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**Conference Paper**

## Rational Expectations and the Stability of Balanced Monetary Development

Beiträge zur Jahrestagung des Vereins für Socialpolitik 2014: Evidenzbasierte Wirtschaftspolitik  
- Session: Monetary and Fiscal Policy: Financial Shocks, Speculative Attacks, and (In)Stability,  
No. F09-V3

**Provided in Cooperation with:**

Verein für Socialpolitik / German Economic Association

*Suggested Citation:* Böhm, Volker (2014) : Rational Expectations and the Stability of Balanced Monetary Development, Beiträge zur Jahrestagung des Vereins für Socialpolitik 2014: Evidenzbasierte Wirtschaftspolitik - Session: Monetary and Fiscal Policy: Financial Shocks, Speculative Attacks, and (In)Stability, No. F09-V3, ZBW - Deutsche Zentralbibliothek für Wirtschaftswissenschaften, Leibniz-Informationszentrum Wirtschaft, Kiel und Hamburg

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# Rational Expectations and the Stability of Balanced Monetary Development\*

by

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## Abstract

The expanding/contracting behavior of monetary macroeconomic models is largely driven by government deficits. Their monetary effects on inflation and monetary growth determine the real value of money (or of government debt) in the long run. Only positive stationary (constant) real values of money guarantees stationary positive levels of output and employment in the long run.

Within a generalized class of *nonlinear monetary macroeconomic models* of the AS–AD type derived from a microeconomic structure with OLG consumers, such economies generically have no stationary equilibria under perfect foresight/rational expectations when tax revenue is income dependent and endogenous (no lump sum taxes) and when the government follows a stationary spending rule. They usually have two balanced stochastic equilibria, an unstable one with positive levels of employment, output, and positive real value of money plus a stable nonmonetary one under hyperinflation (or a monetary bubble). Under the hypothesis of the model, only the stable ones are empirically observable.

The paper shows that these properties are true for a large class of AS-AD models including those with a random budget policy rule whose deficit is zero on average. In contrast, such economies have positive stable balanced stationary equilibria if the government policy has a small strictly positive *nonrandom* demand component in all cases of uncertainty. Among other things, this confirms the long run effectiveness of deficit spending in random economies under rational expectations known from Keynesian theories.

The results are derived using techniques from the theory of random dynamical systems which allows a complete theoretical and numerical analysis of the *dynamics of random time series and their stability* of the *nonlinear stochastic model*.

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\*Preliminary and Incomplete. Please, do not quote

# 1 Introduction

Permanent government deficits in closed monetary macroeconomic models induce expanding quantities of money or government debt. While the necessary temporary equilibrium properties for stability and sustainability of such expanding orbits seem to well e understood, sufficient conditions are often not considered or examined. By sustainability one usually means that the nominally expanding orbits of prices, money, and/or debt (going to infinity) support positive stationary real allocations associated with monetary equilibria. Specifically

## – In monetary models

- stationary policies induce **endogenous government deficits**
- which vary **in size and sign** at each temporary equilibrium

⇒ making monetary expansion/contraction endogenous

⇒ inducing explicit endogenous dynamics of the quantity of money.

## – What are the consequences of such **permanent deficits** in the long run?

- What are the possibilities for **stable** monetary expansion?
- Under rational expectations and perfect competition?
- Which stationary fiscal/monetary policies are **sustainable**?
- Are there *thresh holds* of policy parameters<sup>1</sup> destroying sustainability.

To investigate these question in a properly defined macroeconomic model one needs to have a rich enough model for which the sequential development of

- general temporary equilibrium of closed monetary systems
- with essential heterogeneity of agents
- with fiscal/monetary policy

can be described.

## History and Literature:

- The monetarist tradition: Monetary Rules,  
“The optimal quantity of money” (for example Friedman, 1968, 1969)
- some classical papers in the linear tradition of rational expectations (using the notion from Muth, 1961);
- Lucas (1972) on neutrality of money, nonlinear but nothing on sustainability or stability;
- Blanchard (1979), Blanchard & Kahn (1980) and others raise some doubts about stability in linear environments;

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<sup>1</sup>as conjectured in the debate between Reinhard-Rogoff and Krugman

- few contributions discuss link between real sector and money;
- few contributions use a nonlinear time series approach;

## 2 The AS–AD Model

We take a version of the AS–AD Model based on micro foundations

- with heterogeneous OLG consumers, endogenous labor supply
- with government demand/consumption and a proportional income tax

**to find** a **general structural answer** to the stability question

- for **nonlinear** dynamic macroeconomic models
- in **deterministic and stochastic** environments

**to examine the conditions whether**

- **stable balanced orbits exist under rational expectations**
- **supporting positive stationary real allocations in the long-run.**

**It is shown that there exist** two fiscal policies

- **one is always unstable and one sometimes stable**
- **there exist critical thresh holds beyond which sustainability no longer holds**

**Aggregate Supply** is defined as an *equilibrium consistent* description of the labor market under technological conditions

$$\underbrace{n_f h\left(\frac{w}{p}\right)}_{\text{labor demand}} \stackrel{!}{=} \underbrace{N\left(\frac{w}{p} V\left(\frac{p^e}{p}\right)\right)}_{\text{labor supply}} \implies y^S = AS\left(\frac{p^e}{p}\right)$$

with  $n_f$  identical competitive firms with smooth concave production functions satisfying Inada conditions and globally invertible labor demand function  $h(w/p)$ .  $N$  is the aggregate labor supply function of a finite number worker-consumers with two period lives and time separable utility function (see Blanchard & Fischer, 1989; Böhm, 2012).  $AS$  is a strictly decreasing function of the expected rate of inflation.

**Aggregate Demand** is taken as the *income consistent* description of the equivalence of total income and aggregate commodity expenditure

$$Y = py = pg + M + c_w \text{Net Wages} + c_\pi \text{Net Profits}$$

$$\implies y^D = AD\left(\frac{M}{p}\right) = \frac{M/p + g}{1 - c_w(1 - \tau_w)\frac{\text{Wages}}{py} - c_\pi(1 - \tau_\pi)\frac{\text{Profits}}{py}}$$

$$\implies AD(M/p) \stackrel{!}{=} AS(p^e/p) \implies p = \mathcal{P}(M, p^e) \quad \text{Price Law}$$

As a consequence the condition of temporary equilibrium implies the equilibrium price as a mapping  $p = \mathcal{P}(M, p^e)$  called the price law. Under standard assumptions  $p = \mathcal{P}(M, p^e)$  is homogeneous of degree one and increasing in  $(M, p^e)$ .

### 3 Fiscal Deficit Rules and Monetary Expansion

Consider first the case where the government follows a policy imposing that in each period the deficit is a constant proportion  $\rho > -1$  of tax revenues. For simplicity it is assumed that consumers are identical in their net consumption propensities, i.e.  $c = c_w = c_\pi$  and  $\tau = \tau_w = \tau_\pi$ . In other words, the income distribution among the consumers has no influence on aggregate demand. Thus, if the deficit is a constant fraction  $\rho > -1$  of total tax revenue,

$$(3.1) \quad pg \stackrel{!}{=} (1 + \rho)\tau py, \quad \rho > -1$$

is a constant fraction  $(1 + \rho)\tau$  of output, i.e. there is a [constant fiscal share](#).

Nonnegativity of aggregate income imposes an upper bound

$$\rho < \rho_{\max} := \frac{1 - \tau}{\tau}(1 - c)$$

on the deficit rule  $\rho$ .

This implies a constant growth rate of money  $\rho_M$  equal to

Thus, the deficit rule and aggregate income consistency yield an aggregate demand function

$$(3.2) \quad y = AD(M/p) := \frac{M/p}{(1 - c)(1 - \tau) - \tau\rho}$$

which is unit elastic in the price level. Together with Aggregate Supply derived from competitive labor market clearing

$$y^s = AS(p^e/p),$$

a temporary equilibrium induces a price map  $p = \mathcal{P}(M, p^e)$  solving

$$AS\left(\frac{p^e}{p}\right) = \frac{1}{(1 - c)(1 - \tau) - \tau\rho} \frac{M}{p}.$$

If  $AS$  is globally invertible, the unique temporary equilibrium price level  $p = \mathcal{P}(M, p^e)$  is unique and globally defined the price law [P is strictly increasing, homogeneous of degree one, and concave in  \$\(M, p^e\)\$](#) .

Perfect foresight price dynamics are defined by the inverse of the price law

$$(3.3) \quad p_1^e = \mathcal{P}^e(M, p) := pAS^{-1} \left( \frac{1}{(1-c)(1-\tau) - \tau\rho} \frac{M}{p} \right).$$

One finds that  $\mathcal{P}^e(M, p)$  is **strictly increasing, homogeneous of degree one in  $(M, p)$ , satisfying the convex Inada conditions in  $p$** , i.e.  $\lim_{p^e \rightarrow 0} \mathcal{P}^e(M, p)/p^e = 0$  and  $\lim_{p^e \rightarrow \infty} \mathcal{P}^e(M, p)/p^e = \infty$ .

The dynamics of money balances are given by

$$(3.4) \quad M_1 = M + pg - \tau py = M + \rho \tau py = M \left( \frac{\tau \rho}{(1-c)(1-\tau) - \tau \rho} + 1 \right)$$

which implies a **constant growth rate of money** equal to

$$\rho_M := \frac{M_1}{M} - 1 = \frac{\tau \rho}{(1-c)(1-\tau) - \tau \rho}$$

which is **independent of prices, incomes, and output**. Thus, (3.3) and (3.4) induce a two-dimensional dynamical system under perfect foresight

$$(3.5) \quad \begin{aligned} M_1 &= M + pg - \tau py = \frac{(1-c)(1-\tau)}{(1-c)(1-\tau) - \tau \rho} M \\ p_1 &= \mathcal{P}^e(M, p) := pAS^{-1} \left( \frac{1}{(1-c)(1-\tau) - \tau \rho} \frac{M}{p} \right), \end{aligned}$$

This system is homogeneous of degree one which has no positive fixed points for  $\rho_M \neq 0$ .

$\implies$  **balanced orbits**/paths of monetary growth and of inflation

**Definition 3.1** An orbit  $\{(p_t, M_t)\}_{t=0}^{\infty}$  of (3.5) is called **balanced** if and only if

$$M_{t+1} = (1 + \rho_M)M_t \quad \text{and} \quad p_{t+1} = (1 + \rho_M)p_t.$$

Balanced orbits induce constant allocations. To discuss their existence and stability, one considers the dynamics in intensive form: define  $\tilde{p} := (p/M)$  (the inverse of real money balances!) and consider the one-dimensional dynamical system in  $\tilde{p}$  defined by the time-one map

$$(3.6) \quad \tilde{p}_1 := \frac{p_1}{M_1} = \frac{1}{1 + \rho_M} \mathcal{P}^e(1, p/M) = \frac{1}{1 + \rho_M} \mathcal{P}^e(1, \tilde{p}).$$

This reduced time-one map inherits all the properties of  $\mathcal{P}^e$ , namely strict monotonicity plus the so called convex weak Inada conditions<sup>2</sup> Therefore, (3.6) is a strictly convex map (convex Inada)

<sup>2</sup>A mapping  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to satisfy the convex weak Inada conditions if  $g(x) := f(x)/x$  is strictly increasing with  $\lim_{x \rightarrow 0} g(x) = 0$  and  $\lim_{x \rightarrow +\infty} g(x) = +\infty$ . This is the convex analogue to the concave weak Inada conditions which is often useful in growth theory.

which has exactly one positive fixed point  $\bar{p} > 0$  further fixed points  $p = 0$  and  $p \rightarrow \infty$ . Since aggregate demand is unit elastic in prices, the inverse of the price law  $\mathcal{P}^e$  satisfies

$$\mathcal{P}^e(0, M) = 0 \quad \forall M > 0.$$

Thus, there exists a unique positive fixed point  $\bar{p}$  which defines a unique positive balanced level  $\bar{m} \equiv 1/\bar{p}$  of money balances. Zero is a globally stable fixed point while  $\bar{p}$  is unstable due to convexity. Positive fixed points of (3.6) define balanced paths of (3.5), therefore no orbit  $\{(p_t, M_t)\}$  of (3.5) with  $p_0/M_0 \neq \bar{p}$  converges to the positive balanced path  $\bar{p} > 0$ , and orbits of random extensions of the model do not converge to random balanced orbits either.

**In summary,**

only **three types of long run behavior** under a fixed deficit rule  $\rho > 0$  are possible:

(1) If  $p_0/M_0 = \bar{p}$ , **balanced monetary growth is possible.**

- The real allocation in the economy remains constant over time
- while prices and the quantity of money grow at the same rate  $\rho_M$ .

**However, this is a zero probability event!**

(2) If  $p_0/M_0 > \bar{p}$ , then  $\lim_{t \rightarrow \infty} \frac{M_t}{p_t} = 0$ .

- **all trades, output, and employment tend to zero.**
- in the limit money has no value.

(3) If  $p_0/M_0 < \bar{p}$ , money balances grow at a faster rate than prices

- **all orbits are not balanced, output and employment are not stationary.**

## 4 Stochastic Production and Rational Expectations Dynamics

Consider next the same economy but with a multiplicative Hicks neutral productivity shock  $Z$  of the form

$$y = G(Z, L) := ZF(L), \quad Z \geq 1.$$

The associated random aggregate supply function<sup>3</sup> is of the special form

$$y^S = ZAS \left( \frac{p^e}{pZ} \right).$$

Assume that government policy consists of a choice of the two parameters  $(g, \tau)$  without a budget requirement. This implies an aggregate demand function

$$(4.1) \quad y^D = AD \left( \frac{M}{p}, g \right) := \frac{M/p + g}{1 - c(1 - \tau)}.$$

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<sup>3</sup>where  $F$  satisfies the Inada conditions; see Böhm (2010) for details

Temporary equilibrium is defined by

$$\frac{M/p + g}{1 - c(1 - \tau)} \stackrel{!}{=} ZAS\left(\frac{p^e}{pZ}\right)$$

inducing the [random price law](#)

$$p = \mathcal{P}(M, p^e, Z)$$

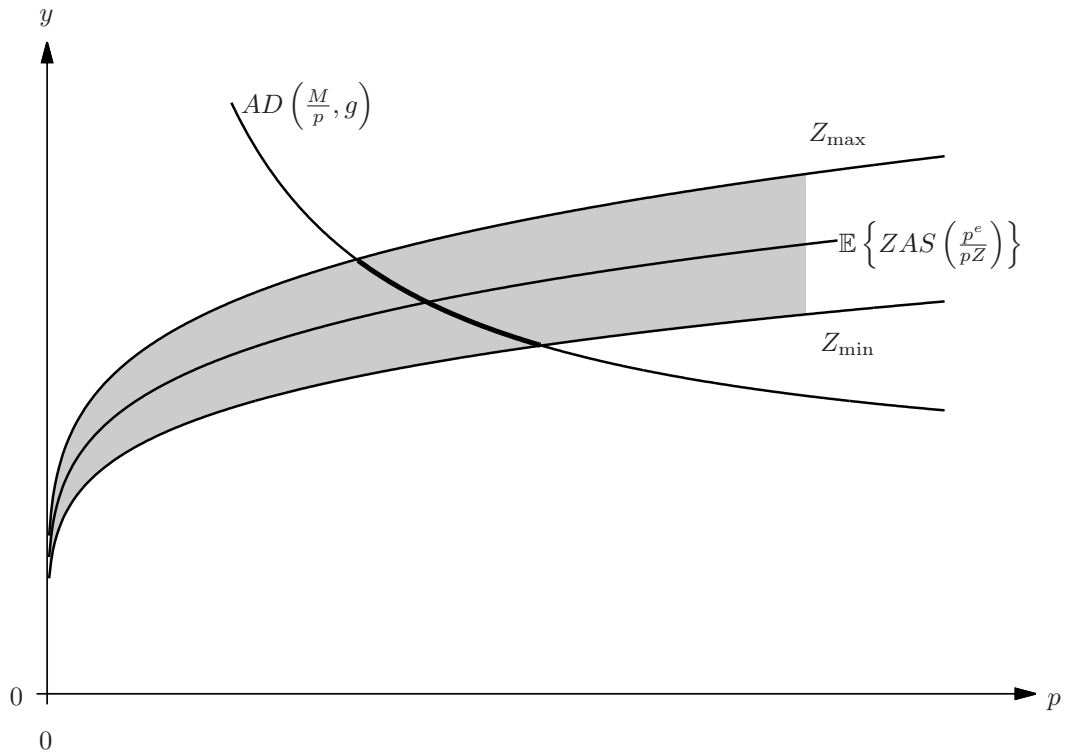


Figure 4.1: [The Role of Random Productivity on Prices and Output](#)

## 4.1 Rational Expectations

- Let  $p_t^e \equiv p_{t,t+1}^e$  denote the [conditional mean forecast in  \$t\$](#)  for prices in  $t + 1$
- the production shocks are distributed on  $\Sigma := [Z_{\min}, Z_{\max}] \subset \mathbb{R}_{++}$
- by a measure  $\mu_t \in \text{Prob}(\Sigma)$ , the set of probability measures on  $\Sigma$
- a [mean-value prediction](#) is called [unbiased](#)<sup>4</sup>
- i.e. “rational expectations prevail”, if the [mean of the prediction error](#)

$$\text{err}(M_t, p_t^e, Z_t, p_{t-1}^e) := p_t - p_{t-1}^e = \mathcal{P}(M_t, p_t^e, Z_t) - p_{t-1}^e$$

<sup>4</sup>proceed as in Böhm & Wenzelburger (2002)



– is equal to zero taken with respect to the true measure  $\mu_t$ , i.e.

$$(\mathbb{E}\mathcal{P})(M_t, p_t^e, \mu_t) := \mathbb{E}_{\mu_t} \{ \mathcal{P}(M_t, p_t^e, Z_t) \} := \int \mathcal{P}(M_t, p_t^e, Z) \mu_t(dZ) \stackrel{!}{=} p_{t-1}^e.$$

**Definition 4.1** A *mean value predictor*  $\psi^* : \mathbb{R}_+^2 \times \text{Prob}(\Sigma) \rightarrow \mathbb{R}$  is called *unbiased* if it solves

$$\mathbb{E}_{\mu_t} \mathcal{P}(M_t, \psi^*(M, p^e, \mu_t), Z_t) = p_{t-1}^e$$

for all  $(M, p^e, \mu_t) \in \mathbb{R}^2 \times \text{Prob}(\Sigma)$ , where  $\text{Prob}(\Sigma)$  is the set of probability measures on  $\Sigma$ .

**Lemma 4.1** There exists a unique globally defined unbiased predictor  $\psi^* : \mathbb{R}_+^2 \times \text{Prob}(\Sigma) \rightarrow \mathbb{R}_+$

$$\psi^*(M, p_{-1}^e, \mu) := (\mathbb{E}\mathcal{P})^{-1}(M, p_{-1}^e, \mu)$$

which is homogeneous of degree one in  $(M, p_{-1}^e)$ , and which satisfies

$$\begin{aligned} \lim_{p_{-1}^e \rightarrow 0} \frac{\psi^*(M, p_{-1}^e, \mu)}{p_{-1}^e} &= 0, \\ \lim_{p_{-1}^e \rightarrow \infty} \frac{\psi^*(M, p_{-1}^e, \mu)}{p_{-1}^e} &\leq Z_{\max} A S^{-1} \left( \frac{g/Z_{\max}}{1 - c(1 - \tau)} \right). \end{aligned}$$

Figure 4.2 displays the graph of the mean price law  $\mathbb{E}\mathcal{P}$  for fixed  $M$  and the basic geometric features of the range of the random price law for the isoelastic case with production shocks distributed on a compact interval  $\Sigma = [Z_{\min}, Z_{\max}]$ . The mean price law is a concave strictly increasing function with a global inverse which is the unbiased predictor  $\psi^*$  whose existence is guaranteed under the assumptions of Proposition ???. When the production shocks are identically and independently distributed across time, i.e. when the measure  $\mu$  of the production noise is the same in every period, then the unbiased predictor is the same *deterministic* function of money balances and expectations alone in each period.

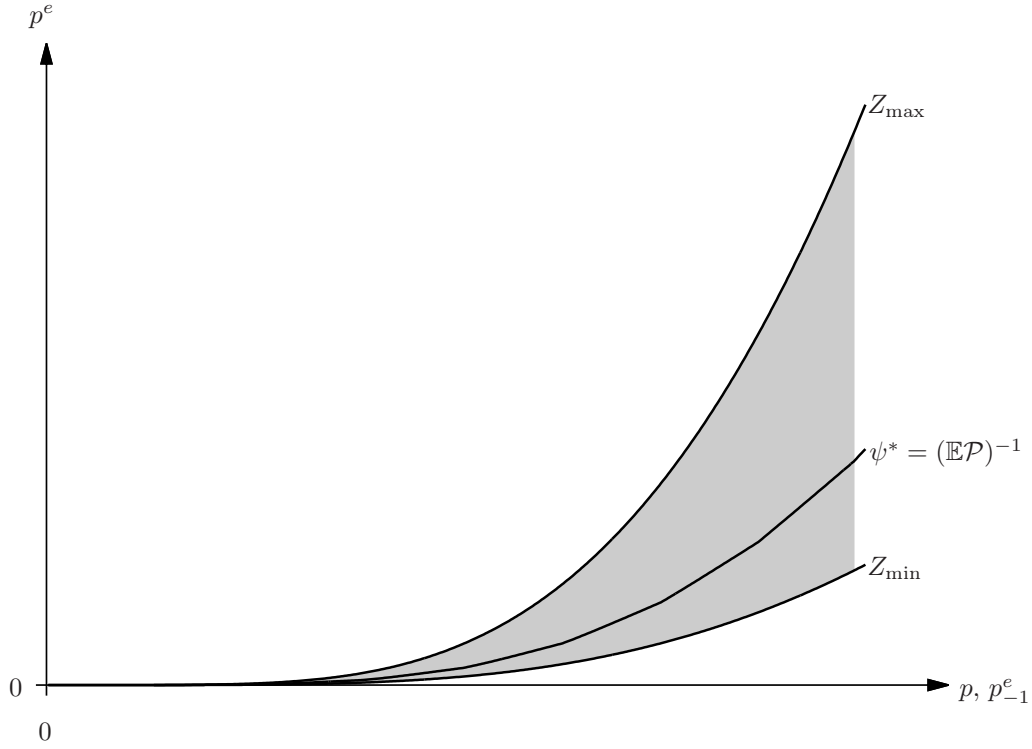


Figure 4.2: Existence of an unbiased predictor  $\psi^*(M, p_{-1}^e, \mu) := (\mathbb{E}\mathcal{P})^{-1}(M, p_{-1}^e, \mu)$

The existence of the unbiased predictor  $\psi^*$  now implies a two dimensional stochastic difference equation which describe the dynamics under rational expectations. In other words,

$\implies$  **Rational expectations dynamics** are given by<sup>5</sup>

$\implies$  a **homogeneous two dimensional random dynamical system** in  $(p_{t-1}^e, M_t)$

$$(4.2) \quad \begin{pmatrix} p_{t,t+1}^e \\ M_{t+1} \end{pmatrix} = \begin{pmatrix} M_t \psi^*(1, p_{t-1,t}^e/M_t) \\ M_t \frac{\tilde{c} - \tau}{\tilde{c}} (1 + g\mathcal{P}(1, \psi^*(1, p_{t-1,t}^e/M_t), Z_t)) \end{pmatrix}$$

$\implies$  for fixed  $Z$ , hyperbolic fixed points do not exist generically.

$\implies$  stationary solutions typically also do not exist for stationarity production shocks  $\{Z_t\}$ .<sup>6</sup>

**Therefore** one considers the dynamics of expected real money balances in intensive form. As in the deterministic case, the dynamics of the economy “in real terms” (or in intensive form) is well defined. Obviously, the ratio of the two maps of the system (4.2) defines the time-one shift  $q_t^e := p_{t-1,t}^e/M_t \mapsto q_{t+1}^e := p_{t,t+1}^e/M_{t+1}$  through the first order stochastic difference

<sup>5</sup>Assume i. i. d. productivity shocks with constant measure  $\mu$ .

<sup>6</sup>The failure is a consequence of the fact that the system (4.2) generically fails to have deterministic fixed points for each level  $Z$ ; see Böhm (2010), Böhm & Claas (2012).

equation given by

$$(4.3) \quad q_{t+1}^e = S(Z_t, q_t^e) := \frac{\tilde{c}}{\tilde{c} - \tau} \frac{\psi^*(1, q_t^e)}{1 + g\mathcal{P}(1, \psi^*(1, q_t^e), Z_t)},$$

– where  $q_t^e := p_{t-1,t}^e/M_t$ .

**Lemma 4.2**

Assume  $g > 0$  and  $Z \in \Sigma = [Z_{\min}, Z_{\max}]$ . Then, the mapping  $S$  is monotonically increasing in  $q^e$  and satisfies

$$\lim_{q^e \rightarrow 0} \frac{S(Z, q^e)}{q^e} = \lim_{q^e \rightarrow \infty} \frac{S(Z, q^e)}{q^e} = 0$$

$$\frac{\partial}{\partial g} S(Z, q^e) < 0 \quad \text{and} \quad \frac{\partial}{\partial Z} S(Z, q^e) > 0.$$

There exist positive levels of government demand  $g^{**} > g^* > 0$  such that

$S(Z, \cdot)$  has no positive fixed point for  $g > g^{**}$ ,

$S(Z, \cdot)$  has at least two positive fixed points for  $g < g^*$ .

Figures 4.3 and 4.4 portray the role of government demand  $g \geq 0$  and  $Z \in \Sigma = [Z_{\min}, Z_{\max}]$  as described in the lemma.

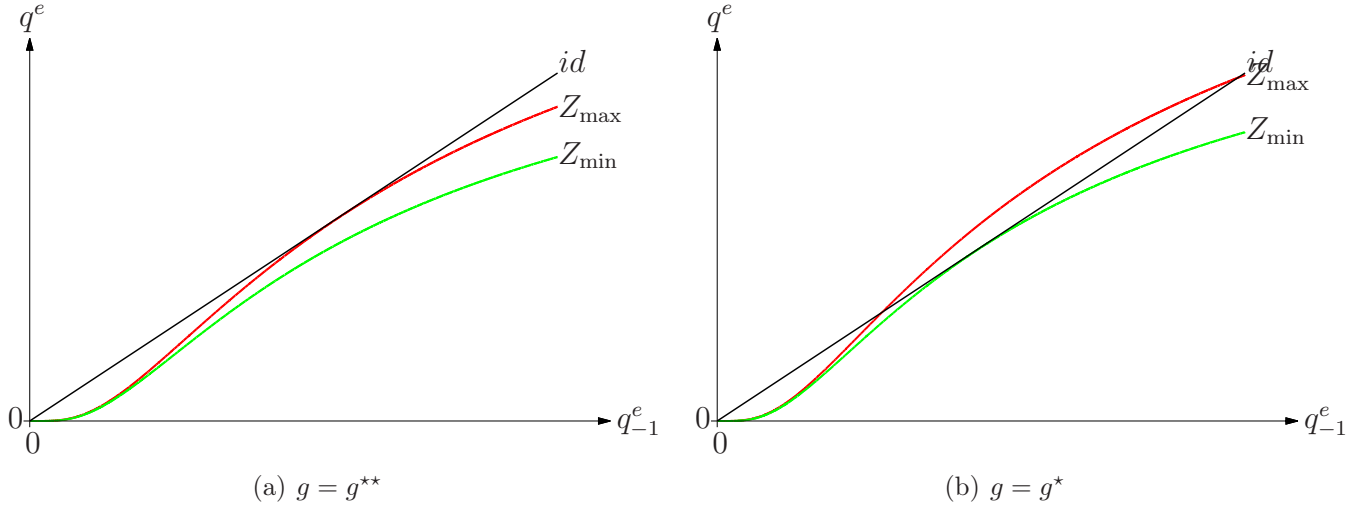
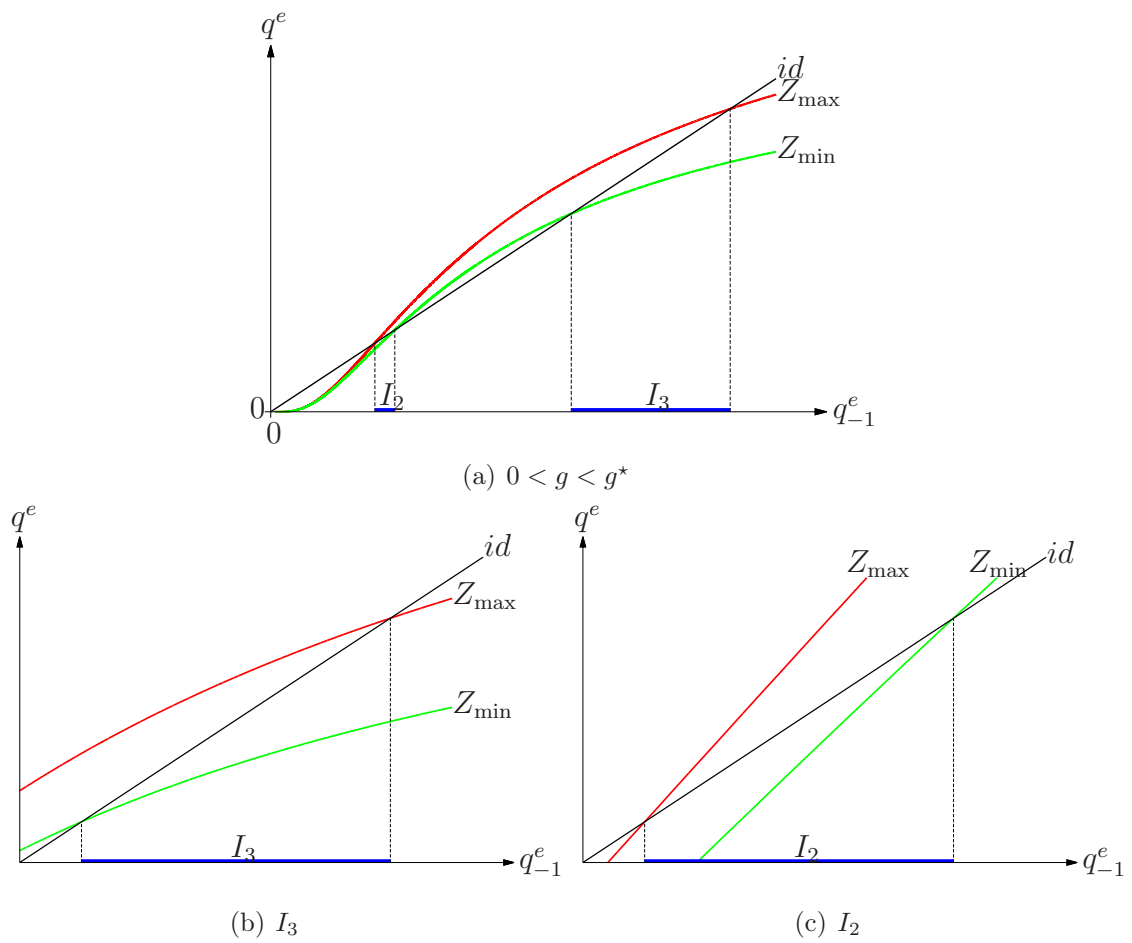


Figure 4.3: The role of  $g$

Figure 4.4: Multiple fixed points and stationary intervals for  $0 < g < g^*$

## Random Dynamics

Consider a stationary Markov process on  $\Sigma$  in its [canonical representation](#) with associated probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and

- let  $\omega := (\dots, Z_{-1}, Z_0, Z_1, \dots) \in \Omega$  denote a realization (a sample path).
- Define  $\vartheta : \Omega \rightarrow \Omega$  to be the so-called left shift on  $\Omega$ , i.e.  $(\vartheta\omega)(s) := \omega(s+1)$  for all  $s \in \mathbb{Z}$  and denote by  $\vartheta^t$  the  $t$ -th iterate of  $\vartheta$ .

**Then:**  $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta^t\})$  is an ergodic dynamical system.

- Define

$$(4.4) \quad S(\vartheta^t \omega) q_t^e := S(Z_t, q_t^e) = q_{t+1}^e.$$

Repeated applications of  $S$  under the perturbation  $\omega$  induce the measurable mapping  $\phi : \mathbb{N} \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$(4.5) \quad \phi(t, \omega, q_0^e) := \begin{cases} S(\vartheta^{t-1} \omega) \circ \dots \circ S(\omega) q_0^e & \text{if } t > 0 \\ q_0^e & \text{if } t = 0 \end{cases}$$

such that  $q_t^e = \phi(t, \omega, q_0^e)$  is the state of the system at time  $t$ .

- The map (4.5) (or equivalently (4.4)) defines a [random dynamical system](#) in the sense of Arnold(1998, Chapters 1 & 2).
- For any initial value  $q_0^e$  and perturbation  $\omega \in \Omega$ , the sequence  $\gamma(q_0^e) := \{\phi(t, \omega, q_0^e)\}_{t=0}^\infty$ ,  $t \in \mathbb{N}$ , defines a stochastic orbit of (4.4).

### Definition 4.2

A **random fixed point** of  $S(\omega) : \mathbb{R} \rightarrow \mathbb{R}$  is a random variable  $q^* : \Omega \rightarrow \mathbb{R}$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  such that

$$q^*(\vartheta\omega) = S(\omega)q^*(\omega) = \varphi(1, \omega, q^*(\omega)) \quad \text{for all } \omega \in \Omega',$$

where  $\Omega' \subset \Omega$  is a  $\vartheta$ -invariant set of full measure,  $\mathbb{P}(\Omega') = 1$ .

It is called *asymptotically stable* if there exists a random neighborhood  $U(\omega) \subset \mathbb{R}$  such that  $\mathbb{P}$ -almost surely

$$\lim_{t \rightarrow \infty} \|\varphi(t, \omega, q_0) - q^*(\vartheta^t \omega)\| = 0 \quad \text{for all } q_0(\omega) \in U(\omega).$$

Finally, it is now possible to formulate the main result describing the role of fiscal policy for the existence of random balanced expansion.

**Theorem 4.1** *Define for any given  $g \geq 0$  and  $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta^t\})$  the associated random dynamical system<sup>7</sup> as*

$$q^e(t, \omega, q_0^e) := S(\vartheta^{t-1}\omega) \circ \dots \circ S(\vartheta\omega) \circ S(\omega)q_0^e.$$

(1)  $q^e(\omega) \equiv 0$  is an asymptotically stable random fixed point.

(2) There exists  $g^{**} > 0$  such that for  $g > g^{**}$ :

- $q^e = 0$  is the unique asymptotically stable random fixed point
- whose basin of attraction contains a non-degenerate compact interval  $[0, \bar{q}] \subset \mathbb{R}$  such that for almost all  $(q_0^e, \omega) \in [0, \bar{q}] \times [Z_{min}, Z_{max}]^{\mathbb{Z}}$

$$\lim_{t \rightarrow \infty} q_t^e = \lim_{t \rightarrow \infty} q^e(t, \omega, q_0^e) = 0$$

(3) There exists  $0 < g^* < g^{**}$  such that for  $0 < g < g^*$ :

- there exist two positive random fixed points

$$q_2^* : \Omega \rightarrow \mathbb{R} \quad \text{and} \quad q_3^* : \Omega \rightarrow \mathbb{R} \quad \text{satisfying} \quad q_2^*(\omega) < q_3^*(\omega)$$

- with invariant measures  $q_2^*\mathbb{P}$  and  $q_3^*\mathbb{P}$
- whose supports are nondegenerate disjoint intervals

$$I_2 := [q_2^e(Z_{min}), q_2^e(Z_{max})], \quad I_3 := [q_3^e(Z_{max}), q_3^e(Z_{min})].$$

(4)  $q_3^*$  is globally stable on  $I_3$  while  $q_2^*$  is unstable.

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<sup>7</sup>see Arnold (1998) and Appendix A for definitions

## 4.2 The Real Economy along Balanced Stochastic Orbits

– Let  $q_3^* \equiv q^* : \Omega \rightarrow I_3$  denote the stable random fixed point, i.e.

$$q^*(\vartheta\omega) = S(\omega)q^*(\omega) := \frac{\psi^*(1, q^*(\omega))}{1 + g\mathcal{P}(1, \psi^*(1, q^*(\omega)), Z(\omega))} \quad \mathbb{P}\text{-a.s.}$$

– with distribution  $q^*\mathbb{P}$  on the compact support  $I_3$ .

**stationary real money balances**  $m^* : \Omega \rightarrow \mathbb{R}_+$  are defined by

$$m^*(\omega) := \frac{1}{\mathcal{P}(1, \psi^*(1, q^*(\omega)), Z(\omega))}.$$

**stationary output**  $y^* : \Omega \rightarrow \mathbb{R}_+$

$$y^*(\omega) = AD(m^*(\omega), g) = \frac{1}{\bar{c}}(m^*(\omega) + g)$$

$\implies$  real balances and output are perfectly correlated  
with identical distributions differing only by a mean shift.

**stationary employment**  $L^* : \Omega \rightarrow \mathbb{R}_+$

$$L^*(\omega) = F^{-1} \left( \frac{AD(m^*(\omega), g)}{Z(\omega)} \right).$$

$\implies$  Government demand has a positive effect on stationary output and employment.

**Output–employment correlation**

$$y^*(\omega) = Z(\omega)F(L^*(\omega))$$

– makes random output a product of two stationary random variables.

$\implies$  comovement of output and employment.

– When  $Z$  is a discrete random variable with two values  $(Z_{min}, Z_{max})$ ,

$$\text{supp}(L^*\mathbb{P}) \subseteq F^{-1} \left( \frac{1}{Z_{min}} \text{supp } y^*\mathbb{P} \right) \cup F^{-1} \left( \frac{1}{Z_{max}} \text{supp } y^*\mathbb{P} \right).$$

**real wage and employment correlation**

– real wage and employment are governed by the marginal product rule,

$$(4.6) \quad \alpha^*(\omega) = Z(\omega)F'(L^*(\omega)),$$

– is the product of two random variables.

$\implies$  they must be negatively but not perfectly correlated.

**stationary rate of inflation:** from the definition

$$\theta_t := \frac{p_t}{p_{t-1}} = \frac{M_t}{M_{t-1}} \frac{\mathcal{P}(1, \psi^*(1, q_t^e), Z_t)}{\mathcal{P}(1, \psi^*(1, q_{t-1}^e), Z_{t-1})}$$

one obtains using (4.2)

$$\begin{aligned} \theta^*(\omega) &:= \frac{\psi^*(1, q^*(\vartheta^{-1}\omega))}{S(\vartheta^{-1}\omega)} \frac{\mathcal{P}(1, \psi^*(1, q^*(\omega), Z(\omega)))}{\mathcal{P}(1, \psi^*(1, q^*(\vartheta^{-1}\omega), Z(\vartheta^{-1}\omega)))} \\ &= \hat{m}^*(\omega) \frac{m^*(\vartheta^{-1}\omega)}{m^*(\omega)} \end{aligned}$$

- ⇒ the stationary rate of inflation equals  
the stationary rate of monetary growth times  
the ratio of stationary real money balances at two successive dates,  
– i.e. it is the product of three stationary random variables  
⇒ inflation rates must show significant serial correlation,

### 4.3 Numerical Results: Stability and Stationarity

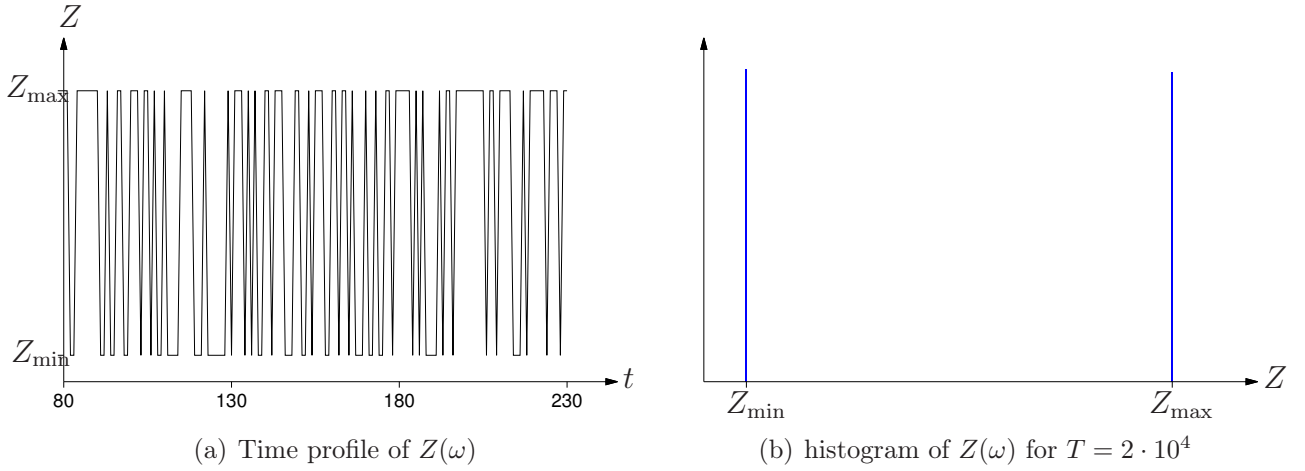
The following diagrams portray the characteristics of convergence to and of the stationary solutions of the economy under a discrete two point production shock for the values of the parameters given in the table when the production function and the labor supply functions are isoelastic. In the implicit integral equation defining the unbiased predictor can be solved numerically which allows to calculate the associated stochastic orbits for a given sample path of the noise. Due to the ergodicity any such orbit will induce the same long run distributions. The parameter  $0 < B < 1$  denotes the elasticity of the production function  $F$  while  $1 + C$  is the elasticity of the labor supply function  $N^8$ .

$Z_{min}$	$Z_{max}$	$B$	$C$	$c$	$\tau$	$g$	$g^*$	$g^{**}$
1.0	1.01	0.6	0.6	0.5	0.7	0.8240	0.8285	0.8328

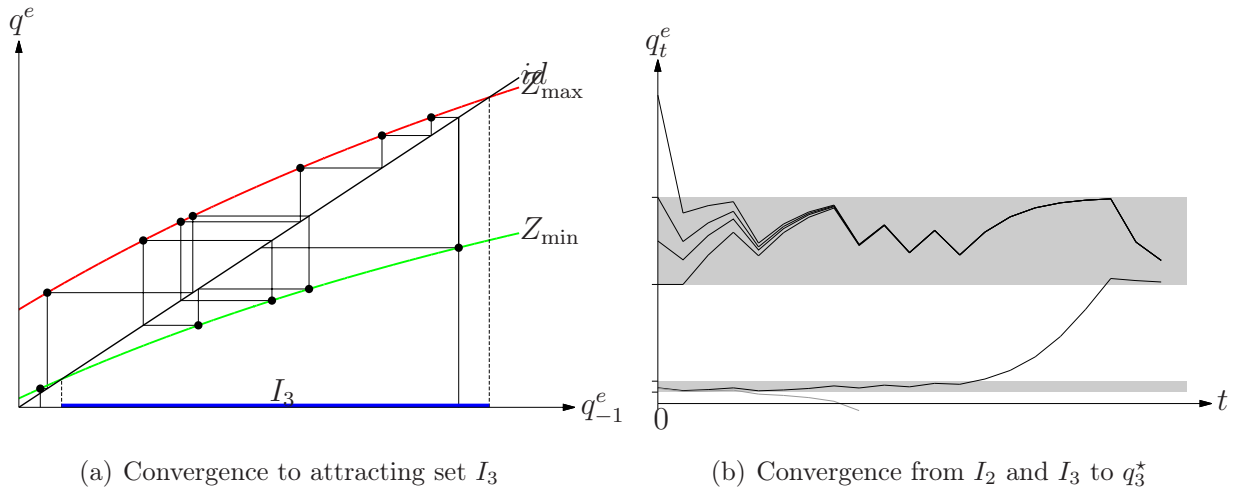
Table 1: Standard parametrization a

<sup>8</sup>I am deeply indebted to Oliver Claas for providing the numerical results.



Figure 4.5:  $Z \sim \mathcal{U}\{Z_{\min}, Z_{\max}\}$  with equal probability

The convergence of orbits of the random dynamical system (4.3) for the stochastic difference equation  $S$  to  $I_3$  and to the stationary solution are shown in Figure 4.6, which displays in panel (a) the associated graph of the two time one maps and an orbit starting outside of  $I_3$ . Panel (b) shows the time profile of six different orbits, five of which converge to  $I_3$  represented by the large shaded interval. Two initial conditions start in  $I_2$  (the small shaded interval) both of which leave this interval in finite time, one diverging the other one converging to  $I_3$ . Figure 4.7 displays the

Figure 4.6: Convergence to  $q^*$ ;  $B = C = .6$ ,  $c = 0.5$ ;  $Z_{\min} = 1.0$ ,  $Z_{\max} = 1.05$ 

main properties of the random fixed point  $q^* : \Omega \rightarrow I_3$  for the stationary solution associated with  $Z(\omega)$ . Subfigure (a) shows a typical phase plot (an attractor in the space  $(q_t^e, q_{t+1}^e) \in I_3 \times I_3$ ) when there are discrete production shocks. All orbit pairs lie on the graphs of two nonlinear time one maps associated with the two values of  $Z$  which are disjoint sets. This reveals in particular that the attractor (or support<sup>9</sup> of the joint distribution) is not a rectangle in  $\mathbb{R}^2$ . This also shows that the dynamics of real expected money balances cannot be approximated well by a one dimensional

<sup>9</sup>i.e.  $\text{supp } \nu \in \mathcal{F}$  of a measure  $\nu$  is the smallest closed subset with full measure, which satisfies  $\nu(\text{supp } \nu) = 1$

AR1 system in  $q^e$ . Nevertheless, because of stationarity, the two piece attractor implies a well defined and structurally simple autocorrelation such that the two marginal distributions (of the

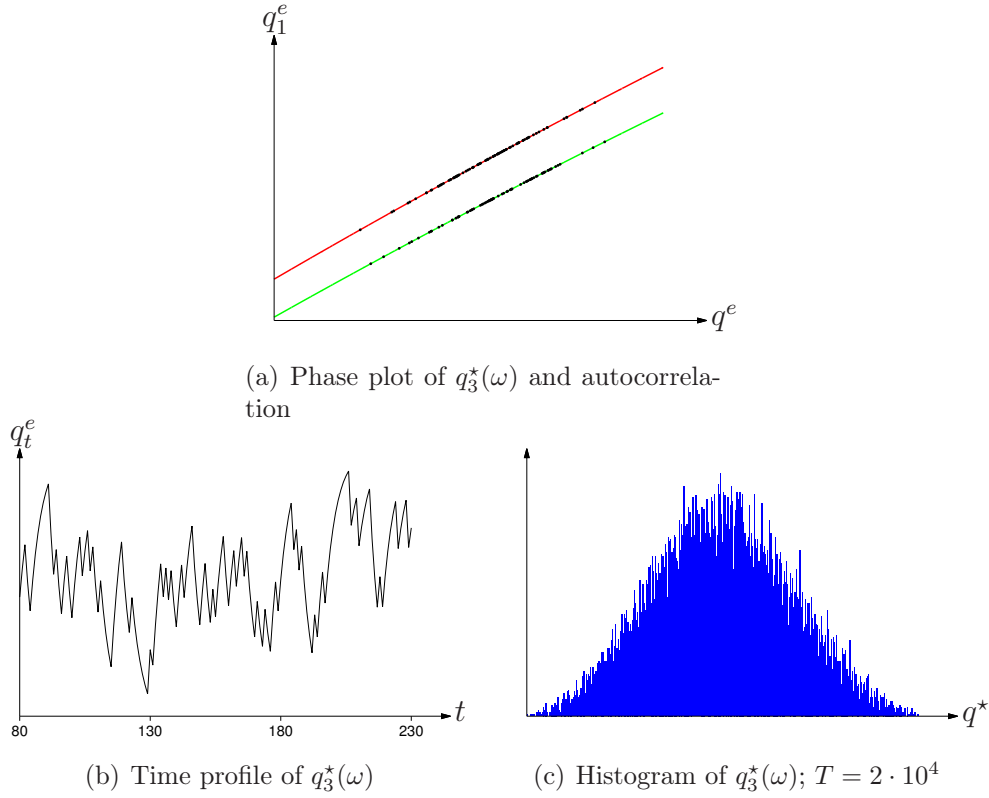


Figure 4.7: Stationary solution  $q^*$ ;  $Z_{min} = 1.0$ ,  $Z_{max} = 1.01$

projections onto the two axis) must be identical, as shown in subfigure (c). The raggedness of the histogram is a typical feature for an IFS, which often does not decrease or become more smooth as the number of iterations becomes large.

Note, however, that in many situations, both fixed points  $q_i^*$ ,  $i = 2, 3$  induce nondegenerate invariant measures  $q_i^* \mathbb{P}$ ,  $i = 2, 3$ , whose supports are the full respective intervals  $I_i$ , even though the production shocks are concentrated on discrete points and the state space representation indicates attractors as subsets of two disjoint graphs. This nondegeneracy is essentially a consequence of the continuity of the time one map  $S$ , preventing discrete stationary solutions to occur under discrete shocks, a feature which occurs in particular for the parameters in table ???. However, for some parameters of the economy determining the slope of the mapping  $S$  and the size of the production shock corresponding to each other in a particular way, the invariant measure may have a very complicated structure with no density and the support may be a Cantor set with Lebesgue measure zero (see Barnsley, 1988, for examples).

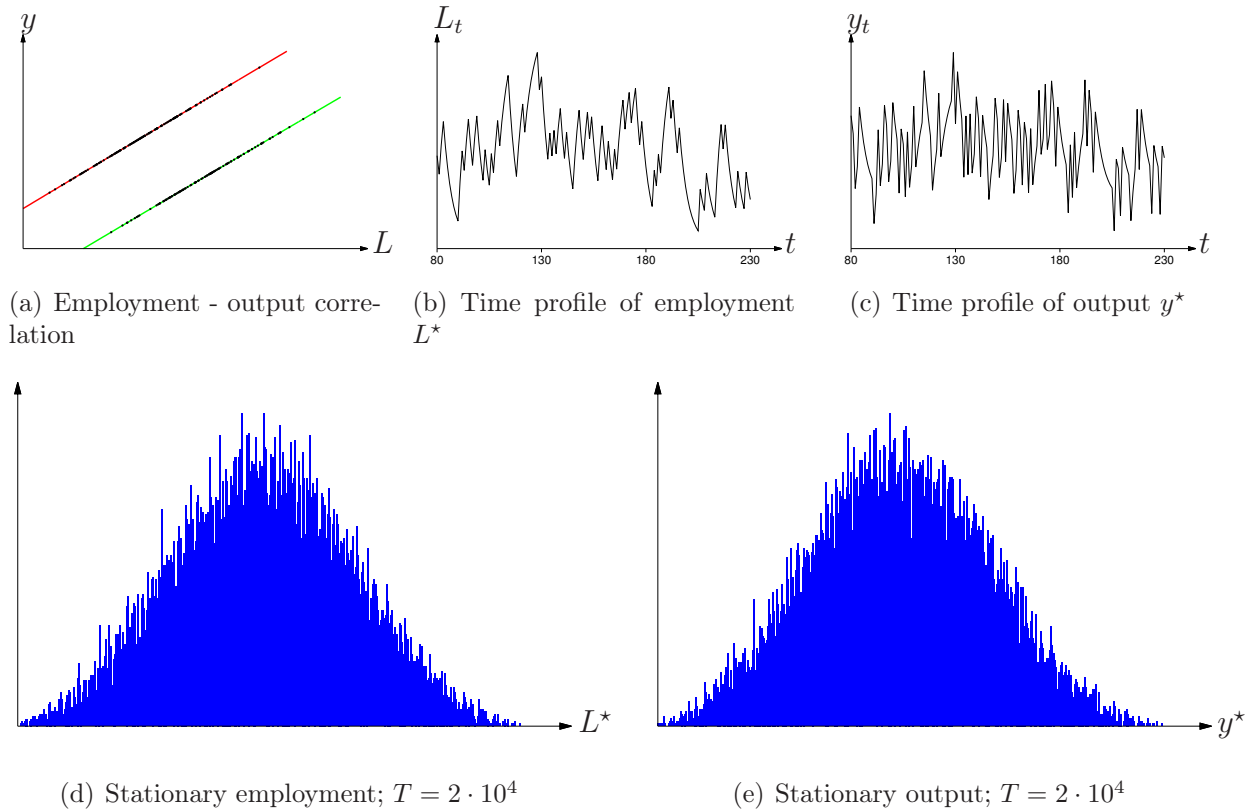


Figure 4.8: Stationary Output and Employment

Figure 4.8 displays the features of the joint distribution of employment and output (subfigure (a)) together with the two time profiles and their histograms which display the typical features of a discrete noise process. The two point distribution of the production shock implies that the joint support of the empirical distribution must be a subset of the two graphs of the production function. At the same time, subfigure (a) displays the typical comovement of employment and output as one would expect.

#### 4.3.1 Employment and the real wage

In a similar fashion one obtains the corresponding diagrams of the statistics between employment and the real wage which shows the distinctive properties of an IFS. The two point distribution of the production shock combined with the marginal product rule of profit maximization implies that the joint support in  $\mathbb{R}_+^2$  must be a subset of the two associated graphs of the marginal product curves (subfigure (a)). In other words, Hicks neutral production shocks induce a negative correlation and not a (positive) empirical comovement of employment and the real wage. Most interestingly, however, one finds that the distribution of the real wage is bimodal and not necessarily symmetric, and that its support consists of two (almost) disjoint intervals. The gap in the support arises jointly because of the size of the production shocks and the slope of the marginal product curve. In other words, the long run dynamics of the real wage fluctuates between the two intervals in a stochastic (nonperiodic) way.

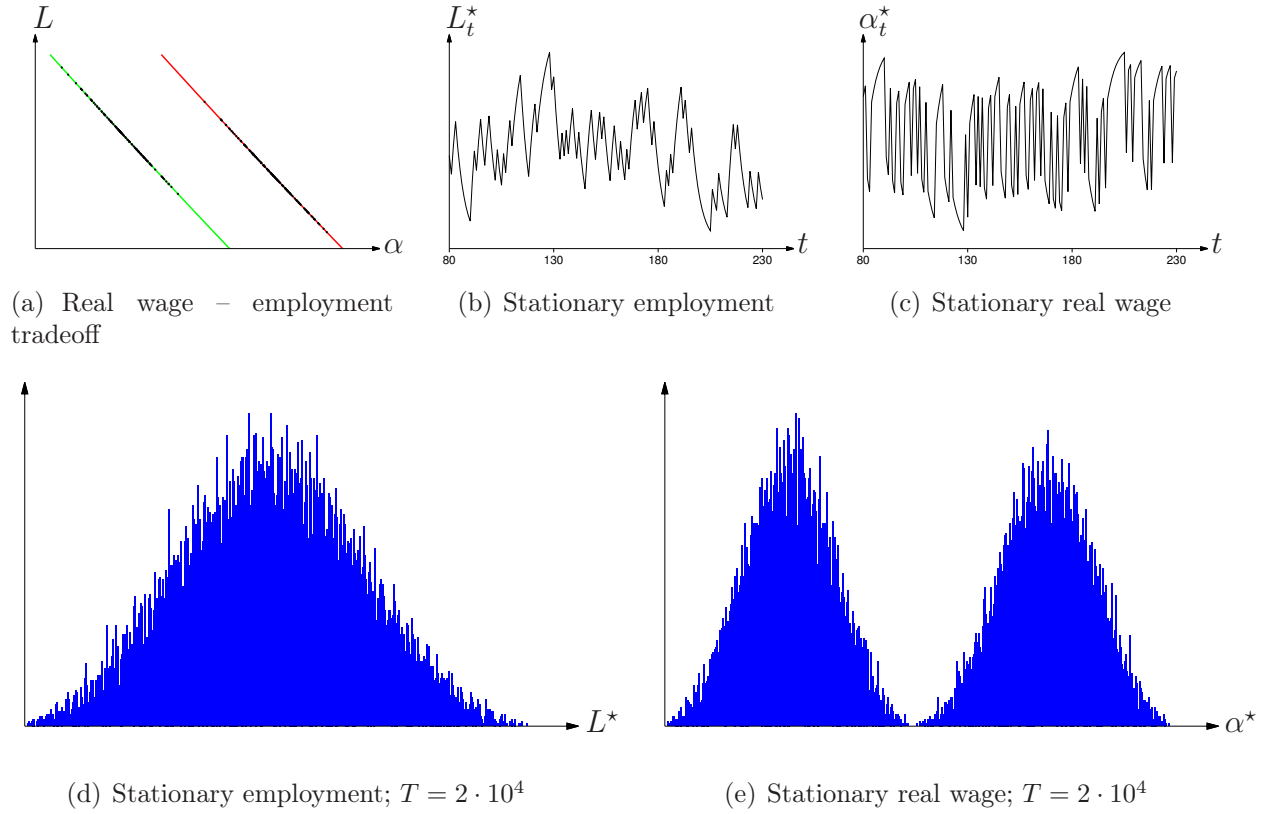
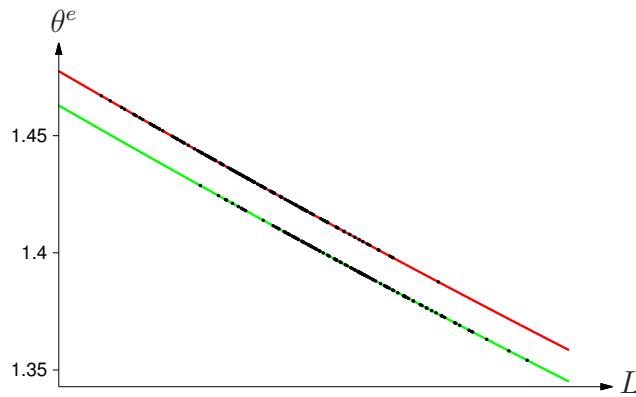


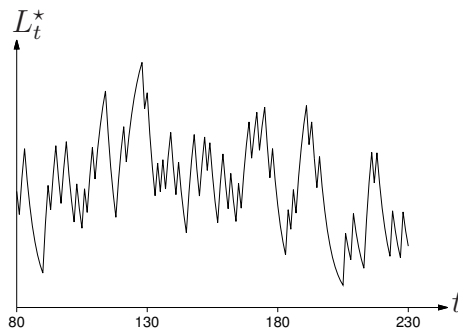
Figure 4.9: Stationary employment and real wage

#### 4.3.2 Employment and expected inflation

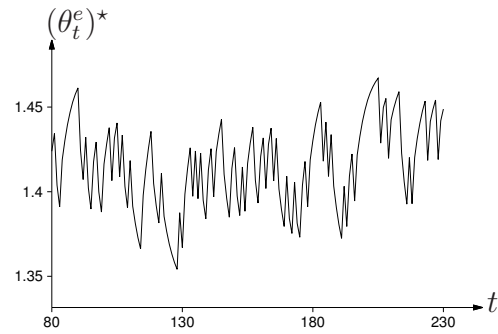
A similar phenomenon occurs the correlation between employment and the expected rate of inflation  $\theta^e = p^e/p_t$ . Since the stationary levels of  $\theta^e$  belong to two small disjoint intervals, its stationary distribution is clearly bimodal and asymmetric.



(a) Employment-expected inflation tradeoff



(b) Stationary employment



(c) Stationary expected inflation

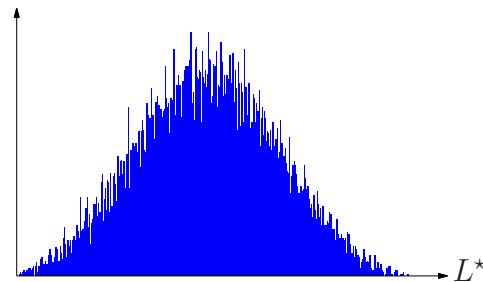
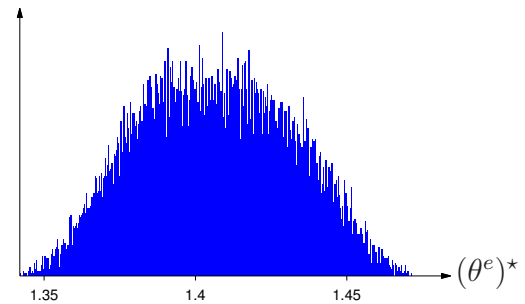
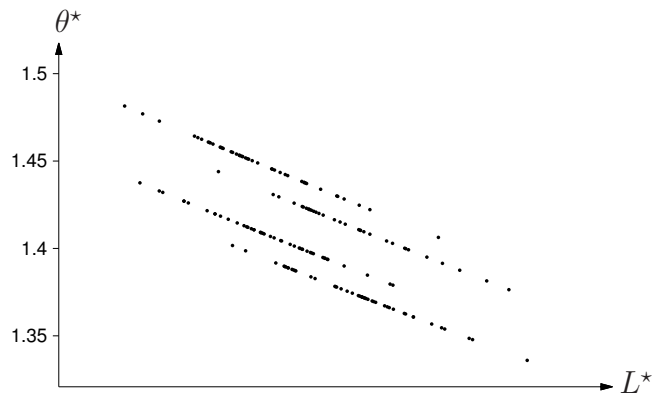
(d) Stationary employment;  $T = 2 \cdot 10^4$ (e) Stationary expected inflation;  $T = 2 \cdot 10^4$ 

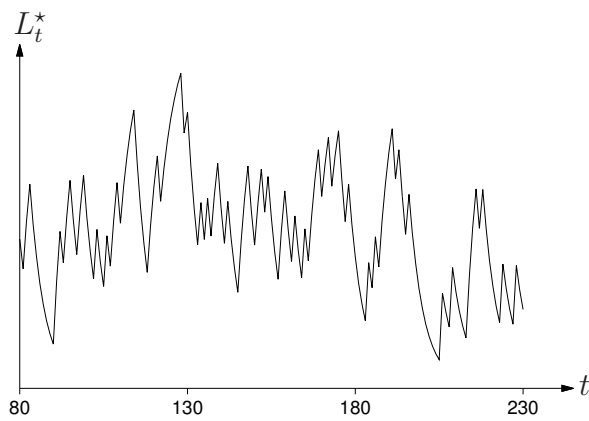
Figure 4.10: Stationary employment and expected inflation

### 4.3.3 Employment and inflation

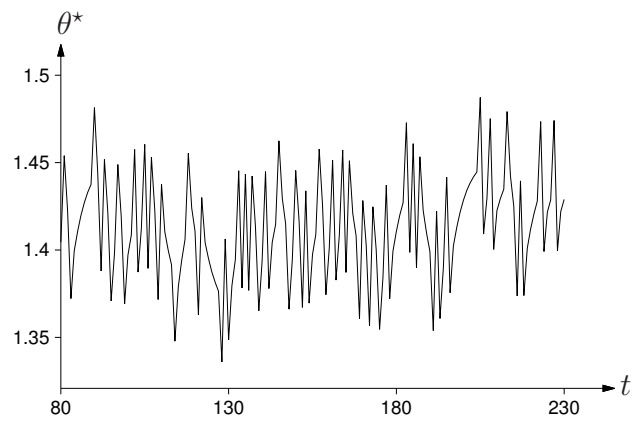
The long run tradeoff between inflation and employment exhibits the typical Markovian structure in the correlation diagram (the Phillips curve) which is an outcome of the central features of the stationary competitive equilibrium under rational expectations.



(a) Employment-inflation tradeoff



(b) Stationary employment



(c) Stationary inflation

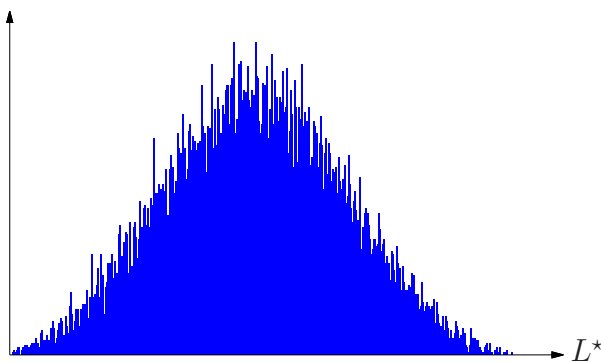
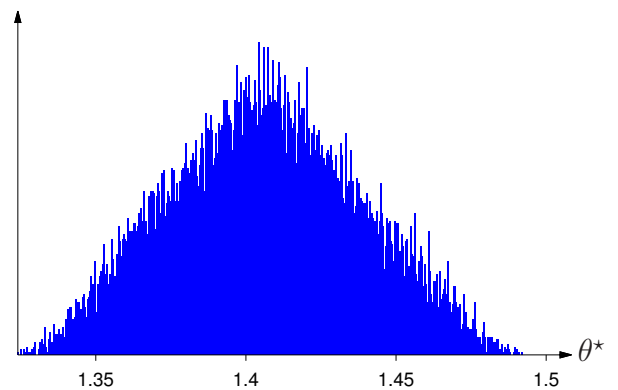
(d) Stationary employment;  $T = 2 \cdot 10^4$ (e) Stationary inflation;  $T = 2 \cdot 10^4$ 

Figure 4.11: Stationary employment and inflation

## 4.4 Convergence and Stability of Balanced Stochastic Orbits

- Convergence in **intensive form** is only a **necessary condition** for convergence to balanced random orbits of money and expectations in  $\mathbb{R}_+^2$ .
- since both variables grow beyond bounds a stability/convergence criterion in the stochastic case has to be redefined.
- fortunately this can be done in an analogous way as in the deterministic case<sup>10</sup>

Thus, for any  $\omega \in \Omega$  and  $(M_0, p_0^e)$ , consider now the two dimensional random difference equation system (4.2)

$$\begin{aligned} M_{t+1} &= M_t \frac{\tilde{c} - \tau}{\tilde{c}} \left( 1 + g\mathcal{P}(1, \psi^*(1, p_t^e/M_t), Z(\vartheta^t \omega)) \right) \\ p_{t+1}^e &= M_t \psi^*(1, p_t^e/M_t) \end{aligned}$$

which generates a **random orbit**  $\{(M_t, p_t^e)\}_{t=0}^\infty$  in  $\mathbb{R}_+^2$  where  $M_t = M(t, \omega, (M_0, p_0^e))$  and  $p_t^e = p^e(t, \omega, (M_0, p_0^e))$ .

### Definition 4.3

Given  $\omega \in \Omega$  and  $(M_0, p_0^e)$ . A random orbit  $\{(\bar{M}_t, \bar{p}_t^e)\} = \{(M(t, \omega, (M_0, p_0^e)), p^e(t, \omega, (M_0, p_0^e)))\}$  is called **balanced**, if there exists a random fixed point  $q^* : \Omega \rightarrow \mathbb{R}_+$  of (4.3)

$$q_{t+1}^e = S(\vartheta^t \omega) q_t^e = \frac{\tilde{c}}{\tilde{c} - \tau} \frac{\psi^*(1, q_t^e)}{1 + g\mathcal{P}(1, \psi^*(1, q_t^e), Z(\vartheta^t \omega))}$$

such that

$$\bar{p}_t^e = \bar{M}_t q^*(\vartheta^t \omega) \quad \text{for all } t \geq 0, \quad \mathbb{P}\text{-a.s.}$$

A balanced orbit is called **asymptotically stable** if for all  $(M_0, p_0^e)$  in a neighborhood of  $(\bar{M}_0, \bar{p}_0^e)$  the distance

$$\Delta_t = \Delta(t, \omega, (M_0, p_0^e)) := p^e(t, \omega, (M_0, p_0^e)) - q^*(\vartheta^t \omega) M(t, \omega, (M_0, p_0^e))$$

satisfies  $\lim_{t \rightarrow \infty} |\Delta_t| = 0$ ,  $\mathbb{P}$ -a.s.

The definition implies the two dimensional auxiliary system in  $(q^e, \Delta)$

$$\begin{aligned} q_{t+1}^e &= S(\vartheta^t \omega) q_t^e \\ \Delta_{t+1} &= \frac{\psi^*(1, q_t^e)}{S(\vartheta^t \omega) q_t^e} \cdot \frac{S(\vartheta^t \omega) q_t^e - S(\vartheta^t \omega) q^*(\vartheta^t \omega)}{q_t^e - q^*(\vartheta^t \omega)} \cdot \Delta_t \end{aligned}$$

which leads to the following theorem.

<sup>10</sup>as in the deterministic situation, (see Deardorff, 1970; Böhm, Pampel & Wenzelburger, 2005; Pampel, 2009; Böhm, 2010; Böhm & Claas, 2012)

**Theorem 4.2** *Let  $S$  be differentiable and increasing with respect to  $q^e$  and let  $q^*$  be an asymptotically stable random fixed point of*

$$q_{t+1}^e = S(\vartheta^t \omega) q_t^e = \frac{\tilde{c}}{\tilde{c} - \tau} \frac{\psi^*(1, q_t^e)}{1 + g\mathcal{P}(1, \psi^*(1, q_t^e), Z(\vartheta^t \omega))}.$$

*For almost all  $\omega \in \Omega$  and any  $q_0^e \in I_3$ , and  $q_0^e \neq q^*(\omega)$  with  $\lim_{t \rightarrow \infty} |q^e(t, \omega, q_0^e) - q^*(\vartheta^t \omega)| = 0$ ,  $\mathbb{P}$ -a.s., the distance  $\Delta_t = p^e(t, \omega, (M_0, p_0^e)) - q^*(\vartheta^t \omega) M(t, \omega, (M_0, p_0^e))$  satisfies*

$$\lim_{t \rightarrow \infty} |\Delta_t| = 0 \quad \text{if}$$

$$\mathbb{E} \log(S'(\omega, q^*(\omega))) + \mathbb{E} \log \frac{\tilde{c} - \tau}{\tilde{c}} (1 + g\mathcal{P}(1, \psi^*(1, q^*(\omega)), Z(\omega))) < 0$$

$$\lim_{t \rightarrow \infty} |\Delta_t| = \infty \quad \text{if}$$

$$\mathbb{E} \log(S'(\omega, q^*(\omega))) + \mathbb{E} \log \frac{\tilde{c} - \tau}{\tilde{c}} (1 + g\mathcal{P}(1, \psi^*(1, q^*(\omega)), Z(\omega))) > 0.$$

There is asymptotic Convergence for values in Table 2 and divergence for the values in Table 1.

**Asymptotic Convergence** for values in Table 2 and

**Divergence** for the values in Table 1.

Figure 4.12 displays the convergence features when the balanced orbit associated with  $q_3^*$  is asymptotically stable, a situation which occurs for the parameters given in Table 2.

**main difference** to the values in Table 1

– lower government demand and higher tax rates.

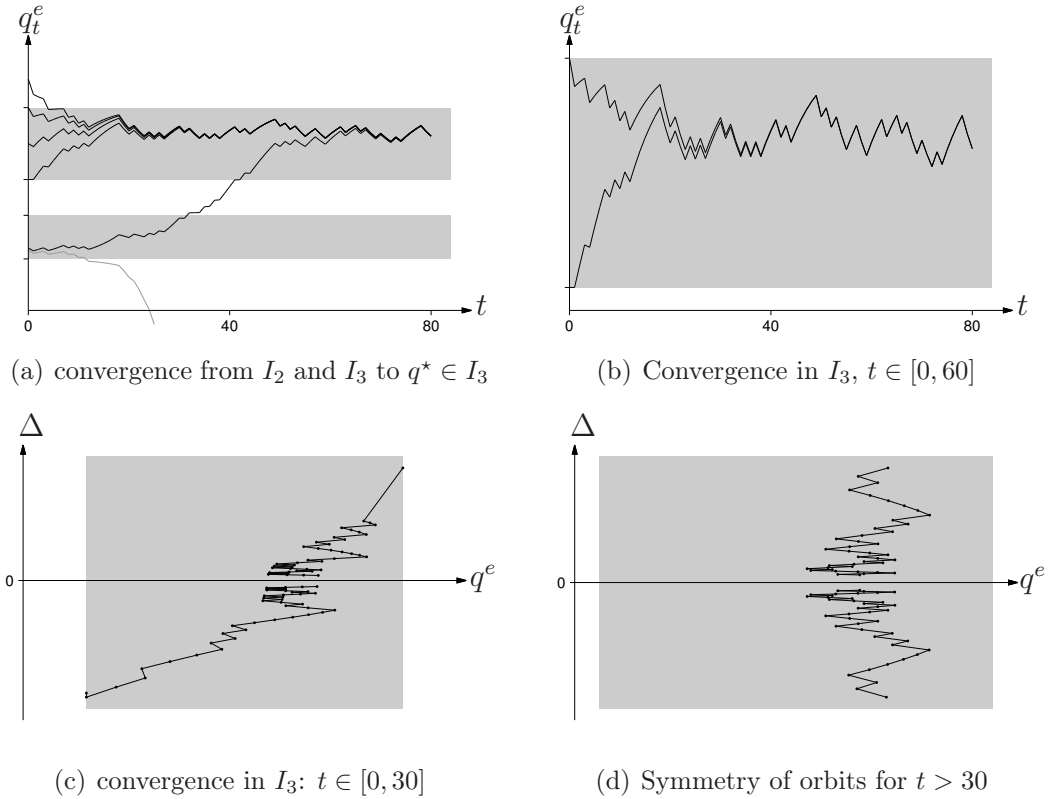
$Z_{min}$	$Z_{max}$	$B$	$C$	$c$	$\tau$	$g$	$g^*$	$g^{**}$
1.0	1.01	0.6	0.6	0.5	0.75	0.8392	0.8400	0.8449

Table 2: Standard parametrization b

$\implies$  lower deficits and thus lower rates of inflation at any one time.

$\implies$  lower growth rates of money at any one time.



Figure 4.12: Convergence in  $(q^e, \Delta)$ -space

## 4.5 Convergence and Growth Rates of Monetary Expansion

To understand convergence, consider the two central equations

$$q_{t+1}^e = S(\vartheta^t \omega) q_t^e = \frac{\tilde{c}}{\tilde{c} - \tau} \frac{\psi^*(1, q_t^e)}{1 + g\mathcal{P}(1, \psi^*(1, q_t^e), Z(\vartheta^t \omega))}$$

$$\Delta_{t+1} = \frac{\tilde{c} - \tau}{\tilde{c}} (1 + g\mathcal{P}(1, \psi^*(1, q_t^e), Z(\vartheta^t \omega))) \cdot \frac{S(\vartheta^t \omega) q_t^e - S(\vartheta^t \omega) q^*(\vartheta^t \omega)}{q_t^e - q^*(\vartheta^t \omega)} \cdot \Delta_t$$

- a skewed random dynamical system in  $(q^e, \Delta)$ , linear in  $\Delta$
- whose coefficient consists of a product of two random variables.
- the first converges to the growth rate of money  $\hat{m}_t := M_t/M_{t-1}$  along  $q_3^*$
- the second term converges to the derivative of  $S$   
since  $\lim_{t \rightarrow \infty} |q_t^e(t, \omega, q_0^e) - q^*(\vartheta^t \omega)| = 0$ ,  $\mathbb{P}$ -a.s.
- $\implies \Delta \rightarrow 0$  if the growth factor  $\hat{m}^*(\vartheta^t \omega) \cdot S'(q_3^*(\vartheta^t \omega))$  of the linear system is sufficiently contracting,  
i.e.
- if and only if  $\mathbb{E} \{ \hat{m}^*(\vartheta^t \omega) \cdot S'(q_3^*(\vartheta^t \omega)) \} < 1$ , which are essentially the conditions stated in

Theorem 4.2.

- this allows permanently positive growth rates of money since

$$S'(q_3^*(\vartheta^t \omega)) < 1, \quad \mathbb{P}\text{-a.s.},$$

given the assumption for  $S$  on  $I_3$ ,

- $\implies$  orbits converging to random balanced monetary orbits do exist where  
the rate of monetary expansion is larger than one along  $\{\vartheta^t \omega\}$ ,  $\mathbb{P}$ -a.s.,
- but it should not make the product with  $S'$  larger than one too often.

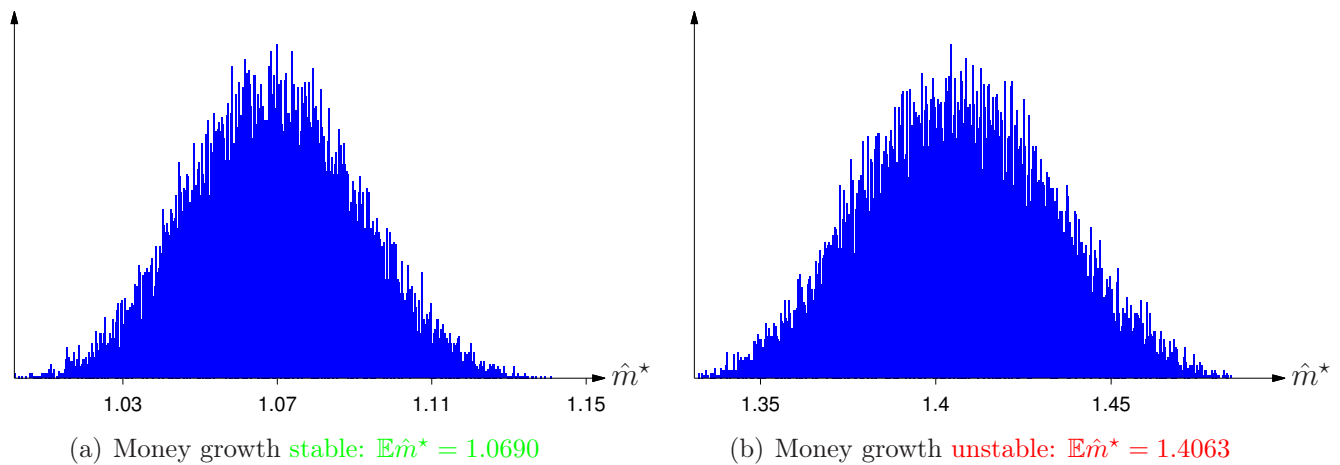


Figure 4.13: Stationary growth rates of money:  $T = 2 \cdot 10^4$

Figure 4.13 displays the histograms of the rates of monetary expansion in the two cases with their respective means, showing clearly the reason for the instability of the balanced orbit in the case of the parameters of Table 1.

## 5 Summary and Conclusions

### Mathematics:

the balanced random evolution of expanding orbits under rational expectations can be analyzed successfully using

- the theory of random dynamical systems in the sense of Arnold (1998)<sup>11</sup>.
- There is no need

- to linearize the random difference equation and analyze approximating stochastic systems<sup>12</sup>,

<sup>11</sup>These methods have been applied successfully in other areas of economics like growth theory and mathematical finance (see Böhm & Wenzelburger, 2005; Böhm & Chiarella, 2005).

<sup>12</sup>a procedure often used in macroeconomic application (as for example in Lettau & Uhlig, 2002)

- to suppress stochastic orbits and
- describe convergence in distributions only<sup>13</sup>.

⇒ the same tools analyze the theoretical and empirical/observable objects

### Economics:

- there may exist two expanding random balanced monetary orbits
- each of which induces bounded positive stationary real solutions
- those with high employment and production are always unstable,
- making them unobservable as time series in the monetary state space

### As in the deterministic case,

- for small government demand the path with low employment can be asymptotically stable
- under additional conditions which are not universally satisfied
- for large government demand all orbits diverge to states with unbounded real money balances.

⇒ a decisive role of autonomous and non stochastic government demand

### Extensions and Implications:

**more assets:** government bonds/debt, stocks, shares, inventory

### noncompetitive markets:

- monopolistic competition in commodity markets
- noncompetitive labor markets: monopolies and bargaining,
- matching: what is the aggregate supply function?

**expectations:** forecasting rules matter! naive, learning, statistical updating

- models of international trade
- disequilibrium trading: sequential opening of markets and rationing

**more micro structure:** agent based?

⇒ all seem to be doable within the same framework!

---

<sup>13</sup>which is a weaker and unobservable concept (as by Bhattacharya & Majumdar, 2004, and others)

## A Appendix: On Random Dynamical Systems

In order to understand the *dynamic* behavior of a system defined by a stochastic difference equation like (4.3), it is necessary to introduce some additional mathematical concepts not commonly used in dynamic macroeconomics. They supply the necessary tools in order to analyze, from a time series perspective, the simultaneous interaction of the *dynamic* forces of a mapping combined with the *stochastic implications of ongoing randomness*. They are also necessary to formulate the main results of this paper.

The classical theory of stochastic processes and the theory of random dynamical systems are two closely related mathematical tools to analyze the evolution of a dynamical system subjected to regular and ongoing exogenous stochastic perturbations. The mathematical literature provides different approaches, depending on the type of questions or characterizations for which an answer is sought. Even the use of the term random dynamical system is not used uniformly in the literature<sup>14</sup>. Depending on the perspective and the objective of the desired properties and results one may be more appropriate than the other. For many economic applications, it seems most natural to use an approach which uses stochastic orbits as the primitive object of investigations, as proposed by Arnold (1998), since these are the typical observable objects in economic empirical work, rather than distributions or Markov kernels which are theoretical concepts typically unobservable empirically as well as generically.

### A.1 Mathematical tools

Following Arnold (1998), let  $\vartheta : \Omega \rightarrow \Omega$  be a measurable mapping on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  which is measure preserving with respect to  $\mathbb{P}$  and whose inverse  $\vartheta^{-1}$  is measurable as well. Assume that  $\mathbb{P}$  is ergodic with respect to  $\vartheta$  and denote by  $\vartheta^t$  the  $t$ -th iterate of the map  $\vartheta$ . The collection  $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta^t\})$  is called an *ergodic metric dynamical system*.

To connect these probabilistic structures with the dynamic properties of the mapping  $S : \Sigma \times X \rightarrow X$ ,  $\Sigma \subset \mathbb{R}^d$ ,  $X \subset \mathbb{R}^k$ , let  $\xi : \Omega \rightarrow \mathbb{R}_+$  denote a measurable map such that the stochastic process  $\{\xi_t\}_{t \in \mathbb{Z}}$  has the so called canonical representation

$$(A.1) \quad \xi_t(\omega) \equiv \xi(\vartheta^t \omega).$$

Finally, using the evaluation map  $\xi : \Omega \rightarrow \Sigma$ ,  $\xi(\omega) := \omega(0) \equiv Z(\omega)$  the stochastic difference equation (4.3) can now be rewritten (using the same symbol  $S$ ) as

$$(A.2) \quad x_{t+1} = S(\vartheta^t \omega) x_t := S(\xi(\vartheta^t \omega), x_t).$$

Therefore, for any initial point  $x_0$ , repeated applications of  $S$  under the perturbation  $\omega$  induce the measurable mapping  $\phi : \mathbb{N} \times \Omega \times X \rightarrow X$  defined by

$$(A.3) \quad \phi(t, \omega, x_0) := \begin{cases} S(\vartheta^{t-1} \omega) \circ \dots \circ S(\omega) x_0 & \text{if } t > 0 \\ x_0 & \text{if } t = 0 \end{cases}$$

<sup>14</sup>For example, Bhattacharya & Majumdar (2004) use the term random dynamical system in a different way than the one adopted here from Arnold (1998)

such that  $x_t = \phi(t, \omega, x_0)$  is the state of the system at time  $t$ . The map (A.3) (or equivalently (A.2)) defines a *random dynamical system* time in the sense of Arnold(1998, Chapters 1 & 2) in forward time. For any initial value  $x_0$  and perturbation  $\omega \in \Omega$ , the sequence  $\gamma(x_0) := \{x_t\}_{t=0}^\infty$  with  $x_t = \phi(t, \omega, x_0)$ ,  $t \in \mathbb{N}$  defines a stochastic orbit of the system  $S$ .

Random fixed points are the respective extension of the concept of a fixed point of deterministic systems to the random case, see Schmalfuß (1996, 1998), Schenk-Hoppé & Schmalfuß (2001). Random fixed points induce stationary orbits (or stationary solutions) of the random dynamical system.

### Definition A.1

A **random fixed point** of  $S(\omega) : X \rightarrow X$  is a random variable  $x^* : \Omega \rightarrow X$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  such that

$$(A.4) \quad x^*(\vartheta\omega) = S(\omega)x^*(\omega) = \varphi(1, \omega, x^*(\omega)) \quad \text{for all } \omega \in \Omega',$$

where  $\Omega' \subset \Omega$  is a  $\vartheta$ -invariant set of full measure,  $\mathbb{P}(\Omega') = 1$ .

It is called *asymptotically stable* if there exists a random neighborhood  $U(\omega) \subset X$  such that  $\mathbb{P}$ -almost surely

$$(A.5) \quad \lim_{t \rightarrow \infty} \|\varphi(t, \omega, x_0) - x^*(\vartheta^t\omega)\| = 0 \quad \text{for all } x_0(\omega) \in U(\omega).$$

The first part of the definition implies that  $x^*(\vartheta^{t+1}\omega) = S(\vartheta^t\omega)x^*(\vartheta^t\omega)$ . In other words, the random fixed point  $x^* : \Omega \rightarrow X$  generates orbits  $\{x^*(\vartheta^t)\}_{t \in \mathbb{N}} = \{x^* \circ \vartheta^t\}_{t \in \mathbb{N}}$ ,  $\omega \in \Omega$  which solve the random difference equation (A.3). In addition, the process  $\{x^* \circ \vartheta^t\}_{t \in \mathbb{N}}$  is stationary and ergodic since  $\vartheta$  is stationary and ergodic.

Let  $x^*\mathbb{P}$  denote the probability distribution of  $x^*$ . Then, if in addition  $\mathbb{E}\|x^*\| < \infty$ , stability and ergodicity together imply that for any  $B \in \mathcal{B}(X)$ ,

$$(A.6) \quad \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=0}^T 1_B(\varphi(t, \omega, x_0(\omega))) = \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=0}^T 1_B(x^*(\vartheta^t\omega)) = x^*\mathbb{P}(B)$$

for all  $x_0(\omega) \in U(\omega)$   $\mathbb{P}$ -almost surely. In other words, the empirical law of an asymptotically converging orbit induces the true stationary probability distribution.

## A.2 Some Examples

It is straightforward to develop an intuitive understanding of these concepts and of the simultaneous interaction of the dynamic forces and the random perturbations from generic one dimensional examples. Let the stochastic difference equation of the type (4.3) be given by a *parameterized* dynamical system  $G : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$  defining a family of mappings

$$(A.7) \quad G(\xi, \cdot) : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto G(\xi, x).$$

Here  $\xi \in \mathbb{R}^m$  is a vector of parameters of the system while  $x$  is the vector of endogenous variables defining the state of the system at any one time.

Then, the one step time change of  $x$  for a *fixed* value of the parameter  $\xi \in \mathbb{R}^m$  is given by

$$(A.8) \quad x_{t+1} = G_\xi(x_t) \quad G_\xi \equiv G(\xi, \cdot),$$

i.e. the *dynamics* follows the rules and the description of a deterministic dynamical system once the value of a *particular*  $\xi$  is given. Assume that  $\xi$  is following a given random path described by

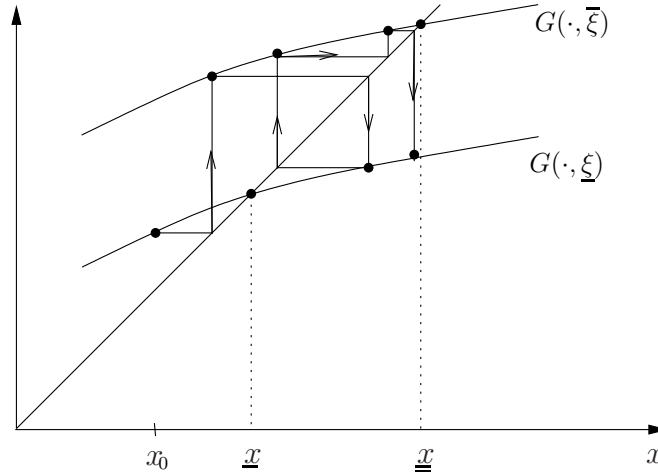


Figure A.1: Orbit for  $(x_0, \underline{\xi})$  converging to the trapping set  $[\underline{x}, \underline{x}]$  for  $\omega = (\dots, \underline{\xi}, \bar{\xi}, \underline{\xi}, \bar{\xi}, \bar{\xi}, \underline{\xi}, \dots)$

$\omega := (\dots, \xi_{s-2}, \xi_{s-1}, \xi_s, \xi_{s+1}, \dots)$ . Then, the change of  $\xi$  over time implies choosing at each iteration for  $x$  a *different mapping*. If, for example,  $G$  is a contraction for all  $\xi$  assuming only two values  $\{\underline{\xi}, \bar{\xi}\}$ , then for any path  $\omega$  of the perturbation the associated evolution of  $x$  can be visualized as in Figure A.1 for any given initial condition  $x_0$ . In this case, the orbit will eventually be trapped in some compact interval  $[\underline{x}, \underline{x}]$ , suggesting that the limiting behavior may be stationary if the perturbation is stationary as well.

In contrast, if  $S$  is an increasing and strictly convex function as one might expect from the deterministic case analyzed in Böhm & Claas (2012), the dynamics could be characterized as in Figure A.2. There could exist two invariant sets: a forward invariant trapping set  $[x_1, x_2]$  (the red interval) which qualifies as the support of a stationary measure of an asymptotically stable random fixed point. This coexists with a second *backward* invariant set (the blue interval) on which a second stationary orbit may exist.

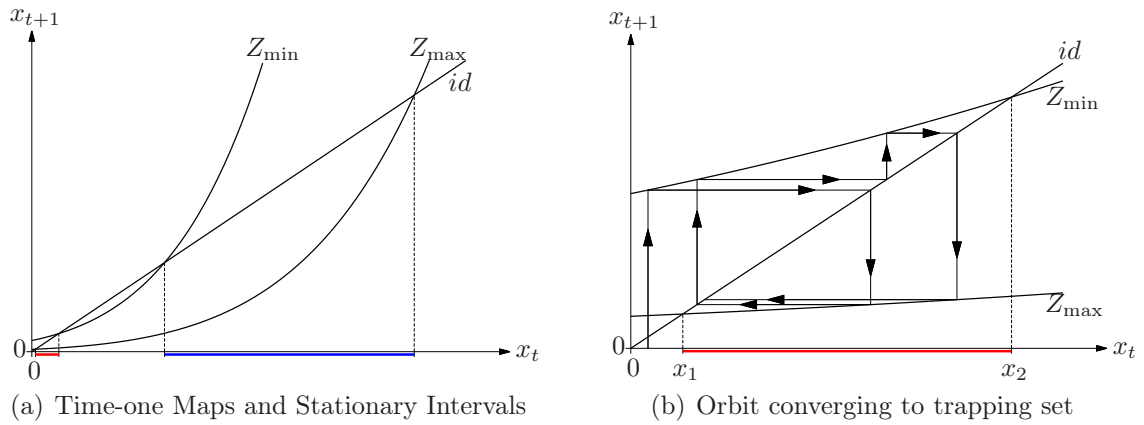


Figure A.2: Discrete production shocks  $\omega = (\dots, Z_{\min}, Z_{\max}, Z_{\min}, Z_{\min}, Z_{\max}, \dots)$

### Convergence to random fixed point

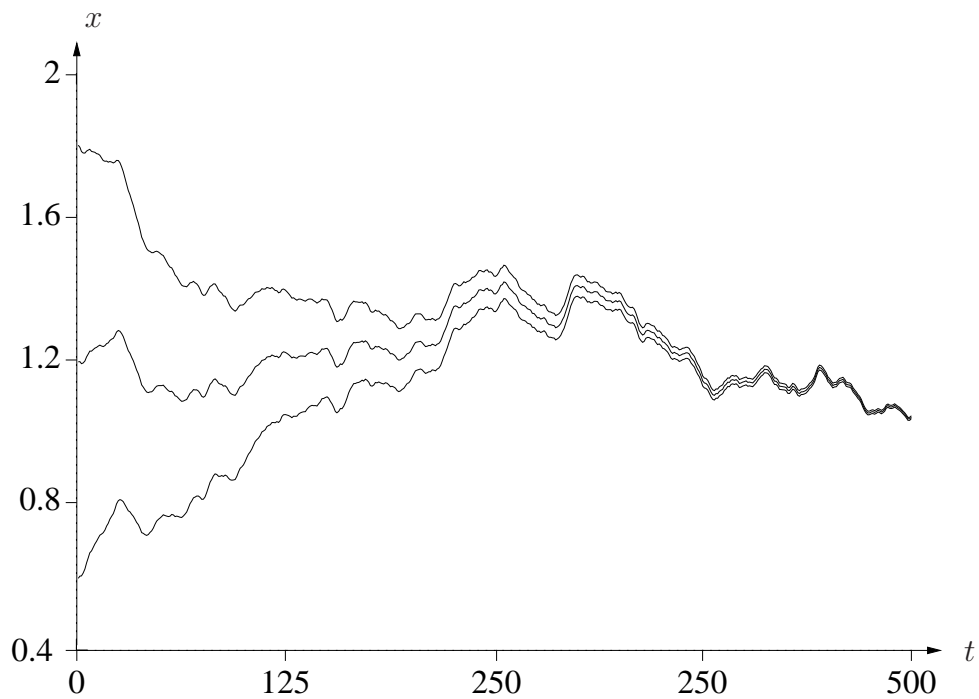


Figure A.3: Asymptotic convergence of three orbits to a random fixed point for given  $\omega$

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