebusch and Swanson (2012).

the Hansen-Jagannathan Bounds from below. The expected utility calibration generates a volatile pricing kernel at the cost of reducing its unconditional mean, as the EIS and risk aversion are inversely correlated in the expected utility, generating Weil's (1989) risk free rate puzzle. Figure 6a shows that given the same value of risk aversion, the calibration with stochastic volatility (baseline calibration) generates a more volatile pricing kernel than the constant volatility calibration. In other words, to generate certain amount of volatility in the pricing kernel, the model with volatility shock appears to need less risk aversion than the model without. This is achieved, however, at the cost of increasing the variance of the log consumption growth. As figure 6b shows, if we hold that variance constant at its empirical counterpart by reducing the homoskedastic component of the productivity growth shock—as Tallarini (2000) does throughout his study, the effect of volatility shock in terms of further increasing the volatility in the pricing kernel vanishes, reiterating the conditional heteroskedastic interpretation of volatility shocks.

This casts doubt on the portability of the results of Bansal and Yaron (2004) and others that identify stochastic volatility as a potential contributor to the resolution of asset pricing puzzles summarized in the pricing kernel's ability to reach the Hansen-Jagannathan bounds. When an endowment setting is abandoned in favor of a production model, the variance of log consumption growth can no longer be held exogenously. Increasing the variance of the volatility process leads to an increase in the variance of the log consumption process. Holding the overall volatility constant by adjusting the homoskedastic component of the productivity process downward counteracts the increased variance of the volatility process nearly completely, leaving the model as well off with regards to the Hansen-Jagannathan bounds as without stochastic volatility.

8 An Extension to Frictional Investment

In this section, we extend the model in section 3 to demonstrate that when the model is no longer frictionless an increase in risk may lead to a fall in output as argued for in Bloom (2009),

Fernández-Villaverde et al. (2011b), Basu and Bundick (2012), Bloom et al. (2012) and Bidder and Smith (2012). To accomplish this, we extend the model in two dimensions. First, we add variable capital utilization with endogenous depreciation to enable households to accumulate their precautionary buffer stock of capital in response to a volatility shock by reducing capital utilization and thereby decreasing depreciation as an alternative to increasing investment.³¹ Second, we impose capital adjustment costs to increase the relative attractiveness of this alternate channel of capital accumulation with respect to increasing investment.³² A numerical analysis of this extended model suggests, when capital adjustment cost is sufficiently high and thus household primarily chooses to decrease utilization rate to build up buffer stock of capital in response to a volatility shock, the resulting decrease in capital for production and the consequential fall in current output outweighs the simultaneous increase in output induced by increased labor input.

The infinitely lived household still seeks to maximize its expected discounted lifetime utility given by the recursive preferences (10) over consumption and labor supply subject to the budget constraint (12). The representative firm now maximizes profits $Y_t - W_t N_t - I_t$ in each period by choosing labor input, investment and the capital utilization rate, subject to the following capital accumulation law and production technology³³

$$(48) \quad Y_t = (u_t K_{t-1})^\alpha (e^{Z_t} N_t)^{1-\alpha}$$

$$(49) \quad K_t = (1 - \delta_t) K_{t-1} + \phi_t K_{t-1}$$

The capital adjustment cost function, ϕ_t in (49) penalizes investment, in units of current capital,

³¹Variable capital utilization allows household to adjust capital in service immediately, as opposed to a time-to-build fashion, in response to shocks that alters the marginal productivity of capital, see Greenwood et al. (1988), Burnside and Eichenbaum (1996) and King and Rebelo (1999) for detailed analysis of models' propagation mechanism in presence of variable capital utilization.

³²See Hayashi (1982) for the theoretical foundation of capital adjustment cost, Jermann (1998) for its application in asset pricing and Baxter and Crucini (1993), Baxter and Crucini (1995) and Baxter and Farr (2005) for its contribution to explaining international trade and business cycles.

³³We follow Uzawa (1969) and introduce adjustment costs associated with investment. See, e.g., Lucas (1967) for an alternative.

for deviating from its frictionless level, and follows Jermann's (1998) specification

$$(50) \quad \phi_t = \phi \left(\frac{I_t}{K_{t-1}} \right) = \frac{b_k}{1 - 1/\xi_k} \left(\frac{I_t}{K_{t-1}} \right)^{1-1/\xi_k} + c_k$$

where b_k and c_k are constants that will be set to ensure that adjustment costs are neutral in the deterministic steady state and ξ_k the elasticity of the investment capital ratio with respect to Tobin's q .³⁴ With variable capital utilization, u_t in (49) and (48), firms can adjust the capital input in production contemporaneously. However, increasing u_t leads to faster capital depreciation and the depreciation function follows Baxter and Farr's (2005) specification

$$(51) \quad \delta_t = \delta(u_t) = \frac{b_u}{1 + \xi_u} u_t^{1+\xi_u} + c_u$$

where b_u and c_u are constants that will be chosen such that capital is fully utilized in the deterministic steady state and ξ_u the elasticity of marginal depreciation with respect to the utilization rate.³⁵

The model is closed by the market clearing condition (16) as before.

The firm's optimal utilization plan in presence of capital adjustment cost, equating the marginal benefit in terms of additional output produced to the marginal cost in terms of additional units of, is capital being worn out

$$(52) \quad \alpha \frac{Y_t}{u_t} = \frac{\delta'_t}{\phi'_t} K_{t-1}$$

The risky rate of return on capital in the presence of both capital adjustment cost and variable utilization is now

$$(53) \quad r_t = \left(\alpha \frac{Y_t}{K_{t-1}} + \frac{1 - \delta_t + \phi_t}{\phi'_t} - \frac{I_t}{K_{t-1}} \right) \phi'_{t-1} - 1$$

We keep the parameters of the baseline model in section 3 at their values stated there (see Table 1), except for the homoskedastic component of the standard deviation of productivity growth, $\overline{\sigma}_z$, which we adjust to match the standard deviation of (log) consumption growth. As do Christiano et al. (2005) and Ríos-Rull et al. (2012), we impose full capital utilization in the deterministic

³⁴I.e., $\xi_k \equiv -(\phi''_t/\phi'_t)/(I_t/K_{t-1})$ where $\phi'_t = \partial\phi_t/\partial(I_t/K_{t-1})$ the marginal capital adjustment cost and the inverse of Tobin's q , and $\phi''_t = \partial^2\phi_t/\partial(I_t/K_{t-1})^2$, see Baxter and Crucini (1995) for example.

³⁵I.e., $\xi_u \equiv u_t \delta''_t/\delta'_t$ where $\delta'_t = \partial\delta_t/\partial u_t$ the marginal capital utilization and $\delta''_t = \partial^2\delta_t/\partial u_t^2$, see Baxter and Farr (2005) for example.

steady state by letting $\bar{u} = 1$, and ensure the adjustment cost does not affect the deterministic steady state by setting $\bar{\phi} = \bar{I}/\bar{K}$ and $\bar{\phi}' = 1$ as also noted in van Binsbergen et al. (2012).

For ξ_k , the literature varies in the range from 0.101 from van Binsbergen et al. (2012), to 0.23 from Jermann (1998),³⁶ to 13.3 from Baxter and Jermann (1999) and to 15 from Baxter and Crucini (1995) with changes in investment becoming less costly as ξ_k increases (See Baxter and Crucini (1993)). We set $\xi_k = 1.5$, making utilization rate a preferred channel of adjusting capital in service in response to a volatility shock as changes in investment is fairly costly with this value. Note that, as the value of ξ_k increases and adjusting capital through investment becomes less costly, the extended model might again predict a boom in response to an increase in risk.

For the elasticity of utilization, ξ_u , Baxter and Farr (2005) examine three values: $\xi_u = 1$, taken from Basu and Kimball (1997),³⁷ 0.1 taken from King and Rebelo (1999), along with a highly elastic case under the value 0.05. We set $\xi_u = 0.1$ and note that the primary concern of the analysis of this extended model, the response of variables to a volatility shock, is qualitatively robust to this highly elastic case as well.

[Figure 7 about here.]

Figure 7 depicts the impulse responses of macroeconomic variables, expressed as deviations from their risk corrected steady states, under a third order approximation to a positive, one standard deviation volatility shock. As in the baseline model (See Figure 3), the household accumulates a buffer stock of capital in response to the increased volatility of future productivity shocks. To accumulate this stock, however, the household decreases the utilization rate, thereby slowing down depreciation. With this additional margin available to the household to accumulate capital, the increase in investment relative to capital, financed by decreasing consumption, is noticeably smaller than that in the model with baseline calibration. In addition, the labor supply equation (17) still

³⁶0.23 is near the lower bound of the empirical range as noted by Christiano et al. (2001)

³⁷Baxter and Farr (2005) note the imprecision of Basu and Kimball's (1997) estimation of this value.

implies an increase in the marginal utility of consumption following the decrease in consumption to finance increased investment and leads to an increase in the marginal utility of leisure, and therefore a decrease in time spend on leisure. Unlike in the model under baseline calibration, the increased labor effort, with productivity having not changed (again, it is only the volatility of the distribution of future productivity shocks that is being shocked), fails to increase current output as the effect on output from the decrease in utilization rate and the consequential under-deployment of capital is stronger. Thus, this model predicts a fall in economic activity following an increase in risk as opposed to a boom predicted by the model with baseline calibration. The volatility shock is persistent but not permanent and as the shock dies out and productivity shocks fail to materialize from their widened distribution, the household winds down its buffer stock of capital by increasing consumption, leisure and the utilization rate, leading to a fall in investment but a rise in output as the effect of the increase in the utilization rate again dominates. Both the duration of the drop in output and its subsequent overshooting are consistent with the results documented by Bloom (2009).

Though, labor and investment still rise in response to a volatility shock here. Additions to the model, such as in Bidder and Smith (2012), with Jaimovich and Rebelo's (2009) preferences and Constantinides's (1990) consumption habit formation, show that a positive volatility shock leads to a simultaneous drop in output, investment, utilization, consumption and labor. Owing to Jaimovich and Rebelo's (2009) preferences, labor supply in their model is largely independent of wealth effects, and thus declines with other macroeconomic quantities. The habit formation slows down consumption adjustment, increasing the persistence and magnitude of a recession. By taking the demand side into account, Basu and Bundick (2012) show this uniform drop in macroeconomic aggregates in response to an increase in risk using a New Keynesian model with a countercyclical markup through sticky prices. On impact of a volatility shock, the increased labor supply as a precautionary reaction reduces firms marginal cost of production and thereby increases

the markup since price is sticky. A higher markup winds down the demand for both consumption and investment goods, leading to a fall in output and employment. Nonetheless, it is noteworthy that adjustment costs and variable capital utilization alone are sufficient to generate the drop in output that Bloom (2009) identified empirically.

[Table 6 about here.]

The third column of Table 6 reports the theoretical standard deviation of the extended model. Comparing to the model with baseline calibration, the standard deviation of the logarithmic investment capital ratio is noticeably smaller, consistent with the interpretation that adjusting capital through investment is very costly in the presence of adjustment cost, leaving investment relatively less volatile. The substantial drop in the standard deviation of the logarithmic investment output ratio reinforces this interpretation as the volatility of investment now contributes much less, through capital, to that of output when the utilization margin is activated.

[Table 7 about here.]

The last two columns of Table 7 report the variance decomposition result of the extended model. Comparing to the results of the model with baseline calibration reported in the fourth and fifth columns, the contribution from the time varying risk adjustment channel to the total variation of all the listed variables, except conditional market price of risk and conditional risk premium, drops dramatically. This is not surprising, as the production side of the extended model is less risky than that of the model with baseline calibration — the presence of variable utilization and adjustment cost highlights the intratemporal substitution effects in response to shocks, i.e., household can adjust capital and thereby output immediately on the impact of shocks, as noted by Greenwood et al. (1988), Burnside and Eichenbaum (1996) and King and Rebelo (1999), and need not wait till next period. This is in contrast to the baseline model, where the capital input

could not be adjusted in response to shifts in risk (fixed utilization rate and time to build capital). Variable utilization and adjustment cost tend to increase the volatility of the pricing kernel as noted in Cochrane (2005),³⁸ by stretching out its time varying risk adjustment channel and thus shifting risk adjustments to risk free rate.

9 Conclusion

We have studied a business cycle model with recursive preferences and stochastic volatility with a third order perturbation approximation to the nonlinear moving average policy function. We use the impulse responses generated by this third order approximation to analyze the propagation mechanism of a volatility shock, and find that for macroeconomic variables, a volatility shock by itself propagates solely through a time-varying risk adjustment channel. For conditional asset pricing variables, this time-varying risk adjustment channel is the only working channel for the transmission of shocks, both to productivity growth and its volatility.

We have derived a closed-form calculation of the theoretical moments of the endogenous variables using a third order approximation. Our calculation of moments lends itself to a decomposition that disentangles the individual contributions of time-varying risk adjustment and amplification channels to the total variance. In our baseline model, we find that adding stochastic volatility alters the composition of variance, making a time-varying risk channel a substantial contributor of variance. For macroeconomic variables, variations that come from the time-varying risk adjustment channel can be explained by the household's precautionary savings desires and, in the presence of stochastic volatility, we find a large portion of variations in macroeconomic variables is driven by precautionary behavior.

Our extended model with frictional investment predicts a drop and subsequent overshooting of output in response to a volatility shock, consistent with empirical findings. Yet, with variable

³⁸The standard deviation of risk free rate of the extend model is smaller than that of the model with baseline calibration, as variable utilization reduces the overall volatility of the extend model, see footnote ??.

capital utilization, the capital input in production can be adjusted contemporaneously in response to shocks, eliminating the importance of the time-varying risk adjustment channel for macroeconomic variables. This finding, skeptical of the importance of stochastic volatility for precautionary behavior in production models, is corroborated by our finding that stochastic volatility contributes to the baseline model's ability to reach the Hansen-Jagannathan bounds only inasmuch as it increases the overall volatility of the model.

In linear approximations, variance decompositions can be applied to study the individual contribution of each shock to the total variance. The channels of risk adjustment and amplification we have derived here are a first step towards a shock-specific decomposition of nonlinear perturbation approximations. This would enable the identification of the individual contributions of each shock, not only to total volatility, but also to individual orders of nonlinearity and risk adjustments.

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A Appendices

A.1 Proof of Lemma 2.1

As (1) is linear in ω_t , only the zeroth and first derivatives with respect to ω_t can be nonzero. As we approximate around the point $\omega_t = 0$, all derivatives of zeroth order with respect to ω_t are zero. Hence, the Taylor expansion of (1) is the product of $\omega_t \frac{\partial e^{\zeta_t} \omega_t}{\partial \omega_t} \Big|_{\omega_t = \zeta_t = 0} \omega_t$ —and the Taylor series of e^{ζ_t} around the point $\zeta_t = 0$, which is given by $\sum_{i=0}^{\infty} \frac{1}{i!} \zeta_t^i$.

A.2 Proof of Proposition 2.2

z_t^{ω} enters into the model (3) through the term $He^{\zeta_t} \omega_t$, which itself enters the model linearly. Hence and following lemma 2.1, terms through $g_{z^{\omega}}$, $g_{zz^{\omega}}$, and $g_{z^2 z^{\omega}}$ are independent of stochastic volatility. Additionally, that z_t^{ω} enters into the model only through the term $He^{\zeta_t} \omega_t$, which enters linearly, means that terms of the form $g_{z^{\omega} 2}$ are given by $(g_{\zeta \omega} \zeta_{t-1} + g_{\eta \omega} \eta_t) \omega_t$ following lemma 2.1 and the first order autoregressive definition of the volatility process ζ_t in (4). These terms dependent on stochastic volatility interact with all the states, z_t , at the next order—accordingly terms of the form $g_{z, z^{\omega} 2}$ are given by $\frac{1}{2} (g_{\zeta \omega z} \zeta_{t-1} + g_{\eta \omega z} \eta_t) \omega_t z_t$ —and with the states of the volatility process, ζ_{t-1} and η_t ,³⁹—hence, terms of the form $g_{z^{\omega} 3}$ are given by $\frac{1}{2} (g_{\zeta^2 \omega} \zeta_{t-1}^2 + 2g_{\zeta \eta \omega} \zeta_{t-1} \eta_t + g_{\eta^2 \omega} \eta_t^2) \omega_t$.

Finally, turning to terms involving the perturbation parameter, σ , terms through $g_{\sigma z^j z^{\omega, k}}$, i.e., first order in σ , are zero following Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), and others. Likewise, g_{σ^3} will be zero following our assumption of normality of all exogenous processes, see Andreasen (2012) for an investigation of the consequences of nonnormality. This leaves terms through, g_{σ^2} , $g_{\sigma^2 z}$, and $g_{\sigma^2 z^{\omega}}$ as claimed in the proposition. That these terms are in general dependent on stochastic volatility will be addressed in the proof of corollary 2.5 as there we will examine a condition under which these terms can explicitly be shown to be independent of stochastic

³⁹But not the shock ω_t , as it enters the model (3) linearly through the term $He^{\zeta_t} \omega_t$ and, as such, only terms first order in it are nonzero, see Lemma 2.1.

volatility.

A.3 Proof of Corollary 2.5

First, the derivatives with respect to ζ_{t-1} . Recall from (4) that $\zeta_t = \rho\zeta_{t-1} + \tau\eta_t$.

$$\begin{aligned} 0 &= \mathcal{D}_{\zeta_{t-1}} \{f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) + He^{\zeta_t} \omega_t\} \\ (54) \quad &= f_{y^+} y_y^+ y_\zeta + f_{y^+} y_\zeta^+ \rho + f_y y_\zeta + He^{\zeta_t} \omega_t \rho \end{aligned}$$

at the deterministic steady state, $y^+ = y$ and $\omega_t = 0$ which gives

$$(55) \quad 0 = (f_{y^+} y_y + f_{y^+} y_\zeta \rho + f_y) y_\zeta \Rightarrow y_\zeta = 0$$

where the invertibility of $(f_{y^+} y_y + f_{y^+} y_\zeta \rho + f_y)$ follows from the stability of y_y and ρ , see Lan and Meyer-Gohde (2013b).

Now differentiate (54) twice with respect to σ , the perturbation parameter,

$$\begin{aligned} 0 &= \mathcal{D}_{\sigma^2 \zeta_{t-1}} \{f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) + He^{\zeta_t} \omega_t\} \\ &= \mathcal{D}_{\sigma^2} \left\{ (f_{y^+} y_y^+ + f_y) y_\zeta + f_{y^+} y_\zeta^+ \rho + He^{\zeta_t} \omega_t \rho \right\} \\ &= \mathcal{D}_{\sigma^2} \left\{ (f_{y^+} y_y^+ + f_y) \right\} y_\zeta + 2 \mathcal{D}_{\sigma} \left\{ (f_{y^+} y_y^+ + f_y) \right\} y_{\sigma \zeta} + (f_{y^+} y_y^+ + f_y) y_{\sigma^2 \zeta} \\ (56) \quad &+ \mathcal{D}_{\sigma^2} \left\{ f_{y^+} \right\} y_\zeta^+ \rho + 2 \mathcal{D}_{\sigma} \left\{ f_{y^+} \right\} \mathcal{D}_{\sigma} \left\{ y_\zeta^+ \right\} \rho + f_{y^+} \mathcal{D}_{\sigma^2} \left\{ y_\zeta^+ \right\} \rho \end{aligned}$$

From (55), $y_\zeta = y_\zeta^+ = 0$ in a steady state; following Jin and Judd (2002) and Schmitt-Grohé and Uribe (2004), first order derivatives with respect to σ are likewise zero— $y_{\sigma \zeta} = 0$. Thus, if $\rho = 0$, the foregoing is⁴⁰

$$(58) \quad (f_{y^+} y_y^+ + f_y) y_{\sigma^2 \zeta} = 0 \Rightarrow y_{\sigma^2 \zeta} = 0$$

Turning finally to the derivatives with respect to η_t . Recall again from (4) that $\zeta_t = \rho\zeta_{t-1} + \tau\eta_t$

$$0 = \mathcal{D}_{\eta_t} \{f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) + He^{\zeta_t} \omega_t\}$$

⁴⁰This begs the question, whether $y_{\sigma^2 \zeta}$ can ever be different from zero; i.e. whether stochastic volatility can ever have an effect through the risk channel. If $\rho \neq 0$, the conditional expectation of (56) at the deterministic steady state is

$$(57) \quad y_{\sigma^2 \zeta} = - (f_{y^+} y_y + f_{y^+} \rho + f_y)^{-1} \left(f_{y^+} y_{\varepsilon \omega^2 \zeta} E_t \left[\varepsilon_{t+1}^{\omega \otimes [2]} \right] + 2 f_{y^+ 2} (y_\omega \otimes y_{\omega \zeta}) E_t \left[\omega_{t+1}^{\otimes [2]} \right] \right) \rho$$

$$(59) \quad = f_{y^+y_y^+} y_\eta + f_{y^+y_\zeta^+} \tau + f_y y_\eta + H e^{\zeta} \omega_t \tau$$

at the deterministic steady state, $y^+ = y$, $\omega_t = 0$, and $y_\zeta = 0$ which gives

$$(60) \quad 0 = (f_{y^+y_y} + f_y) y_\eta \Rightarrow y_\zeta = 0$$

where the invertibility of $(f_{y^+y_y} + f_y)$ follows from the stability of y_y , see Lan and Meyer-Gohde (2013b).

Now differentiate (59) twice with respect to σ , the perturbation parameter,

$$(61) \quad \begin{aligned} 0 &= \mathcal{D}_{\sigma^2 \eta_t} \{ f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) + H e^{\zeta} \omega_t \} \\ &= \mathcal{D}_{\sigma^2} \left\{ (f_{y^+y_y^+} + f_y) y_\eta + f_{y^+y_\zeta^+} \tau + H e^{\zeta} \omega_t \tau \right\} \\ &= \mathcal{D}_{\sigma^2} \left\{ (f_{y^+y_y^+} + f_y) \right\} y_\eta + 2 \mathcal{D}_{\sigma} \left\{ (f_{y^+y_y^+} + f_y) \right\} y_{\sigma \eta} + (f_{y^+y_y^+} + f_y) y_{\sigma^2 \eta} \\ &\quad + \mathcal{D}_{\sigma^2} \left\{ f_{y^+} \right\} y_\zeta^+ \tau + 2 \mathcal{D}_{\sigma} \left\{ f_{y^+} \right\} \mathcal{D}_{\sigma} \left\{ y_\zeta^+ \right\} \tau + f_{y^+} \mathcal{D}_{\sigma^2} \left\{ y_\zeta^+ \right\} \tau \end{aligned}$$

From (60), $y_\eta = 0$ in a steady state, likewise $y_\zeta = y_\zeta^+ = 0$ from (55); following Jin and Judd (2002) and Schmitt-Grohé and Uribe (2004), first order derivatives with respect to σ are likewise zero— $y_{\sigma \zeta} = 0$ and $y_{\sigma \zeta} = 0$. If $\rho = 0$, σ_{t-1} vanishes from the systems of equations and all derivatives with respect to it are equal to zero: $\mathcal{D}_{\sigma^2} \left\{ y_\zeta^+ \right\} = \mathcal{D}_{\sigma} \left\{ y_\zeta^+ \right\} = 0$; accordingly the foregoing is⁴¹

$$(63) \quad (f_{y^+y_y^+} + f_y) y_{\sigma^2 \eta} = 0 \Rightarrow y_{\sigma^2 \eta} = 0$$

A.4 Detrending the Model

Stationary consumption, investment, capital stock and output, denoted by the lower case letters, are defined as follows

$$(64) \quad c_t \equiv \frac{C_t}{e^{Z_t}}, \quad i_t \equiv \frac{I_t}{e^{Z_t}}, \quad k_t \equiv \frac{K_t}{e^{Z_t}}, \quad y_t \equiv \frac{Y_t}{e^{Z_t}},$$

For notational ease in detrending the model, we define a combined shock $\varepsilon_{a,t}$, containing both

⁴¹This begs the question, whether $y_{\sigma^2 \zeta}$ can ever be different from zero; i.e. whether stochastic volatility can ever have an effect through the risk channel. If $\rho \neq 0$, the conditional expectation of (56) at the deterministic steady state is

$$(62) \quad y_{\sigma^2 \eta} = - (f_{y^+y_y} + f_y)^{-1} \left(f_{y^+} y_{\sigma^2 \zeta} \rho + f_{y^+} y_{\varepsilon \omega \zeta} E_t \left[\varepsilon_{t+1}^{\omega \otimes [2]} \right] + 2 f_{y^+2} (y_\omega \otimes y_{\omega \zeta}) E_t \left[\omega_{t+1}^{\otimes [2]} \right] \right) \tau$$

the homoskedastic and heteroskedastic components of the productivity growth shock

$$(65) \quad \varepsilon_{a,t} \equiv \overline{\sigma_z} e^{\sigma_{z,t}} \varepsilon_{z,t}$$

The productivity growth process can therefore be written as

$$(66) \quad a_t \equiv Z_t - Z_{t-1} = \bar{a} + \varepsilon_{a,t}$$

While detrending, the exponential form of the foregoing will be frequently used

$$(67) \quad e^{a_t} = \frac{e^{Z_t}}{e^{Z_{t-1}}} = e^{\bar{a} + \varepsilon_{a,t}}$$

The goal is essentially to substitute C_t , I_t , K_t and Y_t for their stationary counterparts in the relevant model equations. We start with the production function

$$(68) \quad (y_t e^{Z_t}) = (k_{t-1} e^{Z_{t-1}})^\alpha (e^{Z_t} N_t)^{1-\alpha}$$

$$(69) \quad \Rightarrow y_t = \left(\frac{e^{Z_t}}{e^{Z_{t-1}}} \right)^{-\alpha} k_{t-1}^\alpha N_t^{1-\alpha}$$

$$(70) \quad \Rightarrow y_t = e^{-\alpha(\bar{a} + \varepsilon_{a,t})} k_{t-1}^\alpha N_t^{1-\alpha}$$

Detrending the capital accumulation law

$$(71) \quad (k_t e^{Z_t}) = (1 - \delta) (k_{t-1} e^{Z_{t-1}}) + (i_t e^{Z_t})$$

$$(72) \quad \Rightarrow k_t = (1 - \delta) \frac{e^{Z_{t-1}}}{e^{Z_t}} k_{t-1} + i_t$$

$$(73) \quad \Rightarrow k_t = (1 - \delta) e^{-\bar{a} - \varepsilon_{a,t}} k_{t-1} + i_t$$

Detrending the market clearing condition is straightforward as it is a contemporaneous relationship

$$(74) \quad (y_t e^{Z_t}) = (c_t e^{Z_t}) + (i_t e^{Z_t})$$

$$(75) \quad \Rightarrow y_t = c_t + i_t$$

Combing (70), (73) and (75) yields the detrended resource constraint

$$(76) \quad c_t + k_t = e^{-\alpha(\bar{a} + \varepsilon_{a,t})} k_{t-1}^\alpha N_t^{1-\alpha} + (1 - \delta) e^{-\bar{a} - \varepsilon_{a,t}} k_{t-1}$$

Detrending the labor supply equation

$$(77) \quad \frac{\Psi}{1 - N_t} = \frac{1}{c_t e^{Z_t}} (1 - \alpha) (k_{t-1} e^{Z_{t-1}})^\alpha e^{Z_t(1-\alpha)} N_t^{-\alpha}$$

$$(78) \quad \Rightarrow \frac{\Psi}{1-N_t} = (1-\alpha)e^{-\alpha(\bar{a}+\varepsilon_{a,t})} \frac{1}{c_t} k_{t-1}^\alpha N_t^{-\alpha}$$

The risky rate r_t is stationary and we reexpress it in terms of the stationary variables

$$(79) \quad 1+r_t = (1-\delta) + \alpha (k_{t-1} e^{Z_{t-1}})^{\alpha-1} (e^{Z_t} N_t)^{1-\alpha}$$

$$(80) \quad \Rightarrow 1+r_t = (1-\delta) + \alpha k_{t-1}^{\alpha-1} e^{(\bar{a}+\varepsilon_{a,t})(1-\alpha)} N_t^{1-\alpha}$$

We now move to the value function. As the felicity function is logarithmic in nonstationary consumption, removing the trend in consumption will leave a term linear in the level of productivity that when subtracted from V_t gives the stationary value function v_t

$$(81) \quad v_t = V_t - b \ln e^{Z_t} = V_t - b Z_t$$

Substituting the relevant variables for their stationary counterparts yields

$$(82) \quad v_t + b Z_t = \ln(c_t e^{Z_t}) + \psi \ln(1-N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} [v_{t+1} + b Z_{t+1}] \right) \right] \right)$$

$$(83) \quad \Rightarrow v_t = \ln c_t + \psi \ln(1-N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + b \left(Z_{t+1} - \frac{b-1}{b\beta} Z_t \right) \right] \right) \right] \right)$$

It follows that the remaining nonstationarities can be offset if

$$(84) \quad \frac{b-1}{b\beta} = 1$$

which pins down b as

$$(85) \quad b = \frac{1}{1-\beta}$$

Inserting (85) in (83) yields the stationary value function

$$(86) \quad v_t = \ln c_t + \psi \ln(1-N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \varepsilon_{a,t+1}) \right] \right) \right] \right)$$

While stationary, the foregoing value function does not fit in the problem statement (22) in the text, thus can not be implemented directly in perturbation software packages like Dynare. This problem is caused by nonlinear twisting of the expected continuation value, and can be fixed by redefining this conditional expectation as a new variable known in period t . Besides, the twisted expected continuation value is numerically unstable, due to the logarithmic transformation, when

γ approaches zero or becomes very large. To counteract this, we define⁴²

$$(87) \quad \tilde{v}_t \equiv E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} \tilde{\varepsilon}_{t+1} - \bar{v} \right] \right) \right]$$

where \bar{v} denotes the deterministic steady state value of the stationary value function (86) and can be computed as follows

$$(88) \quad \bar{v} = \frac{1}{1-\beta} \left[\ln c + \psi \ln(1-N) + \frac{\beta}{1-\beta} \bar{a} \right]$$

Substituting v_{t+1} in (86) for \tilde{v}_t yields the normalized, stationary value function

$$(89) \quad v_t = \ln c_t + \psi \ln(1-N_t) + \beta \frac{2}{\gamma} \left[\ln \tilde{v}_t + \frac{\gamma}{2} \left(\frac{1}{1-\beta} \bar{a} + \bar{v} \right) \right]$$

With the stationary value function in hand, we reexpress the pricing kernel in terms of stationary variables

$$(90) \quad m_{t+1} = \beta \frac{c_t e^{Z_t}}{c_{t+1} e^{Z_{t+1}}} \frac{\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} Z_{t+1} \right] \right)}{E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} Z_{t+1} \right] \right) \right]}$$

Multiplying both the denominator and numerator of the foregoing with $\exp \left(-\frac{\gamma}{2} \frac{1}{1-\beta} Z_t \right)$, and rearranging yields

$$(91) \quad m_{t+1} = \beta \frac{c_t}{c_{t+1}} e^{-(\bar{a} + \varepsilon_{a,t+1})} \frac{\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \varepsilon_{a,t+1}) \right] \right)}{E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \varepsilon_{a,t+1}) \right] \right) \right]}$$

Writing out the definition of $\varepsilon_{a,t+1}$ yields (21) in the text. Recognizing the expectational term in the previous equation can be replaced by the product $\tilde{v}_t \exp \left(\frac{\gamma}{2} \left[\bar{v} + \frac{1}{1-\beta} \bar{a} \right] \right)$, we substitute it for this product and collect terms

$$(92) \quad m_{t+1} = \beta \frac{c_t}{c_{t+1}} e^{-(\bar{a} + \varepsilon_{a,t+1})} \frac{\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} \varepsilon_{a,t+1} - \bar{v} \right] \right)}{\tilde{v}_t}$$

The period t counterpart of the foregoing follows

$$(93) \quad m_t = \beta \frac{c_{t-1}}{c_t} e^{-(\bar{a} + \varepsilon_{a,t})} \frac{\exp \left(\frac{\gamma}{2} \left[v_t + \frac{1}{1-\beta} \varepsilon_{a,t} - \bar{v} \right] \right)}{\tilde{v}_{t-1}}$$

⁴²Rudebusch and Swanson (2012) adopt, in their companion Mathematica codes, a very similar procedure to improve numerical stability.

A.5 Proof of Proposition 6.1

Rearrange the definition of the second order increment to express the second order approximation as the sum of the first order approximation, the second order increment, and the second order constant risk adjustment

$$(94) \quad y_t^{(2)} = y_t^{(1)} + dy_t^{(2)} + \frac{1}{2}y\sigma^2$$

Applying the expectations operator to the foregoing yields the mean of the second order approximation

$$(95) \quad Ey_t^{(2)} = Ey_t^{(1)} + Eddy_t^{(2)} + \frac{1}{2}y\sigma^2$$

Centering the second order approximation (94) around its mean by subtracting (95) from (94) yields

$$(96) \quad y_t^{(2)} - Ey_t^{(2)} = \left(y_t^{(1)} - Ey_t^{(1)}\right) + \left(dy_t^{(2)} - Eddy_t^{(2)}\right)$$

Noting that the mean of the first order approximation is the deterministic steady state of y_t , i.e., $Ey_t^{(1)} = \bar{y}$, the foregoing can be rewritten as

$$(97) \quad y_t^{(2)} - Ey_t^{(2)} = \left(y_t^{(1)} - \bar{y}\right) + \left(dy_t^{(2)} - Eddy_t^{(2)}\right)$$

Using the definition of the first order increment $dy_t^{(1)} \equiv y_t^{(1)} - \bar{y}$, the foregoing is

$$(98) \quad y_t^{(2)} - Ey_t^{(2)} = dy_t^{(1)} + \left(dy_t^{(2)} - Eddy_t^{(2)}\right)$$

Multiplying the foregoing with its transposition at $t - j$ and noting that $Ey_t^{(2)} = Ey_{t-j}^{(2)}$ and $Eddy_t^{(2)} = Eddy_{t-j}^{(2)}$ yields

$$(99) \quad \begin{aligned} & \left(y_t^{(2)} - Ey_t^{(2)}\right) \left(y_{t-j}^{(2)} - Ey_{t-j}^{(2)}\right)' \\ &= \left[dy_t^{(1)} + \left(dy_t^{(2)} - Eddy_t^{(2)}\right)\right] \left[dy_{t-j}^{(1)} + \left(dy_{t-j}^{(2)} - Eddy_{t-j}^{(2)}\right)\right]' \\ &= dy_t^{(1)} dy_{t-j}^{(1)'} + \left(dy_t^{(2)} dy_{t-j}^{(1)'} - Eddy_t^{(2)} dy_{t-j}^{(1)'}\right) \\ & \quad + \left(dy_t^{(1)} dy_{t-j}^{(2)'} - dy_t^{(1)} Eddy_{t-j}^{(2)'}\right) + \left(dy_t^{(2)} - Eddy_t^{(2)}\right) \left(dy_{t-j}^{(2)} - Eddy_{t-j}^{(2)}\right)' \end{aligned}$$

Applying the expectations operator to the foregoing delivers

$$\begin{aligned}
(100) \quad & E \left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_{t-j}^{(2)} \right)' \right] \\
& = E \left(d y_t^{(1)} d y_{t-j}^{(1)'} \right) + E \left(d y_t^{(2)} d y_{t-j}^{(1)'} \right) - E d y_t^{(2)} E d y_{t-j}^{(1)'} \\
& \quad + E \left(d y_t^{(1)} d y_{t-j}^{(2)'} \right) - E d y_t^{(1)} E d y_{t-j}^{(2)'} + E \left[\left(d y_t^{(2)} - E d y_t^{(2)} \right) \left(d y_{t-j}^{(2)} - E d y_{t-j}^{(2)} \right)' \right]
\end{aligned}$$

To simplify the foregoing, apply the expectations operator to the definition of the first order increment, yielding its mean

$$(101) \quad E d y_t^{(1)} = E y_t^{(1)} - \bar{y}$$

As $E y_t^{(1)} = \bar{y}$, the foregoing implies that the mean of the first order increment is zero

$$(102) \quad E d y_t^{(1)} = 0$$

Using the this result and noting that $E d y_t^{(1)} = E d y_{t-j}^{(1)}$, (100) reduces to

$$\begin{aligned}
(103) \quad & E \left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_{t-j}^{(2)} \right)' \right] \\
& = E \left(d y_t^{(1)} d y_{t-j}^{(1)'} \right) + E \left(d y_t^{(2)} d y_{t-j}^{(1)'} \right) + E \left(d y_t^{(1)} d y_{t-j}^{(2)'} \right) \\
& \quad + E \left[\left(d y_t^{(2)} - E d y_t^{(2)} \right) \left(d y_{t-j}^{(2)} - E d y_{t-j}^{(2)} \right)' \right]
\end{aligned}$$

It then remains to show that

$$(104) \quad E \left(d y_t^{(2)} d y_{t-j}^{(1)'} \right) = 0, \quad E \left(d y_t^{(1)} d y_{t-j}^{(2)'} \right) = 0$$

One way is to use the moving average representation of the order increments. I.e., inserting the moving average representation of the first and second order approximations in the definition of the order increments yields

$$(105) \quad d y_t^{(1)} = \sum_{i=0}^{\infty} y_i \boldsymbol{\varepsilon}_{t-i}$$

$$(106) \quad d y_t^{(2)} = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\boldsymbol{\varepsilon}_{t-j} \otimes \boldsymbol{\varepsilon}_{t-i})$$

Therefore the product of the two order increments, when set in expectation, takes the form of the third moments of the shocks, which is equal to zero under normality.

A.6 Proof of Proposition 6.2

First note that $Ey_t^{(3)} = Ey_t^{(2)}$ under normality⁴³. Given this result, applying the expectations operator to the definition of the third order increment $dy_t^{(3)} \equiv y_t^{(3)} - y_t^{(2)}$ immediately implies $E dy_t^{(3)} = 0$.

Next, rearranging the definition of the third order increment delivers

$$(107) \quad y_t^{(3)} = y_t^{(2)} + dy_t^{(3)}$$

Applying the expectations operator to the foregoing yields

$$(108) \quad Ey_t^{(3)} = Ey_t^{(2)}$$

Centering (107) around its mean by subtracting (108) from (107) gives

$$(109) \quad y_t^{(3)} - Ey_t^{(3)} = y_t^{(2)} - Ey_t^{(2)} + dy_t^{(3)}$$

Multiplying the foregoing with its transposition at $t - j$ and noting $Ey_t^{(3)} = Ey_{t-j}^{(3)}$ and $Ey_t^{(2)} = Ey_{t-j}^{(2)}$ delivers

$$\begin{aligned} \left(y_t^{(3)} - Ey_t^{(3)}\right) \left(y_{t-j}^{(3)} - Ey_{t-j}^{(3)}\right)' &= dy_t^{(3)} dy_{t-j}^{(3)'} + \left(y_t^{(2)} - Ey_t^{(2)}\right) \left(y_{t-j}^{(2)} - Ey_{t-j}^{(2)}\right)' \\ &\quad + dy_t^{(3)} y_{t-j}^{(2)'} - dy_t^{(3)} Ey_{t-j}^{(2)'} + y_t^{(2)} dy_{t-j}^{(3)'} - Ey_t^{(2)} dy_{t-j}^{(3)'} \end{aligned}$$

Applying the expectations operator to the foregoing, noting $E dy_t^{(3)} = 0$, gives

$$(110) \quad \begin{aligned} E \left[\left(y_t^{(3)} - Ey_t^{(3)}\right) \left(y_{t-j}^{(3)} - Ey_{t-j}^{(3)}\right)' \right] &= E_t \left[dy_t^{(3)} dy_{t-j}^{(3)'} \right] + E \left[\left(y_t^{(2)} - Ey_t^{(2)}\right) \left(y_{t-j}^{(2)} - Ey_{t-j}^{(2)}\right)' \right] \\ &\quad + E \left(dy_t^{(3)} y_{t-j}^{(2)'} \right) + E \left(y_t^{(2)} dy_{t-j}^{(3)'} \right) \end{aligned}$$

Rewrite the definition of the second order increment $dy_t^{(2)} \equiv y_t^{(2)} - y_t^{(1)} - \frac{1}{2}y\sigma^2$ as

$$(111) \quad y_t^{(2)} = dy_t^{(2)} + y_t^{(1)} + \frac{1}{2}y\sigma^2 = dy_t^{(2)} + dy_t^{(1)} + \bar{y} + \frac{1}{2}y\sigma^2$$

Given the foregoing expression and noting $E dy_t^{(3)} = 0$, $E \left(y_t^{(2)} dy_{t-j}^{(3)'} \right)$ on the right hand side of

(110) can be rewritten as

$$(112) \quad E \left(y_t^{(2)} dy_{t-j}^{(3)'} \right) = E \left[\left(dy_t^{(2)} + dy_t^{(1)} + \bar{y} + \frac{1}{2}y\sigma^2 \right) dy_{t-j}^{(3)'} \right] = E \left(dy_t^{(1)} dy_{t-j}^{(3)'} \right)$$

⁴³To see this, applying the expectations operator to the second order approximation (30) and comparing the resulting expression with the mean of the third order approximation (25)

Noting that $E \left(dy_t^{(2)} dy_{t-j}^{(3)'} \right)$ is zero under normality.⁴⁴ Analogously, $E \left(dy_t^{(3)} y_{t-j}^{(2)'} \right)$ on the right hand side of (110) can be written as

$$(114) \quad E \left[dy_t^{(3)} y_{t-j}^{(2)'} \right] = E \left(dy_t^{(3)} dy_{t-j}^{(1)'} \right)$$

Inserting the last two equations in (110) yields

$$E \left[\left(y_t^{(3)} - E y_t^{(3)} \right) \left(y_{t-j}^{(3)} - E y_{t-j}^{(3)} \right)' \right] = E_t \left[dy_t^{(3)} dy_{t-j}^{(3)'} \right] + E \left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_{t-j}^{(2)} \right)' \right] \\ + E \left(dy_t^{(1)} dy_{t-j}^{(3)'} \right) + E \left(dy_t^{(3)} dy_{t-j}^{(1)'} \right)$$

A.7 Mean of $dy_t^{(2)}$

The second order increment $dy_t^{(2)}$ can be expressed recursively as

$$(115) \quad dy_t^{(2)} = \alpha dy_{t-1}^{(2)state} + \frac{1}{2} \left[\beta_{22} dy_{t-1}^{(1)state \otimes [2]} + 2\beta_{20} \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t \right) + \beta_{00} \varepsilon_t^{\otimes [2]} \right]$$

Taking expectation of the foregoing yields the following expression of the mean

$$(116) \quad E dy_t^{(2)} = \alpha E \left(dy_{t-1}^{(2)state} \right) + \frac{1}{2} \beta_{22} E \left(dy_{t-1}^{(1)state \otimes [2]} \right) + \frac{1}{2} \beta_{00} E \left(\varepsilon_t^{\otimes [2]} \right)$$

The two remaining unknown terms in the last equation, $E \left(dy_{t-1}^{(2)state} \right)$ and $E \left(dy_{t-1}^{(1)state \otimes [2]} \right)$, can be computed as follows. First, note the state variable block of (115) takes the form

$$(117) \quad dy_t^{(2)state} = \alpha^{state} dy_{t-1}^{(2)state} + \frac{1}{2} \beta_{22}^{state} dy_{t-1}^{(1)state \otimes [2]} + \beta_{20}^{state} \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t \right) + \frac{1}{2} \beta_{00}^{state} \varepsilon_t^{\otimes [2]}$$

Taking expectation of the foregoing and rearranging yields the following expression for the mean of state variable block, noting throughout we use (ns) to denote the number of state variables

$$(118) \quad E dy_t^{(2)state} = (I_{ns} - \alpha^{state})^{-1} \left[\frac{1}{2} \beta_{22}^{state} E \left(dy_{t-1}^{(1)state \otimes [2]} \right) + \frac{1}{2} \beta_{00}^{state} E \left(\varepsilon_t^{\otimes [2]} \right) \right]$$

The problem now reduces to compute $E \left(dy_t^{(1)state \otimes [2]} \right)$. Once it is known, (118) gives the value of $E dy_t^{(2)state}$. Inserting these two values back in (116) yields the mean of the second order increment.

⁴⁴Again consider the moving average representation of the third order increment

$$(113) \quad dy_t^{(3)} = \frac{1}{2} \sum_{i=0}^{\infty} y_{\sigma^2, i} \varepsilon_{t-i} + \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{k, j, i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i})$$

When multiplying with the moving average representation of the second order increment, the result, in expectation, is a sum of the third and fifth moments of shocks, and equal to zero under normality.

To compute $E\left(dy_t^{(1)state\otimes[2]}\right)$, we take the state variable block of the first order increment $dy_t^{(1)state}$ and raise it to the second Kronecker power,

$$(119) \quad dy_t^{(1)state\otimes[2]} = \alpha^{state\otimes[2]} dy_{t-1}^{(1)state\otimes[2]} + (K_{ns,ns} + I_{ns^2}) (\alpha^{state} \otimes \beta_0^{state}) \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t\right) + \beta_0^{state\otimes[2]} \varepsilon_t^{\otimes[2]}$$

where $K_{ns,ns}$ is a $ns^2 \times ns^2$ commutation matrix (See Magnus and Neudecker (1979)). Taking expectation of the foregoing and rearranging the resulting expression yields

$$(120) \quad E\left(dy_t^{(1)state\otimes[2]}\right) = \left(I_{ns^2} - \alpha^{state\otimes[2]}\right)^{-1} \beta_0^{state\otimes[2]} E\left(\varepsilon_t^{\otimes[2]}\right)$$

Inserting (120) and (118) back in (116) yields the mean of the second order increment

$$(121) \quad E dy_t^{(2)} = \frac{1}{2} \left[\alpha (I_{ns} - \alpha^{state})^{-1} \beta_{00}^{state} + \left(\alpha (I_{ns} - \alpha^{state})^{-1} \beta_{22}^{state} + \beta_{22} \right) \left(I_{ns^2} - \alpha^{state\otimes[2]} \right)^{-1} \beta_0^{state\otimes[2]} + \beta_{00} \right] E\left(\varepsilon_t^{\otimes[2]}\right)$$

which is an linear function of the second moments of the exogenous shocks. The coefficients on $E\left(\varepsilon_t^{\otimes[2]}\right)$ in the previous equation corresponds to the infinite sum $\sum_{j=0}^{\infty} y_{j,j}$, noting $y_{j,j} =$

$$\alpha y_{j-1,j-1}^{state} + \beta_{22} (y_{j-1}^{state} \otimes y_{j-1}^{state})$$

$$(122) \quad \sum_{j=1}^{\infty} \alpha y_{j-1,j-1}^{state} = \alpha (I_{ns} - \alpha^{state})^{-1} \beta_{00}^{state}$$

$$(123) \quad \sum_{j=1}^{\infty} \beta_{22} (y_{j-1}^{state} \otimes y_{j-1}^{state}) = \left(\alpha (I_{ns} - \alpha^{state})^{-1} \beta_{22}^{state} + \beta_{22} \right) \left(I_{ns^2} - \alpha^{state\otimes[2]} \right)^{-1} \beta_0^{state\otimes[2]}$$

$$(124) \quad y_{0,0} = \beta_{00}$$

A.8 Second Moments of $dy_t^{(2)}$

If (115) can be cast as a linear recursion, then standard linear methods can be applied to the computation of the second moments. Note $dy_t^{(2)}$, besides being linearly autoregressive in the state variable block of itself $dy_{t-1}^{(2)state}$, is a linear function of all the second order permutations of products of the first order increment $dy_{t-1}^{(1)state}$ and the shocks. This relationship guides the calculations, and we therefore compute the second moments of $dy_t^{(2)state}$ first, then recover the second moments

of variables of interest.⁴⁵

Combing (117) and (119) yields the following linear recursion containing the linear recursion of $dy_t^{(2)state}$

$$(125) \quad X_t^{(2)} = \Theta^{(2)X} X_{t-1}^{(2)} + \begin{bmatrix} \frac{1}{2}\beta_{00}^{state} \\ \beta_0^{state \otimes [2]} \end{bmatrix} E \left(\varepsilon_t^{\otimes [2]} \right) + \Phi^{(2)X} \Xi_t^{(2)}$$

where

$$(126) \quad X_t^{(2)} = \begin{bmatrix} dy_t^{(2)state} \\ dy_t^{(1)state \otimes [2]} \end{bmatrix}$$

$$(127) \quad \Theta^{(2)X} = \begin{bmatrix} \alpha^{state} & \frac{1}{2}\beta_{22}^{state} \\ 0 & \alpha^{state \otimes [2]} \end{bmatrix}$$

$$(128) \quad \Phi^{(2)X} = \begin{bmatrix} \frac{1}{2}\beta_{00}^{state} & \beta_{20}^{state} \\ \beta_0^{state \otimes [2]} & (K_{ns,ns} + I_{ns^2}) (\alpha^{state} \otimes \beta_0^{state}) \end{bmatrix}$$

$$(129) \quad \Xi_t^{(2)} = \begin{bmatrix} \varepsilon_t^{\otimes [2]} - E\varepsilon_t^{\otimes [2]} \\ dy_{t-1}^{(1)state} \otimes \varepsilon_t \end{bmatrix}$$

While the second term on the right hand side of (125) vanishes after centering (125) around its mean, it ensures, by compensating the subtraction of $E \left(\varepsilon_t^{\otimes [2]} \right)$ in $\Xi_t^{(2)}$, that $\Xi_t^{(2)}$ is orthogonal⁴⁶ to $X_{t-1}^{(2)}$

$$(130) \quad E \left(X_{t-1}^{(2)} \Xi_t^{(2)'} \right) = 0$$

With the linear recursion of $X_t^{(2)}$, the second order increment (115) can be recast as the following linear recursion

$$(131) \quad dy_t^{(2)} = \Theta^{(2)} X_{t-1}^{(2)} + \frac{1}{2}\beta_{00} E \left(\varepsilon_t^{\otimes [2]} \right) + \Phi^{(2)} \Xi_t^{(2)}$$

$$\text{where } \Theta^{(2)} = [\alpha \quad \frac{1}{2}\beta_{22}], \quad \Phi^{(2)} = [\frac{1}{2}\beta_{00} \quad \beta_{20}]$$

Noting $E \left(\Xi_t^{(2)} \right) = 0$ by construction.

⁴⁵This procedure is widely adopted to minimize the dimension and improve the speed of the computation. See, e.g., Uhlig's (1999) toolkit, Schmitt-Grohé and Uribe's (2004) software package and Dynare.

⁴⁶This orthogonality condition significantly simplifies the calculation of the autocovariances that followed.

A.8.1 Contemporaneous Covariance

Centering (131) around its mean—by subtracting (??) from (131)—yields the following centered linear recursion of the second order increment

$$(132) \quad \left(dy_t^{(2)} - E dy_t^{(2)} \right) = \Theta^{(2)} \left(X_{t-1}^{(2)} - EX_t^{(2)} \right) + \Phi^{(2)} \Xi_t^{(2)}$$

Multiplying the foregoing with its transposition and applying the expectations operator to the resulting expression yields the contemporaneous variance of the second order increment

$$(133) \quad \Gamma_0^{(2)} = \Theta^{(2)} \Gamma_0^{(2)X} \Theta^{(2)'} + \Phi^{(2)} E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right) \Phi^{(2)'}$$

where

$$(134) \quad \Gamma_0^{(2)X} = E \left[\left(X_t^{(2)} - EX_t^{(2)} \right) \left(X_t^{(2)} - EX_t^{(2)} \right)' \right]$$

$$(135) \quad \Gamma_0^{(2)} = E \left[\left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_t^{(2)} - E dy_t^{(2)} \right)' \right]$$

This requires the contemporaneous variance of $X_t^{(2)}$, i.e., $\Gamma_0^{(2)X}$, as well as $E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right)$. Starting with $\Gamma_0^{(2)X}$, we can proceed by applying the expectations operator to (125) to yield

$$(136) \quad EX_t^{(2)} = \Theta^{(2)X} EX_t^{(2)} + \begin{bmatrix} \frac{1}{2} \beta_{00}^{state} \\ \beta_0^{state \otimes [2]} \end{bmatrix} E \left(\varepsilon_t^{\otimes [2]} \right)$$

Centering the foregoing around its mean yields

$$(137) \quad X_t^{(2)} - EX_t^{(2)} = \Theta^{(2)X} \left(X_{t-1}^{(2)} - EX_t^{(2)} \right) + \Phi^{(2)X} \Xi_t^{(2)}$$

Multiplying the foregoing with its transposition and applying the expectations operator, it follows the unknown contemporaneous variance of $X_t^{(2)}$ solves the following Lyapunov equation⁴⁷

$$(138) \quad \Gamma_0^{(2)X} = \Theta^{(2)X} \Gamma_0^{(2)X} \Theta^{(2)X'} + \Phi^{(2)X} E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right) \Phi^{(2)X'}$$

Thus, $\Gamma_0^{(2)X}$ can be calculated given $E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right)$ and, therefore, $\Gamma_0^{(2)X}$ in (133) too. We require this variance, which is given by

$$(139) \quad E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right) = \begin{bmatrix} (I_{ne^2} + K_{ne,ne}) [E(\varepsilon_t \varepsilon_t') \otimes E(\varepsilon_t \varepsilon_t')] & 0 \\ 0 & \Gamma_0^{(1)X} \otimes E(\varepsilon_t \varepsilon_t') \end{bmatrix}$$

⁴⁷Note $\Gamma_0^{(2)X}$ is of dimension $(ns + ns^2) \times (ns + ns^2)$. For models with a large number of state variables, splitting (138) into four Sylvester equations of smaller size by exploiting the triangularity of $\Theta^{(2)X}$ and solving them one by one is computationally a lot less expensive than solving (138) as a whole. This division also enables exploitation of the symmetry of $\Gamma_0^{(2)X}$ and therefore can avoid redundant computations.

In the right hand side of (139), $\Gamma_0^{(1)X}$ is the state variable block of the contemporaneous variance of the first order approximation (or of the first order increment), and therefore already known from calculations at the first order.

The upper left entry of the right hand side of (139) contains the fourth moment of the shocks and has been simplified using Tracy and Sultan's (1993, p. 344) formula. The two zero entries in (139) are due to the fact that the third moments of the shocks are zero under normality, and $dy_{t-1}^{(1)state}$ is uncorrelated with current shocks.

A.8.2 Autocovariances

Now we turn to the autocovariances of $dy_t^{(2)}$. To start, note that under normality, $\Xi_t^{(2)}$ is serially uncorrelated

$$(140) \quad E \left(\Xi_t^{(2)} \Xi_{t-j}^{(2)'} \right) = 0 \quad \forall j > 0$$

Given the contemporaneous variance $\Gamma_0^{(2)X}$, multiplying (137) with the transposition of (132) and taking expectation yields the contemporaneous variance between the $X_t^{(2)}$ and $dy_t^{(2)}$

$$(141) \quad \Gamma_0^{(2)X,dy} = \Theta^{(2)X} \Gamma_0^{(2)X} \Theta^{(2)'} + \Phi^{(2)X} E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right) \Phi^{(2)'}$$

$$(142) \quad \text{where } \Gamma_0^{(2)X,dy} = E \left[\left(X_t^{(2)} - EX_t^{(2)} \right) \left(dy_t^{(2)} - E dy_t^{(2)} \right)' \right]$$

With all the three contemporaneous variances in hand, the orthogonality (130) and (140) ensures the autocovariance of $dy_t^{(2)}$ can be computed with the following recursive formulae

$$(143) \quad \Gamma_j^{(2)} = \Theta^{(2)} \Gamma_{j-1}^{(2)X,dy}$$

$$(144) \quad \Gamma_j^{(2)X,dy} = \Theta^{(2)X} \Gamma_{j-1}^{(2)X,dy}$$

where

$$(145) \quad \Gamma_j^{(2)} = E \left[\left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_{t-j}^{(2)} \right)' \right]$$

$$(146) \quad \Gamma_j^{(2)X,dy} = E \left[\left(X_t^{(2)} - EX_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_{t-j}^{(2)} \right)' \right]$$

A.9 Second Moments of $dy_t^{(3)}$

The third order increment can be expressed recursively as

(147)

$$\begin{aligned} dy_t^{(3)} = & \alpha dy_{t-1}^{(3)state} + \frac{1}{6} \left[\beta_{333,1} dy_{t-1}^{(1)state \otimes [3]} + \beta_{000} \varepsilon_t^{\otimes [3]} \right] \\ & + \beta_{22} \left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state} \right) + \beta_{20} \left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right) \\ & + \frac{1}{2} \left[\beta_{300} \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t^{\otimes [2]} \right) + \beta_{330,1} \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) + \beta_{\sigma^2 0} \varepsilon_t + \beta_{\sigma^2 1} dy_{t-1}^{(1)state} \right] \end{aligned}$$

Its state variable block takes the form

$$\begin{aligned} (148) \quad dy_t^{(3)state} = & \alpha^{state} dy_{t-1}^{(3)state} + \frac{1}{6} \left[\beta_{333,1}^{state} dy_{t-1}^{(1)state \otimes [3]} + \beta_{000}^{state} \varepsilon_t^{\otimes [3]} \right] \\ & + \beta_{22}^{state} \left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state} \right) + \beta_{20}^{state} \left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right) \\ & + \frac{1}{2} \left[\beta_{300}^{state} \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t^{\otimes [2]} \right) + \beta_{330,1}^{state} \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \right. \\ & \left. + \beta_{\sigma^2 0}^{state} \varepsilon_t + \beta_{\sigma^2 1}^{state} dy_{t-1}^{(1)state} \right] \end{aligned}$$

From the terms on the left hand side of the foregoing, we need to build up two additional recursions, the first in the Kronecker product of the first and second order increments and the second in the triple Kronecker product of the first order increment, to construct the linear recursion containing $dy_t^{(3)state}$ that can be used for calculating moments

(149)

$$\begin{aligned} dy_t^{(2)state} \otimes dy_t^{(1)state} = & \alpha^{state \otimes [2]} \left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state} \right) + \left[\left(\frac{1}{2} \beta_{22}^{state} \right) \otimes \alpha^{state} \right] dy_{t-1}^{(1)state \otimes [3]} \\ & + \left(\alpha^{state} \otimes \beta_0^{state} \right) \left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right) + \left[\left(\frac{1}{2} \beta_{00}^{state} \right) \otimes \beta_0^{state} \right] \varepsilon_t^{state \otimes [3]} \\ & + \left[\left(\beta_{20}^{state} \otimes \alpha^{state} \right) K_{ns*ne,ns} + \left(\frac{1}{2} \beta_{22}^{state} \right) \otimes \beta_0^{state} \right] \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \\ & + \left(\left[\left(\frac{1}{2} \beta_{00}^{state} \right) \otimes \alpha^{state} \right] K_{ne^2,ns} + \beta_{20}^{state} \otimes \beta_0^{state} \right) \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t^{\otimes [2]} \right) \end{aligned}$$

$$\begin{aligned} (150) \quad dy_t^{(1)state \otimes [3]} = & \alpha^{state \otimes [3]} dy_{t-1}^{(1)state \otimes [3]} + \beta_0^{state \otimes [3]} \varepsilon_t^{\otimes [3]} \\ & + \left[\left(K_{ns,ns} \otimes I_{ns} + I_{ns^3} \right) K_{ns^2,ns} + I_{ns^3} \right] \left(\alpha^{state \otimes [2]} \otimes \beta_0^{state} \right) \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \\ & + \left[K_{ns^2,ns} + \left(K_{ns,ns} \otimes I_{ns} + I_{ns^3} \right) \right] \left(\alpha^{state} \otimes \beta_0^{state \otimes [2]} \right) \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t^{\otimes [2]} \right) \end{aligned}$$

Given the foregoing two equations, along with the state variable block of the first order increment

$$(151) \quad dy_t^{(1)state} = \alpha^{state} dy_{t-1}^{(1)state} + \beta_0^{state} \varepsilon_t$$

we construct the following linear recursion

$$(152) \quad X_t^{(3)} = \Theta^{(3)X} X_{t-1}^{(3)} + \Phi^{(3)X} \Xi_t^{(3)}$$

where⁴⁸

$$(153) \quad X_t^{(3)} = \begin{bmatrix} dy_t^{(3)state} \\ dy_t^{(2)state} \otimes dy_t^{(1)state} \\ dy_t^{(1)state \otimes [3]} \\ dy_t^{(1)state} \end{bmatrix}, \quad \Xi_t^{(3)} = \begin{bmatrix} \varepsilon_t^{\otimes [3]} \\ dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \\ dy_{t-1}^{(1)state} \otimes (\varepsilon_t^{\otimes [2]} - E\varepsilon_t^{\otimes [2]}) \\ dy_{t-1}^{(2)state} \otimes \varepsilon_t \\ \varepsilon_t \end{bmatrix}$$

Note there is no need to center $X_t^{(3)}$ before computing its contemporaneous variance as its mean is zero under normality, i.e., $EX_t^{(3)} = 0$. In the third entry of $\Xi_t^{(3)}$, $\varepsilon_t^{\otimes [2]}$ is adjusted using its mean, such that $\Xi_t^{(3)}$ is orthogonal to $X_{t-1}^{(3)}$

$$(154) \quad E \left(X_{t-1}^{(3)} \Xi_t^{(3)'} \right) = 0$$

and it can be shown that $\Xi_t^{(3)}$ is serially uncorrelated

$$(155) \quad E \left(\Xi_t^{(3)} \Xi_{t-j}^{(3)'} \right) = 0 \quad \forall j > 0$$

A.9.1 Contemporaneous Covariance

With linear recursion (152), the third order increment (147) can be cast in a linear recursion⁴⁹

$$(156) \quad dy_t^{(3)} = \Theta^{(3)} X_{t-1}^{(3)} + \Phi^{(3)} \Xi_t^{(3)}$$

Multiplying the foregoing with its transposition and applying the expectations operator to the resulting expression yields the contemporaneous variance of the third order increment

$$(157) \quad \Gamma_0^{(3)} = \Theta^{(3)} \Gamma_0^{(3)X} \Theta^{(3)'} + \Phi^{(3)} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)'}$$

$$(158) \quad \text{where } \Gamma_0^{(3)} = E \left(dy_t^{(3)} dy_t^{(3)'} \right)$$

⁴⁸ $\Theta^{(3)X}$ and $\Phi^{(3)X}$ are specified in section A.12.

⁴⁹ $\Theta^{(3)}$ and $\Phi^{(3)}$ are specified in section A.12.

To compute the yet known contemporaneous variance of $X_t^{(3)}$, i.e., $\Gamma_0^{(3)X}$, we multiply (152) with its transposition and apply the expectations operator to the resulting expression. It follows that $\Gamma_0^{(3)X}$ solves the following Lyapunov equation⁵⁰

$$(159) \quad \Gamma_0^{(3)X} = \Theta^{(3)X} \Gamma_0^{(3)X} \Theta^{(3)X'} + \Phi^{(3)X} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)X'}$$

$$(160) \quad \text{where } \Gamma_0^{(3)X} = E \left(X_t^{(3)} X_t^{(3)'} \right)$$

with $E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right)$ as specified in section A.12.

Given $\Gamma_0^{(3)X}$, multiplying (152) with the transposition of (156) and applying the expectations operator yields the contemporaneous variance between $X_t^{(3)}$ and $dy_t^{(3)}$

$$(161) \quad \Gamma_0^{(3)X,dy} = \Theta^{(3)X} \Gamma_0^{(3)X} \Theta^{(3)'} + \Phi^{(3)X} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)'}$$

$$(162) \quad \text{where } \Gamma_0^{(3)X,dy} = E \left(X_t^{(3)} dy_t^{(3)'} \right)$$

A.9.2 Autocovariances

For the autocovariance of the third order increment, the orthogonality (154) and $\Xi_t^{(3)}$ being serially uncorrelated, i.e., (155), ensure that it can be computed with the following recursive formulae

$$(163) \quad \Gamma_j^{(3)} = \Theta^{(3)} \Gamma_{j-1}^{(3)X,dy}$$

$$(164) \quad \Gamma_j^{(3)X,dy} = \Theta^{(3)X} \Gamma_{j-1}^{(3)X,dy}$$

where

$$(165) \quad \Gamma_j^{(3)} = E \left(dy_t^{(3)} dy_{t-j}^{(3)'} \right)$$

$$(166) \quad \Gamma_j^{(3)X,dy} = E \left(X_t^{(3)} dy_{t-j}^{(3)'} \right)$$

A.10 Second Moments between $dy_t^{(1)}$ and $dy_t^{(3)}$

First rewrite the linear recursion of the first order increment (26) using $X_t^{(3)}$

$$(167) \quad dy_t^{(1)} = [0 \ 0 \ 0 \ \alpha] X_{t-1}^{(3)} + [0 \ 0 \ 0 \ 0 \ \beta_0] \Xi_t^{(3)}$$

⁵⁰Note that (159) is a Lyapunov equation of dimension $(ns + ns^2 + ns^3 + ns) \times (ns + ns^2 + ns^3 + ns)$. By exploiting the triangularity of $\Theta^{(3)X}$ and the symmetry of $\Gamma_0^{(3)X}$, that large Lyapunov equation can be split and reduced to 10 Sylvester equations of dimension up to $ns^3 \times ns^3$.

Multiplying the foregoing with the transposition of the linear recursion of the third order increment (156), and applying the expectations operator to the resulting expression yields the contemporaneous covariance between $dy_t^{(1)}$ and $dy_t^{(3)}$

$$(168) \quad \Gamma_0^{(1),(3)} = [0 \ 0 \ 0 \ \alpha] \Gamma_0^{(3)X} \Theta^{(3)'} + [0 \ 0 \ 0 \ 0 \ \beta_0] E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)'}$$

$$(169) \quad \text{where } \Gamma_0^{(1),(3)} = E \left(dy_t^{(1)} dy_t^{(3)'} \right)$$

The autocovariance, $\Gamma_j^{(1),(3)}$, can be computed using the following recursive formula

$$(170) \quad \Gamma_j^{(1),(3)} = [0 \ 0 \ 0 \ \alpha] \Gamma_{j-1}^{(3)X,dy}$$

A.11 Variance Decomposition

The decomposition the variance of the third order approximation follows directly from the decomposition of the third order increment. Defining

$$(171) \quad dy_t^{(3)} \equiv dy_t^{(3)amp} + dy_t^{(3)risk}$$

Multiplying the foregoing with its transposition and applying the expectations operator, a variance decomposition immediately follows

$$(172) \quad \Gamma_0^{(3)} = \Gamma_0^{(3)amp} + \Gamma_0^{(3)risk} + \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)'$$

where

$$(173) \quad \Gamma_0^{(3)amp} = E \left(dy_t^{(3)amp} dy_t^{(3)amp'} \right)$$

$$(174) \quad \Gamma_0^{(3)risk} = E \left(dy_t^{(3)risk} dy_t^{(3)risk'} \right)$$

$$(175) \quad \Gamma_0^{(3)amp,risk} = E \left(dy_t^{(3)amp} dy_t^{(3)risk'} \right)$$

Proposition (6.2) in the text implies the contemporaneous variance of the variables of interest takes the form

$$(176) \quad \Gamma_0^{y^{(3)}} = \Gamma_0^{y^{(2)}} + \Gamma_0^{(3)} + \Gamma_0^{(1),(3)} + \left(\Gamma_0^{(1),(3)} \right)'$$

Inserting the decomposed $\Gamma_0^{(3)}$, i.e., (172), in the previous equation yields the decomposition of

the contemporaneous variance of the variables of interest

$$(177) \quad \Gamma_0^{y^{(3)}} = \Gamma_0^{y^{(2)}} + \Gamma_0^{(3)amp} + \Gamma_0^{(3)risk} + \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)' \\ + \Gamma_0^{(1),(3)} + \left(\Gamma_0^{(1),(3)} \right)'$$

Note the decomposition (177) is not yet complete as the cross-contemporaneous variance $\Gamma_0^{(1),(3)}$ can be further broken down into two parts⁵¹

$$(178) \quad \Gamma_0^{(1),(3)} = E \left(dy_t^{(1)} dy_t^{(3)'} \right) \\ = E \left[dy_t^{(1)} \left(dy_t^{(3)amp} + dy_t^{(3)risk} \right)' \right] \\ = E \left(dy_t^{(1)} dy_t^{(3)amp'} \right) + E \left(dy_t^{(1)} dy_t^{(3)risk'} \right) \\ = \Gamma_0^{(1)amp,(3)amp} + \Gamma_0^{(1)amp,(3)risk}$$

Inserting the foregoing in (177) yields the complete variance decomposition

$$(179) \quad \Gamma_0^{y^{(3)}} = \Gamma_0^{y^{(2)}} + \Gamma_0^{(3)amp} + \Gamma_0^{(3)risk} + \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)' \\ + \Gamma_0^{(1)amp,(3)amp} + \Gamma_0^{(1)amp,(3)risk} \\ + \left(\Gamma_0^{(1)amp,(3)amp} + \Gamma_0^{(1)amp,(3)risk} \right)'$$

Letting $\Gamma_0^{y^{(3)amp}}$ collect the contribution from all amplification channels of all three orders, $\Gamma_0^{y^{(3)risk,amp}}$ collects all interaction between amplification and time-varying risk adjustment channels and $\Gamma_0^{y^{(3)risk}}$ collects the contribution from the time-varying risk adjustment channel

$$(180) \quad \Gamma_0^{y^{(3)amp}} = \Gamma_0^{y^{(2)}} + \Gamma_0^{(3)amp} + \Gamma_0^{(1)amp,(3)amp} + \left(\Gamma_0^{(1)amp,(3)amp} \right)'$$

$$(181) \quad \Gamma_0^{y^{(3)risk,amp}} = \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)' + \Gamma_0^{(1)amp,(3)risk} + \left(\Gamma_0^{(1)amp,(3)risk} \right)'$$

$$(182) \quad \Gamma_0^{y^{(3)risk}} = \Gamma_0^{(3)risk}$$

Inserting the foregoing in (179) yields (44) in the text. Note the first order amplification effect reported in Table 5 is included in (180). In particular, it is included in $\Gamma_0^{y^{(2)}}$. As implied by

⁵¹In (178), $\Gamma_0^{(1)amp,(3)amp}$ is used to denote $E \left(dy_t^{(1)} dy_t^{(3)amp'} \right)$ as there is only amplification effects in the first order increment $dy_t^{(1)}$.

proposition 6.1, the contemporaneous variance of the second order approximation takes the form

$$(183) \quad \Gamma_0^{y^{(2)}} = \Gamma_0^{y^{(1)}} + \Gamma_0^{(2)}$$

where $\Gamma_0^{y^{(1)}}$ captures the first order amplification effect.

To compute the individual terms in (179), first note $dy_t^{(3)amp}$ collects all amplification effects and $dy_t^{(3)risk}$ collects the time-varying risk adjustment effect in the third order increment

$$(184) \quad \begin{aligned} dy_t^{(3)amp} = & \alpha dy_{t-1}^{(3)amp,state} + \frac{1}{6} \left[\beta_{333,1} dy_{t-1}^{(1)state \otimes [3]} + \beta_{000} \varepsilon_t^{\otimes [3]} \right] \\ & + \beta_{22} \left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state} \right) + \beta_{20} \left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right) \\ & + \frac{1}{2} \left[\beta_{300} \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t^{\otimes [2]} \right) + \beta_{330,1} \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \right] \end{aligned}$$

$$(185) \quad dy_t^{(3)risk} = \alpha dy_{t-1}^{(3)risk,state} + \frac{1}{2} \beta_{\sigma^2 0} \varepsilon_t + \frac{1}{2} \beta_{\sigma^2 1} dy_{t-1}^{(1)state}$$

We start with constructing an auxiliary vector $X_{t-1}^{(3D)}$ for this decomposition

$$(186) \quad X_t^{(3D)} = \begin{bmatrix} dy_t^{(3)amp,state} \\ dy_t^{(3)risk,state} \\ dy_t^{(2)state} \otimes dy_t^{(1)state} \\ dy_t^{(1)state \otimes [3]} \\ dy_t^{(1)state} \end{bmatrix}$$

With the foregoing auxiliary vector, $dy_t^{(3)amp}$ and $dy_t^{(3)risk}$ can be cast as linear recursions

$$(187) \quad dy_t^{(3)amp} = \Theta^{(3)amp} X_{t-1}^{(3D)} + \Phi^{(3)amp} \Xi_t^{(3)}$$

$$(188) \quad dy_t^{(3)risk} = \Theta^{(3)risk} X_{t-1}^{(3D)} + \Phi^{(3)risk} \Xi_t^{(3)}$$

where

$$(189) \quad \Theta^{(3)amp} = \left[\alpha \quad 0 \quad \beta_{22} \quad \frac{1}{6} \beta_{333,1} \quad \frac{1}{2} \beta_{300}^{state} \left(I_{ns} \otimes E \varepsilon_t^{\otimes [2]} \right) \right]$$

$$(190) \quad \Theta^{(3)risk} = \left[0 \quad \alpha \quad 0 \quad 0 \quad \frac{1}{2} \beta_{\sigma^2 1} \right]$$

$$(191) \quad \Phi^{(3)amp} = \left[\frac{1}{6} \beta_{000} \quad \frac{1}{2} \beta_{330,1} \quad \frac{1}{2} \beta_{300} \quad \beta_{20} \quad 0 \right]$$

$$(192) \quad \Phi^{(3)risk} = \left[0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{2} \beta_{\sigma^2 0} \right]$$

Multiplying (187) with its transposition and applying the expectations operator yields the contemporaneous variance $\Gamma_0^{(3)amp}$, which collects the contribution of amplification channels to the

total variance of the third order increment

$$(193) \quad \Gamma_0^{(3)amp} = \Theta^{(3)amp} E \left(X_{t-1}^{(3D)} X_{t-1}^{(3D)'} \right) \Theta^{(3)amp'} + \Phi^{(3)amp} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)amp'}$$

where $E \left(X_{t-1}^{(3D)} X_{t-1}^{(3D)'} \right)$ can be computed using the following relationship

$$(194) \quad X_t^{(3)} = A^D X_t^{(3D)}$$

where

$$(195) \quad A^D = \begin{bmatrix} I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

therefore

$$(196) \quad E \left(X_{t-1}^{(3D)} X_{t-1}^{(3D)'} \right) = A^{D+} E \left(X_{t-1}^{(3)} X_{t-1}^{(3)'} \right) A^{D+'} = A^{D+} \Gamma_0^{(3)X} A^{D+'}$$

where A^{D+} denotes the Moore-Penrose inverse of A^D and $\Gamma_0^{(3)X}$ is already known. Then $\Gamma_0^{(3)amp}$ can be computed using

$$(197) \quad \Gamma_0^{(3)amp} = \left(\Theta^{(3)amp} A^{D+} \right) \Gamma_0^{(3)X} \left(\Theta^{(3)amp} A^{D+} \right)' + \Phi^{(3)amp} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)amp'}$$

Likewise, the contemporaneous variance $\Gamma_0^{(3)risk}$ collects the contribution of the time-varying risk adjustment channel to the total variance of the third order increment, and can be computed using

$$(198) \quad \Gamma_0^{(3)risk} = \left(\Theta^{(3)risk} A^{D+} \right) \Gamma_0^{(3)X} \left(\Theta^{(3)risk} A^{D+} \right)' + \Phi^{(3)risk} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)risk'}$$

$\Gamma_0^{(3)amp,risk}$ and its transposition collects the contribution of the interaction between the amplification and time-vary risk adjustment channels to the total variance of the third order increment, and can be computed using

$$(199) \quad \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)' = \Gamma_0^{(3)} - \Gamma_0^{(3)amp} - \Gamma_0^{(3)risk}$$

To compute $\Gamma_0^{(1)amp,(3)amp}$, multiply (167) with the transposition of (187) and apply the expectations operator to the resulting expression to yield

$$(200) \quad \Gamma_0^{(1)amp,(3)amp} = [0 \ 0 \ 0 \ \alpha] \Gamma_0^{(3)X} \left(\Theta^{(3)amp} A^{D+} \right)' + [0 \ 0 \ 0 \ 0 \ \beta_0] E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)amp'}$$

As $\Gamma_0^{(1),(3)}$ was already computed in section A.10, $\Gamma_0^{(1)amp,(3)risk}$ can be obtained by subtracting

the foregoing from $\Gamma_0^{(1),(3)}$.

A.12 Coefficient Matrices

This section contains explicit expressions for several coefficient matrices left implicit above.

$$\Theta^{(3)} = \left[\alpha \quad \beta_{22} \quad \frac{1}{6}\beta_{333,1} \quad \frac{1}{2}\beta_{300} \left(I_{ns} \otimes E \boldsymbol{\varepsilon}_t^{\otimes [2]} \right) + \frac{1}{2}\beta_{\sigma^2_1} \right]$$

$$\Phi^{(3)} = \left[\frac{1}{6}\beta_{000} \quad \frac{1}{2}\beta_{330,1} \quad \frac{1}{2}\beta_{300} \quad \beta_{20} \quad \frac{1}{2}\beta_{\sigma^2_0} \right]$$

$$\Theta^{(3)X} = \begin{bmatrix} \alpha^{state} & \beta_{22}^{state} & \frac{1}{6}\beta_{333,1}^{state} & \frac{1}{2} \left[\beta_{300}^{state} \left(I_{ns} \otimes E \varepsilon_t^{\otimes [2]} \right) + \beta_{\sigma^2 1}^{state} \right] \\ 0 & \alpha^{state \otimes [2]} & \left(\frac{1}{2} \beta_{22}^{state} \right) \otimes \alpha^{state} & \left(\left[\left(\frac{1}{2} \beta_{00}^{state} \right) \otimes \alpha^{state} \right] K_{ne^2, ns} + \beta_{20}^{state} \otimes \beta_0^{state} \right) \left(I_{ns} \otimes E \varepsilon_t^{\otimes [2]} \right) \\ 0 & 0 & \alpha^{state \otimes [3]} & \left[K_{ns^2, ns} + (K_{ns, ns} \otimes I_{ns} + I_{ns^3}) \right] \left(\alpha^{state} \otimes \beta_0^{state \otimes [2]} \right) \left(I_{ns} \otimes E \varepsilon_t^{\otimes [2]} \right) \\ 0 & 0 & 0 & \alpha^{state} \end{bmatrix}$$

$$\Phi^{(3)X} = \begin{bmatrix} \frac{1}{6}\beta_{000}^{state} & \frac{1}{2}\beta_{330,1}^{state} & \dots \\ \left(\frac{1}{2} \beta_{00}^{state} \right) \otimes \beta_0^{state} & \left(\beta_{20}^{state} \otimes \alpha^{state} \right) K_{ns*ne, ns} + \left(\frac{1}{2} \beta_{22}^{state} \right) \otimes \beta_0^{state} & \dots \\ \beta_0^{state \otimes [3]} & \left[(K_{ns, ns} \otimes I_{ns} + I_{ns^3}) K_{ns^2, ns} + I_{ns^3} \right] \left(\alpha^{state \otimes [2]} \otimes \beta_0^{state} \right) & \dots \\ 0 & 0 & \dots \end{bmatrix}$$

$$\begin{bmatrix} \dots & \frac{1}{2}\beta_{300}^{state} & \beta_{20}^{state} & \frac{1}{2}\beta_{\sigma^2 0}^{state} \\ \dots & \left[\left(\frac{1}{2} \beta_{00}^{state} \right) \otimes \alpha^{state} \right] K_{ne^2, ns} + \beta_{20}^{state} \otimes \beta_0^{state} & \alpha^{state} \otimes \beta_0^{state} & 0 \\ \dots & \left[K_{ns^2, ns} + (K_{ns, ns} \otimes I_{ns} + I_{ns^3}) \right] \left(\alpha^{state} \otimes \beta_0^{state \otimes [2]} \right) & 0 & 0 \\ \dots & 0 & 0 & \beta_0^{state} \end{bmatrix}$$

$$\begin{aligned}
E \left(\underline{\Xi}_t^{(3)} \underline{\Xi}_t^{(3)'} \right) = & \\
& \left[\begin{array}{cc}
E \left(\boldsymbol{\varepsilon}_t^{\otimes [3]} \boldsymbol{\varepsilon}_t^{\otimes [3]'} \right) & E \left[\boldsymbol{\varepsilon}_t^{\otimes [3]} \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \boldsymbol{\varepsilon}_t \right)' \right] \quad \dots \\
E \left[\left(dy_{t-1}^{(1)state \otimes [2]} \otimes \boldsymbol{\varepsilon}_t \right) \boldsymbol{\varepsilon}_t^{\otimes [3]'} \right] & E \left[\left(dy_{t-1}^{(1)state \otimes [2]} \otimes \boldsymbol{\varepsilon}_t \right) \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \boldsymbol{\varepsilon}_t \right)' \right] \quad \dots \\
0 & 0 \quad \dots \\
E \left[\left(dy_{t-1}^{(2)state} \otimes \boldsymbol{\varepsilon}_t \right) \boldsymbol{\varepsilon}_t^{\otimes [3]'} \right] & E \left[\left(dy_{t-1}^{(2)state} \otimes \boldsymbol{\varepsilon}_t \right) \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \boldsymbol{\varepsilon}_t \right)' \right] \quad \dots \\
E \left(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t^{\otimes [3]'} \right) & E \left[\boldsymbol{\varepsilon}_t \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \boldsymbol{\varepsilon}_t \right)' \right] \quad \dots
\end{array} \right. \\
& \dots \quad \quad \quad 0 \quad \quad \quad \dots \\
& \dots \quad \quad \quad 0 \quad \quad \quad \dots \\
& \dots E \left[\left(dy_{t-1}^{(1)state} \otimes \left(\boldsymbol{\varepsilon}_t^{\otimes [2]} - E \boldsymbol{\varepsilon}_t^{\otimes [2]} \right) \right) \left(dy_{t-1}^{(1)state} \otimes \left(\boldsymbol{\varepsilon}_t^{\otimes [2]} - E \boldsymbol{\varepsilon}_t^{\otimes [2]} \right) \right)' \right] \quad \dots \\
& \dots \quad \quad \quad 0 \quad \quad \quad \dots \\
& \dots \quad \quad \quad 0 \quad \quad \quad \dots \\
& \dots \quad \quad \quad E \left[\boldsymbol{\varepsilon}_t^{\otimes [3]} \left(dy_{t-1}^{(2)state} \otimes \boldsymbol{\varepsilon}_t \right)' \right] \quad \quad \quad E \left(\boldsymbol{\varepsilon}_t^{\otimes [3]} \boldsymbol{\varepsilon}_t' \right) \\
& \dots E \left[\left(dy_{t-1}^{(1)state \otimes [2]} \otimes \boldsymbol{\varepsilon}_t \right) \left(dy_{t-1}^{(2)state} \otimes \boldsymbol{\varepsilon}_t \right)' \right] \quad \quad \quad E \left[\left(dy_{t-1}^{(1)state \otimes [2]} \otimes \boldsymbol{\varepsilon}_t \right) \boldsymbol{\varepsilon}_t' \right] \\
& \dots \quad \quad \quad 0 \quad \quad \quad 0 \\
& \dots E \left[\left(dy_{t-1}^{(2)state} \otimes \boldsymbol{\varepsilon}_t \right) \left(dy_{t-1}^{(2)state} \otimes \boldsymbol{\varepsilon}_t \right)' \right] \quad \quad \quad E \left[\left(dy_{t-1}^{(2)state} \otimes \boldsymbol{\varepsilon}_t \right) \boldsymbol{\varepsilon}_t' \right] \\
& \dots \quad \quad \quad E \left[\boldsymbol{\varepsilon}_t \left(dy_{t-1}^{(2)state} \otimes \boldsymbol{\varepsilon}_t \right)' \right] \quad \quad \quad E \left(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' \right)
\end{aligned}$$

A.13 Computing Elements in $E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right)$

For every nonzero entry of $E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right)$ in section A.12, the terms inside the expectations operator are either i) second, fourth, or sixth moments of the shocks, or ii) the product of these moments with the state variable block of the order increments, i.e., $dy_{t-1}^{(2)state}$ and $dy_{t-1}^{(1)state}$. The fourth and sixth moments of the shocks can be computed using Tracy and Sultan's (1993, p. 344-345) formulae. E.g., for sixth moments in the form $E \left(\varepsilon_t^{\otimes [3]} \varepsilon_t^{\otimes [3]'} \right)$, applying the mixed Kronecker product rule yields

$$(201) \quad E \left(\varepsilon_t^{\otimes [3]} \varepsilon_t^{\otimes [3]'} \right) = E \left(\varepsilon_t \varepsilon_t' \otimes \varepsilon_t \varepsilon_t' \otimes \varepsilon_t \varepsilon_t' \right)$$

then Tracy and Sultan's (1993) Theorem 3 (repeated here) can be applied directly

$$(202) \quad E \left(\varepsilon_t \varepsilon_t' \otimes \varepsilon_t \varepsilon_t' \otimes \varepsilon_t \varepsilon_t' \right) = [E \left(\varepsilon_t \varepsilon_t' \right)]^{\otimes [3]} \left[K + (K_{ne} \otimes K_{ne,ne}) + (K_{ne,ne} \otimes K_{ne}) + K_{ne,ne^2} (K_{ne,ne} \otimes K_{ne}) \right] \\ + K \left([vec \left(E \left(\varepsilon_t \varepsilon_t' \right) \right) vec' \left(E \left(\varepsilon_t \varepsilon_t' \right) \right)] \otimes E \left(\varepsilon_t \varepsilon_t' \right) \right) K$$

where

$$(203) \quad K = I_{ne^3} + K_{ne,ne^2} + K_{ne^2,ne}$$

is a sum of commutation matrices (See Magnus and Neudecker (1979)).

For the fourth moment in the form $E \left(\varepsilon_t^{\otimes [3]} \varepsilon_t' \right)$, Jinadasa and Tracy's (1986, p. 404) formula (repeated here) can likewise be applied directly

$$(204) \quad E \left(\varepsilon_t^{\otimes [3]} \varepsilon_t' \right) = E \left(\varepsilon_t \varepsilon_t' \right) \otimes vec \left(E \left(\varepsilon_t \varepsilon_t' \right) \right) + vec \left(E \left(\varepsilon_t \varepsilon_t' \right) \right) \otimes E \left(\varepsilon_t \varepsilon_t' \right) + (I_{ne} \otimes K_{ne,ne}) \left[vec \left(E \left(\varepsilon_t \varepsilon_t' \right) \right) \otimes E \left(\varepsilon_t \varepsilon_t' \right) \right]$$

For the entries in the form of a product between the moments and the state variable block of order increments, use the property of the Kronecker product of column vectors and the mixed Kronecker product rule to rearrange until they are in the form of a (Kronecker) product of two clusters: one cluster contains the state variable block of the order increments only, and the other contains (the product of) shocks only. As all the order increments of the last period are uncorrelated

with the current shocks, the expected value of the two clusters can be computed separately. E.g. (205)

$$\begin{aligned} E \left[\left(dy_{t-1}^{(1)state\otimes[2]} \otimes \boldsymbol{\varepsilon}_t \right) \boldsymbol{\varepsilon}_t^{\otimes[3]'} \right] &= E \left[dy_{t-1}^{(1)state\otimes[2]} \otimes \boldsymbol{\varepsilon}_t \otimes \boldsymbol{\varepsilon}_t^{\otimes[3]'} \right] = E \left[dy_{t-1}^{(1)state\otimes[2]} \otimes \left(\boldsymbol{\varepsilon}_t \otimes \boldsymbol{\varepsilon}_t^{\otimes[3]'} \right) \right] \\ &= E \left[dy_{t-1}^{(1)state\otimes[2]} \otimes \left(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t^{\otimes[3]'} \right) \right] = E \left(dy_{t-1}^{(1)state\otimes[2]} \right) \otimes E \left(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t^{\otimes[3]'} \right) \end{aligned}$$

where $E \left(dy_{t-1}^{(1)state\otimes[2]} \right)$ was computed in section A.8 and $E \left(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t^{\otimes[3]'} \right)$ can be computed using the transposed version of (204).

In fact, many nonzero entries in $E \left(\boldsymbol{\Xi}_t^{(3)} \boldsymbol{\Xi}_t^{(3)'} \right)$ can be recycled from the calculations in section A.8 and therefore need not to be computed again. E.g., the block entry in the second row and second column of $E \left(\boldsymbol{\Xi}_t^{(3)} \boldsymbol{\Xi}_t^{(3)'} \right)$ can be written as

$$(206) \quad E \left[\left(dy_{t-1}^{(1)state\otimes[2]} \otimes \boldsymbol{\varepsilon}_t \right) \left(dy_{t-1}^{(1)state\otimes[2]} \otimes \boldsymbol{\varepsilon}_t \right)' \right] = E \left(dy_{t-1}^{(1)state\otimes[2]} dy_{t-1}^{(1)state\otimes[2]'} \right) \otimes E \left(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' \right)$$

The first term on the right hand side of the foregoing can be recycled from $\Gamma_0^{(2)X}$ as the lower right entry (the block entry in the second row and second column) of $\Gamma_0^{(2)X}$ takes the form

$$(207) \quad \Gamma_{0,22}^{(2)X} = E \left[\left(dy_{t-1}^{(1)state\otimes[2]} - E dy_{t-1}^{(1)state\otimes[2]} \right) \left(dy_{t-1}^{(1)state\otimes[2]} - E dy_{t-1}^{(1)state\otimes[2]} \right)' \right]$$

$$(208) \quad = E \left(dy_{t-1}^{(1)state\otimes[2]} dy_{t-1}^{(1)state\otimes[2]'} \right) - E \left(dy_{t-1}^{(1)state\otimes[2]} \right) E \left(dy_{t-1}^{(1)state\otimes[2]'} \right)'$$

therefore

$$(209) \quad E \left(dy_{t-1}^{(1)state\otimes[2]} dy_{t-1}^{(1)state\otimes[2]'} \right) = \Gamma_{0,22}^{(2)X} + E \left(dy_{t-1}^{(1)state\otimes[2]} \right) E \left(dy_{t-1}^{(1)state\otimes[2]'} \right)'$$

Some entries of $E \left(\boldsymbol{\Xi}_t^{(3)} \boldsymbol{\Xi}_t^{(3)'} \right)$ are zero as they contain one or some of terms equal to zero under normality: the odd moments of the exogenous shocks, $E \left(dy_t^{(1)state} \right)$, $E \left(dy_t^{(1)state\otimes[3]} \right)$ and $E \left(dy_t^{(1)state\otimes[5]} \right)$.

Table 1: Parameter Values: Common to All Three Calibrations

Parameter	β	ψ	χ	α	δ	\bar{a}	ρ_σ	τ
Value	0.9926	2.9869	100	0.339	0.021	0.004	0.9	0.15

See Tallarini (2000) and the main text.

Table 2: Parameter Values: Calibrating Homoskedastic Volatility

	Baseline	Constant Volatility	Expected Utility	Extended Model
$\bar{\sigma}_a$	0.009824769	0.011588754	0.0115	0.0225

$\bar{\sigma}_a$ calibrated to keep the standard deviation of $\Delta \ln(c) = 0.0055$

Table 3: Mean Comparison

Variable	<i>Det.S.S.*</i>	Baseline Calibration	Constant Volatility Calibration	Tallarini (2000)
$\log(k)$	2.0841	2.1373	2.1581	2.1584
i	0.2002	0.2106	0.2146	0.2160
$\log(c)$	-0.5672	-0.5542	-0.5491	-0.5499
$\log(y)$	-0.2649	-0.2417	-0.2326	-0.2319
$\log(N)$	-1.4675	-1.4597	-1.4566	-1.4563
R^f	1.1493	1.0470	1.0070	1.011
R	1.1493	1.0532	1.0156	1.022

* The deterministic steady state value
See Table 5 and 8, Tallarini (2000).

Table 4: Standard Deviation Comparison

Variable	Baseline Calibration	Constant Volatility Calibration	Tallarini (2000)	Data
$\Delta \log(c)$	0.0055	0.0055	0.0055	0.0055
$\Delta \log(y)$	0.0096	0.0100	0.0095	0.0104
$\Delta \log(i)$	0.0240	0.0223	0.0224	0.0279
$\log(c) - \log(y)$	0.0154	0.0150	0.0147	0.0377
$\log(i) - \log(y)$	0.0425	0.0404	0.0403	0.0649

See Table 7, Tallarini (2000).

Table 5: Variance Decomposition in Percentage

	Constant Volatility Calibration		Baseline Calibration	
	1st order amp.	risk adjustment	1st order amp.	risk adjustment
<i>cmpr</i>	0	100	0	100
<i>erp</i>	0	100	0	100
<i>rp</i>	109.03	0.60	80.68	8.51
$\log(k)$	96.35	0.02	75.08	1.37
<i>i</i>	95.28	0.02	57.19	30.25
$\log(c)$	96.65	0.01	75.88	2.62
$\log(y)$	95.03	0.04	44.54	36.99
$\log(N)$	97.82	0.01	66.25	18.61

For each calibration, the columns may not add up to 100 due to the omission of 2nd and 3rd order amplification and cross effects.

Table 6: Standard Deviation Comparison

Variable	Baseline Calibration	Extended Model	Data
$\Delta\log(c)$	0.0055	0.0055	0.0055
$\Delta\log(y)$	0.0096	0.0058	0.0104
$\Delta\log(i)$	0.0240	0.0068	0.0279
$\log(i) - \log(k)$	0.0707	0.0368	-
$\log(c) - \log(y)$	0.0154	0.0018	0.0377
$\log(i) - \log(y)$	0.0425	0.0052	0.0649

Table 7: Variance Decomposition in Percentage

	Constant Volatility Calibration		Baseline Calibration		Extended Model	
	1st order amp.	risk adj.	1st order amp.	risk adj.	1st order amp.	risk adj.
<i>cmpr</i>	0	100	0	100	0	100
<i>erp</i>	0	100	0	100	0	100
<i>rp</i>	109.03	0.60	80.68	8.51	79.45	0.95
<i>rf</i>	105.56	0.14	80.35	0.86	14.83	81.54
$\log(k)$	96.35	0.02	75.08	1.37	80.03	0.21
<i>i</i>	95.28	0.02	57.19	30.25	78.39	1.03
$\log(c)$	96.65	0.01	75.88	2.62	80.02	0.23
$\log(y)$	95.03	0.04	44.54	36.99	80.09	0.21
$\log(N)$	97.82	0.01	66.25	18.61	78.76	1.44

For each calibration, the columns may not add up to 100 due to the omission of 2nd and 3rd order amplification and cross effects.

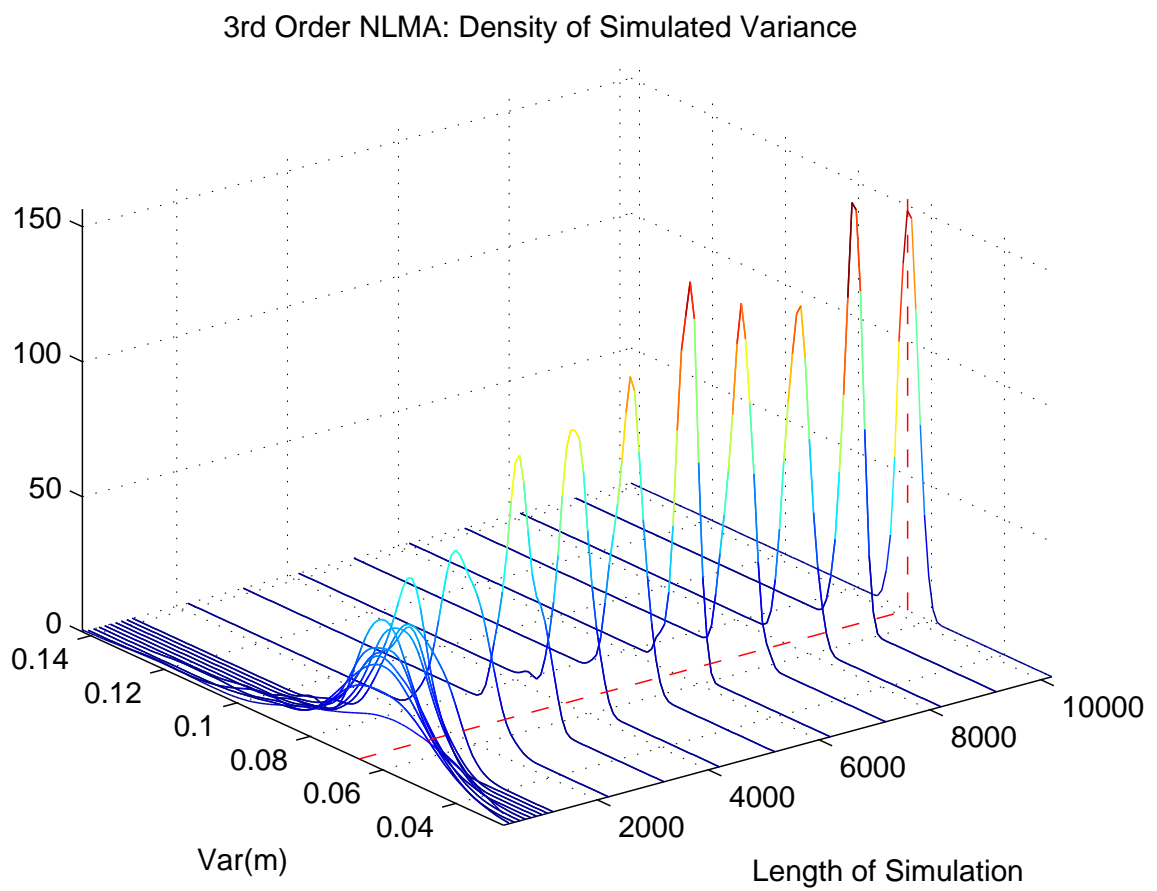


Figure 1: Monte Carlo Consistency of Moment Calculations, Example of m_t , Baseline Model of Section 3

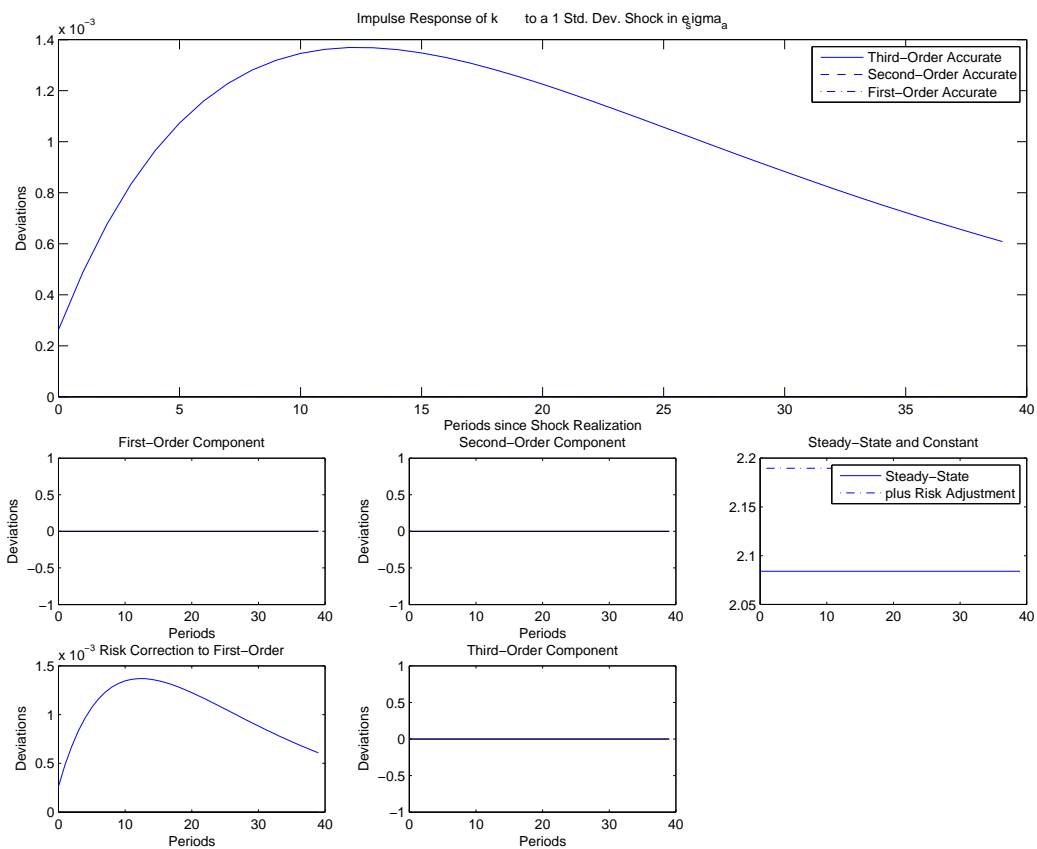


Figure 2: Capital IRF: Volatility Shock, Baseline Model of Section 3

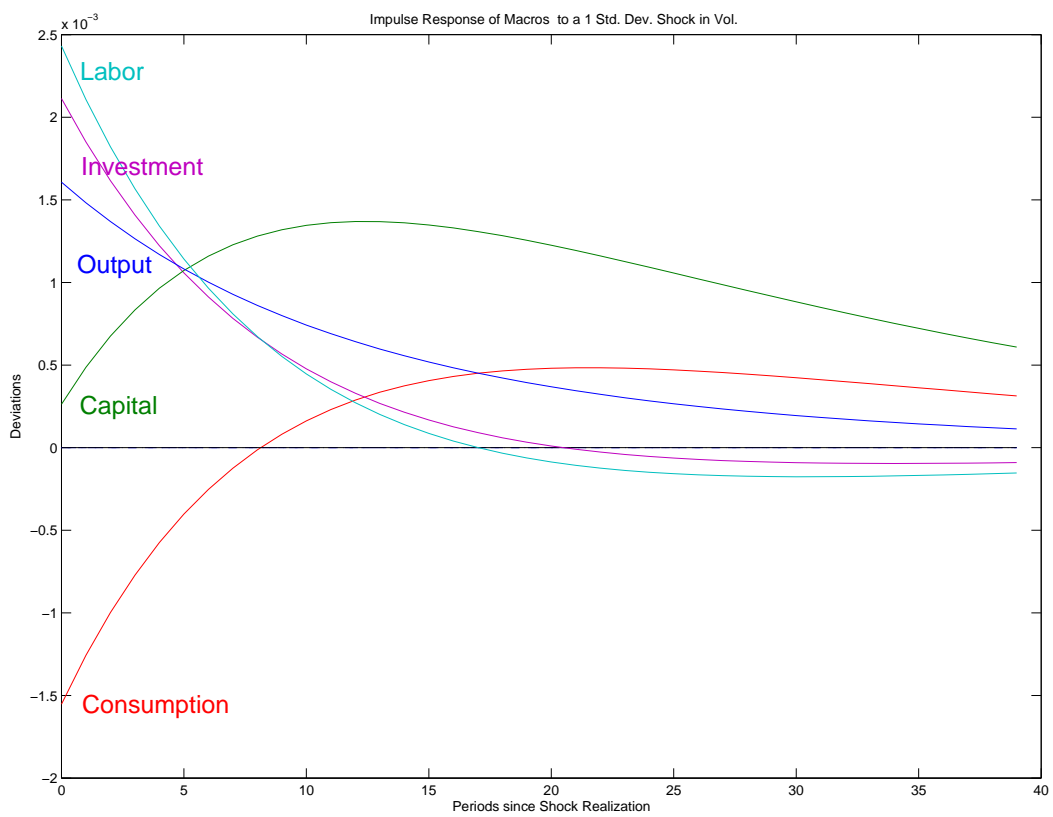
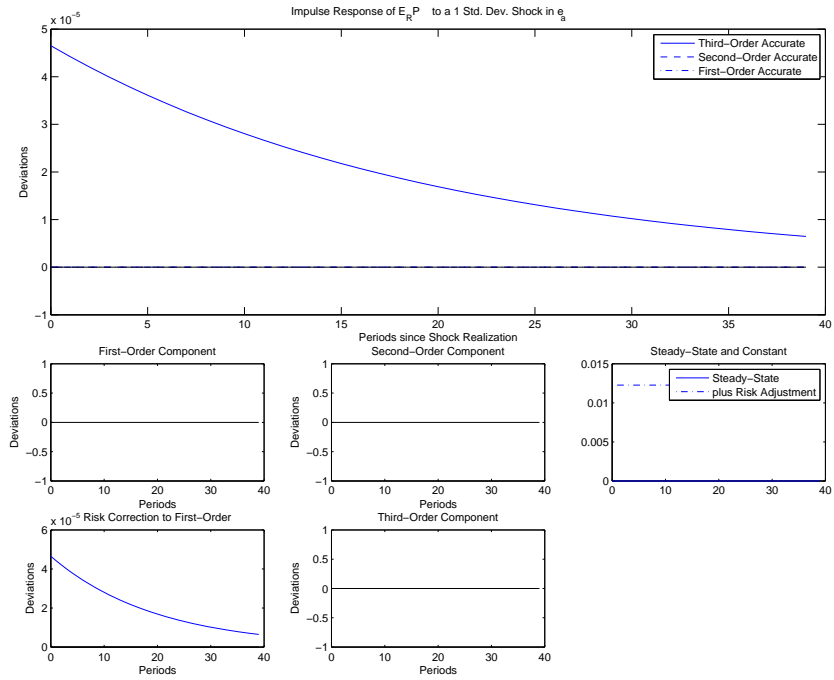
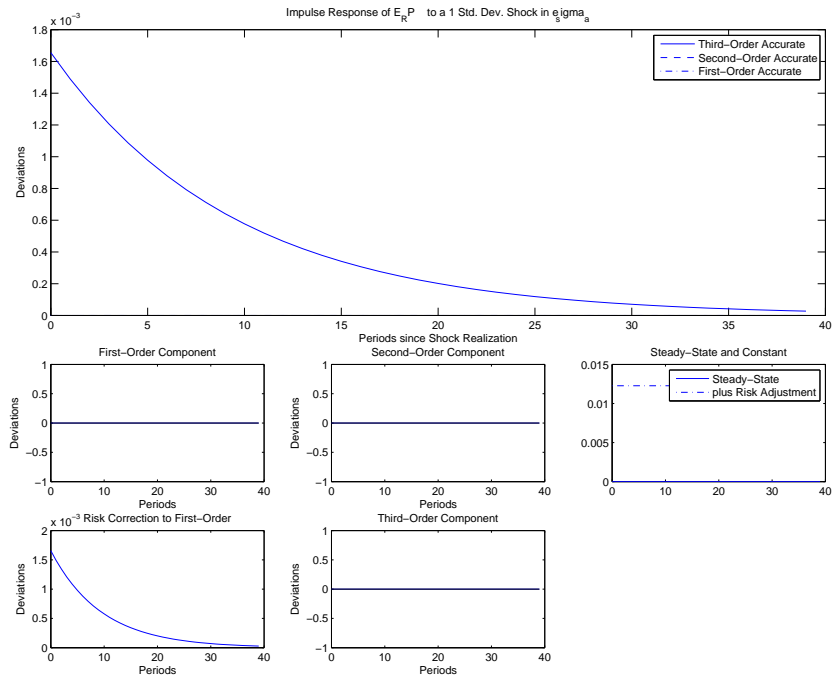


Figure 3: Macro IRFs: Volatility Shock, Baseline Model of Section 3

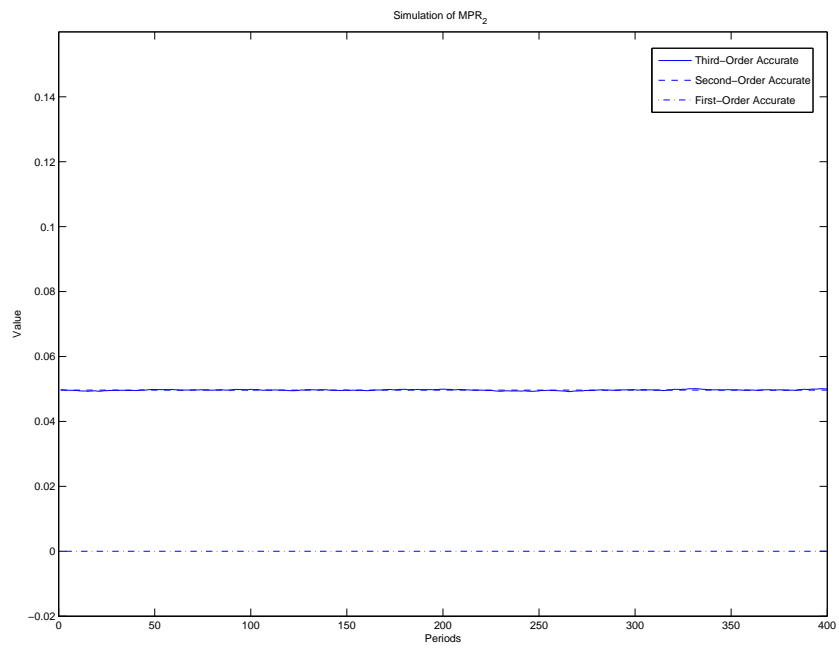


(a) Growth Shock

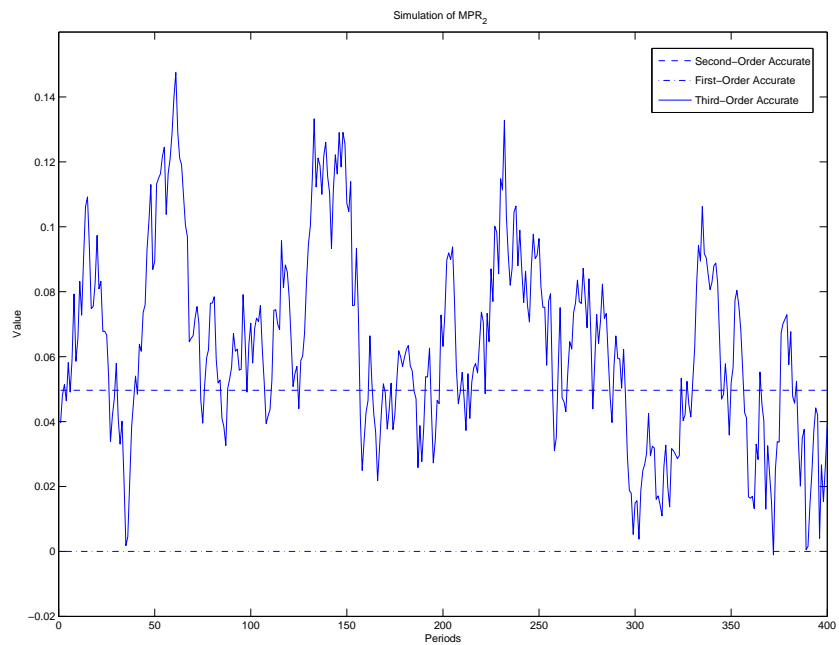


(b) Volatility Shock

Figure 4: Expected Risk Premium IRFs: Volatility and Growth Shocks, Baseline Model of Section 3

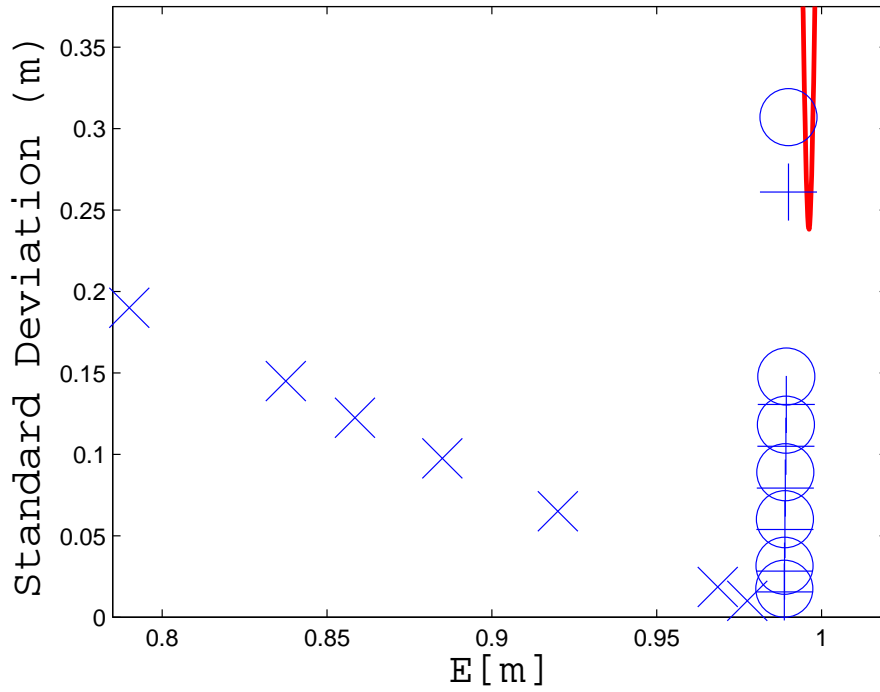


(a) Constant Volatility Calibration

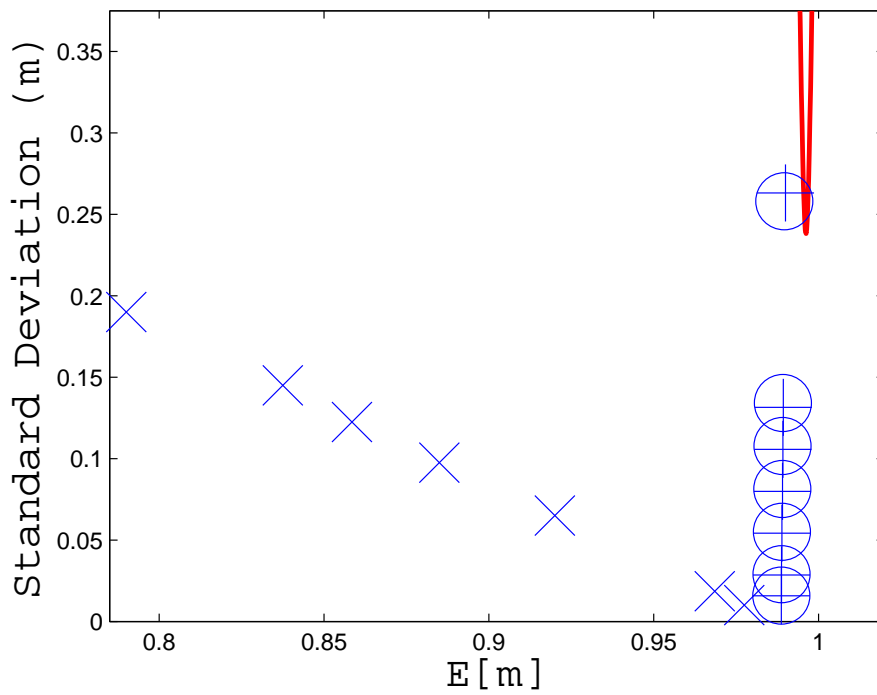


(b) Baseline (Stochastic Volatility) Calibration

Figure 5: Simulated Squared Conditional Market Price of Risk, Baseline and Constant Volatility Model of Section 3



(a) $\bar{\sigma}_a = 0.0115$ for All Three Calibrations



(b) $\bar{\sigma}_a$ Calibrated to Match $\Delta \log(c) 0.0055$

Figure 6: Stochastic Volatility and the Hansen-Jagannathan Bounds
 ×: Expected Utility; +: Constant Volatility; ○: Baseline (Stochastic Volatility)

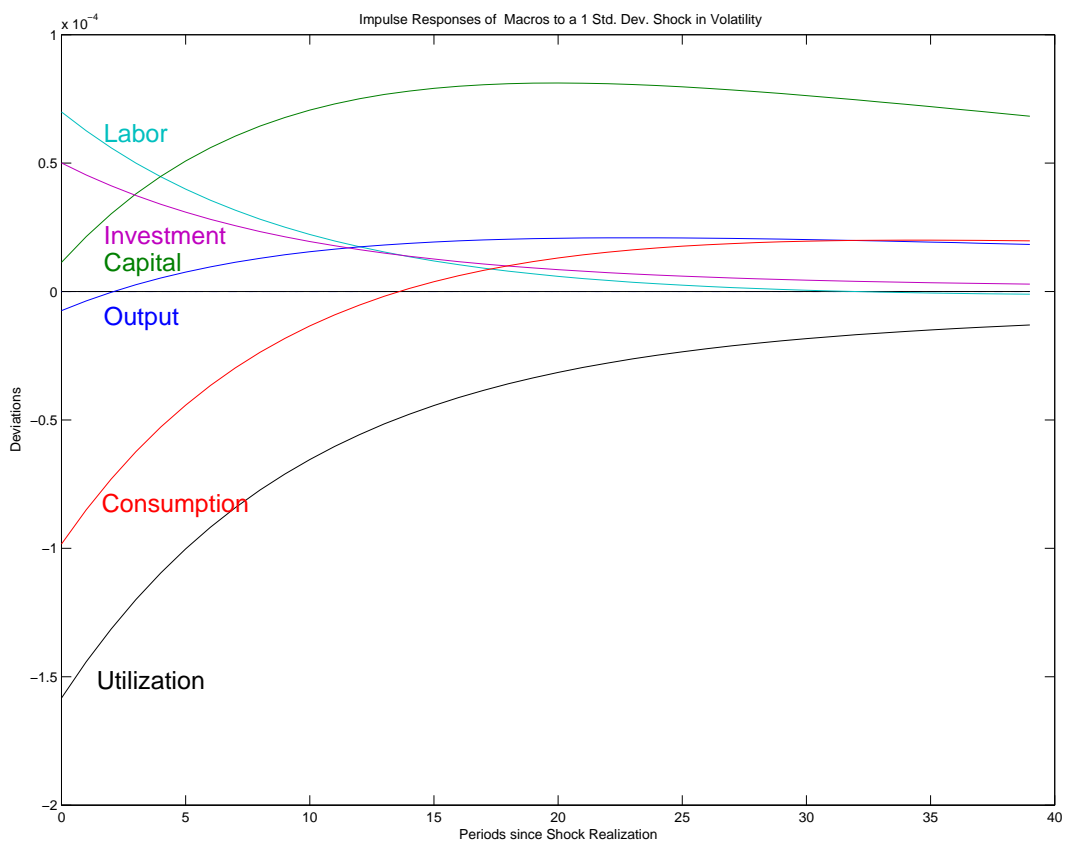


Figure 7: Macro IRFs: Volatility Shock, Extended Model of Section 7