

Belau, Julia

Conference Paper

Consequences of Connection Failure - Centrality and the Importance for Cohesion

Beiträge zur Jahrestagung des Vereins für Socialpolitik 2014: Evidenzbasierte Wirtschaftspolitik
- Session: Game Theory and Welfare Economics, No. B12-V2

Provided in Cooperation with:

Verein für Socialpolitik / German Economic Association

Suggested Citation: Belau, Julia (2014) : Consequences of Connection Failure - Centrality and the Importance for Cohesion, Beiträge zur Jahrestagung des Vereins für Socialpolitik 2014: Evidenzbasierte Wirtschaftspolitik - Session: Game Theory and Welfare Economics, No. B12-V2, ZBW - Deutsche Zentralbibliothek für Wirtschaftswissenschaften, Leibniz-Informationszentrum Wirtschaft, Kiel und Hamburg

This Version is available at:

<https://hdl.handle.net/10419/100368>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

Consequences of Connection Failure - Centrality and the Importance for Cohesion*

Julia Belau

University of Dortmund (TU) and Ruhr Graduate School in Economics
Faculty of Economics and Social Science
Chair of Microeconomics
Vogelpothsweg 87, 44227 Dortmund, Germany.

Abstract

This paper suggests a new approach for centrality measures for general (weighted) networks taking into account the importance for cohesion and relative power of connections. While existing literature either ignores the importance for cohesion or measures it by analyzing consequences arising from the failure of whole nodes, this approach analyzes consequences of tie failures. Using cooperative game theory, we assign weights to every tie of the network where the cooperative game accounts for the cohesion of the network. These weights are combined with the weights of the original network where emphasis for the latter and for cohesion can be regulated individually. Then, the degree measure and Eigenvector measure are applied. This provides the first centrality approach accounting for cohesion and relative importance/power of connections. We provide axiomatic characterizations for the degree-based measures in the case of binary networks and discuss computational complexity. Furthermore, we give examples discussing the drawbacks of existing measures in contrast to our suggested one and as a political application, we show how our approach can be used to forecast government formation by the case of the state parliament election in Hamburg, Germany.

KEYWORDS: Centrality; Cohesion; Government Formation; Network; Shapley; Banzhaf; Failure

JEL classification numbers: C71, D72, D85, H11

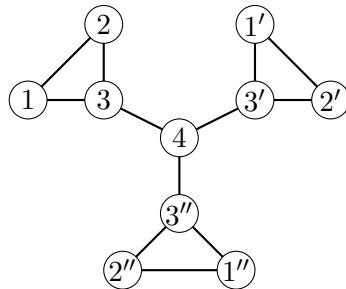
*Financial support by the Ruhr Graduate School in Economics is gratefully acknowledged.
Email: julia.belau@tu-dortmund.de

1. Introduction

There is a large literature on centrality measures, mostly applied for social networks, economic networks or also political networks. Centrality is often used to identify top key nodes, those nodes in the network being most *important* for the network. But what means important? Mostly, we mean importance for cohesion of either the whole network or a set of essential nodes. A reasonable and convincing approach for the analysis of importance for cohesion is the analysis of consequences of failure. Application domains are for example energy networks or political networks: consequenses of failure are capacity overloads or blackouts for energy networks or a (partly or complete) breakdown of trading routes/networks or an Economic and Monetary Union. These example are of recent importance considering increasing blackout probabilities due to outdated reactors as for example in France or the recent nuclear phaseout in Germany. However, while existing literature only analyzes failure consequences of whole nodes, already the failure of a certain connection can seperate a whole network into pieces. The application of analyzing failure consequences of connections seems even more relevant for political or economic networks: oil pipelines can break (connection failure) without a breakdown of the whole gas province (node failure), a bilateral trading agreement can be broken without a whole country leaving the trading union, political parties can stop bilateral coalitional negotiations due to incompatibilities without leaving the political spectrum or a country might stop monetary flows for bail-out packages without leaving the European Union. Hence, consequences of connection failure should be taken into account for a measure for identifying top key nodes or (relative) coalitional power by means of relative importance for cohesion of the whole.

Existing centrality measures either generally ignore the importance of cohesion or bear other drawbacks. As an example, consider the following communication network: Imagine there are three chairholders (3 , $3'$ and $3''$), each having two phd students ($1/2$, $1'/2'$ and $1''/2''$). The three chairholders do not know each other personally, but have a common friend (4) through which they can get in contact to each other. The communication network is presented in Figure 1.

Figure 1: Communication Network



Now, we are interested in the relative power of each participant in the network. The most popular centrality measures are Freeman (1978)'s degree, closeness and betweenness measures or Bonacich (1972)'s eigenvector centrality. Centrality according to these approaches (normalized for comparability issues) is presented in Table 1.

Table 1: Normalized Centralities

Participant	1,2	3	4
Degree	8.33	12.50	12.50
Closeness	8.45	11.56	14.64
Betweenness	0	21.21	36.36
Eigenvector	8.10	12.26	14.63

While having the advantage of very low computational complexity, the degree measure only takes into account the direct links of a node and not the whole network, that is, generally lacks to take into account importance for cohesion. In the example we see, that there is no difference between the centrality of the chairholders and the common friend, who is connecting the whole network, as they all have the same number of connections. The closeness measure, which counts the length of shortest paths, does account for cohesion of the whole network, but provides an unreasonable allocation in terms of failure: in the example, the relative distance of centrality between the chairholders and the common friend is very small and the boundary nodes (phd students) still obtain a relatively high centrality. Furthermore, the closeness measure bears some problems for weighted networks as "length" of a shortest path is difficult to measure correctly in this case. For a further discussion on drawbacks of closeness, also see Gómez et al. (2003).

Usually, the betweenness measure is suggested to be the most suitable measure for cohesion. However, there are examples showing that the betweenness measure might distribute unreasonable payoffs, most likely in smaller networks or large networks with a small number of connections: In the example, the boundary nodes obtain a betweenness-centrality of zero. One could argue that these nodes should obtain some centrality greater than zero as their existence actually "creates" the centrality of the other nodes: if a boundary node fails, the betweenness measure of the corresponding chairholder decreases.

As an alternative to the original betweenness measure, Freeman et al. (1991) suggest a betweenness measure based on network flow. Still, in the example, all boundary nodes would obtain a centrality of zero. Another approach is Eigenvector centrality (Bonacich, 1972) which aims to measure the influence of a node on the network (this idea is for example the basis of Google's PageRank). However, for the example above, this approach yields (as the closeness approach) relatively small distances between the centralities of the nodes.

Beside the above mentioned "classic" centrality measures, there exist game theoretic

approaches that use the Shapley (for example Gómez et al., 2003 or Suri and Narahari, 2008) or the Banzhaf value (for example Grofman and Owen, 1982). These measures take into account consequences of failure by analyzing marginal contributions, that is, the surplus a certain node creates when entering a coalition (which could be seen as the negative of the failure of this node for each coalition). The network structure is taken into account by using functions for the underlying game which consider (shortest) paths present in the network depending on the presence or absence of a node. However, this only accounts for the failure of a whole node with all its connections while already the failure of one connection influences (shortest) paths and even might split the whole network. In the paper, we will present a political example which shows, that these game theoretic approaches are not always suitable while the analysis of connection failure solves the problem. Furthermore, these approaches have no extension for weighted networks.

In this paper, we suggest a new approach for centrality measures accounting for importance of cohesion of a node by the relative importance of its connections. For this, we analyze consequences of link failure using two approaches from cooperative game theory, the Shapley value and the Banzhaf value, generalized for a game which accounts for the links of a network, the so-called arc game. We assign the corresponding Shapley or Banzhaf values to all links of the network to capture consequences of failure, that is, the importance for cohesion of connections. We generalize the approach for weighted networks by combining these values with the existing weights of the network in place where we implement an emphasis parameter which allows us to regulate emphasis for importance of cohesion or centrality by means of political, economic or social weights individually. Then, we apply Freeman (1978)'s degree measure and Bonacich (1972)'s Eigenvector measure. This is what we call *Cohesion Indices*.

Considering the example from above again, centrality according to the new approach using the Shapley value is represented in Table 2. Here, relative distances are more plausible and also, boundary nodes do not obtain a centrality of zero.

Table 2: Normalized Cohesion-Centralities

Participant	1,2	3	4
Cohesion-Shapley-Degree	4.24	14.55	30.91
Cohesion-Shapley-Eigenvector	2.41	18.19	30.99

We provide axiomatic characterizations for the degree-based measures in the case of binary networks and analyze computational complexity. Furthermore, we give an example which discusses exclusiveness of links and as a political application, we apply our approach in the case of the state parliament elections (*Bürgerschaftswahl*) 2001 in Hamburg, Germany.

The paper is organized as follows: the following section gives definitions and notations. Section 3 provides our new approach: the Cohesion-Indices. In section 4 we discuss

computational complexity and properties, section 5, provides an example where we also compare our approach to existing measures and discuss the difference in the use of the Shapley or Banzhaf value with respect to exclusiveness of links. Finally, section 6 concludes.

2. Preliminaries

2.1. Cooperative games

A *cooperative game with transferable utility* (TU-game) (N, v) consists of a (non-empty and finite) player set $N = \{1, \dots, n\}$ (individuals, political parties, countries, firms,...) and a *characteristic function* $v \in \mathbb{V}_N := \{v : 2^N \rightarrow \mathbb{R} | v(\emptyset) = 0\}$ where $v(K)$ describes the social, economic or political possibilities of $K \subseteq N$. A TU-game is called *simple* if $v(K) \in \{0, 1\}$ for all $K \subseteq N$.

A *binary (social, economic or political) network* is a binary graph $g(N, x)$ consisting of nodes $N = \{1, \dots, n\}$ and edges (or links or ties) between the nodes, where x_{ij} denotes the indicator whether a link between node i and j exists or not, that is, $x_{ij} = 1$ if link ij exists in network g and $x_{ij} = 0$ otherwise. Let \mathbb{G}_N^b denote the set of all binary networks $g(N, x)$. We say that nodes i and j are *connected* in the network g if there exists a path $ih_1, \dots, h_kj \in g$, $h_1, \dots, h_k \in N$. Connected nodes form components of a network g and these components build a partition on the node/player set N . We denote this partition by $\mathcal{G}(N, g)$; $G_i := G_i(N, g) \in \mathcal{G}(N, g)$ is the component of all players connected with player $i \in N$. A TU-game is called *zero-normalized* if $v(\{i\}) = 0$ for all $i \in N$. Note that every game can be zero-normalized, hence, without loss of generality, we can restrict ourselves to the set of zero-normalized games, denoted by \mathbb{V}_N^0 . Meessen (1988) and Borm et al. (1992) assign to every zero-normalized cooperative game with a network structure (N, v, g) the *link-game* (or *arc game*) in which the links in the network are the players:

$$v^N : \{\lambda \in g\} \rightarrow \mathbb{R}, v^N(g') := v^{g'}(N) = \sum_{S \in \mathcal{G}(N, g')} v(S) \quad \forall g' \subseteq g$$

where v^g is the *graph-restricted game* introduced by Myerson (1977). Note that if v were not zero-normalized, v^N might not be a value function as $v^N(\emptyset) = v^{\emptyset}(N) = \sum_{i \in N} v(\{i\}) \neq 0$.

An *allocation rule for cooperative games with network structures* is a function $Y : N \times \mathbb{V}_N \times \mathbb{G}_N^b \rightarrow \mathbb{R}^n$, assgning a payoff $Y_i(N, v, g)$ to player $i \in N$ for every TU-game (N, v) with network structure g . An allocation rule Y is called *feasible* if $\sum_{i \in N} Y_i \leq \sum_{G \in \mathcal{G}(N, g)} v(G)$.

In the commonly used definition of feasibility, literature uses $v(N)$ instead of $\sum_{G \in \mathcal{G}(N, g)} v(G)$. The use of $v(N)$ is suitable if the cooperative game has no inner structure or if g is connected, because then $\sum_{G \in \mathcal{G}(N, g)} v(G) = v(N)$ holds.

Two of the most popular allocation rules for cooperative games without a network

structure are the *Shapley value* (Shapley, 1953) Sh and the *Banzhaf value* Ba , introduced by Banzhaf (1952) for simple games and extended to general games by Owen (1975):

$$Sh_i(N, v) = \sum_{K \subseteq N \setminus \{i\}} \frac{|K|!(|N| - 1 - |K|)!}{|N|!} [v(K \cup \{i\}) - v(K)]$$

$$Ba_i(N, v) = \sum_{K \subseteq N \setminus \{i\}} \frac{1}{2^{|N|-1}} [v(K \cup \{i\}) - v(K)]$$

As the Banzhaf value might not be feasible, we will often refer to the normalized Banzhaf value

$$\overline{Ba}_i(N, v) = Ba_i(N, v) \cdot \frac{v(N)}{\sum_{j \in N} Ba_j(N, v)}$$

Note that we use $v(N)$ here as the Banzhaf value is designed for games without inner structure.

2.2. Weighted graphs and Freeman (1978)'s centrality measures

To model differences between the links, weighted graphs are used: A *weighted (social, economic or political) network* $g(N, w)$ is a binary network endowed with a weight w_{ij} for every link ij . If $w_{ij} = 0$, we say that link ij does not exist in network g . Hence, a binary network is a weighted network with weights being 0 or 1. Let \mathbb{G}_N denote the set of all weighted networks $g(N, w)$. Note that $\mathbb{G}_N^b \subset \mathbb{G}_N$.

A *centrality measure* is a function $C : \mathbb{G}_N \rightarrow \mathbb{R}^n$ which assigns an index to every node depending on how "central" this node is in network g . The most popular centrality measures for binary networks are Freeman's 1978 degree, closeness and betweenness. The *degree measure* counts the number of direct connections. This has been extended for weighted networks by Barrat et al. (2004), Newman (2004) and Opsahl et al. (2008):

$$C_i^d(g(N, w)) = \sum_{j \in N \setminus \{i\}} w_{ij}$$

Closeness and Betweenness are designed by shortest paths. Closeness is measured by the lengths of the shortest paths from a node to all other nodes (closeness to other nodes) while betweenness counts how often a node is on a shortest paths between two other nodes (betweenness of nodes) relative to all shortest paths.

A *binary shortest path* between node i and j is defined by $d(i, j) := \min(x_{ih_1} + \dots + x_{h_k j})$ where h_1, \dots, h_k are the intermediate nodes that have to be passed between i and j . Dijkstra (1959) suggested an algorithm to find shortest paths in weighted networks where weights are transmission costs. To implement this in a general weighted network (where a "good" link usually has a high weight instead of low costs), Brandes (2001) and Newman (2001)

invert weights to interpret them as costs, that is,

$$d^w(i, j) = \min\left(\frac{1}{w_{ih_1}} + \dots + \frac{1}{w_{h_k j}}\right),$$

which can also be interpreted as the fastest path between two nodes if weights represent the speed of for example information flow or passing speed in road networks. They use this to extent the *closeness* and *betweenness measures* for weighted networks:

$$C_i^c(g(N, w)) = \left[\sum_{j \in N \setminus \{i\}} d^w(i, j) \right]^{-1}$$

$$C_i^b(g(N, w)) = \sum_{(j, k) \in N \setminus \{i\} \times N \setminus \{i\}, j \neq k} \frac{|d^w(j, k)(i)|}{|d^w(j, k)|}$$

where $|d^w(j, k)|$ is the number of shortest paths between j and k and $|d^w(j, k)(i)|$ the number of those of them going through i . For notational reasons we will write

$$\sum_{j \neq i \neq k}^N := \sum_{(j, k) \in N \setminus \{i\} \times N \setminus \{i\}, j \neq k}$$

Beside Freeman (1978)'s centrality measures, Bonacich (1972) introduces the *eigenvector centrality*. The idea of this approach is, that the centrality of a node should be proportional to the centralities of the node's neighbors. Consider the *adjacency matrix* $A(g)_{ij} := (w_{ij})_{ij}$ corresponding to network g . Then, the eigenvector centrality $C_i^{EV}(g)$ of node i in network g is given by the i^{th} entry of the eigenvector corresponding to the largest eigenvalue of A , that is, the (unique) nonnegative solution of

$$A \cdot C^{EV} = \lambda \cdot C^{EV}.$$

3. Cohesion-Indices

Let $g := g(N, w)$ be a network. For simplicity, we will only consider connected networks. Furthermore, to avoid problems in normalizations, we will exclude empty networks (i.e., networks where do not exist any connections). Let $v \in \mathbb{V}_N$ be a characteristic function accounting for cohesion. As a simple example one could consider the so-called *unanimity game* corresponding to T :

$$u_T(K) = \begin{cases} 1 & , \text{ if } T \subseteq K \\ 0 & , \text{ otherwise} \end{cases}$$

where $T \subseteq N$ is the set of essential nodes. We will discuss the unanimity games and other possible such v in the following section.

Now consider the corresponding binary network $g_b := g(N, x)$ and, using the link-game

of v, v^N , assign weights to every link in g_b , that is, create a new weighted network out of the binary network by means of the importance for cohesion. Weights are assigned by either the Shapley value or the Banzhaf value.

Case 1: g is a binary network

Define a new weighted network $g(N, \tilde{w}(v, Y))$ ($Y = Sh$ or $Y = Ba$) with

$$\begin{aligned}\tilde{w}_{ij}(v, Sh) &:= Sh_{ij}(g_b, v^N) \text{ or} \\ \tilde{w}_{ij}(v, Ba) &:= \overline{Ba}_{ij}(g_b, v^N), \text{ respectively}\end{aligned}$$

Case 2: $g = g(N, w)$ is a binary network

In this case, we combine the new weights with the original weights. For this, we need to match the size of the two different weights. As numerical size of indices or weights does not change ranks and relative distances, we normalize w to $v(N)$:

$$\overline{w}_{ij} := \frac{w_{ij}}{\sum_{\lambda \in g} w_{\lambda}} \cdot v(N)$$

Note that $Sh(g_b, v^N)$ and $\overline{Ba}(g_b, v^N)$ are already normalized to $v(N)$ as due to efficiency of the Shapley value and normalization of the Banzhaf value we have

$$\sum_{ij \in g} Sh_{ij}(g_b, v^N) = v^N(g_b) = \sum_{ij \in g} \overline{Ba}_{ij}(g_b, v^N) \text{ and } v^N(g_b) = v(N)$$

where the latter drops from the fact that we assumed g to be connected. Note that for the unanimity game we have $u_T(N) = 1$ for all $T \subseteq N$.

Then, define the new weighted network $g(N, \tilde{w}(v, \alpha, Y))$ ($Y = Sh$ or $Y = Ba$) with

$$\begin{aligned}\tilde{w}_{ij}(v, \alpha, Sh) &:= \alpha \overline{w}_{ij} + (1 - \alpha) Sh_{ij}(g_b, v^N) \text{ or} \\ \tilde{w}_{ij}(v, \alpha, Ba) &:= \alpha \overline{w}_{ij} + (1 - \alpha) \overline{Ba}_{ij}(g_b, v^N), \text{ respectively}\end{aligned}$$

where $\alpha \in [0, 1]$ is the parameter regulating the emphasis of social/economic/political weights and importance for cohesion. Note that

$$\sum_{\lambda \in g} \tilde{w}_{\lambda}(v, \alpha, Y) = \alpha \underbrace{\sum_{\lambda \in g} \overline{w}_{\lambda}}_{=v(N)} + (1 - \alpha) \underbrace{\sum_{\lambda \in g} Y_{\lambda}(g, v^N)}_{=v(N)} = v(N).$$

Remark 1 (α for binary networks). Note that the definition of \tilde{w} in the case of binary networks coincides with the definition for weighted networks for $\alpha = 0$. Hence, for simplicity, we will use the more general definition and set $\alpha = 0$ whenever $g(N, w)$ actually is a binary network.

Definition 1 (Cohesion-Index). A cohesion-index is a function $C : \mathbb{G}_N \times \mathbb{V}_N^0 \rightarrow \mathbb{R}$, where $v \in \mathbb{V}_N^0$ accounts for cohesion.

Remark 2. Note that in contrast to "classical" centrality measures, cohesion-indices also have a cohesion game as an argument.

Now we apply Freeman's centrality measures:

Definition 2 (Cohesion-Degree-Index). For every network $g(N, w)$, the Shapley-Cohesion-Degree-index CD_j^{Sh} and the Banzhaf-Cohesion-Degree-index CD_j^{Ba} are given by

$$CD_i^Y(g, v) := C_i^d(g(N, \tilde{w}(v, \alpha, Y))) = \sum_{j \in N \setminus \{i\}} \tilde{w}_{ij}(v, \alpha, Y)$$

where $Y = Sh$ or $Y = \overline{Ba}$, respectively.

For comparability (and in order to define a feasible/efficient allocation rule), we normalize the indices. This is by multiplication with

$$\frac{v(N)}{\sum_{j \in N} CD_j^{Sh}(g, v)} \text{ or } \frac{v(N)}{\sum_{j \in N} CD_j^{Ba}(g, v)}, \text{ respectively.}$$

Note that we have

$$\begin{aligned} \sum_{j \in N} CD_j^Y(g, v) &= \sum_{j \in N} \sum_{k \in N \setminus \{j\}} \left[\alpha \overline{w}_{jk} + (1 - \alpha) Y_{jk}(g_b, v^N) \right] \\ &= 2 \cdot \left[\alpha \sum_{\lambda \in g} \overline{w}_\lambda + (1 - \alpha) \sum_{\lambda \in g} Y_\lambda(g_b, v^N) \right] \\ &= 2 \cdot \sum_{\lambda \in g} \tilde{w}_\lambda(v, \alpha, Y) = 2 \cdot v(N) \end{aligned}$$

Remark 3. For $\alpha = 0$, the normalized Shapley-Cohesion-Degree index coincides with the Position value, introduced by Meessen (1988) and further analyzed by Borm et al. (1992) and Slikker (2005).

Remark 4 (Relation to Closeness and Betweenness). Our approach uses the corresponding Shapley/Banzhaf value of links, that is, we calculate how important a certain connection is for the cohesion of a network. This is done by taking into account all other connections of the network, hence, these values capture the idea of (relative) necessity of connections which is the basic idea of betweenness. Taking into account the actual value captures the idea of closeness. Hence, one could argue that the Cohesion-Degree indices are sufficient as the position between other nodes and relative necessity are captured by the weights for importance of cohesion. This can further be regulated by the emphasis parameter α .

However, applying closeness and betweenness might still be of interest and we provide the formal definitions for completeness.

Definition 3 (Cohesion-Closeness-Index). *For every network $g(N, w)$, the Shapley-Cohesion-Closeness-index CC^{Sh} and the Banzhaf-Cohesion-Closeness-index CC^{Ba} are given by*

$$CC_i^Y(g, v) := C_i^c(g(N, \tilde{w}(v, \alpha, Y))) = \left[\sum_{j \in N \setminus \{i\}} d^{\tilde{w}(v, \alpha, Y)}(i, j) \right]^{-1}$$

where $Y = Sh$ or $Y = \overline{Ba}$, respectively.

Normalization is done by

$$\frac{v(N)}{\sum_{j \in N} CC_j^{Sh}(g, v)} \text{ or } \frac{v(N)}{\sum_{j \in N} CC_j^{Ba}(g, v)}, \text{ respectively.}$$

Definition 4 (Cohesion-Betweenness-Index). *For every network $g(N, w)$, the Shapley-Cohesion-Betweenness-index CB^{Sh} and the Banzhaf-Cohesion-Betweenness-index CB^{Ba} are given by*

$$CB_i^Y(N, v, g) := C_i^b(g(N, \tilde{w}(v, \alpha, Y))) = \sum_{j \neq i \neq k}^N \frac{|d^{\tilde{w}(v, \alpha, Y)}(j, k)(i)|}{|d^{\tilde{w}(v, \alpha, Y)}(j, k)|}$$

where $Y = Sh$ or $Y = \overline{Ba}$, respectively.

Normalization (for $|N| > 2$) is done by

$$\frac{v(N)}{\sum_{j \in N} CB_j^{Sh}(g, v)} \text{ or } \frac{v(N)}{\sum_{j \in N} CB_j^{Ba}(g, v)}, \text{ respectively.}$$

Note that Opsahl et al. (2010) mention that the implementation of Dijkstra's algorithm (inverting weights) is unproblematic if we are only interested in identifying the shortest path in a weighted graph as for the betweenness measure but that this becomes problematic in interpretation of the numerical results if we take into account the actual "length" of such a path of least costs/least resistance as it is done for the closeness measure. We avoid this problem as we normalize the indices.

Remark 5. *For $\alpha = 1$, the normalized Cohesion indices coincide with Freeman's centrality measures.*

Definition 5 (Cohesion-Eigenvector-Index). *For every network $g(N, w)$, the Shapley-Cohesion-Eigenvector-index CEV^{Sh} and the Banzhaf-Cohesion-Eigenvector-index CEV^{Ba} are given by*

$$CEV_i^Y(N, v, g) := C_i^{EV}(g(N, \tilde{w}(v, \alpha, Y)))$$

where $Y = Sh$ or $Y = \overline{Ba}$, respectively.

That is, CEV^Y is given by the unique nonnegative solution of

$$(\tilde{w}(v, \alpha, Y)_{ij})_{ij} \cdot CEV^Y = \lambda \cdot CEV^Y.$$

Again, one can normalize by $\sum_{j \in N} \frac{v(N)}{CEV_j^Y(g, v)}$.

4. Properties of Cohesion-Indices

4.1. Axiomatic Characterizations

In this section, we provide axiomatic characterizations of the two Cohesion-Degree-Indices for binary networks ($\alpha = 0$). For simplicity, we define:

Definition 6 (Banzhaf Position value). *For any TU game with a network structure (N, v, g) , the Banzhaf Position value π^{Ba} is given by*

$$\pi^{Ba}(N, v, g)_i := \sum_{\lambda \in g_i} \frac{1}{2} Ba_\lambda(g, v^N).$$

We first introduce an axiom which could be seen as the driving force of the (Shapley) Position value (cf. Borm et al. (1992)): A game (N, v, g) is called *link anonymous* if $\exists f : \{0, 1, \dots, |g|\} \rightarrow \mathbb{R}$ such that $v^N(g') = f(|g'|)$ for all $g' \subseteq g$. An allocation rule for network structures satisfies the *Degree-Property (DEG)*, if for all link anonymous games (N, v, g) there exists $\alpha \in \mathbb{R}$ such that

$$Y(N, v, g) = \alpha \cdot C^d(g).$$

In a link anonymous game, we have $v^N(g' \setminus \lambda) = v^N(g' \setminus \lambda')$ for all $\lambda, \lambda' \in g' \subseteq g$, hence, all links obtain the same marginal contributions and are therefore equally important within the network. One could argue, that in this case the strength of a node should be measured by its degree. If an allocation rule is used for centrality issues, **DEG** seems a favourable axiom.

Example 1. Consider $N = \{1, 2, 3\}$, $g = \{13, 23\}$ and the unanimity game $u_{\{1, 2\}}$. Note that for

$$f : \{0, 1, 2, 3\} \rightarrow \mathbb{R}, f(x) := \begin{cases} 1, & \text{if } x = 3 \\ 0, & \text{else} \end{cases}$$

it holds that $u_{\{1, 2\}}^N(g') = f(|g'|)$ for all $g' \subseteq g$, that is, (N, v, g) is link anonymous. The Shapley value, the normalized Banzhaf value and the Myerson value are given by

$$Sh(N, v, g) = \left(\frac{1}{2}, \frac{1}{2}, 0\right) = \overline{Ba}(N, v, g), \mu(N, v, g) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

while the Degree-measure is given by $C^d(g) = (1, 1, 2)$, hence, (DEG) is not satisfied by the Shapley value, the Banzhaf value and the Myerson value.

On the other hand, the (Shapley) Position value and the Banzhaf Position value are given by

$$\pi^{Sh}(N, v, g) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right), \pi^{Ba}(N, v, g) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{4}\right).$$

that is, they are proportional to the Degree-measure.

Borm et al. (1992) show that the (Shapley) Position value satisfies **DEG**.

Lemma 1. *The Banzhaf Position value satisfies **DEG**.*

Find the proof in the appendix.

The next axiom is used to characterize the (Shapley) Position value on cycle-free graphs (cf. Borm et al. (1992)): A link λ is called *superfluous*, if $v^N(g' \cup \lambda) = v^N(g')$ for all $g' \subseteq g$, that is, if λ is a Nullplayer in (g, v^N) . An allocation rule for network structures Y satisfies the *Superflous link Property (SLP)* if $Y(n, v, g) = Y(N, v, g \setminus \lambda)$ for all superflous links λ .

Borm et al. (1992) show, that the (Shapley) position value is on cycle-free graphs uniquely determined by **DEG**, **SLP** and the following axioms:

- Additivity **A**: $Y(N, v + w, g) = Y(N, v, g) + Y(N, w, g)$ for all $v, w \in \mathbb{V}_N$
- Component Efficiency **CE**: $\sum_{i \in C} Y_i(N, v, g) = v(C)$ for all $C \in \mathcal{G}(N, g)$

As a difference to the (Shapley) Position value, the Banzhaf Position value does not satisfy component efficiency. To close the emerging gap, we define the following axiom:

Definition 7. *Component Link Banzhaf Efficiency (CLBE)* An allocation rule for network structures Y satisfies **CLBE** if we have for $C \in \mathcal{G}(N, g)$:

$$\sum_{i \in C} Y_i(N, v, g) = \sum_{\lambda \in g|_C} Ba_\lambda(g|_C, v^N|_C).$$

CLBE states, that the total link-power (due to Banzhaf) of a component should be distributed among players within the component. For connected networks, this is in the same spirit as Banzhaf Efficiency, the crucial axiom of the characterization of the Banzhaf value, just with respect to link contributions.

Theorem 1. *The Banzhaf Position value is uniquely determined by **A**, **DEG**, **SLP** and **CLBE** for all cycle-free graphs.*

Find the proof in the appendix.

Remark 6. *On cycle-free graphs the following holds.*

1. *The Harsanyi-divident representation of π^{Ba} is*

$$\pi_i^{Ba}(N, v, g) = \sum_{T \subseteq 2^N \setminus \{\emptyset\}} \frac{\lambda_T(v)}{2^{|g|_{H(T)}}} C_i^d(g|_{H(T)}),$$

where $\lambda_T(v)$ is the Harsanyi divident of v corresponding to T . Hence, the Banzhaf position value is given by the degree measures of the connected hulls of v 's basis representations, weighted accordingly.

2. The normalized Banzhaf Position value of any unanimity game is given by

$$\overline{\pi_i^{Ba}}(N, \beta u_T, g) := \frac{\pi_i^{Ba}(N, \beta u_T, g)}{\sum_{j \in N} \pi_j^{Ba}(N, \beta u_T, g)} \cdot v(N) = \frac{|g_i|_{H(T)}}{2 \cdot |g|_{H(T)}} \cdot \beta$$

which coincides with the (Shapley) Position value $\pi_i^{(Sh)}(N, \beta u_T, g)$. This is due to the fact, that on cycle-free graphs, only the connected hull matters and there, all links are equally important.

Slikker (2005) introduces the following axiom which, together with component efficiency **CE**, uniquely determines the (Shapley) Position value for general graphs: An allocation rule for network structures Y satisfies *Balanced Link Contributions* (**BLC**) if we have

$$\sum_{\lambda \in g_j} [Y_i(N, v, g) - Y_i(N, v, g \setminus \lambda)] = \sum_{\lambda \in g_i} [Y_j(N, v, g) - Y_j(N, v, g \setminus \lambda)]$$

for all $i, j \in N$, $i \neq j$, and $v \in \mathbb{V}_N^0$.

BLC states that the total threat of a player towards another player should be equal to the reverse total threat. Using **BLC**, we find the following general characterization of the Banzhaf Position value:

Theorem 2. *The Banzhaf Position value π^{Ba} is uniquely determined by **BLC** and **CLBE**.*

Find the proof in the appendix.

4.2. Cohesion-Games and General Properties

In order to calculate relative importance for cohesion of essential nodes, we only explicitly considered unanimity game so far. The unanimity game is simple in calculations and a natural and intuitive way to define a game accounting for cohesion of a *specific* set of nodes. Michalak et al. (2013) discuss other possibilities for the characteristic function which are more general as they do not rely on a specific set of nodes; here, the number of nodes that are reachable by path from a certain coalition is counted:

$$v(K) = \# \text{ of nodes in } K \text{ and those (directly) connected to } K$$

where one could either analyze direct connections or paths with at most k steps. In order to match this value function with our framework, we have to zero-normalize the function and transform the weights correspondingly. This can be done for any characteristic function by the following transformation:

Transformation 1. For any characteristic function $v \in \mathbb{V}_N$, proceed as following

Step 1: Zero-normalization:

$$v_0(K) := v(K) - \sum_{i \in N} v(\{i\}) \cdot u_{\{i\}}(K)$$

Step 2: Transformation of weights: Set

$$Y_{ij}(g_b, v^N) := Y_{ij}(g_b, v_0^N) + v^N(\{ij\}),$$

where $Y = Sh, \overline{Ba}$.

Following the idea of reachability, we define another possible cohesion game which directly accounts for the network structure:

Definition 8 (Cohesion Game). Given a network $g \in \mathbb{G}_N^b$, we define the corresponding cohesion game $c : \mathbb{G}_N \rightarrow \mathbb{R}$ as follows:

$$c(g, K) := \sum_{i \in K} c_i(g) \quad \forall K \subseteq N$$

where $c_i(g) := \# \text{ of nodes reachable from } i \text{ by path in } g$

Note, that we do not need a zero-normalization for the cohesion game (as $c(\emptyset) = 0$) or a transformation via the link-game as it directly works on networks (that is, $c \in \mathbb{V}_{\mathbb{G}_N}$).

Remark 7. Note that the unanimity games form a basis of \mathbb{V}_N and both the Shapley value and the Banzhaf value are linear in the characteristic function, hence, we can express any $v \in \mathbb{V}_N$ by unanimity games. Therefore, we will first focus on unanimity games.

Computational complexity for calculating shortest path using Dijkstra (1959)'s algorithm has widely been discussed. Hence, we will only discuss computational complexity for calculating importance for cohesion. Taking the unanimity game as a basis, the calculation is mainly about checking whether a subnetwork is connected or not.

Theorem 3 (Computational Complexity of weights). For $v = u_N$, computational complexity is at most

$$|g| \cdot \sum_{k=|T|-1}^{|g|-1} \binom{|g|-1}{k}$$

connectivity checks for the Banzhaf value and for the Shapley value it is the same connectivity checks, just that for every "yes"-labeled item, one number has to be stored.

Find the proof in the appendix.

Remark 8. NP-Hardness Connectivity checks are made by depth-first search algorithms. It is well known, that worst case performance of a depth-first-search is $O(|N|)$, that is,

not NP-hard. However, the number of connectivity checks needed might get of higher than polynomial complexity.

Theorem 4 (Proportionality). *Let g be a cycle-free connected binary graph/network (i.e., a minimal spanning tree) and consider the unanimity game βu_T , $T \subseteq N$, $|T| > 1$. Then, we have for $Y = Sh$ or $Y = Ba$*

$$\begin{aligned} \overline{CD}_i^Y(g, \beta u_T) &= \overline{C^d}_i(g|_{H(T)}), & \overline{CC}_i^Y(g, \beta u_T) &= \overline{C^c}_i(g|_{H(T)}) \\ \overline{CB}_i^Y(g, \beta u_T) &= \overline{C^b}_i(g|_{H(T)}) \text{ and} & \overline{CEV}_i^Y(g, \beta u_T) &= \overline{C^{EV}}_i(g|_{H(T)}) \end{aligned}$$

where $H(T)$ is the connected hull of T , that is, set of nodes that are essential to connect T .

Find the proof in the appendix.

Theorem 5. *If g has star structure with $i \in N$ being the hub, then*

$$\begin{aligned} CD_i^Y(g, v) &= \max_{g' \in \mathbb{G}_N} CD_i^Y(g', v) \text{ and} \\ CB_i^Y(g, v) &= \max_{g' \in \mathbb{G}_N} CB_i^Y(g', v) \end{aligned}$$

for $Y = Sh$ or $Y = \overline{Ba}$ and any zero normalized v .

Proof. We know that the sum of weights over all links is $v(N)$ (for both the Shapley and the normalized Banzhaf value). Hence, for the Cohesion-Degree-indices the maximal value obtainable is $v(N)$ which is the case for hub i . As the index is maximal, the same holds for its normalization.

The Cohesion-Betweenness-indices are maximal if $|d^{\tilde{w}}(j, k)(i)| = |d^{\tilde{w}}(j, k)|$ for all (j, k) , that is, if i lies on all shortest paths between any two nodes. This is the case if i is the hub of a star (there is only one path between any two nodes). As the index is maximal, the same holds for its normalization. \square

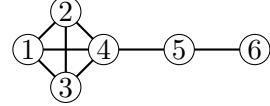
Remark 9. *For the closeness measure for weighted graphs, maximality is not clear as the problem of inverted weights as costs of paths of least resistance occurs. Here, not the actual weights are taken into account, one takes the sum of inverses which is unproportional to the corresponding sum of weights. Hence, one could argue that the closeness measure might not be a good option in all applications.*

5. Exclusiveness of Links, Top Nodes and Application

5.1. Example: Exclusiveness of Links and Identifying Top Nodes

Let $N = \{1, 2, 3, 4, 5, 6\}$ and consider the binary network:

$$g = \{12, 13, 14, 23, 24, 34, 45, 56\}$$



Let $v = u_N$, that is, we are interested in cohesion of the whole network. Weights are displayed in Table 3.

Table 3: Weights

λ	12, 13, 14, 23, 24, 34	45, 56
Sh_λ	17/420	53/140
Ba_λ	10/136	38/136

Results for the Degree measure and Eigenvector centrality and the corresponding Cohesion indices are given in Table 4.

Table 4: Classical Centrality Measures and Cohesion Indices (in share of 100%)

Y	1	2	3	4	5	6
Degree and Cohesion-Degree						
\overline{C}^d ($\alpha = 1$)	18.75	18.75	18.75	25.00*	12.50	6.25
\overline{CD}^{Sh} ($\alpha = 0$)	6.07	6.07	6.07	25.00	37.86*	18.93
\overline{CD}^{Ba} ($\alpha = 0$)	11.03	11.03	11.03	25.00	27.94*	13.97
Eigenvector and Cohesion-Eigenvector						
\overline{C}^{EV} ($\alpha = 1$)	21.65	21.65	21.65	23.74*	8.56	2.76
\overline{CEV}^{Sh} ($\alpha = 0$)	2.43	2.43	2.43	27.48	38.28*	26.93
\overline{CEV}^{Ba} ($\alpha = 0$)	7.04	7.04	7.04	25.38	31.87*	21.61

Top-2-nodes are bold face, * identifies Top-node

This example does not only show numerical differences, also the ranking between nodes changes. Consider the top 2 nodes: in contrast to the Degree or Eigenvector measure ($\alpha = 1$), node 5 is under the top 2 nodes for the Cohesion-Indices ($\alpha = 0$) and even the

top node changes. From an interpretative point of view, it is reasonable that nodes 4 and 5 should obtain the highest value in terms of centrality and cohesion as node 4 is the most central while 5 is essential for cohesion.

One could argue that node 6 and node 1 (or 2 or 3) should obtain the same payoff as there is no structural reason (as connection node versus boundary node) to treat them differently. This can be regulated by the emphasis parameter α . If α decreases, one could argue that, as cohesion emphasis $(1 - \alpha)$ increases, node 6 should obtain a higher payoff than 1 (or 2 or 3) due to the "exclusiveness" of node 6: there is only one connection to node 6, hence an exclusive connection which could be seen as more important than other connections. If emphasis for centrality increases, 1 (or 2 or 3) should obtain a higher payoff than 6 due to a higher centrality in terms of connections (or original weights for weighted graphs).

Remark 10 (Gametheoretic Approaches). *Suri and Narahari (2008) and Gómez et al. (2003) suggest to identify top nodes by computing the network-restricted Shapley value (Myerson, 1977) of the nodes (or its difference to the original Shapley value) while no specific game is suggested. Considering the game u_N and our example, both the Shapley value and the network-restricted Shapley value assign an equal value of $1/6$ to every node. This stands in contradiction to centrality (4 and 5 are more central than 1, 2 and 3 and all more central than 6) and furthermore the difference between the connections 1-4 and 5-6 (exclusiveness) is ignored. One can argue that the drawbacks arise due to the too simple form of the unanimity game we use. However, independently of the characteristic function, the concepts suggested above only analyze failure of a whole node with all its connections at once.*

5.2. Application: State Parliament Elections

As an application for political networks, let us consider the state parliament elections (*Bürgerschaftswahl*) in Hamburg, Germany in 2001. After the election, there were five parties obtaining seats in the parliament¹, namely the Social Democratic Party "SPD", the Christian Democratic Union "CDU", the Conservative Law and Order Party "Schill", the Green/Alternative Party "Grüne" and the Free Democratic Party "FDP". The distribution of seats was according to Table 5. To build the government, a coalition needs

Table 5: Results of the state parliament elections in Hamburg, 2001

party	SPD	CDU	Schill	Grüne	FDP	\sum
seats	46	33	25	11	6	121
seat share	0.38	0.27	0.21	0.09	0.05	1

source: Statistical Office of Hamburg and Schleswig-Holstein

¹Due to regulations, parties obtaining a vote share less than 5% are not going to be in the parliament.

at least 50% of the seats². In the end, the government was built by the coalition $\{\text{CDU}, \text{Schill}, \text{FDP}\}$ ³. We are now interested in the question, if this outcome could have been forecasted by the use of centrality measures.

For notational reasons, let us denote the parties by $N = \{1, 2, 3, 4, 5\}$ (i.e., SPD is player 1, CDU is player 2,...). The situation can be modelled by a simple weighted voting game: assign to every player $i \in N$ the corresponding seat share s_i . Then, the characteristic function is given by

$$v(K) := \begin{cases} 1 & , \text{ if } \sum_{i \in K} s_i \geq 0.5 \\ 0 & , \text{ otherwise} \end{cases}.$$

Minimal winning coalitions are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$. Applying the Shapley value or the normalized Banzhaf value leads to a distribution of power and resulting coalitional power displayed in Table 6, "Unrestricted Case".

In fact, a coalition between "CDU" and "Grüne" as well as between "Schill" and "Grüne" was excluded by the parties due to ideological/political incompatibilities. This means, that (minimal winning) coalition $\{2, 3, 4\}$ would not materialize as well as all $K \subseteq N$ such that $\{2, 4\} \subseteq K$ or $\{3, 4\} \subseteq K$ would not be winning coalitions. To account for these incompatibilities, we restrict the space of potential swings⁴: any set including $\{2, 3\}$ or $\{3, 4\}$ is excluded as potential swing for $i \in \{1, 5\}$, any set including $\{4\}$ is excluded for $i \in \{2, 3\}$ and any set including $\{2\}$ or $\{3\}$ is excluded for $i = 4$. Due to this restriction, Shapley-based approaches are not suitable anymore as the weights of the marginal contributions would be unappropriate, hence, we will only use the Banzhaf-based approaches from now on⁵. Distribution of power and resulting coalitional power for restricted case is displayed in Table 6, "Restricted Case"⁶.

As we see, the coalitions obtaining the highest coalitional power are $\{1, 2\}$ and $\{1, 3\}$ in the unrestricted case (for both the Shapley and the normalized Banzhaf value) and $\{1, 2\}$, $\{1, 3\}$ and $\{1, 4, 5\}$ in the restricted case, which does not explain the resulting coalition $\{2, 3, 5\}$.

Now, take a look at the corresponding political network where we could interpret a link as a potential coalitional negotiation. We could argue that relative political power of a party should depend on the seats obtained *and* the position within the political network. Taking into account the incompatibilities, the network could be modelled as presented by Figure 2.

Applying the classical centrality measures (without accounting for the weighted voting

²Parties are assumed to vote en bloc

³For completeness one should note that this coalition broke 2 years later due to upcoming personal issues between the leaders of "CDU" and "Schill". However, these issues have not been known after the elections and hence, can not be taken into account for forecasting issues.

⁴Here, $K \subseteq N \setminus \{i\}$ is a swing for i iff $v(K \cup \{i\}) - v(K) = 1$.

⁵Note that for voting games, one usually uses the Banzhaf approach to avoid the different weights of swings.

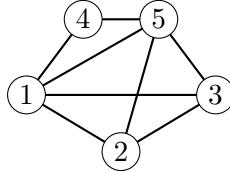
⁶For further details consult the appendix, Table 8.

Table 6: Gametheoretic Approaches

Distribution of Power (normalized to 100 %)					
party	1	2	3	4	5
Unrestricted Case					
Sh_i	40.00	23.33	23.33	6.67	6.67
Ba_i	38.46	23.08	23.08	7.69	7.69
Restricted Case					
Ba_i	40.00	20.00	20.00	6.67	13.33
Resulting Coalitional Power					
coalition	{1, 2}	{1, 3}	{1, 4, 5}	{2, 3, 4}	{2, 3, 5}
Unrestricted Case					
$\sum Sh_i$	63.33	63.33	53.34	53.33	53.33
$\sum Ba_i$	61.54	61.54	53.84	53.85	53.85
Restricted Case					
$\sum Ba_i$	60.00	60.00	60.00	-	53.33

(Highest coalitional power is bold face.)

Figure 2: Political Network



game) is not appropriate as in this case, there is no difference between parties 1 and 5 while the relative power of party should depend on the seats obtained.⁷ The Cohesion-Indices account for both the seat share and the position within the political network. We restrict the set of potential swings for a connection containing 2 or 3 to the set of connections not containing 4 and for a connection containing 4 to the set of connections not containing 2 or 3.⁸ Find the distribution of individual and coalitional power dis-

⁷However, only Eigenvector centrality uniquely selects coalition {2, 3, 5} obtaining the highest coalitional power. For completeness, find the values for the classic approaches in Table 9, appendix.

⁸One could argue that this is too restrictive and connections containing 4 can still be part of a swing for a connection containing 2 or 3 as long as no connected component resulting in excluded coalitions is build (and correspondingly for swings for connections containing 2 or 3). Note that this less restrictive case the order of highest individual and coalitional power does not change. For completeness, find details for this case in the appendix, Table 8 and Table 10.

played in Table 7.⁹

Table 7: Cohesion Centrality Approaches

Distribution of Power (normalized to 100 %)					
party	1	2	3	4	5
CD^{Ba}_i	30.00	20.00	20.00	6,67	23.33
CEV^{Ba}_i	27.04	21.81	21.81	7.28	22.05
Resulting Coalitional Power					
coalition	{1, 2}	{1, 3}	{1, 4, 5}	{2, 3, 4}	{2, 3, 5}
$\sum CD^{Ba}_i$	50.00	50.00	60.00	-	63.33
$\sum CEV^{Ba}_i$	48.85	48.85	56.37	-	65.67

(Highest individual and coalitional power are bold face.)

We see that, taking into account both centrality and relative importance for the voting game (cohesion), the coalition $\{2, 3, 5\}$ is uniquely selected with respect to highest coalitional power. Hence, our new approach could have been considered for forecasting issues leading to the coalition which was actually built. Note that the other approaches either do not forecast the actual coalition or are not suitable as the seat share was not taken into account. and Schill.

6. Conclusion

We suggest a new approach for centrality measures of networks, the cohesion-indices, accounting for the importance for cohesion and relative power of connections using cooperative game theory. In contrast to existing (cooperative) game theoretic approaches, we analyze consequences of connection failures rather than failures of whole nodes. This makes our approach suitable in applications in for example energy networks or political networks: oil pipelines can break (connection failure) without a breakdown of the whole gas province (node failure), a bilateral trading agreement can be broken without a whole country leaving the trading union or, as in the political example we used, political parties can have bilateral incompatibilities without leaving the whole political spectrum. Consequences of connection failure can be capacity overloads resulting in blackouts for energy networks or a (partly or complete) breakdown of an Economic and Monetary Union due to a failure of certain trading routes, monetary flows or upcoming incompatibilities between parties. Furthermore, our approach is suitable for weighted networks.

Beside the definition of cohesion-indices and the political application of this approach in the case of the state parliament elections in Hamburg, Germany 2001, we also provide

⁹For further details consult the appendix, Table 8.

axiomatic characterizations for the Cohesion-Degree approaches for both cycle-free and general binary networks, discussed specific cohesion games and analyzed computational complexity.

For further research, we plan to analyze axiomatic characterizations for the Cohesion-Eigenvector indices and a generalization of the Cohesion-Degree characterization to weighted networks. Furthermore, an application for weighted networks might be of interest.

References

Banzhaf, J., 1952. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review* 19 (2), 317–343.

Barrat, A., Barthélémy, M., Pastor-Satorras, R., Vespignani, A., 2004. The architecture of complex weighted networks. *Proceedings of the National Academy of Sciences* 101 (11), 3747–3752.

Bonacich, P., 1972. Factoring and weighing approaches to status scores and clique identification. *Journal of Mathematical Sociology* 2(1), 113–120.

Borm, P., Owen, G., Tijs, S., 1992. On the position value for communication situations. *SIAM Journal of Discrete Mathematics* 5, 305–320.

Brandes, U., 2001. A faster algorithm for betweenness centrality. *Journal of Mathematical Sociology* 25, 163–177.

Dijkstra, E. W., 1959. A note on two problems in connexion with graphs. *Numerische Mathematik* 1, 269–271.

Freeman, L. C., 1978. Centrality in social networks: Conceptual clarification. *Social Networks* 1, 215–239.

Freeman, L. C., Borgatti, S. P., White, D., 1991. Centrality in valued graphs: A measure of betweenness based on network flow. *Social Networks* 13, 141–154.

Gómez, D., González-Arangüena, E., Manuel, C., Owen, G., Del Pozo, M., Tejada, J., 2003. Centrality and power in social networks: A game theoretic approach. *Mathematical Social Sciences* 46(1), 27–54.

Grofman, B., Owen, G., 1982. A game theoretic approach to measuring degree of centrality in social networks. *Social Networks* Vol 4(3), 213–224.

Meessen, R., 1988. Communication games. Master's thesis, Department of Mathematics, University of Mijmegen, The Netherlands.

Michalak, T., Aadithya, K., Szcześniak, P., Ravindran, B., Jennings, N., 2013. Efficient computation of the shapley value for game-theoretic network centrality. *Journal of Artificial Intelligence Research* 46, 607–650.

Myerson, R., 1977. Graphs and cooperations in games. *Mathematics of Operation Research* 2, 225–229.

Newman, M. E. J., 2001. Scientific collaboration networks. ii. shortest paths, weighted networks, and centrality. *Physical Review E* 64, 016132.

Newman, M. E. J., 2004. Analysis of weighted networks. *Physical Review E* 70, 056131.

Opsahl, T., Agneessens, F., Skvoretz, J., 2010. Node centrality in weighted networks: Generalizing degree and shortest paths. *Social Networks* 32 (3), 245–251.

Opsahl, T., Colizza, V., Panzarasa, P., Ramasco, J. J., 2008. Prominence and control: The weighted rich-club effect. *Physical Review Letters* 101 (168702).

Owen, G., 1975. Multilinear extensions and the banzhaf value. *Naval Research Logistic Quarterly* 22, 741–750.

Owen, G., 1986. Values of graph restricted games. *SIAM Journal of Algebraic Discrete Methods* 7, 210–220.

Shapley, L., 1953. A value for n-person games. In: Kuhn, H., Tucker, A. (Eds.), *Contributions to the Theory of Games*, Volume II, Princeton University Press, 307–317.

Slikker, M., 2005. A characterization of the position value. *International Journal of Game Theory* 33 (4), 504–514.

Suri, N., Narahari, Y., 2008. Determining the top-k nodes in social networks using the shapley value. *AAMAS 2008: Proceedings of the Seventh International Joint Conference on Autonomous Agents and Multi-Agent Systems*, 1509–1512.

A. Appendix

A.1. Proofs for Section 4

Proof of Lemma 1:

The Banzhaf Position value satisfies **DEG**.

Proof. Let (N, v, g) be link anonymous, that is, there exists $f : \{0, 1, \dots, |g|\} \rightarrow \mathbb{R}$ such that $f(|g'|) = v^N(g')$ for all $g' \subseteq g$. For $g = \emptyset$, we have $\pi_i^{Ba}(N, v, g) = 0 = C^d(g)$. Let $g \neq \emptyset$ and

$\lambda \in g$.

$$\begin{aligned}
Ba_\lambda(g, v^N) &= \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} (v^N(g' \cup \lambda) - v^N(g')) = \frac{1}{2^{|g|-1}} \sum_{i=0}^{|g \setminus \lambda|} \sum_{\substack{g' \subseteq g \setminus \lambda \\ |g'|=i}} (f(i+1) - f(i)) \\
&= \frac{1}{2^{|g|-1}} \underbrace{\sum_{i=0}^{|g|-1} \binom{|g|-1}{i} (f(i+1) - f(i))}_{:=A \in \mathbb{R} \text{ independent of } \lambda} \\
\Rightarrow \pi_i^{Ba}(N, v, g) &= \sum_{\lambda \in g_i} \frac{A}{2^{|g|}} = \frac{A}{2^{|g|}} |g_i| = \underbrace{\frac{A}{2^{|g|}}}_{:=\alpha \in \mathbb{R}} C^d(g)
\end{aligned}$$

□

Proof of Theorem 1:

The Banzhaf Position value is uniquely determined by **A**, **DEG**, **SLP** and **CLBE** for all cycle-free graphs.

Proof. Existence: **A** is clear and **DEG** has been shown in Lemma 1.

- **SLP:** Let λ be superfluous in game (N, v) . By the Banzhaf value satisfying the Nullplayer axiom, we then have $Ba_\lambda(g, v^N) = 0$. For $\nu \neq \lambda \in g$ we have

$$\begin{aligned}
Ba_\nu(g, v^N) &= \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \nu} [v^N(g' \cup \nu) - v^N(g')] \\
&= \frac{1}{2^{|g|-1}} \left[\sum_{\substack{g' \subseteq g \setminus \nu \\ \lambda \in g'}} \underbrace{[v^N(g' \cup \nu) - v^N(g')]}_{=v^N((g' \setminus \lambda) \cup \nu) - v^N(g' \setminus \lambda)} + \sum_{g' \subseteq (g \setminus \lambda) \setminus \nu} [v^N(g' \cup \nu) - v^N(g')] \right] \\
&= \frac{2}{2^{|g|-1}} \sum_{g' \subseteq (g \setminus \lambda) \setminus \nu} [v^N(g' \cup \nu) - v^N(g')] = Ba_\nu(g \setminus \lambda, v^N)
\end{aligned}$$

and hence, $\pi^{Ba}(N, v, g) = \pi^{Ba}(N, v, g \setminus \lambda)$.

- **CLBE:** Let $C \in \mathcal{G}(N, g)$.

$$\begin{aligned}
\sum_{i \in C} \pi_i^{Ba}(N, v, g) &= \sum_{i \in C} \sum_{\lambda \in g_i} \frac{1}{2} \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} (v^N(g' \cup \lambda) - v^N(g')) \\
&= \sum_{\lambda \in g|_C} \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} (v^N(g' \cup \lambda) - v^N(g')) \\
&= \sum_{\lambda \in g|_C} \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} \underbrace{[v^N((g' \cap g|_C) \cup \lambda) - v^N(g' \cap g|_C)]}_{\text{for } \lambda \in g|_C, \text{ marginal contributions are not effected by connections outside } C} \\
&= \sum_{\lambda \in g|_C} \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} \underbrace{[v^N(g'|C \cup \lambda) - v^N(g'|_C)]}_{\text{the same for all } g', g'' \subseteq g \setminus \lambda \text{ such that } g'|_C = g''|_C} \\
&= \sum_{\lambda \in g|_C} \frac{1}{2^{|g|-1}} \sum_{\tilde{g} \subseteq (g|_C) \setminus \lambda} 2^{|g|_{N \setminus C}|} [v^N(\tilde{g} \cup \lambda) - v^N(\tilde{g})] \\
&\stackrel{|g|=|g|_{N \setminus C}|+|g|_C|}{=} \sum_{\lambda \in g|_C} \frac{1}{2^{|g|_C|-1}} \sum_{\tilde{g} \subseteq (g|_C) \setminus \lambda} [v^N(\tilde{g} \cup \lambda) - v^N(\tilde{g})] \\
&= \sum_{\lambda \in g|_C} Ba_\lambda(g|_C, v^N|_C)
\end{aligned}$$

Uniqueness¹⁰: Let Y satisfy **A**, **DEG**, **SLP** and **CLBE** and g be cycle free. By **A**, it is sufficient to show that $Y(N, \beta u_T, g) = \pi(N, \beta u_T, g)$ for all $\beta \in \mathbb{R}$ and $T \in 2^N$ such that $|T| \geq 2$. Let such β, T be arbitrary but fixed.

Case 1: $\nexists C \in \mathcal{G}(N, g)$ such that $T \subseteq C$.

That is, there exists $i, j \in T$ being unconnected in g and hence, $\beta u_T^N(g') = 0$ for all $g' \subseteq g$. Therefore, $Ba_\lambda(g, \beta u_T) = 0$ for all $\lambda \in g$ and $\pi_i^{Ba}(N, \beta u_T, g) = 0$ for all $i \in N$. On the other hand, if $\beta u_T(g') = 0$ for all $g' \subseteq g$, every $\lambda \in g$ is superfluous, and hence, by **SLP**, we have $Y(N, \beta u_T, g) = Y(N, \beta u_T, g \setminus \lambda_1) = Y(N, \beta u_T, g \setminus \{\lambda_1, \lambda_2\}) = \dots = Y(N, \beta u_T, \emptyset)$.

Trivially, the game $(N, \beta u_t, \emptyset)$ is link anonymous and hence, by **DEG**, there exists $\alpha \in \mathbb{R}$ such that

$$\begin{aligned}
Y_i(N, \beta u_T, g) &= Y_i(N, \beta u_t, \emptyset) = \alpha \cdot C_i^d(\emptyset) = 0 \quad \forall i \in N \\
\Rightarrow Y(N, \beta u_T, g) &= \pi^{Ba}(N, \beta u_T, g)
\end{aligned}$$

Case 2: $\exists C \in \mathcal{G}(N, g)$ such that $T \subseteq C$.

Consider the (unique) *connected hull* (cf. Owen (1986)) of T , $H(T)$, given by

$$H(T) := \bigcap \{S \mid T \subseteq S \subseteq C \text{ such that } g|_S \text{ is connected subgraph}\}$$

¹⁰We follow the idea of the proof for the (Shapley) position value of Borm et al. (1992).

As g is cycle-free, $H(T)$ is the minimal set of nodes that are essential to connect T .¹¹

$$\Rightarrow \beta u_T^N(g') = \begin{cases} \beta & , \text{ if } g|_{H(T)} \subseteq g' \\ 0 & , \text{ otherwise} \end{cases}$$

If $\lambda \notin g|_{H(T)}$, we have $Ba_\lambda(g, \beta u_T^N) = 0$. For $\lambda \in g|_{H(T)}$, we have

$$\begin{aligned} Ba_\lambda(g, \beta u_T^N) &= \frac{1}{2^{|g|-1}} \sum_{g' \subseteq g \setminus \lambda} (\beta u_T^N(g' \cup \lambda) - \beta u_T^N(g')) \\ &\stackrel{\text{cf. existence CLBE}}{=} \frac{1}{2^{|g|_{H(T)}|-1}} \sum_{g' \subseteq g|_{H(T)} \setminus \lambda} \underbrace{(\beta u_T^N(g' \cup \lambda) - \beta u_T^N(g'))}_{\begin{cases} \beta & , \text{ if } g' = g|_{H(T)} \setminus \lambda \\ 0 & , \text{ otherwise} \end{cases}} \\ &= \frac{\beta}{2^{|g|_{H(T)}|-1}} \end{aligned}$$

and therefore, it holds that

$$\begin{aligned} Ba_\lambda(g, \beta u_T^N) &= \begin{cases} \frac{\beta}{2^{|g|_{H(T)}|-1}} & , \text{ if } \lambda \in g|_{H(T)} \\ 0 & , \text{ otherwise} \end{cases} \\ &= Ba_\lambda(g|_C, \beta u_T^N) \\ &= Ba_\lambda(g|_{H(T)}, \beta u_T^N) \end{aligned}$$

$$\begin{aligned} \Rightarrow \pi_i^{Ba}(N, \beta u_T, g) &= \sum_{\lambda \in g_i \cap g|_{H(T)}} \frac{1}{2} \cdot \frac{\beta}{2^{|g|_{H(T)}|-1}} = \frac{|g_i|_{H(T)}| \cdot \beta}{2^{|g|_{H(T)}|}} \\ &= \frac{\beta}{2^{|g|_{H(T)}|}} \cdot C_i^d(g|_{H(T)}) \end{aligned}$$

On the other hand, all links $\lambda \notin g|_{H(T)}$ are superfluous in $(N, \beta u_T)$, hence, by **SLP**, $Y(N, \beta u_T, g) = Y(N, \beta u_T, g|_{H(T)})$. The game $(N, \beta u_T, g|_{H(T)})$ is link anonymous (all links have the same number of swings, namely, one) with

$$f : \{0, 1, \dots, |g|_{H(T)}\} \rightarrow \mathbb{R}, f(x) := \begin{cases} \beta & , \text{ if } x = |g|_{H(T)} \\ 0 & , \text{ otherwise} \end{cases}$$

hence, by **DEG**, there exists $\alpha \in \mathbb{R}$ such that

$$Y_i(N, \beta u_T, g) \stackrel{\text{SLP}}{=} Y_i(N, \beta u_T, g|_{H(T)}) \stackrel{\text{DEG}}{=} \alpha \cdot C_i^d(g|_{H(T)}) \quad (*)$$

It directly follows that

$$Y_i(N, \beta u_T, g) = 0 \quad \forall i \in N \setminus H(T) \quad (**) \quad \text{(*)}$$

¹¹Note that cycle-freeness is essential here: if there is more than one path connecting T , the intersection is empty on the disjoint parts of the connecting paths.

By (**) and **CLBE**, we have

$$\begin{aligned}
\sum_{i \in H(T)} Y_i(N, \beta u_T, g) &\stackrel{(*)}{=} \sum_{i \in C} Y_i(N, \beta u_T, g) \\
&\stackrel{\text{CLBE}}{=} \sum_{\lambda \in g|_C} Ba_\lambda(g|_C, \beta u_T^N) \\
&= \sum_{\lambda \in g|_{H(T)}} \frac{\beta}{2^{|g|_{H(T)}|-1}} \\
&= \frac{|g|_{H(T)} \cdot \beta}{2^{|g|_{H(T)}|-1}}
\end{aligned}$$

On the other hand, by (*), we have

$$\sum_{i \in C} Y_i(N, \beta u_T, g) = \alpha \sum_{i \in C} C_i^d(g|_{H(T)}) = \alpha \cdot 2 \cdot |g|_{H(T)} |$$

Combining this, we get

$$\alpha \cdot 2 \cdot |g|_{H(T)} | = \frac{|g|_{H(T)} | \cdot \beta}{2^{|g|_{H(T)}|-1}} \Leftrightarrow \alpha = \frac{\beta}{2^{|g|_{H(T)}|}}$$

and hence

$$Y_i(N, \beta u_T, g) = \frac{\beta}{2^{|g|_{H(T)}|}} C_i^d(g|_{H(T)}) = \pi_i^{Ba}(N, \beta u_T, g)$$

which finishes the proof. □

Proof of Theorem 2:

The Banzhaf Position value π^{Ba} is uniquely determined by **BLC** and **CLBE**.

Proof. Existence: We have already shown that π^{Ba} satisfies **CLBE**. To see **BLC**, we follow Slikker (2005), who states that there exists a unique linear combination of link-unanimity games which represents any link game $v^N : 2^{|g|} \rightarrow \mathbb{R}$: for any v^N exist $\beta_{g'}$, $g' \subseteq g$, such that

$$v^N(\tilde{g}) = \sum_{g' \subseteq g} \beta_{g'} u_{g'}^N(\tilde{g}).$$

For $(g, \beta_{g'} u_{g'}^N)$ we get

$$Ba_\lambda(g, \beta_{g'} u_{g'}^N) = \begin{cases} \frac{\beta_{g'}}{2^{|g'||-1}} & , \text{ if } \lambda \in g' \\ 0 & , \text{ if } \lambda \notin g' \end{cases}$$

and using this, we have

$$\begin{aligned}
\pi_i^{Ba}(N, v, g) &= \sum_{\lambda \in g_i} \frac{1}{2} Ba_\lambda(g, v^N) \\
&\stackrel{\text{Ba satisfies A}}{=} \sum_{\lambda \in g_i} \frac{1}{2} \sum_{g' \subseteq g} Ba_\lambda(g, \beta_{g'} u_{g'}^N) \\
&= \sum_{g' \subseteq g} \sum_{\lambda \in g_i} \frac{1}{2} \underbrace{Ba_\lambda(g, \beta_{g'} u_{g'}^N)}_{\begin{cases} \frac{\beta_{g'}}{2^{|g'|-1}}, & \text{if } \lambda \in g' \cap g_i \\ 0, & \text{if } \lambda \notin g' \cap g_i \end{cases}} \\
&= \sum_{g' \subseteq g} \sum_{\lambda \in g'_i} \frac{\beta_{g'}}{2^{|g'|}} = \sum_{g' \subseteq g} \beta_{g'} \frac{|g'_i|}{2^{|g'|}}
\end{aligned}$$

and therefore

$$\begin{aligned}
\sum_{\lambda \in g_j} [\pi_i^{Ba}(N, v, g) - \pi_i^{Ba}(N, v, g \setminus \lambda)] &= \sum_{\lambda \in g_j} \left(\sum_{g' \subseteq g} \beta_{g'} \frac{|g'_i|}{2^{|g'|}} - \sum_{g' \subseteq g \setminus \lambda} \beta_{g'} \frac{|g'_i|}{2^{|g'|}} \right) \\
&= \sum_{\lambda \in g_j} \sum_{\substack{g' \subseteq g \\ \lambda \in g'}} \beta_{g'} \frac{|g'_i|}{2^{|g'|}} \\
&= \sum_{g' \subseteq g} \sum_{\lambda \in g'_j} \beta_{g'} \frac{|g'_i|}{2^{|g'|}} \\
&= \sum_{g' \subseteq g} \beta_{g'} \frac{|g'_j| \cdot |g'_i|}{2^{|g'|}} \\
&\stackrel{\text{backwards}}{=} \sum_{\lambda \in g_i} [\pi_j^{Ba}(N, v, g) - \pi_j^{Ba}(N, v, g \setminus \lambda)]
\end{aligned}$$

for all $i \in N$.

Now suppose Y, W being two allocation rules satisfying **BLC** and **CLBE**. We prove uniqueness by induction over $|g|$.

Induction basis [IB]: For $|g| = 0$, that is, $g = \{\emptyset\}$, we have that $\mathcal{G}(N, g) = \{i\}_{i \in N}$ and hence, by **CLBE**, we have

$$Y_i(N, v, g) = \sum_{i \in C} Y_i(N, v, g) = \sum_{\lambda \in \{\emptyset\}} Ba_\lambda(g, v^N) = 0 = \sum_{i \in C} W_i(N, v, g) = W_i(N, v, g).$$

Now suppose that $Y(N, v, g) = W(N, v, g)$ for all g such that $|g| = k$, $k \geq 0$ (induction hypothesis [IH]).

Consider g such that $|g| = k+1$. As $k+1 \geq 1$, there exists $i, j \in N$, $i \neq j$ such that $j \in C_i(N, g)$ (for all l with $|C_l(N, g)| = 1$, we have $Y_l(N, v, g) = W_l(N, v, g)$ by **CLBE**). By **BLC**, we have

for all $i, j \in C_i(N, g)$:

$$\begin{aligned}
\sum_{\lambda \in g_j} Y_i(N, v, g) - \sum_{\lambda \in g_i} Y_j(N, v, g) &\stackrel{\text{BLC}}{=} \sum_{\lambda \in g_j} Y_i(N, v, g \setminus \lambda) - \sum_{\lambda \in g_i} Y_j(N, v, g \setminus \lambda) \\
&\stackrel{[\text{IH}]}{=} \sum_{\lambda \in g_j} W_i(N, v, g \setminus \lambda) - \sum_{\lambda \in g_i} W_j(N, v, g \setminus \lambda) \\
&\stackrel{\text{BLC}}{=} \sum_{\lambda \in g_j} W_i(N, v, g) - \sum_{\lambda \in g_i} W_j(N, v, g) \\
\Leftrightarrow |g_j|Y_i(N, v, g) - |g_i|Y_j(N, v, g) &= |g_j|W_i(N, v, g) - |g_i|W_j(N, v, g) \tag{*}
\end{aligned}$$

Summing up $(*)$ over all $j \in C$ yields:

$$\begin{aligned}
\sum_{j \in C} [|g_j|Y_i(N, v, g) - |g_i|Y_j(N, v, g)] &= \sum_{j \in C} [|g_j|W_i(N, v, g) - |g_i|W_j(N, v, g)] \\
\Leftrightarrow Y_i(N, v, g) \underbrace{\sum_{j \in C} |g_j|}_{:= A(j) > 0 \text{ by } |C_i| \geq 1} - |g_i| \sum_{j \in C} Y_j(N, v, g) &= W_i(N, v, g) \underbrace{\sum_{j \in C} |g_j|}_{:= A(j) > 0 \text{ by } |C_i| \geq 1} - |g_i| \sum_{j \in C} W_j(N, v, g) \\
\stackrel{\text{CLBE}}{\Rightarrow} A(j) \cdot Y_i(N, v, g) - |g_i| \sum_{\lambda \in g|_C} Ba_\lambda(g|_C, v^N|_C) &= A(j) \cdot W_i(N, v, g) - |g_i| \sum_{\lambda \in g|_C} Ba_\lambda(g|_C, v^N|_C) \\
\Leftrightarrow Y_i(N, v, g) &= W_i(N, v, g) \quad \forall i \in N
\end{aligned}$$

which finishes the proof. \square

Proof of Theorem 3:

For $v = u_N$, computational complexity is at most

$$|g| \cdot \sum_{k=|T|-1}^{|g|-1} \binom{|g|-1}{k}$$

connectivity checks for the Banzhaf value and for the Shapley value it is the same connectivity checks, just that for every "yes"-labeled item, one number has to be stored.

Proof. In general, the computational effort for calculating the Shapley or the Banzhaf value is high (at least P-complete) as it increases disproportionately with the number of nodes in the network of interest. But, as u_T^N is a simple game, the Shapley and the Banzhaf value can be calculated via swings: $g' \subseteq g \setminus \{\lambda\}$ is *swing* for $\lambda \in g$ iff $u_T^N(g' \cup \lambda) - u_T^N(g') = 1$. For the Shapley value, swings have to be weighted according to their cardinality, for the Banzhaf value, one only has to count the total number of swings and devide this by the number of potential swings, namely, $2^{|g|-1}$.

The effort of identifying swings in u_T^N further decreases as g' is a swing for $\lambda = ij$ iff $T \subseteq G \in \mathcal{G}(N, g' \cup \{ij\})$ while $\nexists G \in \mathcal{G}(N, g')$ with $T \subseteq G$. That is, iff $g' \cup \{ij\}$ connects T while g' does not. For all $\lambda \in g$, there are $2^{|g|-1}$ subnetworks of $g \setminus \{\lambda\}$. Of interest are those of the $2^{|g|-1}$ subnetworks, that do not connect the nodes in T but the nodes will be connected when adding λ , that is, we can restrict the analysis of the $2^{|g|-1}$ subnetworks to those with at least $|T| - 1$

links. The number of subnetworks that have to be checked is at most:

$$\sum_{k=0}^{|g|-1} \binom{|g|-1}{k} - \sum_{k=0}^{|T|-2} \binom{|g|-1}{k} = 2^{|g|-1} - \sum_{k=0}^{|T|-2} \binom{|g|-1}{k} = \sum_{k=|T|-1}^{|g|-1} \binom{|g|-1}{k}$$

Unfortunately it is well known that there is no closed formula for the number of all subsets of a certain set with cardinality greater than some border. Note that the number of subnetworks of interest further decreases as only those that include all nodes in T are of interest.

We end up with the fact that calculating the Shapley and the Banzhaf value are counting problems: For the Banzhaf value, one has to count for any $\lambda \in g$ how many of the subnetworks of interest do not connect T but will when adding λ . For the Shapley value, also the cardinality of these subnetworks is of interest. Computational complexity is at most checking

$$|g| \sum_{k=|T|-1}^{|g|-1} \binom{|g|-1}{k}$$

networks, where a connectivity check (g' does not connect T but $g' \cup \lambda$ does: yes/no) is sufficient; for the Shapley value, one further has to save the information about the cardinality of those g' with label "yes". \square

Proof of Theorem 4:

Let g be a cycle-free connected binary graph/network (i.e., a minimal spanning tree) and consider the unanimity game u_T , $T \subseteq N$, $|T| > 1$. Then, we have for $Y = Sh$ or $Y = Ba$

$$\begin{aligned} \overline{CD}_i^Y(g, u_T) &= \overline{C^d}_i(g|_{H(T)}), & \overline{CC}_i^Y(g, u_T) &= \overline{C^c}_i(g|_{H(T)}) \\ \overline{CB}_i^Y(g, u_T) &= \overline{C^b}_i(g|_{H(T)}) \text{ and} & \overline{CEV}_i^Y(g, u_T) &= \overline{C^{EV}}_i(g|_{H(T)}) \end{aligned}$$

where $H(T)$ is the connected hull of T , that is, set of nodes that are essential to connect T .

Proof. Let g be a minimal connected binary graph/network and $v = u_T$. Then, following the uniqueness proof of Theorem 1 (and similar for the (Shapley) Position value), we have

$$Y_\lambda(g, \beta u_T^N) = \begin{cases} \text{constant} & , \text{ for } \lambda \in g|_{H(T)} \\ 0 & , \text{ for } \lambda \notin g|_{H(T)} \end{cases}$$

for $Y = Sh$ or $Y = Ba$. Hence, in network $g_{H(T)}$, all weights \tilde{w}_λ are constant. As the scale of weights does not matter for ranks and relative distances of the indices, weights can be rescaled to 1 which leads the original (binary) network, restricted on $H(T)$, that is, $g_{H(T)}$. This proves proportionality to Freeman's centrality measures as well as Eigenvector centrality. \square

A.2. Supplementary Material Political Example, Section 5.2

Table 8: Swings and Banzhaf values for parties and connections (both cases)

For parties (nodes)			
party i	swings for i	# of swings	$\bar{B}a_i$ (to 100%)
1	{2}, {3}, {2, 3}, {2, 5}, {3, 5}, {4, 5}	6	40.00
2	{1}, {1, 5}, {3, 5}	3	20.00
3	{1}, {1, 5}, {2, 5}	3	20.00
4	{1, 5}	1	6.67
5	{1, 4}, {2, 3}	2	13.33
Σ		15	100%
For connections (links)			
link λ	swings for λ	# of swings	$\bar{B}a_\lambda$ (to 100%)
12	$\emptyset, \{15\}, \{23\}, \{25\}, \{35\}, \{15, 23\}$ [+{45}, {23, 45}]	6 [8]	20.00 [17.39]
13	$\emptyset, \{15\}, \{23\}, \{25\}, \{35\}, \{15, 23\}$ [+{45}, {23, 45}]	6 [8]	20.00 [17.39]
14	{15}, {45} [+{15, 23}, {23, 45}]	2 [4]	6.67 [8.70]
15	{14}, {45}, {25}, {35} [+{14, 23}, {23, 45}]	4 [6]	13.33 [13.04]
23	{25}, {35} [+{14, 25}, {14, 35}]	2 [4]	6.67 [8.70]
25	{15}, {23}, {35}, {15, 23} [+{14, 23}, {14, 35}]	4 [6]	13.33 [13.04]
35	{15}, {23}, {25}, {15, 23} [+{14, 23}, {14, 25}]	4 [6]	13.33 [13.04]
14	{14}, {15} [+{14, 23}, {15, 23}]	2 [4]	6.67 [8.70]

(extra swings for less restrictive case in parentheses.)

Table 9: Classic Centrality Approaches

Distribution of Power (normalized to 100 %)					
party	1	2	3	4	5
C^d_i	25.00	18.75	18.75	12.50	25.00
$\overline{C^c}_i$	23.44	18.75	18.75	15.62	23.44
$\overline{C^b}_i$	50.00	0.00	0.00	0.00	50.00
$\overline{C^{EV}}_i$	23.13	19.91	19.91	13.92	23.13
Resulting Coalitional Power					
coalition	{1, 2}	{1, 3}	{1, 4, 5}	{2, 3, 4}	{2, 3, 5}
$\sum \overline{C^d}_i$	43.75	43.75	62.50	-	62.50
$\sum \overline{C^c}_i$	42.19	42.19	62.50	-	60.94
$\sum \overline{C^b}_i$	50.00	50.00	100.00	-	50.00
$\sum \overline{C^{EV}}_i$	43.04	43.04	60.18	-	62.95

(Highest individual and coalitional power are bold face.)

Table 10: Cohesion Centrality Approaches, less restrictive case

Distribution of Power (normalized to 100 %)					
party	1	2	3	4	5
CD^{Ba}_i	28.26	19.57	19.57	8.70	23.91
CEV^{Ba}_i	25.62	21.21	21.21	9.60	22.35
Resulting Coalitional Power					
coalition	{1, 2}	{1, 3}	{1, 4, 5}	{2, 3, 4}	{2, 3, 5}
$\sum CD^{Ba}_i$	47.83	47.83	60.87	-	63.05
$\sum CEV^{Ba}_i$	46.83	46.83	57.57	-	64.77

(Highest individual and coalitional power are bold face.)