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FAIR AND EFFICIENT LOTTERIES OVER INDIVISIBLE GOODS

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ABSTRACT. We study the problem of assigning indivisible goods to individuals where each is to receive one object. To guarantee fairness in the absence of monetary compensation, we consider random assignments and analyse various equity criteria for such lotteries. In particular, we find that sd-no-envy (as championed by the Probabilistic Serial) is incompatible with the sd-core from equal division. As an alternative, we present a Walrasian mechanism, whose outcomes are sd-efficient, group sd-envy-free, lie in the sd-core from equal division and satisfy the sd-equal-division-lower-bound.

Keywords: Probabilistic Serial; Sd-efficiency; Sd-no-envy; Sd-core from equal division

JEL codes: C70, D63

1. MOTIVATION

Efficiency in itself is rarely sufficient to determine a unique allocation of resources. In order to remedy this problem and to arrive at a compromise, we frequently apply formal fairness criteria, often in the spirit of equity. However, where resources are indivisible, equity with respect to deterministic allocations is ruled out. Applying a lottery may then allow us to restore fairness at least from an ex-ante perspective. While the design of optimal lotteries has received a lot of attention in recent years, most of the work concentrates on their efficiency and incentive properties (i.e. is it optimal for participants to reveal their true preferences). In this paper, we try to complement the literature by taking a closer look at the original motivation for applying a lottery mechanism and ask “when is a lottery fair?”. For this, we draw on the rich literature on fair allocation and adapt various equity criteria to random assignments.

2. TECHNICALITIES

We consider the problem of allocating n objects to n agents. Each agent $i \in \mathcal{I}$ is to receive one object $o \in \mathcal{O}$ and holds strict ordinal preferences \succ_i over the set of objects \mathcal{O} :

$$o(1, i) \succ_i o(2, i) \succ_i \dots \succ_i o(n, i)$$

where $o(l, i)$ denotes the l -most preferred object of individual i . A lottery p assigns object o to agent i with probability p_o^i . It is feasible, if $\forall o \in \mathcal{O} : \sum_{i \in \mathcal{I}} p_o^i = 1$. The collection $p^i = (p_o^i)_{o \in \mathcal{O}}$ is referred to as the individual lottery of agent i ; throughout this paper we restrict attention to individual lotteries such that $\forall i \in \mathcal{I} : \sum_{o \in \mathcal{O}} p_o^i = 1$.¹

To extend agents preferences to (individual) lotteries,² we rely on first order stochastic dominance: $p^i \succeq_i \tilde{p}^i$ iff

$$\forall k \in \{1, \dots, n\} : \sum_{l=1}^k p_{o^{(l,i)}}^i \geq \sum_{l=1}^k \tilde{p}_{o^{(l,i)}}^i,$$

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¹We assume that there is no outside option preferred to any of the objects $o \in \mathcal{O}$.

²We do not consider any consumption externalities, so preferences only depend on the individual component.

i.e. an agent weakly prefers lottery p iff it guarantees her a weakly higher chance of receiving her most preferred object *and* a weakly higher chance of receiving the most or second most preferred object *and* ... so on. Iff at least one of the inequalities is strict, we write $p^i \succ_i \tilde{p}^i$. Note that first order stochastic dominance only induces a partial ordering over lotteries - this will require us to differentiate between weak and strong versions of prominent equity criteria.

Finally, a lottery p is sd-efficient³ unless there exists another feasible lottery $\tilde{p} \neq p$ such that $\tilde{p}^i \succeq_i p^i$, for all $i \in \mathcal{I}$. [Bogomolnaia and Moulin \[2001\]](#) refer to this concept as *ordinal efficiency*, to highlight the coarse informational underpinning of our preference relation.

3. EQUITY CRITERIA

Envy-Freeness is arguably the most prominent equity criterion if we rule out interpersonal comparability. To check whether an allocation is envy-free, we need to compare agents individual allocations - no agent should then prefer anyone else's lottery.

Definition 1. Formally, a lottery p is

- *sd-envy-free* iff $\forall i, j \in \mathcal{I} : p^i \succeq_i p^j$.
- *weakly sd-envy-free* iff $\nexists i \neq j \in \mathcal{I} : p^j \succ_i p^i$.

As mentioned before, the difference between the strong and weak notion arises out of the incompleteness of preferences. Weak sd-envy-freeness is satisfied iff there exist (row-)vectors of Bernoulli utilities $u^i = (u_o^i)_{o \in \mathcal{O}}$, consistent with agents preferences over sure objects⁴, such that each lottery p is envy-free with respect to u , i.e. if $\forall i \neq j : u^i \cdot p^i \geq u^i \cdot p^j$. Strict sd-envy-freeness is satisfied whenever p is envy-free with respect to *all* possible Bernoulli utilities u_o^i that are consistent with individuals preferences over sure objects. From the perspective of a social planner who assumes that agents evaluate lotteries as expected utility maximisers, requiring sd-envy-freeness can then be seen as an attempt to ensure equity in the face of the limited information available. The most prominent allocation mechanism that satisfies sd-envy-freeness is the Probabilistic Serial Mechanism, proposed by [Bogomolnaia and Moulin \[2001\]](#), where agents are simultaneously “eating” probability shares, starting with their most preferred object until it is exhausted, then moving down to their second most preferred object, and so on.

Another natural yardstick to measure individuals shares is equal division, denoted as $(\frac{1}{n})$, i.e. the individual lottery that grants an agent each object with probability $\frac{1}{n}$.

Definition 2. A lottery p satisfies the

- *sd-equal-division-lower-bound* iff $\forall i \in \mathcal{I} : p^i \succeq_i (\frac{1}{n})$.
- *weak sd-equal-division-lower-bound* iff $\nexists i \in \mathcal{I} : (\frac{1}{n}) \succ_i p^i$.

Again, the weak notion is satisfied iff the equal division lower bound is met for some consistent Bernoulli utilities u_o^i , while the strong notion requires that it is met for all possible u_o^i that are consistent with agents preferences over sure objects.

Observe that any lottery that is sd-envy-free also meets the sd-equal-division-lower-bound - as each agent receives one of her k -most preferred objects with a higher probability than anyone else, she also receives it with a probability weakly greater than $\frac{k}{n}$:

$$\forall i, k : \sum_{l=1}^k p_{o_l, i}^i \geq \sum_{l=1}^k \frac{1}{n} \sum_{j \in \mathcal{I}} p_{o_l, i}^j = \sum_{l=1}^k \frac{1}{n} = \frac{k}{n}$$

The two weak notions, however, are logically independent; neither implies the other. To see this, consider the following two examples.

³'sd' stands for 'stochastic dominance'.

⁴I.e., $\forall i : u_{o(1,i)}^i > u_{o(2,i)}^i > \dots > u_{o(n,i)}^i > 0$.

Example 1. Consider 3 agents $\{1, 2, 3\}$ who share the same preferences over objects $a \succ b \succ c$. Then the following lottery

	a	b	c
1:	$\frac{11}{20}$	0	$\frac{9}{20}$
2:	$\frac{3}{20}$	$\frac{14}{20}$	$\frac{3}{20}$
3:	$\frac{6}{20}$	$\frac{6}{20}$	$\frac{8}{20}$

is weakly sd-envy-free but fails to meet the weak sd-equal-division-lower-bound as agent 3 would prefer the uniform lottery.

Example 2. Suppose that the preferences of agent $i \in \{1, 2\}$ are given as $a \succ_i b \succ_i c$ while the third agents most preferred object is c . Then the following lottery

	a	b	c
1:	$\frac{2}{3}$	$\frac{1}{3}$	0
2:	$\frac{1}{3}$	$\frac{2}{3}$	0
3:	0	0	1

meets the sd-equal-division-lower-bound but fails to be weakly sd-envy-free as $p^1 \succ_2 p^2$.

In addition, there are various equity criteria for groups of agents, perhaps most notably the core from equal division - see for example [Thomson \[2011\]](#).

Definition 3. A group of agents $G \subset \mathcal{I}$ may object to a lottery \tilde{p} if there is an alternative lottery p such that

- $\forall o \in \mathcal{O} : \sum_{i \in G} p_o^i = \frac{|G|}{n}$ and
- $\forall i \in G : p^i \succeq \tilde{p}^i$ and $\exists j \in G : p^j \succ \tilde{p}^j$.

If there is no such objection that can be raised against a lottery, the lottery is said to be in the *sd-core (from equal division)*.

Note that an inefficient lottery would be blocked by the grand coalition. Hence, any lottery in the (strong) sd-core is sd-efficient.

The sd-core extends the idea of an equal-division-lower-bound to group comparisons. In the same way, we can extend envy-freeness to take by comparing the (aggregate) share of one group with the share of other groups. This leads us to the concept of group envy-freeness [[Thomson, 2011](#)],[[Varian, 1974](#)],[[Vind, 1971](#)]⁵.

Definition 4. A group of agents $G \subset \mathcal{I}$ may object to a lottery \tilde{p} if there is another group $G' \subset \mathcal{I}$ and a lottery p such that

- $\forall o \in \mathcal{O} : \sum_{i \in G} p_o^i = \frac{|G|}{|G'|} \sum_{j \in G'} \tilde{p}_o^j$ and
- $\forall i \in G : p^i \succeq \tilde{p}^i$ and $\exists j \in G : p^j \succ \tilde{p}^j$

If there is no objection that can be raised against a lottery, the lottery is said to be *group sd-envy-free*.

Note that as we allow for the case $|G| > |G'|$, the alternative lottery p may not be feasible. Restricting objections, e.g. by requiring p to be feasible or demanding that G and G' are disjoint, enlarges the set of group sd-envy-free lotteries. However, as our main finding on group sd-envy-freeness will be a possibility result, working with the most restrictive notion only strengthens our result.

Group envy-freeness is typically satisfied in a walrasian equilibrium with equal incomes [[Varian, 1974](#)]. When allocating indivisible goods as we do here, [Hylland and Zeckhauser \[1979\]](#)

⁵Varian and Vind use the term Coalition Fairness.

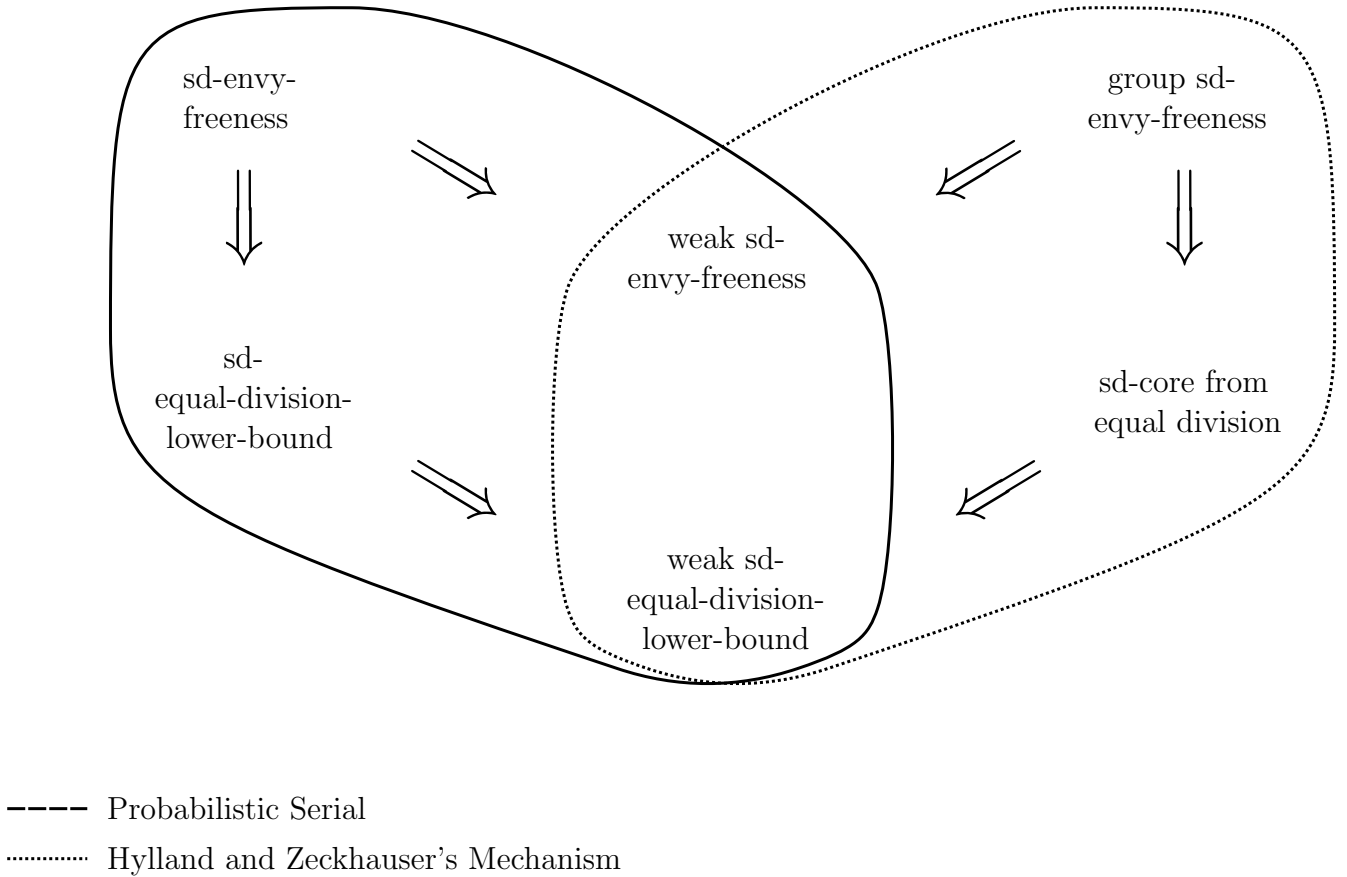


FIGURE 1. Logical relations between equity criteria and 2 sd-efficient mechanisms

show how general equilibrium theory can be used to derive a lottery that is efficient and, so we add, group envy-free when the initial incomes are equal. In contrast to our setup, they assume that agents evaluate lotteries according to some vNM-utilities $(u_i)_{i \in \mathcal{I}}$. However, any lottery constructed via their mechanism, will be sd-efficient and group sd-envy-free, provided that $(u_i)_{i \in \mathcal{I}}$ and $(\succ_i)_{i \in \mathcal{I}}$ agree on the ranking of objects $o \in \mathcal{O}$. The reason is simple: any trade (resp. objection) that would make everyone (resp. members of G) better off with respect to first order stochastic dominance would also yield an increase in individuals expected utility.

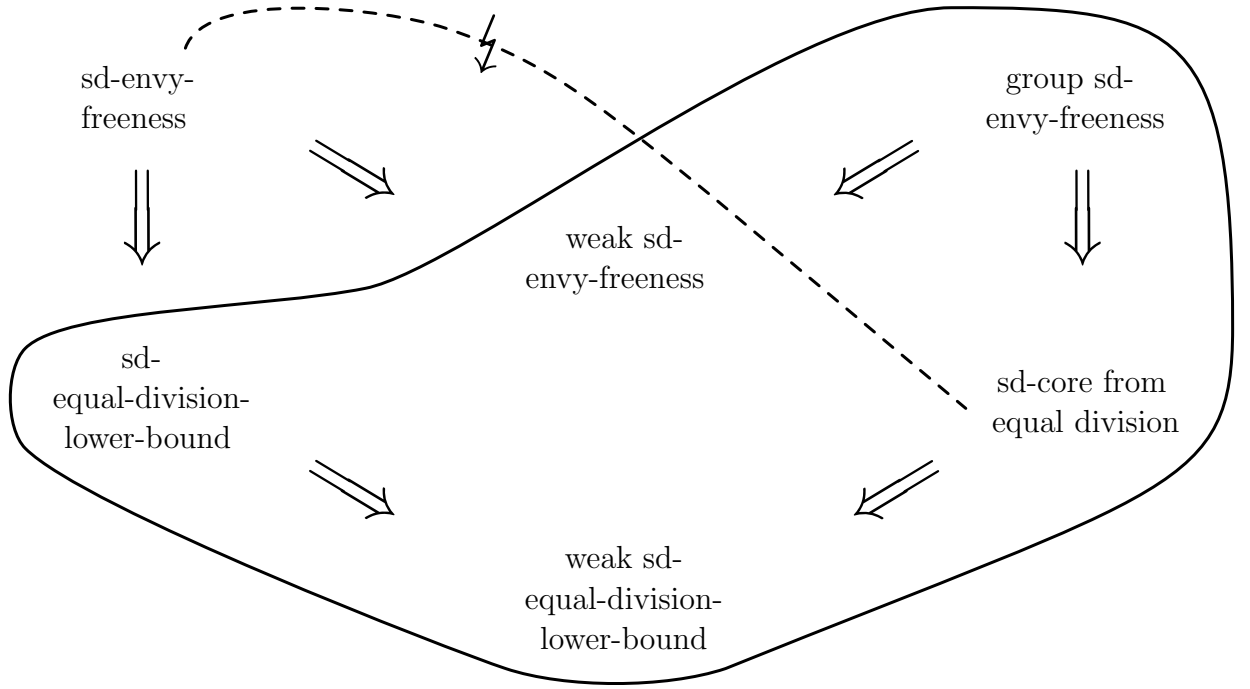
Note that group sd-envy-freeness implies weak sd-envy-freeness (just choose G and G' as singletons). Moreover, any lottery that is group sd-envy-free is also in the sd-core - just chose $G' = \mathcal{I}$ as comparison group. Figure 1 summarises all logical relations and exemplary mechanisms discussed so far. See the Appendix for an example where Hylland and Zeckhauser's mechanism does not satisfy the strict equal division lower bound⁶. The fact that probabilistic serial does not, in general, yield lotteries that are in the sd-core will follow from our Proposition 1.

4. MAIN RESULTS

In light of Figure 1, we may ask whether there exists an efficient mechanism that is able to satisfy all of the above equity concepts. The following Proposition answers that question in the negative.

Proposition 1. *For all $n \geq 4$, there exist preference profiles $(\succ_i)_{i \in \mathcal{I}}$ such that lottery simultaneously satisfies sd-envy-freeness and lies in the sd-core from equal division.*

⁶Of course, if one assumes that agents vNM-utilities are known, as Hylland and Zeckhauser do, the sd-equal-division-lower-bound loses much of its appeal.



- - - - Proposition 1 - Impossibility Result

— Proposition 2 - Possibility Result

FIGURE 2. Main Results

In the following we give a proof for the case $n = 4$; the general case is referred to the Appendix. Consider 4 agents with preferences $b \succ_1 a \succ_1 c \succ_1 d$, $a \succ_2 c \succ_2 b \succ_2 d$, $a \succ_3 b \succ_3 d \succ_3 c$, $a \succ_4 b \succ_4 c \succ_4 d$. Then sd-envy-freeness and sd-efficiency (the latter is a necessary condition for a lottery to lie in the sd-core) are sufficient to pin down the following unique lottery.

	a	b	c	d
1:	0	60/108	28/108	20/108
2:	36/108	0	52/108	20/108
3:	36/108	24/108	0	48/108
4:	36/108	24/108	28/108	20/108

The zero entries are a consequence of efficiency: Suppose for example that $p_a^2 > 0$. Then $p_b^1 < 1$ and hence $p_b^i > 0$ for some $i \in \{2, 3, 4\}$. But then agent 1 could trade some of her probability share in a against some of i 's share in b , leading to a pareto improvement. Analogously, we find that $p_b^2 = p_c^3 = 0$. The remaining entries then follow from strict sd-envy-freeness: agent 2, 3 and 4 need to receive weakly more of object a than anybody else, and so receive $1/3 = 36/108$ each. Agents 1, 3 and 4 need to receive weakly more of objects a and b than anybody else, so each receives $60/108$ in sum. Entries in the c column are computed analogously and the d entries are chosen so that the sum over individual probabilities is 1.

However, the following lottery constitutes a blocking proposal from the point of view of agents $i \in \{1, 2, 3\}$. Agent 1 receives her most preferred object with a higher probability while the probability of receiving her third-most preferred object decreases. Agent 2 is able to shift some probability mass from 4th to 2nd position, and agent 3 from 2nd to 1st.

	a	b	c	d
1:	0	66/108	29/108	13/108
2:	36/108	0	52/108	20/108
3:	45/108	15/108	0	48/108
4:	27/108	27/108	27/108	27/108

This proves proposition 1 for the case $n = 4$. In the appendix, we add a general example for the case of $n \geq 5$.

Accepting the disconcerting news of Proposition 1, we may ask whether group equity criteria are at least compatible with the sd-equal-division-lower-bound. Here the answer is yes.

Proposition 2. *For all n and all preference profiles $(\succ_i)_{i \in \mathcal{I}}$, there exist lotteries that are group sd-envy-free and that meet the sd-equal-division-lower-bound.*

Note that a fortiori, such lotteries will be sd-efficient, weakly sd-envy-free and lie in the sd-core from equal division.

We will proof Proposition 2 by establishing existence of a competitive equilibrium from equal incomes and suitably constrained consumption sets. Let us define the set of admissible price vectors as $Q = \{(q_o)_{o \in \mathcal{O}} \in \mathbb{R}^n | \forall o \in \mathcal{O} : q_o \geq -1, \sum_{o \in \mathcal{O}} q_o = 1\}$ ⁷ and fix individual incomes at $1/n$.⁸ Consumption sets of agents are given by

$$C^i = \{(p_o^i)_{o \in \mathcal{O}} | \forall o \in \mathcal{O} : p_o^i \geq 0, \forall k : \sum_{l=1}^k p_{o(n+1-l,i)}^i \leq k/n\}.$$

Recall that $o(l, i)$ denotes the l -most preferred object of individual i , so individuals are restricted to consume at most $1/n$ of their least preferred object, a total of $2/n$ of their two least favourite objects... and not more than 1 of their n least favourite objects, i.e. not more than 1 of all objects. If $\sum_{o \in \mathcal{O}} p_o^i = 1$, as will be the case in equilibrium, the constraints on consumption ensure that the sd-equal-division-lower-bound is met. Unfortunately \succ_i , i.e. the preference relation given by first order stochastic dominance, is not continuous. So in order to apply a fixed point argument for continuous correspondences, we augment each \succ_i by choosing a (row-)vector of compatible Bernoulli utilities $u^i = u_{o \in \mathcal{O}}^i$,

$$u_{o(1,i)}^i > u_{o(2,i)}^i > \dots > u_{o(n,i)}^i > 0.$$

The precise choice of u^i is not important for our existence result. What matters is, that each u^i and \succ_i agree on the ranking of sure objects. This guarantees that $u^i \cdot p^i > u^i \cdot \tilde{p}^i$ whenever $p^i \succ_i \tilde{p}^i$. We are now set to formulate the main Lemma towards a proof of Proposition 2.

Lemma 1. *For any choice of Bernoulli utilities $(u^i)_{i \in \mathcal{I}}$, there exists a competitive equilibrium from equal incomes, i.e. a tuple $(p, q) \in C^1 \times \dots \times C^m \times Q$ such that*

- $\forall i \in \mathcal{I}; \tilde{p}^i \in C^i : q \cdot p^i \leq 1/n$ and $(u^i \cdot \tilde{p}^i > u^i \cdot p^i \Rightarrow q \cdot \tilde{p}^i > 1/n)$ (preference maximisation)
- $\forall o \in \mathcal{O} : \sum_{i \in \mathcal{I}} p_o^i = 1$ (feasibility / non-wastefulness)

The proof of Lemma 1 is referred to the Appendix.

Proof of Proposition 2. Assured of its existence, consider an equilibrium $(p, q) \in C^1 \times \dots \times C^m \times Q$.

Claim 1. *The associated lottery p meets the sd-equal-division-lower-bound.*

The constraint $\sum_{o \in \mathcal{O}} p_o^i \leq 1$ imposed on each individual lottery $p^i \in C^i$, together with the equilibrium condition $\forall o \in \mathcal{O} : \sum_{i \in \mathcal{I}} p_o^i = 1$ guarantee that $\sum_{o \in \mathcal{O}} p_o^i = 1$ for all i . Hence, by construction of C^i , p satisfies the sd-equal-division-lower-bound.

⁷The price vector q is taken to be a row vector, while individual lotteries p^i take the form of column vectors.

⁸This income corresponds to the value of the average bundle $(\frac{1}{n})$ at any price vector $q \in Q$.

Claim 2. *The associated lottery p is group sd-envy-free.*

Assume otherwise, i.e. there are groups G, G' and an alternative lottery \tilde{p} , such that

$$\forall i \in G : \tilde{p}^i \succeq_i p^i \quad \text{and} \quad \sum_{i \in G} \tilde{p}^i = \frac{|G|}{|G'|} \sum_{j \in G'} p^j.$$

Due to Claim 1, we know that $\tilde{p}^i \in C^i$ for all $i \in G$. From the non-wastefulness property, $\forall o \in \mathcal{O} : \sum_{i \in \mathcal{I}} \tilde{p}_o^i = 1$, we know that each individual spends its whole budget, i.e. $\forall i \in \mathcal{I} : q \cdot p^i = 1/n$. But then $u_i(\tilde{p}^i) \geq u_i(p^i)$, implies that $q \cdot \tilde{p}^i \geq q \cdot p^i = 1/n$ for all $i \in G$ while for at least on $j \in G$, $u_i(\tilde{p}^i) > u_i(p^i)$ and $q \cdot \tilde{p}^i > 1/n$. However, this contradicts our initial assumption $\sum_{i \in G} \tilde{p}^i = \frac{|G|}{|G'|} \sum_{j \in G'} p^j$, which implies that

$$\sum_{i \in G} q \cdot \tilde{p}^i = \frac{|G|}{|G'|} \sum_{j \in G'} q \cdot p^j \leq |G|^{1/n},$$

where the inequality follows from individual budget constraints, i.e. $q \cdot p^j \leq \frac{1}{n}$, for all $j \in \mathcal{I}$. \square

5. APPENDIX

Proof of Proposition 1. It remains to show how our example can be extended to the case of $n \geq 5$. To do so, add objects o_5, o_6, \dots, o_n and assume that all agents agree on their ranking, i.e. $a, b, c, d \succ_i o_5 \succ_i \dots \succ_i o_n$, for all $i \in \mathcal{I}$. Moreover, let us add $n - 4$ copies of agent 4, such that each agent $j \in \{4, \dots, n\}$ holds preferences $a \succ_j b \succ_j c \succ_j d \succ_j o_5 \succ_j \dots \succ_j o_n$. The unique lottery \tilde{p} that satisfies sd-efficiency and strict sd-envy-freeness can then be calculated, as

	a	b	c	d	o_k
1:	0	$\frac{2}{n-1} - \frac{1}{(n-1)^2}$	$\frac{1}{n-1} - \frac{1}{(n-1)^2} + \frac{1}{(n-1)^3}$	$\frac{4}{n} - \frac{3}{n-1} + \frac{2}{(n-1)^2} - \frac{1}{(n-1)^3}$	$\frac{1}{n}$
2:	$\frac{1}{n-1}$	0	$\frac{2}{n-1} - \frac{2}{(n-1)^2} + \frac{1}{(n-1)^3}$	$\frac{4}{n} - \frac{3}{n-1} + \frac{2}{(n-1)^2} - \frac{1}{(n-1)^3}$	$\frac{1}{n}$
3:	$\frac{1}{n-1}$	$\frac{1}{n-1} - \frac{1}{(n-1)^2}$	0	$\frac{4}{n} - \frac{2}{n-1} + \frac{1}{(n-1)^2}$	$\frac{1}{n}$
j :	$\frac{1}{n-1}$	$\frac{1}{n-1} - \frac{1}{(n-1)^2}$	$\frac{1}{n-1} - \frac{1}{(n-1)^2} + \frac{1}{(n-1)^3}$	$\frac{4}{n} - \frac{3}{n-1} + \frac{2}{(n-1)^2} - \frac{1}{(n-1)^3}$	$\frac{1}{n}$

where $k = 5, \dots, n$ and $j = 4, \dots, n$. The following lottery p constitutes a blocking proposal by agents 1, 2 and 3:

	a	b	c	d	o_k
1:	0	$\frac{2}{n} + \frac{1}{(n-1)^2}$	$\frac{7}{n} - \frac{5}{n-1} + \frac{2}{(n-1)^2} - \frac{1}{(n-1)^3}$	$-\frac{5}{n} + \frac{5}{n-1} - \frac{3}{(n-1)^2} + \frac{1}{(n-1)^3}$	$\frac{1}{n}$
2:	$\frac{4}{n} - \frac{2}{n-1}$	0	$-\frac{4}{n} + \frac{5}{n-1} - \frac{2}{(n-1)^2} + \frac{1}{(n-1)^3}$	$\frac{4}{n} - \frac{3}{n-1} + \frac{2}{(n-1)^2} - \frac{1}{(n-1)^3}$	$\frac{1}{n}$
3:	$-\frac{1}{n} + \frac{2}{n-1}$	$\frac{1}{n} - \frac{1}{(n-1)^2}$	0	$\frac{4}{n} - \frac{2}{n-1} + \frac{1}{(n-1)^2}$	$\frac{1}{n}$
j :	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$

It is straightforward, to check that

$$\sum_{i=1}^3 p_a^i = \sum_{i=1}^3 p_b^i = \sum_{i=1}^3 p_c^i = \sum_{i=1}^3 p_d^i = \frac{3}{n}.$$

To see that $p^1 \succ_1 \tilde{p}^1$, observe that

$$p_b^1 - \tilde{p}_b^1 = \frac{2}{n} + \frac{1}{(n-1)^2} - \frac{2}{n-1} + \frac{1}{(n-1)^2} = \frac{2(n-1)^2 + n - 2n(n-1) + n}{n(n-1)^2} = \frac{2}{n(n-1)^2} > 0$$

and

$$\begin{aligned}
p_d^1 - \tilde{p}_d^1 &= -\frac{5}{n} + \frac{5}{n-1} - \frac{3}{(n-1)^2} + \frac{1}{(n-1)^3} - \frac{4}{n} + \frac{3}{n-1} - \frac{2}{(n-1)^2} + \frac{1}{(n-1)^3} \\
&= \frac{-5(n-1)^3 + 5n(n-1)^2 - 3n(n-1) + n - 4(n-1)^3 + 3n(n-1)^2 - 2n(n-1) + n}{n(n-1)^3} \\
&= \frac{-9(n-1)^3 + 8n(n-1)^2 - 5n(n-1) + 2n}{n(n-1)^3} \\
&= \frac{-n^3 + 6n^2 - 12n + 9}{n(n-1)^3} \\
&= \begin{cases} -\frac{26}{320} & , n = 5 \\ \frac{1}{n(n-1)^3} \underbrace{(-n^3 + 6n^2 - 12n + 9)}_{\leq 0} < 0 & , n \geq 6 \end{cases}
\end{aligned}$$

Hence agent 1 now receives more of her most and less of her least preferred object. To see that agent 2 is better off, note that $p_b^2 = \tilde{p}_b^2$ and $p_d^2 = \tilde{p}_d^2$ while

$$p_a^2 - \tilde{p}_a^2 = \frac{4}{n} - \frac{3}{(n-1)} = \frac{4(n-1) - 3n}{n(n-1)} = \frac{n-4}{n(n-1)} > 0.$$

Finally, agent 3 strictly prefers p^3 as well, given that $p_c^3 = \tilde{p}_c^3$ and $p_d^3 = \tilde{p}_d^3$ while

$$p_a^3 - \tilde{p}_a^3 = \frac{1}{n-1} - \frac{1}{n} > 0.$$

□

Proof of Lemma 1. We rely on the following fixed point theorem, see [Mas-Colell \[1992\]](#).

Let $X_1 \times X_2 \times \dots \times X_H = X \subset \mathbb{R}^m$ be the Cartesian product of non-empty, compact, convex sets. Assume that $F_h : X \rightarrow X_h, h = 1, \dots, H$ are convex-valued correspondences with an open graph. Then there exists an $x \in X$ such that, for every h , we have either $x_h \in F_h(x)$ or $F_h(x) = \emptyset$.

To apply the theorem, we identify each C^i with X_i and Q with $X_H, H = n + 1$. It is easy to check that all X_i are non-empty, compact and convex. Next, for any $i = 1, \dots, n$ define

$$F_i(x) = \begin{cases} \{\bar{p}^i \in C^i | q \cdot \bar{p}^i < 1/n\}, & \text{if } q \cdot p^i > 1/n. \\ \{\bar{p}^i \in C^i | q \cdot \bar{p}^i < 1/n \text{ and } u^i \cdot \bar{p}^i > u^i \cdot p^i\}, & \text{if } q \cdot p^i \leq 1/n. \end{cases}$$

where $x = (x_1, x_2, \dots, x_H) = (p^1, p^2, \dots, p^n, q)$. Surely, F_i is convex valued. Moreover, by construction, $x_i \in F_i(x)$ is ruled out. To see that F_i has an open graph, consider $y_i \in F_i(x)$. If $x_H \cdot x_i > 1/n$ then $x_H \cdot y_i < 1/n$ and both inequalities continue to hold in a small neighbourhood of x and y_i . If $x_H \cdot x_i \geq 1/n$, then $x_H \cdot y_i < 1/n$ and $u^i \cdot y_i > u^i \cdot x_i$. Consider x' and y'_i close to x, y_i . The two strict inequalities still hold, $x'_H \cdot y'_i < 1/n$ and $u^i \cdot y'_i > u^i \cdot x'_i$, so if $x'_H \cdot x'_i \geq 1/n$, then $y'_i \in F_i(x'_i)$. If on the other hand $x'_H \cdot x'_i < 1/n$, the fact that $x'_H \cdot y'_i < 1/n$ is sufficient; in either case $y'_i \in F_i(x'_i)$. Next, consider a market agent setting prices q according to

$$F_H(x) = \left\{ \bar{q} \in Q | \bar{q} \cdot \left(\sum_{i \in \mathcal{I}} p^i - (1) \right) > q \cdot \left(\sum_{i \in \mathcal{I}} p^i - (1) \right) \right\}$$

where (1) denotes the total supply. The economic logic is familiar; the price of commodities in excess supply is raised, while in the face of excess demand, the price is reduced. It is easy to check that F_H is convex-valued and open. Moreover, as for F_i , we find that $x_H \in F_H(x)$ is ruled out. Hence, from our fixed-point theorem, we know that there exists x such that $F_h(x) = \emptyset$, for all $h = 1, 2, \dots, H$.

To see that $x = (p^1, p^2, \dots, p^n, q)$ satisfies our preference maximisation condition, observe first, that $q \cdot p^i \leq 1/n$ as otherwise $F_i(x) = \{\bar{p}^i \in C^i | q \cdot \bar{p}^i < 1/n\} \neq \emptyset$. Second, assume there exists

another $\bar{p}^i \in C^i$ that is affordable ($q \cdot \bar{p} \leq 1/n$) and preferred over p^i ($u^i \cdot \bar{p}^i > u^i \cdot p^i$). But then, $\hat{p}^i = (1 - \varepsilon)\bar{p}^i$ costs even less and is still preferred over p^i . Hence, $\hat{p}^i \in F_i(x) \neq \emptyset$.

Finally, consider feasibility. If there is positive excess supply for each good, i.e. $\forall o : \sum_{i \in \mathcal{I}} p_o^i < 1$, then every agent spend her complete income $1/n$ - otherwise, she must be satiated, i.e. able to afford a full share of her most preferred object at cost below $1/n$. But then, there is at least one good not in excess supply. Hence, we know that

$$\forall o : \sum_{i \in \mathcal{I}} p_o^i < 1 \Rightarrow q \cdot \left(\sum_{i \in \mathcal{I}} p^i - (1) \right) = 0.$$

Then, it would be possible to distribute the remaining probability shares among the agents without increasing the aggregate cost. But if there is no cost increase in the aggregate, there is at least one affected agent for whom the cost does not increase individually - yet her utility would increase which is in conflict with our previous conclusion on preference maximisation. Having ruled out positive excess supply in each good, consider the case $\sum_{i \in \mathcal{I}} p^i \neq (1)$ while there is one o , such that $\sum_{i \in \mathcal{I}} p_o^i \geq 1$. Then $\max_{q'} q' \cdot (\sum_{i \in \mathcal{I}} p^i - (1))$ has a positive value: (i) if there exists a good o' in positive excess supply, set $q'_{o'} = -1$, $q'_o = 2$ and all other prices to zero. Then $q' \cdot (\sum_{i \in \mathcal{I}} p^i - (1)) > 0$. If on the other hand (ii) there is no good in positive excess supply, then by $\sum_{i \in \mathcal{I}} p^i \neq (1)$ there has to be a good o' in positive excess demand. Set $q'(o') = 1$ and all other prices to zero, and $q' \cdot (\sum_{i \in \mathcal{I}} p^i - (1)) > 0$. Hence, as $q \in \operatorname{argmax}_{q'} q' \cdot (\sum_{i \in \mathcal{I}} p^i - (1))$, we find that $q \cdot (\sum_{i \in \mathcal{I}} p^i - (1)) > 0$. But this violates the first part of our preference maximisation condition: $q \cdot p^i \leq 1/n$. Thus, we conclude that $\sum_{i \in \mathcal{I}} p^i = (1)$. \square

Example 3. To see that the outcome of Hylland and Zeckhausers mechanism may not meet the sd-equal-division-lower-bound, suppose that agents utilities are given as $u_1(a) = 5$, $u_i(a) = 4$ for $i \in \{2, 3\}$. The remaining objects b and c yield the same utility for all agents; $u_i(b) = 2$ and $u_i(c) = 1$ for $i \in \{1, 2, 3\}$. Then the following lottery

	a	b	c
1:	$\frac{1}{2}$	0	$\frac{1}{2}$
2:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
3:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

is a walrasian equilibrium with equal incomes $1/3$ and prices $q_a = 2/3$, $q_b = 1/3$ and $q_c = 0$. Thus, it is (sd-)efficient, as well as group (sd-)envy-free. However, it doesn't meet the sd-equal-division-lower-bound, as $p_a^{2,3} < \frac{1}{3}$.

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