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Moreno, Othón M.

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Othón M. Moreno
Banco de México

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Working Paper 2014-02

Consumption of Durable Goods under Ambiguity*

Othón M. Moreno†
Banco de México

Abstract
The focus of this paper is to analyze the effect that ambiguity will have on the buyer's reservation price and the value of the option to purchase the durable good with an embedded option to resell it. The agent is assumed to be risk neutral and ambiguity averse. The problem is formulated as an optimal stopping problem with multiple priors in continuous time with infinite horizon. Uncertainty comes from prices, which is summarized in a state variable that follows a Brownian motion. Preferences have a multiple-prior utility representation where the set of priors consist of a family of Brownian motions with unknown drift and common variance. We show that the direction of the change in the buyer's reservation price depends on the parametrization of the model and that the value of the embedded option is decreasing in the perceived level of ambiguity.
Keywords: Ambiguity, optimal stopping, embedded option, durable goods.
JEL Classification: C61, D81, D91.

Resumen
El objetivo de este documento es analizar el efecto que tiene la ambigüedad sobre el precio de reserva de un comprador y sobre el valor de la opción de adquirir un bien durable con una opción de reventa implícita (*embedded*). El agente se supone neutral al riesgo y adverso a la ambigüedad. El problema es formulado como un problema de suspensión óptima con múltiples distribuciones previas en tiempo continuo con un horizonte infinito. La incertidumbre proviene de los precios futuros, la cual está resumida en una variable de estado que sigue un movimiento browniano. Las preferencias poseen una representación de utilidad con múltiples distribuciones previas donde el conjunto de distribuciones consiste en una familia de movimientos brownianos con tendencias no observables y varianza común. Mostramos que la dirección del cambio en el precio de reserva del comprador depende de la parametrización del modelo y que el valor de la opción implícita es decreciente en el nivel percibido de ambigüedad.
Palabras Clave: Ambigüedad, suspensión óptima, opción implícita, bienes durables.

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† Dirección General de Investigación Económica. Email: omoreno@banxico.org.mx.

1 Introduction

One of the most important decisions for a household is the purchase of a durable good. Consider, for example, the decision of buying a car; in this case, the decision maker usually devotes a significant amount of time and contemplates a relatively large amount of information before making a decision. In addition to the current price and the utility derived from the consumption of the good, the agent needs to consider future spot prices as well as the possibility of reselling the good in order to make an optimal choice. However, as originally argued by Knight (1921), the agent may not have enough information to assess the distribution of future prices precisely. This lack of information can be the result of relatively few data available in a new market, unobservable characteristics of the good, or any other irreducible uncertainty idiosyncratic to the good in question. In this paper, we approach the problem of consumption of a durable good with the option to resell it at any moment as an optimal stopping problem where the agent chooses the time of purchase and resale of the good in order to maximize her expected present value at time zero. Furthermore, we develop the model in an environment where the decision maker is subjected to *ambiguity* or *Knightian uncertainty*. We are mostly concerned with the effect that changes in the level of perceived ambiguity have on the buying reservation price and the corresponding timing of the purchase. In this type of decisions, the resale motive will play an important role when choosing a probability measure when forming expectations about future prices and, thus, affecting both the resale and buying reservation prices.

In order to give some intuition about the problem analyzed, allow us to frame the discussion around two particular markets where the resale option is of relevance when making purchase decisions. First, let us return to our example of the decision to buy a vehicle[1]. When considering to buy a car a consumer will prefer a lower price, *ceteris paribus*. Thus, there is an incentive to "delay" the purchase and wait for the price to fall sufficiently low. However, the possibility of reselling the car makes higher prices more attractive as the resale value is, usually, a function of the spot market price of the good. The resale motive adds value to owning the car, which increases the price the consumer is willing to pay for it and "rushes" the purchase decision, relative to the timing without resale. When we add ambiguity to this problem, the agent is *ex ante* unable to form a unique belief about the distribution of price increments. Ambiguity aversion will create a discrepancy between the distribution of prices used to evaluate the resale and purchase decisions. The combined effect that ambiguity has on the incentives to rush and delay the purchase of this good is the main

[1]The possibility of reselling a car is so important that some companies even use the resale values as advertising strategies

topic of the paper.

A second type of markets where the option to resell the good is of particular relevance is housing. When considering to buy a house, consumers face the same incentives to rush or delay the purchase as in the car example. We make particular notice of the decision to buy a house since it is, perhaps, the most important economic decision made by households and the rise and burst of price bubbles in these markets have proven to be an important source of economic instability. Furthermore, one of the main results of the paper can be easily interpreted in the context of price bubbles, for which house prices have become the canonical example. In our model we show that it is possible to observe an increase in prices due to an increase in ambiguity. Thus even if the possibility of a bursting bubble would imply a decrease in market prices, consumers could increase their reservation price as long as processes with high price increments remain a possibility.

It is not hard to motivate the presence of ambiguity in this type of decisions. In fact, Giboa (2013) argues that this particular information structure is more common than the unique prior used in traditional decision making models. In our two examples, the particular distributions of future prices depend on many factors including aggregate behavior of the economy, particularities of the market in question and even characteristics of individual goods. Requiring the agent to incorporate all this information in a unique probability measure might be asking too much, even if she is an expert. The sources of ambiguity can be easily exemplified in the housing market where price dynamics depend on the economic cycle, particular patterns of migration due to changes in labor markets, housing policies implemented at all government levels, etc. The same argument can be easily made for the car example. Furthermore, the sources of ambiguity can be also related to idiosyncratic characteristics of the good such a quality or particular attributes that may become more or less valuable in the future. As we will discuss latter in the article, we model the resale price as a fraction of the current spot price, thus any ambiguity present in the distribution of future prices will prevail in the process for the resale price as well.

The idea of modeling economic problems as timing problems is not new. In fact, optimal timing has been used extensively to analyze a wide variety of economically relevant questions such as the firm's entry and exit decisions, and policy implementation of a planner. This stopping time approach, or Real Options as introduced by McDonald and Siegel (1986) and Dixit and Pindyck (1994) since it relies on the methods developed to price financial options, can be applied to a broad class of problems sharing three main characteristics: some degree of irreversibility of the decision, ongoing uncertainty, and freedom over the timing of the decision. These characteristics are present in our problem of consumption of durable goods, allowing us to exploit the vast literature on optimal stopping times.

The solution to this type of models takes the form of a reservation value for the state variable, called the exercise threshold, at which the decision maker is indifferent between the termination payoff – received at the moment the option is exercised – and the continuation payoff – the value associated with the possibility of exercising the option in the future. In our case, the solution is characterized by two exercise thresholds: the *buying threshold*, i.e. the exercise threshold for an agent considering to buy the durable good, and the *resale threshold*, i.e. the exercise threshold for an agent considering to sell the good, where the state variable is a transformation of the market price. The optimal time to buy (resell) the durable good is the first time the spot price crosses the buyer's (seller's) reservation price.

In order to model ambiguity averse preferences we use the multiple-prior expected utility representation developed by Gilboa and Schmeidler (1989). In general, under this utility representation, the agent calculates the expectation of future payments using the worst possible probability measure in a set of priors. The set of priors is constructed using the κ-ignorance specification proposed by Chen and Epstein (2002) in their analysis of the multiple-prior representation in continuous time within the context of investment and consumption in an asset pricing model. In particular, this set is built by generating equivalent probability measures perturbed by a bounded stochastic process. The resulting set satisfies the requirements of the *Gilboa-Schmeilder* representation as well as the important rectangularity condition that guarantees time consistency. Additionally, each probability measure in the set corresponds to an underlying Brownian motion with unknown drift term and common variance. For a treatment of this utility representation in discrete time see Epstein and Wang (1994) and Epstein and Schneider (2003).

Developing a theoretical framework for optimal stopping with multiple-priors has been an active research topic in recent years. Riedel (2009) and Miao and Wang (2011) are concerned with a theory of optimal stopping in discrete time while Bayraktar and Yao (2011), Trevino-Aguilar (2012), and Cheng and Riedel (2013) present alternative models in continuous time. In our work we borrow a crucial result from Cheng and Riedel that allow us to find the *worst-case scenario* for the buyer's problem in a fairly simple way.

This paper is related to the work of Miao and Wang (2011), Nishimura and Ozaki (2004), Nishimura and Ozaki (2007), and Schroder (2011) who also examine the effect of ambiguity on the exercise thresholds. Nishimura and Ozaki (2007) and Schroder (2011) use the multiple-priors representation in continuous time to analyze the investment decision of a firm who is subjected to ambiguity regarding the returns on investment. In their results, ambiguity decreases the value of the option to invest and delays investment. In Schroder (2011), the firm uses both the worst and best case scenarios when calculating the optimal stopping time using a generalization of the *Gilboa-Schmeilder* representation (see Ghirardato et al. (2004)).

Nishimura and Ozaki (2004) apply the Choquet expected utility model (see Schmeidler (1989)) in a discrete time job search model. They show that ambiguity expedites the exercise of the option, shortening the time the agent engages in search. Miao and Wang (2011) reconcile these seemingly contrasting results using a multiple priors model in discrete time. They analyze the firm's entry and exit decisions and show that ambiguity affects the exercise thresholds differently, depending on whether or not uncertainty is resolved after exercising the option. If uncertainty is not resolved at the moment of exercise, as in the investment problem, the optimal exercise time is delayed. Alternatively, if uncertainty is fully resolved after the option exercise the presence of ambiguity rushes the exercise of the option.

All the applications we have discussed so far focus on simple financial options, i.e. American *call* and *put* options, or their equivalent interpretation in the Real Options context. Recent papers, however, have turn their attention into more exotic options. Cheng and Riedel (2013) apply their model to the barrier option and the American straddle.[2] They show that ambiguity makes the agent change the probability measure used to evaluate the options. For the barrier option the lowest mean return is used to evaluate the option before it knocks in and the highest mean return is used afterwards. For the American straddle the agent constantly changes the probability measure as the underlying moves in the interval between two exercise thresholds. Chudjakow and Vorbrink (2009) analyze several types of exotic options in discrete time that can be constructed by embedding simple options. As in Cheng and Riedel (2013), the worst-case measure can vary over time as the agent transitions from one simple option into the other.

There are alternative approaches to incorporate ambiguity, of course. Hansen and Sargent (2001) and Barillas et al. (2009), for example, propose a method based on robust control theory. Instead of using a multiple-prior representation, they introduce a zero-sum game where a malevolent player chooses a bounded level of entropy that distorts the decision maker beliefs. As a result, they obtained a Bellman equation with a correction term that accounts for model misspecifications. For a discussion comparing the two approaches see Epstein and Schneider (2003) and Hansen et al. (2006). Ju and Miao (2012) and Klibanoff et al. (2009) use a recursive smooth ambiguity model where the utility function is represented by the composition of two functions, one characterizing the risk preferences and the other the ambiguity preferences.

We follow a procedure similar to Chudjakow and Vorbrink (2009) (for exotic financial options under ambiguity) and Boyarchenko and Levendorskii (2010) (for a firm's entry prob-

[2]In a barrier option the contract is only implemented when the underlying hits a predetermined price, otherwise it remains worthless. In the American straddle the contract is implemented over the difference between the underlying and the strike price, regardless of the sign.

lem with and embedded option to exit with known drift) and decompose our buying and reselling problem into two simpler embedded options. Our embedded option, however, is more complex than the stradle and barrier options analyzed by Chudjakow and Vorbrink (2009), thus requiring a simplifying assumption on the way ambiguity is resolved discussed in detail in Section 3.[3] Specifically, the resale decision is modeled as an option to abandon a stream of payoffs including the utility generated by the consumption of the durable good and the opportunity cost of selling it at a fraction of the spot price. First, we solve this problem for an arbitrary drift parameter, which is revealed to the agent at the moment of purchase, and obtain the value of the resale option by evaluating it at the worst-case scenario, i.e. the most unfavorable drift parameter for the seller. Then, we solve the problem for an option to acquire a one time payment equal to the value of the resale option minus the price of the good and find the worst-case scenario to evaluate the option. We find that the agent evaluates the embedded option using two different drifts, the resale value of the good and its optimal resale price is computed using the lowest possible drift in the ignorance interval generated by the κ-ignorance specification, while the optimal buyer's reservation price is computed using the highest possible drift in a similar way to Cheng and Riedel (2013) and Chudjakow and Vorbrink (2009).

Additionally, we examine the effect that an increase in ambiguity has on the buyer's reservation price. This analysis, however, is done numerically since the lack of a closed form solution to the optimal buyer's reservation price prevent us to obtain analytic expressions for the derivatives. We show that the direction of the change in the buyer's reservation price depends on the particular parametrization of the model and provide some intuition on the relevant determinants of the effect. Our result is consistent with Miao and Wang (2011) who find an indefinite effect in the optimal exercise threshold as a response to an increase in ambiguity. However, Miao and Wang (2011) use two different sources of ambiguity, while in our model the source of qualitatively different effects is that, as a result of the change in the drift considered to evaluate the embedded options, the positive effect of ambiguity on the buyer's reservation price can be dominated by a negative effect product of lower expected resale values. Intuitively, the benefit of owning the durable good has two sources: the utility generated from the use of it and its resale value. When the expected mean increments used to evaluate the resale decision is "too low", the resale motive plays a smaller role in the *ex ante* purchase decision as the agent considers a relatively pessimistic process to calculate the seller's reservation price. Therefore, further reductions on these originally low mean

[3]Consider, for example, the barrier option where a standard American option "kicks in" the first time the price of the underlying hits a predetermined level. In this case, the stopping time at which the option becomes valuable is not an endogenous decision of the agent as it is specified in the contract.

increments will be dominated by higher expected price increments used to evaluate the buying decision, also due to the increase in ambiguity. This intuition is similar to the case when the fraction of the spot market price at which the good can be resold is close to zero, where the resale motive plays a lesser role as well. In this case, the effect of an increase of ambiguity in the resale decision gets smaller and it is dominated by increases in the expected price increments used to evaluate the buying decision. Furthermore, we found that higher levels of perceived ambiguity decrease the value of the embedded option. This last result is consistent with what has been previously found in the literature.

The rests of the paper is organized as follows. In the next section we formalize the stopping time problem described above and solve for the exercise thresholds and value of the options for the unique prior case. Section 3 discusses in detail the construction of the set of priors, elaborates on the solution concept used to obtain the buyer's reservation price and value of the option under ambiguity, and states our main result. Finally, section 4 concludes. All proofs are collected in the appendix.

2 The basic model

In this section we analyze the buying decision of a durable good that can be resold at any point in time as an optimal stopping problem. First, we elaborate on the basic setting for the case of a unique prior and introduce some regularity conditions as well as general definitions and previous results that will be used throughout the paper. We then extend the model to allow for multiple priors in section 3.

Consider an infinite-horizon optimal stopping problem in continuous time where a risk neutral agent faces the option to buy a durable good which provides a discounted utility at the time of purchase denoted by U. We assume for simplicity that the good does not depreciate over time. Additionally, it is possible to resell the good at a fraction, φ, of the spot market price with $0 < \varphi \leq 1$. The agent discounts future payoff flows using the discount rate $q > 0$ and preferences are represented by a time-additive expected utility.

Let $(\Omega, P, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0})$ be a filtered probability space, and $(X_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$ with respect to P for its filtration $(\mathscr{F}_t)_{t\geq 0}$ – satisfying the usual properties – that solves the stochastic differential equation

$$dX_t = \mu dt + \sigma dW_t;$$

where μ is the drift parameter, σ^2 is the diffusion parameter and dW_t is the increment of a

standard Wiener process.[4] We use the stochastic process X_t to characterize the uncertainty about future prices for the durable good with $p_t = e^{x_t}$. Equivalently, we could specify the stochastic process in terms of p_t. In this case p_t follows a geometric Brownian motion with drift $(\mu + \frac{1}{2}\sigma^2)$ and diffusion parameter σ^2.

The agent's problem is to determine both the timing of purchase and resale of the durable good in order to maximize her expected present value at time zero. Moreover, the expected resale value must be taken into account when considering the purchase of the good. Therefore, the agent is presented with two binary choices. The first choice is whether to *stop* and exercise the option to buy the durable good at the current price obtaining the discounted utility U and the embedded resale option or *continue* for one more period and face the same decision in the future. The second choice corresponds to the problem once the agent has ownership over the durable good; whether to *stop* and resell the good at a fraction φ of the current market price, hence forgoing the utility provided by the durable good, or *continue* and face the same choice in the future.

We start our analysis of the solution to the problem stated above in the following way: First, assuming that the good has been purchased already, we find the optimal resale threshold x^*, i.e. the value of x at which the agent is indifferent between selling the good and continue to hold it, which allow us to calculate the value of the resale option. Then, we proceed to calculate the optimal buyer's threshold, i.e. the value of x at which the agent is indifferent between continue waiting and purchasing the durable good with the embedded resale option, and the value of the embedded option.

2.1 The Resale Decision

Once the good has been purchased, at each time t the agent receives a constant stream of payoffs from holding the good equal to qU. However, by holding the durable good, the agent forgoes the payment received from selling it, therefore incurring an opportunity cost equal to the stream of payments whose expected present value is φe^{x_t}. In order to calculate the stream of payoffs let us introduce the following notation for the *expected present value operator*[5]: for any function f, let

$$\mathcal{E}f(x) = \mathbb{E}_x\left(\int_0^\infty e^{-qt} f(X_t)dt\right).$$

[4]The analysis can be extended to more general Levy processes with some extra assumptions (see Boyarchenko and Levendorskii (2007)).

[5]This operator is known in the theory of stochastic processes as the *resolvent* of the process.

It is possible to obtain the stream of payoffs whose EPV is φe^{x_t} using the previous equation and the moment generating function of the Brownian motion described above. However, it is convenient to use the equality $\mathcal{E}f(x) = (q-L)^{-1}f(x)$, where L is the infinitesimal generator of the Brownian motion defined as:

$$L = \frac{\sigma^2}{2}\partial^2 + \mu\partial.$$

Using the EPV-operator we can get the stream of payments, $g(x)$, associated to the resale price by $g(x) = (q-L)(\varphi e^{x_t})$. The profit flow generated from holding the durable good at time t can now be obtained by subtracting $g(x)$ from the consumption payoff received. Let denote this profit flow by:

$$\pi(x) = qU - (q - \psi(1))\varphi e^x, \tag{1}$$

where $\psi(z) = \frac{\sigma^2}{2}z^2 + \mu z$ is the *Levy exponent* of the process X_t.

Let β^+ and β^- be the positive and negative roots of $q - \psi(z)$, respectively. In order to guarantee that the solution to our problem is well defined we need we need to set a restriction on the drift and diffusion terms of the Brownian motion such that $\beta^+ > 1$. Using standard algebraic manipulations it can be shown that this condition is satisfied as long as

$$\mu < q - \frac{\sigma^2}{2}. \tag{2}$$

For the remainder of the paper we will assume that the relationship between μ, σ and q stated above holds.[6]

[6]This restriction comes from the need to guarantee that the function $(e^{\zeta^- x} + e^{\zeta^+ x})^{-1}\pi(x)$ is bounded. Formally, there exist a constant C such that

$$\left|(e^{\zeta^- x} + e^{\zeta^+ x})^{-1}\pi(x)\right| \leq C$$

a.e. for all x, for all $z \in [\zeta^-, \zeta^+]$ such that $q - \psi(z) > 0$, for some $\zeta^- \leq 0 \leq \zeta^+$. This regularity condition assures that the expected value operators used to calculate the optimal stopping time and option value are bounded. In other words, we need $\pi(x)$ not to grow too fast, with respect to the stochastic process, X_t, as x approaches positive or negative infinity. Note that, as x tends to negative infinity, $\pi(x)$ approaches the constant term qU and is always possible to find C satisfying the regularity condition. As x tends to positive infinity the absolute value of $\pi(x)$ grows unbounded. However, it is easy to see that the left hand side of the restriction remains bounded as long as $\zeta^+ \geq 1$. Therefore, we need to restrict the parameters in the model such that $q - \psi(z) > 0$ for all $z \in [1, \zeta^+]$, or in other words, that the positive root of the characteristic equation is greater than one.

The stopping problem described above is then given by the following equation[7]:

$$V_2(x) = \max_{\tau \geq 0} \mathbb{E}\left(\int_0^{\tau} e^{-qt} \pi(x) dt \right). \tag{3}$$

To reiterate, the problem consists of choosing a random time, $\tau \geq 0$, at which the good will be sold that maximizes the expected present value of the stream of payoffs $\pi(x)$. The solution to equation (3) comes in the form of a reservation value for the random variable x, known in the Real Options literature as the *exercise threshold*, which divides the real line into two regions. For values of x above its reservation value it is optimal for the consumer to abandon the stream of payoffs, thus getting a utility of zero thereafter. We refer to this region as the *termination region*. For values of x below the reservation value the agent finds profitable to hold on to the durable good and receives the stream $\pi(x)$ together with the expected value of the option to sell the good further in the future. We refer to this region as the *continuation region*. Note that, as the spot market price for the durable good increases, i.e. the state variable x increases, the opportunity cost of holding the good becomes larger, and the continuation payoff becomes negative. Should x rise sufficiently high, it may become optimal to sell the good.

Denote the reservation value at which it is optimal to resell the good by x^*, and the optimal stopping time $\tau_{x^*} = \inf\{t \geq 0 | x_t \geq x^*\}$ as the solution to equation (3). So it will be optimal to resell the good the first time x_t crosses x^* from below.

The resale threshold and option value that solves for equation (3) can be obtained by applying Theorem 11.6.5 in Boyarchenko and Levendorskii (2007). Below we state a simplified version of the theorem for our particular case and notation. Before applying this theorem, however, we need to verify that our particular profit flow is decreasing in x and has a zero at some x. First, $\partial_x \pi(x) < 0$ since the restriction on the parameters of the model imposed by equation (2) is equivalent to $q > \psi(1)$ and $\varphi > 0$ by assumption. Direct inspection of equation (1) suffices to see that $\pi(x)$ is negative for sufficiently large values of x and, as x tends to negative infinity, $\pi(x)$ approaches $qU > 0$ thus crossing zero at some point.

Theorem 2.1 (Boyarchenko-Levendorskii)**.** *For $\pi(x)$ as defined in equation (1):*

(a) equation $F(x) = 0$ has a unique solution at x^* with

$$F(x) = -\beta^- \int_{-\infty}^{0} e^{-\beta^- y} \pi(x + y) dy;$$

[7]The subindex used for the value of the stream of payoffs with the option to resell it, $V_2(x)$, denotes the second stage of a backward induction solution concept. In a similar fashion we use $V_1(x)$ in the next section to refer to the value of the option to acquire this payoff as the first stage of the problem.

(b) τ_{x^*} *is an optimal stopping time; and*

(c) the value of the stream of payoffs with the option to abandon it is given by:

$$V_2^*(x) = q^{-1}\beta^+ \int_0^{x^*-x} e^{-\beta^+ y} \left(F(x+y)\right) dy.$$

For a proof of this theorem see Theorem 11.6.5 in Boyarchenko and Levendorskii (2007). Directly applying Theorem 2.1 to our stream of payoffs, $\pi(x)$, give us the optimal resale threshold, x^*, and the value of the option to resell the durable good. The result is presented in the following corollary. Note that equation (4) is written in terms of e^{x^*} which directly give us the optimal seller's reservation price.

Corollary 2.2. *For* $\pi(x_t) = qU - (q - \psi(1))\varphi e^{x_t}$*, the optimal resale threshold,* x^**, is given by*

$$e^{x^*} = \frac{qU\left(\beta^- - 1\right)}{\varphi\beta^-(q - \psi(1))}. \tag{4}$$

For $x < x^*$ *the value of the stream with the option to resell is*

$$V_2^*(x) = U - \varphi e^x + \frac{U}{(\beta^+ - 1)} e^{\beta^+(x - x^*)}, \tag{5}$$

and $V_2^*(x) = 0$ *for* $x \geq x^*$*.*

2.2 The Buying Decision

With the optimal resale threshold and value of the stream of payoffs derived from owning the durable good in hand we now turn to the agent's buying decision. At the moment of purchase, the decision maker is entitled to a stream of payoffs whose value is given by $V_2^*(x)$ as defined in equation (5). Therefore, the agent faces the option of acquiring the payoff $V_2^*(x)$ at a price e^x. Define the instantaneous payoff of buying the durable good as $\Pi(x_t) = V_2^*(x_t) - e^x$, or equivalently as

$$\Pi(x_t) = U - (1 + \varphi)e^{x_t} + \frac{U}{(\beta^+ - 1)} e^{\beta^+(x_t - x^*)}. \tag{6}$$

Similarly to the previous step we need to verify that $\Pi(x)$ satisfies some regularity conditions for our choice of parameters μ, σ and q. In particular, our solutions are well defined

if for all N, there exist C such that

$$\sum_{0\leq s\leq 2}\left|e^{-\zeta^{-}x}\Pi^{(s)}(x)\right|\leq C \tag{7}$$

on the interval $(-\infty, N]$.

Since the instantaneous payoff function $\Pi(x)$ is due when a certain boundary is crossed from above – as it will be optimal to buy the good once the price falls sufficiently low – we need to impose a bound in a neighborhood of negative infinity. However, a simple examination of equation (6) suffices to verify that at negative infinity $\Pi(x)$ is bounded by U, and the first and second derivatives are bounded by zero so we do not need extra requirements on the parameters of the model to guarantee that the solution to our problem is well defined.

The buying problem is formally characterize by the equation:

$$V_1(x)=\max_{\tau\geq 0}\mathbb{E}\left(e^{-q\tau}\Pi(x_\tau)\right). \tag{8}$$

For the buying decision, in the termination region the agent acquires a instantaneous payoff, $\Pi(x)$, which includes the stream of utilities derived from the consumption of the durable good and its embedded resale value. In the continuation region the agent receives no payment, obtaining the expected value of waiting to exercise the option to buy the good. Additionally, as the spot market price for the durable good decreases, i.e. the state variable x decreases, the instantaneous payoff increases. Intuitively, a lower value of x reduces the price that the agent needs to pay in order to acquire the durable good. Furthermore, a reduction in x implies a lower expected resale value, reducing $V_2^*(x)$ as well. This result is formalize in the following lemma. Therefore, should x fall sufficiently low it may become optimal to acquire the instantaneous payoff $\Pi(x)$. Denote the threshold at which it is optimal to exercise this option by x_*, and its corresponding stopping time $\tau_{x_*}=\inf\{t\geq 0|x_t\leq x_*\}$. Note that it is never optimal to buy the good for $x\geq x^*$ since in this interval the acquired value is zero, i.e. $V_2^*(x|x\geq x^*)=0$, as stated in Corollary 2.2. Thus, it must be the case that $x_*<x^*$, and the value of the durable good with the option to resell is given by $\Pi(x)$ for all $x<x_*$.

Lemma 2.3. *$\Pi(x)$ as defined in equation (6) is a decreasing function of x for all $x\leq x_*$.*

In order to find the optimal buying threshold and option value that solves the problem defined by equation (8) we make use of Theorem 11.5.6 in Boyarchenko and Levendorskii (2007). Similarly as above, we present a simplified version of the Theorem relevant to our particular case and notation.

Theorem 2.4 (Boyarchenko-Levendorskii)**.** *Assume that there exists x_* such that*

(i) $\Pi(x) - (\beta^-)^{-1}\Pi'(x) > 0$ *for all* $x < x_*$*, and*

(ii) $\Pi(x) - (\beta^-)^{-1}\Pi'(x) < 0$ *for all* $x > x_*$*.*

Then

(a) x_* is an optimal threshold;

(b) τ_{x_*} is an optimal stopping time; and

(c) the value of the option with payoff $\Pi(x)$ is given by

$$V_1^*(x) = -\beta^- \int_{-\infty}^{x_*-x} e^{-\beta^- y}(\Pi(x+y) - (\beta^-)^{-1}\Pi'(x+y))dy.$$

A proof is provided by Boyarchenko and Levendorskii (2007) Theorem 11.5.6. Unlike the case of resale, the previous Theorem does not guarantee the existence of the optimal threshold, x_*, so we need to verify that conditions *(i)* and *(ii)* are satisfied for our function $\Pi(x)$.

Lemma 2.5. *Let $G(x) = \Pi(x) - (\beta^-)^{-1}\Pi'(x)$, with $\Pi(x)$ defined as in equation (6). Then, $G(x)$ has a zero at some $x_* < x^*$. Furthermore, x_* is unique and $G(x)$ changes sign as x passes x_* with $G(x) > 0$ for all $x < x_*$.*

Applying Theorem 2.4 and using the result in Lemma 2.5 we can obtain the optimal buying threshold and value of the option to buy the durable good with embedded resale option.

Corollary 2.6. *For $\Pi(x_t) = V_2^*(x_t) - e^{x_t}$, the optimal buying threshold, x_*, is given implicitly by*

$$U - \frac{e^{x_*}(1+\varphi)(\beta^- - 1)}{\beta^-} + \frac{U(\beta^- - \beta^+)}{(\beta^+ - 1)\beta^-} e^{\beta^+(x_* - x^*)} = 0. \tag{9}$$

For $x > x_*$ the value of the option to buy the durable good is

$$V_1^*(x) = \Pi(x_*)e^{\beta^-(x - x_*)}, \tag{10}$$

and $V_1^*(x) = \Pi(x)$ for $x \leq x_*$.

Proof. Equation (9) is obtained using Lemma 2.5 and $G(x)$ defined therein evaluated at x_*. Equation (10) follows from directly calculating the integral in Theorem 2.4

$$V_1^*(x) = \left[U - e^{x_*}(1+\varphi) + \frac{U}{(\beta^+ - 1)} e^{\beta^+(x_* - x^*)}\right] e^{\beta^-(x - x_*)},$$

and recognizing the term in brackets as $\Pi(x_*)$ in equation 6. ■

The buyer's reservation price, e^{x_*}, can be calculated numerically from equation (9). Figure 2.1 shows the value of the option to buy the durable good and the optimal buying threshold. This concludes our analysis of the basic setup. We will come back latter to the results obtained in this section and compare them with those of the multiple priors case, which is the main focus of our work.

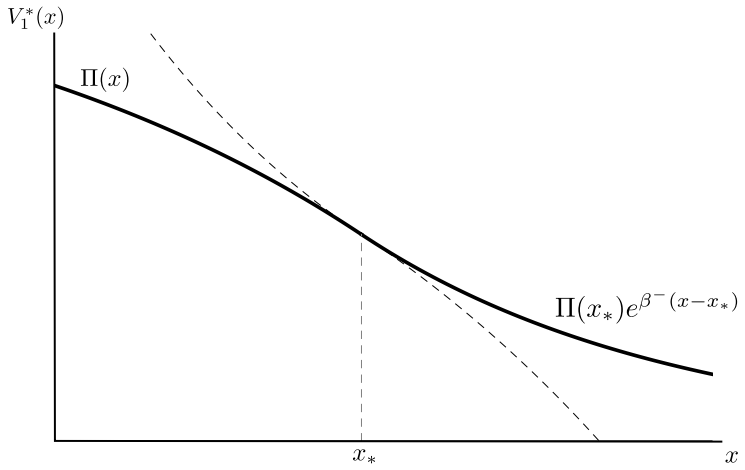


Figure 2.1: *Value of the option to buy a durable good, $V_1^*(x)$*

3 The multiple-priors model

In this section we extend the previous model by adding ambiguity about the drift of the underlying Brownian motion. First, we describe the structure of ambiguity in the model and discuss the importance of our simplifying assumption. Next, we construct a suitable set of probability measures to account for the uncertainty about the drift. We then elaborate on our solution concept and present our main result.

In the previous section we assumed that the agent knows (at least subjectively) the probability measure underlying the stochastic process X_t. Alternatively, we now allow the agent to be uncertain about the particular measure governing the state space and considering, instead, a set of probability measures denoted by $\mathscr{P}$. It can be argued that this type of

uncertainty is more common in realistic decision problems. We incorporate ambiguity to the model by using the multiple-priors utility representation proposed by Gilboa and Schmeidler (1989). Under the *Gilboa-Schmeilder* representation of preferences, and ambiguity averse agent evaluates the expectation of future payments using the *worst case scenario* measure. Formally,

$$\mathbb{E}f(x_t) = \inf_{P \in \mathscr{P}} \left(\mathbb{E}^P f(x_t) \right).$$

for a suitable set of priors, $\mathscr{P}$. In the following subsection we discuss in detail the structure of $\mathscr{P}$ and the the technical reasons for our simplifying assumption of ambiguity resolving once the durable good is bought.

3.1 The set of priors

The set of priors, $\mathscr{P}$, is constructed using the κ-*ignorance* specification proposed by Chen and Epstein (2002). In this type of models the agent is assumed to be uncertain about the drift term of the underlying Brownian motion while the variance term is observable. In order to account for ambiguity, the agent considers a family of Brownian motions whose drift parameters take any value in the interval $[\nu - \sigma\kappa, \nu + \sigma\kappa]$, thus ambiguity is parameterized by a constant $\kappa > 0$ that generates an interval for the drift parameters centered at an arbitrary value, ν, hence the name.

In particular, let $(\Omega, P^\nu, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0})$ be a filtered probability space, and $(X_t)_{t\geq 0}$ be a Brownian motion defined on $(\Omega, \mathscr{F}, P^\nu)$ where $(\mathscr{F}_t)_{t\geq 0}$ is the filtration generated by X_t satisfying the usual conditions augmented by the P^ν-null sets of $\mathscr{F}$. P^ν is only used as a reference measure to generate a set equivalent probability measures with respect to it (and each other) and should not be interpreted as the true probability measure. We generate $\mathscr{P}$ as the set of probability measures Q *mutually absolutely continuous* with respect to P^ν, which is the probability measure characterizing a Brownian motion with drift ν and variance σ^2. Define $\mathscr{D}_\kappa$ as the set of all real-valued processes $\theta = (\theta_t)_{0\geq t\geq T}$ with $|\theta_t| \leq \kappa$. Let B_t be a Brownian motion under P^ν. Now define,

$$z_t = \exp\left\{ \int_0^t \theta_s dB_s ds - \frac{1}{2}\int_0^t \theta_s^2 ds \right\}$$

for some $\theta \in \mathscr{D}_\kappa$. For each $T > 0$, $Q_\theta(F) = \mathbb{E}_P(z_T \mathbf{1}_F)$ defines a probability measure *a.c.* with respect to P^ν. From Girasanov's theorem

$$\tilde{B}_t = B_t + \int_0^t \theta_t ds,$$

where $\tilde{B}_t$ is a Brownian motion under Q_θ with instantaneous drift $(\nu - \sigma\theta_t)$ and variance σ^2. The set of priors $\mathscr{P}$ is constructed as the set of Q_θ for all $\theta \in \mathscr{D}_\kappa$.

In this way we can model uncertainty over the drift of the underlying Brownian motion, and therefore the agent needs to consider any drift in the interval $M = [\nu - \sigma\kappa, \nu + \sigma\kappa]$ at all times, we will refer to this interval as the *ignorance interval*. In order to guarantee that our solutions under ambiguity are well defined we need to impose a restriction on the parameters in the ignorance interval similar to the one stated in equation (2). Namely, the highest drift under κ-*ignorance* should satisfy

$$\nu + \sigma\kappa < q - \frac{\sigma^2}{2}. \tag{11}$$

We can interpret κ as a measure of the degree of ambiguity. As κ tends to zero, the interval of drifts considered by the agent shrinks around ν, and the drift used to evaluate the expectations gets closer the the drift of the actual Brownian motion generating the state process. As κ increases the set of drifts gets larger and the worst-case scenario drift can be, in general farther away from the actual drift. Finally, Chen and Epstein (2002) showed that a set of priors constructed in this manner satisfies the rectangularity condition, which is sufficient for the dynamic consistency of the problem.[8]

For rectangularity to hold, however, it is important for the instantaneous drift of the Brownian motion under Q_θ to be time varying and stochastic whenever ambiguity is present. Thus we need to restrict the time at which the agent is subjected to ambiguity in order to be able to embed our two decisions and solve the problem in a way similar to the previous section. In particular, we assume that ambiguity is only present at the time when the decision maker is considering to buy the durable good, and once the good is purchased the actual probability measure, denoted by $Q \in \mathscr{P}$, is revealed thus resolving ambiguity. When facing the resale decision, the agent still experiences uncertainty over future prices, but this uncertainty can be assessed using the now observed measure. We restrict Q even further by requiring it to be the probability measure underlying a Brownian motion with constant drift. We do not formally introduce any specific reason for which ambiguity is resolved in this seemingly odd fashion. Nevertheless, one can find economic motivations that justify this assumption. Consider, for example, the case where the price distributions depend on the quality of the good. During the buying stage of the problem quality is unobservable, thus giving room for ambiguity to arise. Once the buying decision is made, the quality of

[8]For a formal discussion on the rectangulariy condition see Epstein and Schneider (2003). In a nutshell, this restriction requires that any possible measure describing beliefs about the "next step", at any point in time, must be included in the original set of priors. Thus, one can view this original set of priors as the set generated by the one-step-ahead conditional measures.

the good is known, allowing the agent to pin down the particular probability distribution of market prices.

If we want to naturally fit our simplifying assumption on the way ambiguity is resolved to the two examples discussed in the introduction, we would need all sources of ambiguity to be related to idiosyncratic characteristics of the good, which will be immediately learned once the good is "experienced". Any other sources of ambiguity such as the macroeconomic environment, in the housing example, or general perceptions of the quality of the good, in the car example, will fail to resolve ambiguity in the way we assume it. This is, admittedly, a weak spot in our analysis. However, we concentrate mostly on the effect that ambiguity will have on the buyer's reservation price and the value of the option to purchase the durable good. These two effects take place before the purchase decision is made, when the agent is *ex ante* subjected to ambiguity even for the resale decision. An alternative would be to assume that ambiguity is not resolved after buying the good and the agent solves the resale problem under ambiguity first; this solution, including the agent's choice of the measure to evaluate the resale decision, is embedded in the purchase decision which is solved taking the "worst case measure" for the second stage as given. The solution to this problem would be equivalent to the one presented in this paper. However, it is not clear that this behavioral assumption is dynamically consistent nor that it is not. We chose to make the assumption so it ensures dynamic consistency at the cost of losing generality. To the extent of the author's knowledge, there is yet no way to solve the problem in full generality, but we argue that the solution presented here is a good candidate, or at least an approximation, for the general solution.

To see why resolving ambiguity in this manner is so crucial to obtain a solution to the model let $\Pi(x|\mu)$ be the *ex post* instantaneous payoff as defined in equation (6) for a particular realization of the (constant) drift for the Brownian motion generated by Q at the point of purchase, x. The *ex ante* value of the instantaneous payoff received at the moment of buying the durable good can then be obtained by evaluating $\Pi(x|\mu)$ at each μ in the ignorance interval and selecting the drift that minimizes it. That is

$$\tilde{\Pi}(x) = \min_{\mu \in M} \Pi(x|\mu). \tag{12}$$

Let $V(x)$ be the the value of the embedded stopping time problem. Under the *Gilboa-Schmeilder* representation and assuming ambiguity fully resolves after the good is purchased we have:

$$V(x) = \max_{\tau \geq 0} \min_{P \in \mathscr{P}} \mathbb{E}_0^P \left(e^{-q\tau_P} \tilde{\Pi}(X_{\tau_P}^P) \right), \tag{13}$$

Time consistency of the model is preserved since at the moment of purchase ambiguity is

resolved, and the resale problem of the embedded option reduces to the standard case with single priors discussed in subsection 2.1. Therefore, the buying decision can be modeled as the multiple priors version of acquiring an instantaneous payoff $\tilde{\Pi}(x)$.

3.2 The solution concept

Assuming that the agent observes the actual drift of the underlying Brownian motion generating the process X_t at the moment of purchase we can obtain the optimal value of the stream $\pi(x|\mu)$ and the optimal resale threshold, $x^*_{(\mu)}$, as in subsection 2.1. In order to make the dependence of our solutions on the *ex post* value of the drift term evident, denote $\beta^+_{(\mu)}$ and $\beta^-_{(\mu)}$ as the positive and negative roots of the characteristic equation $q - \psi_{(\mu)}(z)$. By means of Corollary 2.2, the optimal resale threshold, $x^*_{(\mu)}$, is characterized by:

$$e^{x^*_{(\mu)}} = \frac{qU\left(\beta^-_{(\mu)} - 1\right)}{\varphi\beta^-_{(\mu)}(q - \psi_{(\mu)}(1))}, \tag{14}$$

and the value of the stream with the option to resell in the continuation region $x < x^*_{(\mu)}$ is given by:

$$V^*_2(x|\mu) = U - \varphi e^x + \frac{U}{\left(\beta^+_{(\mu)} - 1\right)} e^{\beta^+_{(\mu)}\left(x - x^*_{(\mu)}\right)}. \tag{15}$$

As in the previous section, the *ex post* instantaneous payoff of buying a durable good with an option to resell it is simply given by:

$$\Pi(x|\mu) = V^*_2(x|\mu) - e^x. \tag{16}$$

The *ex ante* version of equation (16) is then constructed as the worst-case scenario for $\Pi(x|\mu)$ over all possible vaues of μ in the ignorance interval as denoted in equation (12).

Lemma 3.1. *$\Pi(x|\mu)$ as defined in equation (16) is a strictly increasing function of μ for all $\mu \in [\nu - \sigma\kappa, \nu + \sigma\kappa]$.*

Therefore, the problem stated in equation (12) has a corner solution at $\underline{\mu} = \nu - \sigma\kappa$, and the *ex ante* value of the instantaneous payoff received at the moment of buying the durable good is given by:

$$\tilde{\Pi}(x) = U - (1 + \varphi)e^x + \frac{U}{\left(\beta^+_{(\underline{\mu})} - 1\right)} e^{\beta^+_{(\underline{\mu})}\left(x - x^*_{(\underline{\mu})}\right)}. \tag{17}$$

Now let us revisit the option to acquire an instantaneous payoff equal to $\tilde{\Pi}(x)$ under ambiguity. Using the *Gilboa-Schmeilder* representation of preferences, the optimal time to

buy a durable good with the possibility of reselling it is given by the stopping time τ_* that solves for

$$\mathscr{V}_1(x) = \max_{\tau} \min_{P \in \mathscr{P}} \mathbb{E}^P \left(e^{-q\tau} \tilde{\Pi}(x) \right). \tag{18}$$

Cheng and Riedel (2013) produced a result that can be applied to solve the problem in equation (18), provided $\tilde{\Pi}(x)$ is monotone in the state variable. To put it in terms of the discussion in Cheng and Riedel, what we have in equation (18) is a simple real option problem with a put-like payoff. Below we present an abbreviated version of their result for reference, but first let us establish a couple of properties of $\tilde{\Pi}(x)$ that allow us to use this result in our model.

Clearly, $\tilde{\Pi}(x)$ is a continuous function of x. Additionally, the restriction on the parameter values imposed by equation (11) guarantees that $\tilde{\Pi}(x)$ is bounded in the sense of equation (7) for all μ in the ignorance interval and in particular for $\underline{\mu}$. This restriction on the parameter values is, in fact, stronger than the "growing condition" used by Cheng and Riedel. Lemma 2.3 established that $\Pi(x)$ is decreasing for all $x < x_*$, independently of the value of μ, thus $\tilde{\Pi}(x)$ is also decreasing in x for said interval.

Theorem 3.2 (Cheng-Riedel)**.** *Denote by v^κ the value function of the standard optimal stopping problem with payoff function $\tilde{\Pi}(x)$ under the measure P^κ. Then the optimal stopping time under κ-ignorance has value function $V_t = v^\kappa(t, X_t)$ where v^κ is the value function of the classical optimal stopping problem under the measure P^κ with the least favorable drift of the underlying process X_t.*[9]

For a proof see Theorem 4.1 in Cheng and Riedel (2013). The previous theorem states that, for the case of the κ-ignorance specification, the solution to the optimal stopping problem under ambiguity is the standard solution for a single prior evaluated at the worst possible drift in the ignorance interval. Therefore, we can characterize the optimal buying threshold and option value as in equations (9) and (10) evaluated at the drift that minimizes $V_1^*(x)$.

Denote $\lambda^+_{(\eta)}$ and $\lambda^-_{(\eta)}$ as the positive and negative roots of the characteristic equation $q - \xi(z)$ where $\xi(z) = \frac{\sigma^2}{2} + \eta z$ and η is the drift of the Brownian motion characterized by P^κ, and define y_* as the optimal buying threshold under ambiguity. Using Corollary 2.6 we can directly establish that, for $x > y_*$ the value of the option to buy the durable good is

$$\mathscr{V}_1^*(x) = \tilde{\Pi}(y_{*(\eta)}) e^{\lambda^-_{(\eta)}(x - y_{*(\eta)})}, \tag{19}$$

[9] The original Theorem 4.1 in Cheng and Riedel (2013) is stated in terms of an increasing payoff function. Here, we present the clearly analogous result for a decreasing function that fits better for our purposes.

with $y_{*(\eta)}$ given implicitly by

$$U - \frac{e^{y_{*(\eta)}}(1+\varphi)\left(\lambda^{-}_{(\eta)}-1\right)}{\lambda^{-}_{(\eta)}} + \frac{U\left(\lambda^{-}_{(\eta)}-\beta^{+}_{(\underline{\mu})}\right)}{\left(\beta^{+}_{(\underline{\mu})}-1\right)\lambda^{-}_{(\eta)}} e^{\beta^{+}_{(\underline{\mu})}\left(y_{*(\eta)}-x^{*}_{(\underline{\mu})}\right)} = 0. \tag{20}$$

The only remaining step to have a complete characterization of the value of the option and optimal buying threshold under ambiguity is to find η associated to the measure P^{κ} that minimizes $\mathscr{V}_1^*(x)$. From an economic point of view, it seems natural to infer that the highest possible drift is the most unfavorable since higher expected increments of the market price are clearly not aligned with the interests of an agent considering to buy the durable good. Our intuition is, in fact, correct and the result is formalized in the following Lemma.

Lemma 3.3. *$\mathscr{V}_1^*(x)$ as defined in equation (19) is a decreasing function of the drift term, η, for all $x > y_*$.*

Consequently, the value for the option to buy the durable good and the optimal buying threshold under ambiguity are given by equations (19) and (20) with $\bar{\eta} = \nu + \sigma\kappa$, i.e. the highest drift in the ignorance interval.

We now summarize the main result of the section and harmonize some notation to make the dependence of the *ex ante* option value and optimal buying threshold on the level of perceived ambiguity clear. Define $\tilde{\beta}^{\pm} = \beta^{\pm}_{(\underline{\mu})}$ with $\underline{\mu} = \nu - \sigma\kappa$, and $\tilde{\lambda}^{\pm} = \lambda^{\pm}_{(\bar{\eta})}$ with $\eta = \nu + \sigma\kappa$. Similarly, let $\tilde{y}_* = y_{*(\bar{\eta})}$ and $\tilde{x}^* = x^*_{(\underline{\mu})}$.

Result 3.4. *The optimal buying threshold under ambiguity, $\tilde{y}_*$, is given implicitly by*

$$U - \frac{(1+\varphi)\left(\tilde{\lambda}^{-}-1\right)}{\tilde{\lambda}^{-}} e^{\tilde{y}_*} + \frac{U\left(\tilde{\lambda}^{-}-\tilde{\beta}^{+}\right)}{\left(\tilde{\beta}^{+}-1\right)\tilde{\lambda}^{-}} e^{\tilde{\beta}^{+}(\tilde{y}_*-\tilde{x}^*)} = 0. \tag{21}$$

For $x > \tilde{y}_*$ the value of the option to buy the durable good is

$$\mathscr{V}_1^*(x) = \tilde{\Pi}(\tilde{y}_*) e^{\tilde{\lambda}^{-}(x-\tilde{y}_*)}, \tag{22}$$

and $\mathscr{V}_1^*(x) = \tilde{\Pi}(x)$ for $x \leq \tilde{y}_*$.

Intuitively, at time zero the pessimistic agent presumes that the *ex post* mean increments of the resale price will be at its lowest. This is consistent with ambiguity aversion since the utility received from consuming the good is constant and the only source of uncertainty after purchase is the resale value of the durable good. Before purchasing the good, however, the most unfavorable process for the agent is the one with the highest mean price increments,

regardless of the realized process at the time of purchase. In this sense, at time zero – or at any time before the good is bought – the agent uses two drifts to evaluate the embedded option of buying a durable good with the possibility of reselling it at some point in the future: (1) the highest possible drift when evaluation the option to buy the good, and (2) the lowest possible drift to compute the resale value as well as the optimal resale threshold. Chudjakow and Vorbrink (2009) and Cheng and Riedel (2013) find a similar change in the drift used to evaluate their exotic options as the problem transitions from one simple option to another.

3.3 Changes in the level of ambiguity

Now we analyze the effect that an increase in the level of ambiguity has on the optimal buyer's and seller's reservation price, as well as on the value of the option to buy the durable good. We start our analysis by determining the effect that a change on $\underline{\mu}$ have on the buyer's and seller's optimal reservation prices.

Proposition 3.5. *The optimal resale threshold, $\tilde{x}^*$, is an increasing function of $\underline{\mu}$.*

Intuitively, as $\underline{\mu}$ increases the agent considers the option to resell the good using a process with higher expected price increments. This has no effect on the termination payoff and increases the value of the option to resell in its continuation region, thus increasing the value of waiting to exercise the option. This in turn will increase the optimal resale threshold. Since the durable good is going to be sold the moment the market price crosses the reservation resale price from below, a higher resale threshold implies a longer waiting time (in expectation) to resell the durable good. From our Result 3.4 we have that, under ambiguity, the *ex ante* drift used to evaluate the option to resell the good is given by $\underline{\mu} = \nu - \sigma\kappa$. Therefore, an increase in the level of ambiguity, κ, will reduce $\underline{\mu}$ and the seller's reservation price, $e^{\tilde{x}^*}$.

Let us now turn to the effect that an increase in κ has on the buyer's optimal reservation price, $e^{\tilde{y}_*}$, and the value of the option to buy the durable good, $\mathscr{V}_1^*(x)$ in its continuation region. Note that $\tilde{y}_*$ is a function of both $\bar{\eta}$ and $\underline{\mu}$. Thus, in order to evaluate the total effect that a change in ambiguity has on the buyer's optimal threshold we first need to establish the partial effects of $\bar{\eta}$ and $\underline{\mu}$ on $\tilde{y}_*$.

Proposition 3.6. *The optimal buying threshold, $\tilde{y}_*$, is an increasing function of both $\bar{\eta}$ and $\underline{\mu}$.*

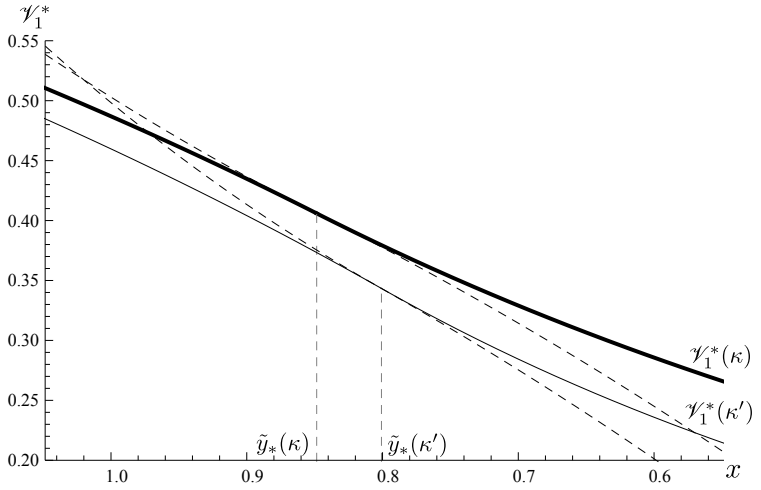
In order to see the intuition behind Proposition 3.6 notice that for higher values of $\bar{\eta}$ the agent uses the process with greater expected price increments to evaluate the decision of buying the durable good. This perceived environment of increasing prices, relative to the

price increments for lower ambiguity levels, makes optimal to increase the reservation price for the good in question, thus increasing $\tilde{y}_*$. On the other hand, a decrease in $\underline{\mu}$ reduces the value of the option in its termination region by lowering the prospect of a high resale price for the durable good. The optimal reservation price will be lower due to the lower (expected) resale value of the good, therefore reducing the optimal buying threshold, $\tilde{y}_*$. An increase in ambiguity will then have two contrasting partial effects: (1) an increase in the optimal exercise threshold due to changes in $\bar{\eta}$, and (2) a decrease in $\tilde{y}_*$ through changes in $\bar{\mu}$.

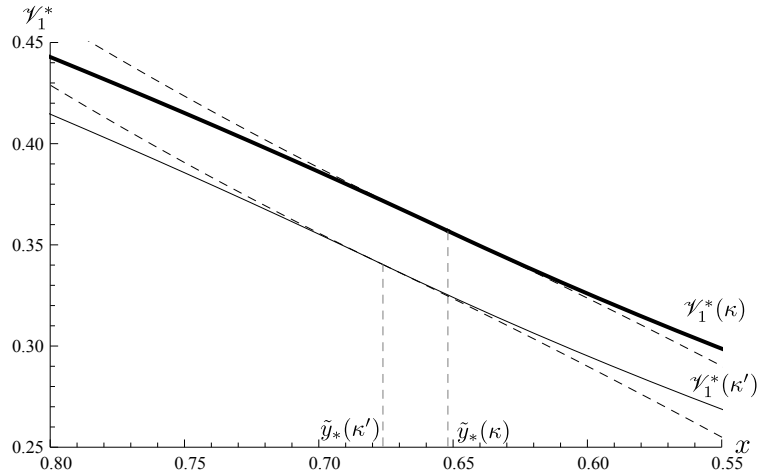
Similar to previous results in the literature (see Miao and Wang (2011)), the direction of the change in the buyer's optimal reservation price as a response to an increase in ambiguity depends on the particular parametrization of the model. That is, the partial positive effect of an increase in $\bar{\eta}$ on $\tilde{y}_*$ due to a change in ambiguity dominates the negative effect of an increase in $\underline{\mu}$ only for a subset of parameters. Figure 3.1 presents two particular parametrizations for which an increase in ambiguity has opposite effects on the optimal buying threshold.

Characterizing the regions for the parameter values that will guarantee a particular direction of the change in the optimal buyer's reservation price is quite complicated due to the lack of a close form solution for $\tilde{y}_*$. However, we can identify some regularities by computing the value of the optimal buying threshold numerically. In particular, we conjecture that $\tilde{y}_*$ is increasing in κ when $\underline{\mu}$ is "low enough" with respect to the initial level of ambiguity. Figure 3.2 presents the region at which $\tilde{y}_*$ is decreasing in κ for a particular set of parameter values. In Appendix B, we vary the values of σ, φ, and q to examine the result at "extreme" values of these parameters.

From an economic point of view, the total benefit of owning the durable good has two sources: the utility generated from the use of it and its resale value. When $\underline{\mu}$ is low, the resale motive plays a smaller role in the *ex ante* purchase decision as the agent considers a relatively pessimistic process to calculate the seller's reservation price. Therefore, further reductions to the originally low mean increments of the resale price will be dominated by higher expected price increments used to evaluate the buying decision. This intuition is similar to the case when the fraction of the spot market price at which the good can be resold, φ, is close to zero. In this case, the effect of an increase of ambiguity in the termination region gets smaller, and the decreasing pressure on $\tilde{y}_*$ through changes in $\underline{\mu}$ becomes less relevant. At the extreme case, when $\varphi = 0$, our model collapses to that of Nishimura and Ozaki (2007) and Miao and Wang (2011) where, in the absence of a resale option, the agent faces a simple perpetual American put-like option with instantaneous payoff $U - e^{x_t}$. Similarly, for a given $\underline{\mu}$, a higher initial κ implies a relatively higher $\bar{\eta}$. If this $\bar{\eta}$ is "too high", further increases will be dominated by the lower expected price increments used to compute the *ex ante* resale



(a) $\tilde{y}_*$ increasing in κ ($\nu = 0$, $\kappa = 0$, $\kappa' = 0.4$)



(b) $\tilde{y}_*$ decreasing in κ ($\nu = 0.3$, $\kappa = 0$, $\kappa' = 0.1$)

Figure 3.1: *Change in $\tilde{y}_*$ as a response to an increase in κ ($\sigma = 1$, $q = 1$, $\varphi = 0.5$).*

value of the good, making $\tilde{y}_*$ decreasing in κ.

To finalize this section, we examine the effects of an increase in ambiguity on the value of the option to buy the durable good both in its termination and continuation regions. First, in the termination region the value of the option is given by $\tilde{\Pi}(x)$, which is independent of $\bar{\eta}$ by assumption. As shown in Lemma 3.1, $\tilde{\Pi}(x)$ is increasing in $\underline{\mu}$, thus an increase in ambiguity will reduce the value of the option in its termination region since an increase in κ reduces the value of $\underline{\mu} = \nu - \sigma\kappa$.[10]

In order to analyze the effect of an increase in κ in the continuation region we first need to establish the following result.

[10]The result in Lemma 3.1 holds for all μ in the ignorance interval, and in particular for $\underline{\mu}$.

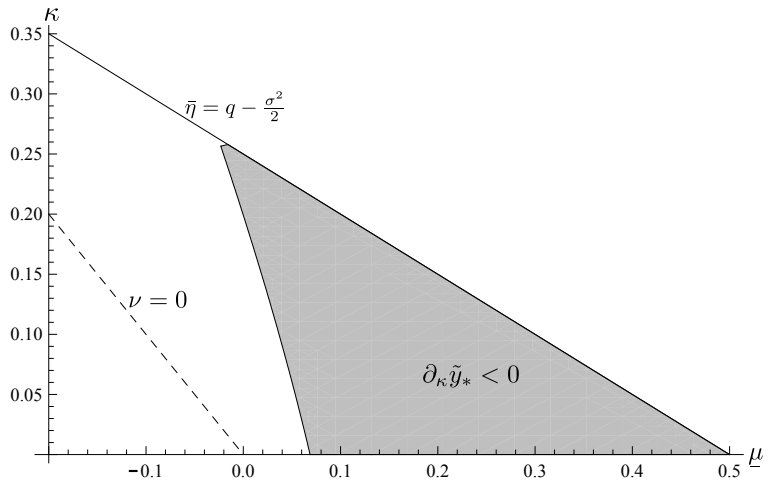


Figure 3.2: *Region Plot for* $\partial_\kappa \tilde{y}^* < 0$ *(*$\sigma = 1$*,* $q = 1$*,* $\varphi = 0.5$*).*

Proposition 3.7. *The value of the option to buy a durable good,* $\mathscr{V}_1^*(x)$*, is an increasing function of* $\underline{\mu}$ *for all* x *in the continuation region,* $x > \tilde{y}_*$*.*

Using our results from Lemma 3.3 and Proposition 3.7 we can see that the partial effects that a change in the level of ambiguity has on the value of the option in the continuation region coincide. That is, $\mathscr{V}_1^*$ decreases as a result of increases in κ, since it is decreasing in $\bar{\eta}$ and increasing in $\underline{\mu}$. As the level of perceived ambiguity increases the agent uses a process with higher expected price increments. This reduces the value of the option to buy the good as the agent expects to pay more for a good whose utility derived from its consumption is independent of the market price. The *ex ante* resale decision is made using lower expected price increments which negatively affect the resale value of the good.

4 Conclusion

We have developed a model to analyze the option to buy a durable good with an embedded option to resell it at any point in time at a fraction of the spot market price where the agent is ambiguous regarding the drift of the Brownian motion characterizing the process of the state variable before the moment of purchase. We assume that ambiguity is resolved after the good is bought, and the resale decision is made with a well known probability distribution of future prices. We find that while the agent is considering to buy the durable good she uses two drifts to evaluate the embedded option: (1) the highest possible drift to compute the value of the option to buy the good and the optimal buying threshold, and (2) the lowest possible drift to calculate the *ex ante* resale value and optimal resale threshold. In addition we use computational methods to analyze the effect that an increase in the perceived level

ambiguity has on the buyer's reservation price. We show that the direction of the change in the buyer's reservation price depends on the particular parametrization of the model. Furthermore, the change in the buying threshold due to an increase in ambiguity is greater as the fraction of the spot market price at which the agent can resell the good decreases, and when this fraction gets closer to zero our problem gets closer to the perpetual American put-like option. As the last result, an increase in ambiguity reduces the value of the option in both its continuation and termination regions.

A common critique to any model that uses the utility representation proposed by Gilboa and Schmeilder is that they do not allow for smoother attitudes towards ambiguity. In order to assess this valid point, it would be necessary to extend the model by using the utility representations proposed by Ju and Miao (2012) and Klibanoff et al. (2009). The results in Cheng and Riedel (2013) and Boyarchenko and Levendorskii (2010) that we used in this work can be applied to more general Levy processes, thus there is room to generalize our model in that direction. Finally, another natural extension of the model is to remove the assumption that ambiguity vanishes at the moment of purchase or to formally incorporate it into the model by allowing the utility derived from the consumption of the durable good to be a function of the state variable or even the drift of the Brownian motion directly.

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A Proofs of Lemmata and Propositions

Proof of Corollary 2.2

From the definition of $\pi(x)$ in equation 1 we have:

$$\frac{\partial \pi(x)}{\partial x} = -(q - \psi(1))\varphi e^x.$$

With $\psi(1) = \frac{\sigma^2}{2} + \mu$. Thus the derivative of $\pi(x)$ with respect to x is negative, provided $q > \psi(1)$, which is the same restriction imposed by equation (2). $\pi(x)$ changes sign as:

$$\lim_{x\to\infty} \pi(x) = -\infty, \text{ and}$$
$$\lim_{x\to-\infty} \pi(x) = qU > 0.$$

Equation (4) follows from direct evaluation of the integral defining $W(x)$ in Theorem 2.1. By doing a few simple manipulations and rearrangement we write

$$V_2^* = U - Ue^{\beta^+(x-x^*)} - \varphi\left(e^x - e^{\beta^+(x-x^*)+x^*}\right)\frac{(q-\psi)}{q}\left(\frac{\beta^-}{\beta^- - 1}\right)\left(\frac{\beta^+}{\beta^+ - 1}\right).$$

Using the fact that $\frac{q}{(q-\psi)} = \left(\frac{\beta^-}{\beta^- - 1}\right)\left(\frac{\beta^+}{\beta^+ - 1}\right)$ we can further simplify V_2^* to

$$V_2^* = U - Ue^{\beta^+(x-x^*)} - \varphi e^x + e^{\beta^+(x-x^*)}\varphi e^{x^*}$$

and equation (5) follows by substituting equation (4) in the previous expression. ■

Proof of Lemma 2.3

From equation (6) we have:

$$\partial_x \Pi(x) = -(1+\varphi)e^x + \frac{U\beta^+}{\beta^+ - 1}e^{\beta^+(x-x^*)} \tag{23}$$

Using equation (4), rewrite equation 23 as:

$$\partial_x \Pi(x) = -(1+\varphi)e^x\left[\frac{1}{e^{x^*}}\frac{qU(\beta^- - 1)}{\varphi\beta^-(q-\psi(1))}\right] + \frac{U\beta^+}{\beta^+ - 1}e^{\beta^+(x-x^*)}.$$

From the fact that $\frac{q}{q-\psi(1)} = \frac{\beta^+}{\beta^+ - 1}\frac{\beta^-}{\beta^- - 1}$, the previous expression can rewritten as:

$$\partial_x \Pi(x) = \frac{U\beta^+}{\beta^+ - 1}\left(Z^{\beta^+} - \frac{1+\varphi}{\varphi}Z\right). \tag{24}$$

with $Z = e^{x-x^*}$. Since $\beta^+ > 1$, Z^{β^+} is a convex function of Z while $\frac{1+\varphi}{\varphi}Z$ is linear in Z. At $x = x^*$, $Z = 1$ and

$$\left(Z^{\beta^+} - \frac{1+\varphi}{\varphi}Z\right) = 1 - \frac{1+\varphi}{\varphi} < 0$$

since $\varphi > 0$ by definition. Furthermore,

$$\lim_{x\to-\infty}\left(Z^{\beta^+} - \frac{1+\varphi}{\varphi}Z\right) = 0$$

Since a convex function and a linear function cross in at most two points, it must be the case that $\left(Z^{\beta^+} - \frac{1+\varphi}{\varphi}Z\right) < 0$ for all $x \leq x^*$, which establishes the result from equation (24) ■

Proof of Lemma 2.5

Using equation (6), and after some manipulations we have

$$G(x) = U - \frac{e^x(1+\varphi)\left(\beta^- - 1\right)}{\beta^-} + \frac{U\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-} e^{\beta^+(x-x^*)} \tag{25}$$

Let x_* be a candidate solution for $G(x_*) = 0$. Using equation (4) we can then write:

$$U - \frac{e^{x_*}(1+\varphi)\left(\beta^- - 1\right)}{\beta^-}\left(\frac{1}{e^{x^*}}\right)\left(\frac{qU}{(q-\psi)\varphi}\frac{(\beta^- - 1)}{\beta^-}\right) + \frac{U\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-} e^{\beta^+(x_*-x^*)} = 0,$$

which can be simplified to

$$U - \frac{\left(\beta^- - 1\right)\beta^+}{\beta^-\left(\beta^+ - 1\right)}\left(\frac{(1+\varphi)U}{\varphi}\right)e^{x_*-x^*} + \frac{U\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-} e^{\beta^+(x_*-x^*)} = 0.$$

The previous expression is equivalent to:

$$1 - \frac{\left(\beta^- - 1\right)\beta^+}{\beta^-\left(\beta^+ - 1\right)}\left(\frac{(1+\varphi)}{\varphi}\right)e^{x_*-x^*} + \frac{\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-} e^{\beta^+(x_*-x^*)} = 0. \tag{26}$$

Set $\delta = x_* - x^*$ and rewrite equation (26) in the form of $e^\delta F(\delta)$ with

$$F(\delta) = e^{-\delta} - \frac{\left(\beta^- - 1\right)\beta^+}{\beta^-\left(\beta^+ - 1\right)}\left(\frac{(1+\varphi)}{\varphi}\right) + \frac{\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-} e^{\left(\beta^+ - 1\right)\delta}.$$

Note that $F(\delta) = 0$ has a negative solution at δ^* and changes sign as δ crosses δ^* if and only if $G(x_*) = 0$ at some $x_* < x^*$ and changes sign at x_*.

Claim A.1. *At $\delta = 0$, $F(0) < 0$.*

Proof. Evaluating $F(\delta)$ at zero,

$$\begin{aligned} 1 - \frac{\left(\beta^- - 1\right)\beta^+}{\beta^-\left(\beta^+ - 1\right)}\left(\frac{1+\varphi}{\varphi}\right) + \frac{\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-} &< 0 \\ \frac{\left(\beta^+ - 1\right)\beta^- + \left(\beta^- - \beta^+\right)}{\left(\beta^- - 1\right)\beta^+} &< \frac{1+\varphi}{\varphi} \\ 1 &< \frac{1+\varphi}{\varphi} \end{aligned}$$

Which is clearly satisfied for $\varphi > 0$. ■

Hence $F(\delta)$ is negative in a left neighborhood of 0 since F is a continuous function. The restriction in the parameters stated in equation (2) imply that $\beta^+ > 1$. Therefore

$$\lim_{\delta \to -\infty} F(\delta) = +\infty.$$

Again, by continuity of the function, $F(\delta)$ must have a zero $\delta^* < 0$. Furthermore,

$$\frac{\partial^2 F(\delta)}{\partial \delta^2} = e^{-\delta} + \frac{e^{\delta\left(-1+\beta^+\right)}\left(\beta^- - \beta^+\right)\left(\beta^+ - 1\right)}{\beta^-} > 0,$$

thus $F(\delta)$ is convex, which guarantees that F changes sign as it passes δ^*, and the uniqueness of δ^*. ■

Proof of Corollary 2.6

Equation (9) is obtained using Lemma 2.5 and equation (25) therein evaluated at x_*. Equation (10) follows from directly calculating the integral in Theorem 2.4

$$V_1^*(x) = \left[U - e^{x_*}(1+\varphi) + \frac{U}{(\beta^+ - 1)} e^{\beta^+(x_* - x^*)}\right] e^{\beta^-(x - x_*)},$$

and recognizing the term in brackets as $\Pi(x_*)$ in equation (6). ■

Proof of Lemma 3.1

First, let us establish the following result:

Claim A.2. $\partial_\mu x^*_{(\mu)} = -\frac{\partial_\mu \beta^+_{(\mu)}}{\left(\beta^+_{(\mu)} - 1\right)\beta^+_{(\mu)}}$

Proof. Using the fact that $\frac{q}{(q - \psi_\mu(1))} = \frac{\beta^+_{(\mu)}}{\left(\beta^+_{(\mu)} - 1\right)} \frac{\beta^-_{(\mu)}}{\left(\beta^-_{(\mu)} - 1\right)}$ rewrite equation (14) as:

$$e^{x^*_{(\mu)}} = \frac{U\beta^+_{(\mu)}}{\varphi\left(\beta^+_{(\mu)} - 1\right)} \tag{27}$$

Taking the derivative with respect to μ on both sides of the previous expression we have:

$$\partial_\mu x^*(\mu) = -\left(\frac{1}{\left(\beta^+_{(\mu)} - 1\right)}\right) \frac{U \partial_\mu \beta^+_{(\mu)}}{\varphi\left(\beta^+_{(\mu)} - 1\right)} e^{-x^*_{(\mu)}}$$

The result follows from substituting $e^{-x^*_{(\mu)}}$ using equation (27) in the previous expression. ■

Taking the derivative of equation (16) with respect to μ we obtain:

$$\partial_\mu \Pi(x|\mu) = \left(\left(x - x^*_{(\mu)}\right)\partial_\mu \beta^+_{(\mu)} - \beta^+_{(\mu)}\left[\partial_\mu x^*_{(\mu)} + \frac{\partial_\mu \beta^+_{(\mu)}}{\left(\beta^+_{(\mu)} - 1\right)\beta^+_{(\mu)}}\right]\right) \frac{U e^{\beta^+_{(\mu)}\left(x - x^*_{(\mu)}\right)}}{\left(\beta^+_{(\mu)} - 1\right)}$$

From Claim A.2, the term in brackets is equal to zero and

$$\partial_\mu \Pi(x|\mu) = \left(\left(x - x^*_{(\mu)}\right)\partial_\mu \beta^+_{(\mu)}\right) \frac{U e^{\beta^+_{(\mu)}\left(x - x^*_{(\mu)}\right)}}{\left(\beta^+_{(\mu)} - 1\right)} > 0$$

since the option to sell the durable good is evaluated in its continuation region, $x < x^*_{(\mu)}$, $\beta^+_{(\mu)} > 1$ by assumption, and it is easy to verify that $\partial_\mu \beta^+_{(\mu)} < 0$ from its definition. ■

Proof of Lemma 3.3

Before proceeding with the proof, let us establish a useful result.

Claim A.3.

$$\partial_\eta \Pi(y_*) - \lambda^-_{(\eta)} \Pi(y_*) \partial_\eta y_* = 0. \tag{28}$$

Proof. Combining equations (15) and (16)

$$\Pi(y_*) = U - (1+\varphi)e^{y_*} + \frac{U}{\left(\beta^+_{(\bar{\mu})} - 1\right)} e^{\beta^+_{(\bar{\mu})}(y_* - y^*)}. \tag{29}$$

Taking the derivative with respect to μ we have:

$$\partial_\eta \Pi(y_*) = \left(\frac{e^{\beta^+_{(\bar{\mu})}(y_* - y^*)} U \beta^+_{(\bar{\mu})}}{\beta^+_{(\bar{\mu})} - 1} - (1+\varphi)e^{y_*} \right) \partial_\eta y_*$$

Plugging it in equation (28)

$$\left(\frac{e^{\beta^+_{(\bar{\mu})}(y_* - y^*)} U \beta^+_{(\bar{\mu})}}{(\beta^+_{(\bar{\mu})} - 1)\lambda^-_{(\eta)}} - \frac{(1+\varphi)e^{y_*}}{\lambda^-_{(\eta)}} - \Pi(y_*) \right) \lambda^-_{(\eta)} \partial_\eta y_* = 0.$$

Substituting $\Pi(y_*)$ using equation (29), and after some basic algebra, we can write the previous expression as:

$$\left(U - \frac{e^{y_*}(1+\varphi)\left(\lambda^-_{(\eta)} - 1\right)}{\lambda^-_{(\eta)}} + \frac{U\left(\lambda^-_{(\eta)} - \beta^+_{(\bar{\mu})}\right)}{\left(\beta^+_{(\bar{\mu})} - 1\right)\lambda^-_{(\eta)}} e^{\beta^+_{(\bar{\mu})}(y_* - y^*)} \right) \lambda^-_{(\eta)} \partial_\eta y_* = 0.$$

By equation (20) we can immediately see that the term in parenthesis is equal to zero. ■

Rewrite $\mathscr{V}_1^*$ in equation (19) as a function of an arbitrary drift $\eta \in [\nu - \sigma\kappa, \nu + \sigma\kappa]$

$$\mathscr{V}_1^*(x|\bar{\mu}) = \Pi(y_*) e^{\lambda^-_{(\eta)}(x - y_*)}. \tag{30}$$

Applying the chain rule to equation (30)

$$\begin{aligned} \partial_\eta \mathscr{V}_1^*(x|\bar{\mu}) &= e^{\lambda^-_{(\eta)}(x-y_*)} \left(\partial_\eta \Pi(y_*) + \Pi(y_*) \left((x - y_*)\partial_\eta \lambda^-_{(\eta)} - \lambda^-_{(\eta)} \partial_\eta y_* \right) \right) \\ &= e^{\lambda^-_{(\eta)}(x-y_*)} \left(\partial_\eta \Pi(y_*) - \lambda^-_{(\eta)} \Pi(y_*) \partial_\eta y_*) + (x - y_*)\Pi(y_*)\partial_\eta \lambda^-_{(\eta)} \right), \end{aligned}$$

and using Claim A.3

$$\partial_\eta \mathscr{V}_1^*(x|\bar{\mu}) = e^{\lambda^-_{(\eta)}(x-y_*)} \left((x - y_*)\Pi(y_*)\partial_\eta \lambda^-_{(\eta)} \right) < 0$$

for all $x > y_*$. The latter follows from the fact that $\Pi(y_*) > 0$ since, by definition, it is the value of the termination payoff evaluated at the exercise threshold and it will never be optimal to exercise an option with negative instant payoff, and we can easily verify that $\partial_\eta \lambda^-_{(\eta)} < 0$. ■

Proof of Proposition 3.5

Equation (14) give us the close form solution for y^*. By taking the derivative with respect to $\bar{\mu}$ we obtain:

$$\partial_{\bar{\mu}} y^* = \frac{\partial_{\bar{\mu}} \beta^-_{(\bar{\mu})}}{\beta^-_{(\bar{\mu})}\left(\beta^-_{(\bar{\mu})} - 1\right)} + \frac{1}{(q - \chi(1))}.$$

Using the definition $\beta^- = -\frac{\mu}{\sigma^2} - \frac{\sqrt{\mu^2 + 2q\sigma^2}}{\sigma^2}$ we can directly calculate the value of the derivative of y^*, which is given by:

$$\partial_{\bar{\mu}} y^* = \frac{\sigma^2}{2q\sigma^2 + \bar{\mu}^2 - \bar{\mu}\sqrt{\bar{\mu}^2 + 2q\sigma^2} - \sigma^2\sqrt{\bar{\mu}^2 + 2q\sigma^2}},$$

We want to show that $\partial_{\bar{\mu}} y^* > 0$, or equivalently that

$$\begin{aligned} 2q\sigma^2 + \bar{\mu}^2 - \bar{\mu}\sqrt{\bar{\mu}^2 + 2q\sigma^2} - \sigma^2\sqrt{\bar{\mu}^2 + 2q\sigma^2} &> 0 \\ 2q\sigma^2 + \bar{\mu}^2 &> \sqrt{\bar{\mu}^2 + 2q\sigma^2}\left(\bar{\mu} + \sigma^2\right) \\ \sqrt{\bar{\mu}^2 + 2q\sigma^2} &> \left(\bar{\mu} + \sigma^2\right). \end{aligned}$$

If $\left(\bar{\mu} + \sigma^2\right) < 0$, the proof is complete as the right hand side of the previous equation must be positive. On the other hand, if $\left(\bar{\mu} + \sigma^2\right)$ is positive we can square both sides of the previous equation and see that the condition is verified as long as $2q > 2\bar{\mu} + \sigma^2$, which corresponds to the restriction on parameters set by equation (2). ■

Proof of Proposition 3.6

Before proving the statement in the proposition we establish a necessary and sufficient condition on $(\tilde{y}_* - \tilde{x}^*)$ for $\partial_{\bar{\eta}} \tilde{y}_* > 0$ and $\partial_{\underline{\mu}} \tilde{y}_* > 0$.

Claim A.4. $\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-} e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} < 1$.

Proof. From equation (21) we have:

$$\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-} e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} = \frac{(1+\varphi)\left(\tilde{\lambda}^- - 1\right)\left(\tilde{\beta}^+ - 1\right)}{U\tilde{\lambda}^-} e^{\tilde{y}_*} - \left(\tilde{\beta}^+ - 1\right)$$

Using equation (27) we obtain:

$$\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-} e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} = \left(\frac{1+\varphi}{\varphi}\right) \frac{\tilde{\beta}^+\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-} e^{\tilde{y}_* - \tilde{x}^*} - \left(\tilde{\beta}^+ - 1\right).$$

Let $Z(y) = e^{y - \tilde{x}^*}$, and rewrite the previous expression as

$$\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-} \left(Z\left(\tilde{y}_*\right)\right)^{\tilde{\beta}^+} = \left(\frac{1+\varphi}{\varphi}\right) \frac{\tilde{\beta}^+\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-} Z\left(\tilde{y}_*\right) - \left(\tilde{\beta}^+ - 1\right) \tag{31}$$

Note that as y tends to negative infinity, $Z(y)$ approaches zero, and $Z(y) < 1$ since the solution to equation (21) must be less than $\tilde{x}^*$. Additionally, the left hand side of the equation (31) is a

convex function of Z, as $\tilde{\beta}^+ > 1$, while the right hand side is a linear function of Z, and these functions cross only once in the interval $(0,1)$ (see the proof of of Lemma 2.5). At $Z = 0$,

$$\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-}(Z)^{\tilde{\beta}^+} > \left(\frac{1+\varphi}{\varphi}\right)\frac{\tilde{\beta}^+\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-}Z - \left(\tilde{\beta}^+ - 1\right).$$

Let Z' be the value of Z such that $\frac{(\tilde{\lambda}^- - \tilde{\beta}^+)}{\tilde{\lambda}^-}(Z')^{\tilde{\beta}^+} = 1$. Then, if

$$\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-}(Z')^{\tilde{\beta}^+} < \left(\frac{1+\varphi}{\varphi}\right)\frac{\tilde{\beta}^+\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-}Z' - \left(\tilde{\beta}^+ - 1\right),$$

it must be the case that $\left(Z\left(\tilde{y}_*\right)\right)^{\tilde{\beta}^+} < 1$. Therefore, the proof will be complete if we can show that

$$\left(\frac{1+\varphi}{\varphi}\right)\frac{\tilde{\beta}^+\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-}\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right)^{1/\tilde{\beta}^+} - \left(\tilde{\beta}^+ - 1\right) > 1,$$

or equivalently:

$$\left(\frac{1+\varphi}{\varphi}\right)\left(\frac{\tilde{\lambda}^- - 1}{\tilde{\lambda}^-}\right) > \left(\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-}\right)^{1/\tilde{\beta}^+} \tag{32}$$

As $\tilde{\beta}^+$ approaches 1, equation (32) reduces to

$$\left(\frac{1+\varphi}{\varphi}\right)\left(\frac{\tilde{\lambda}^- - 1}{\tilde{\lambda}^-}\right) > \frac{\tilde{\lambda}^- - 1}{\tilde{\lambda}^-}$$

which is satisfied for $\varphi > 0$. The left hand side of equation (32) is independent of $\tilde{\beta}^+$, so all that remains to verify is that the right hand side is decreasing in $\tilde{\beta}^+$.

$$\frac{\partial}{\partial\tilde{\beta}^+}\left(\left(\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-}\right)^{1/\tilde{\beta}^+}\right) = -\frac{\left(\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-}\right)^{1/\tilde{\beta}^+}\left[\tilde{\beta}^+ + \left(\tilde{\lambda}^- - \tilde{\beta}^+\right)\log\left(\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-}\right)\right]}{\left(\tilde{\beta}^+\right)^2\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)},$$

which is negative if and only if the term in brackets is less than zero, i.e. if

$$\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right) < \frac{\tilde{\beta}^+}{\tilde{\lambda}^- - \tilde{\beta}^+}. \tag{33}$$

Assume $\tilde{\beta}^+ \geq \tilde{\lambda}^-(1-e)$. Then $\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right) \leq \frac{1}{e}$ and $\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right) \leq -1$. Since $\tilde{\lambda}^- < 0$ and $\tilde{\beta}^+ > 1$, $\frac{\tilde{\beta}^+}{\tilde{\lambda}^- - \tilde{\beta}^+} \in (-1, 0)$ and equation (33) is satisfied. Assume $\tilde{\beta}^+ < \tilde{\lambda}^-(1-e)$. Since $\frac{\tilde{\beta}^+}{\tilde{\lambda}^- - \tilde{\beta}^+}$ is decreasing in $\tilde{\beta}^+$,

$$\lim_{\tilde{\beta}^+ \to \left(\tilde{\lambda}^-(1-e)\right)^-}\left(\frac{\tilde{\beta}^+}{\tilde{\lambda}^- - \tilde{\beta}^+}\right) = -1 + \frac{1}{e}, \text{ and}$$

$$\lim_{\tilde{\beta}^+ \to \left(\tilde{\lambda}^-(1-e)\right)^-}\left(\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right)\right) = -1.$$

Hence, equation (33) holds at the upper bound of $\tilde{\beta}^+$. As $\tilde{\beta}^+$ tends to 1, equation (33) approaches

$$\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - 1}\right) < \frac{1}{\tilde{\lambda}^- - 1}.$$

In order to see that the previous condition is satisfied for all $\tilde{\lambda}^- < 0$, define $f(\tilde{\lambda}^-) = \log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - 1}\right) - \frac{1}{\tilde{\lambda}^- - 1}$. Clearly,

$$\lim_{\tilde{\lambda}^- \to -\infty} f(\tilde{\lambda}^-) = 0, \text{ and}$$

$$\frac{df(\tilde{\lambda}^-)}{d\tilde{\lambda}^-} = \frac{1}{\tilde{\lambda}^-(\tilde{\lambda}^- - 1)^2} < 0.$$

Therefore, equation (33) also holds at the lower bound of $\tilde{\beta}^+$. Finally, it suffices to show that $\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - 1}\right)$ is decreasing in $\tilde{\beta}^+$ to guarantee that equation (33) holds throughout the interval $(1, \tilde{\lambda}^-(1-e))$.

$$\frac{\partial}{\partial \tilde{\beta}^+}\left(\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right)\right) = \frac{1}{\tilde{\lambda}^- - \tilde{\beta}^+} < 0.$$

■

Now we proof the statements in Proposition 3.6. First, we show that $\partial_{\bar{\eta}} \tilde{y}_* > 0$. Equation (21) (implicitly) defines the exercise threshold $\tilde{y}_*$. Taking the implicit derivative with respect to $\bar{\eta}$ and after some rearrangement of terms we obtain:

$$\left(\frac{e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} U \tilde{\beta}^+ \left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\left(\tilde{\beta}^+ - 1\right)\tilde{\lambda}^-} - \frac{e^{\tilde{y}_*}(1+\varphi)\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-}\right) \partial_{\bar{\eta}} \tilde{y}_* =$$

$$\left(\frac{e^{\tilde{y}_*}(1+\varphi)\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-} - \frac{\tilde{\beta}^+ e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} U \left(\tilde{\lambda}^- - 1\right)}{\left(\tilde{\beta}^+ - 1\right)\tilde{\lambda}^-}\right) \frac{\partial_{\bar{\eta}} \tilde{\lambda}^-}{\tilde{\lambda}^-\left(\tilde{\lambda}^- - 1\right)}$$

Using equation (20) we can replace $\frac{(1+\varphi)\left(\tilde{\lambda}^- - 1\right)e^{\tilde{y}_*}}{\tilde{\lambda}^-}$ in both sides of the previous expression which allow us to rewrite the derivative of $\tilde{x}^*$ with respect to $\bar{\eta}$ as:

$$\partial_{\bar{\eta}} \tilde{y}_* = \left(\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-} e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} - 1\right)^{-1} \left(1 - e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)}\right) \frac{\partial_{\bar{\eta}} \tilde{\lambda}^-}{\tilde{\lambda}^-\left(\tilde{\lambda}^- - 1\right)} \tag{34}$$

The first term in the right hand side of the previous expression is negative from Claim A.4. Since $(\tilde{y}_* - \tilde{x}^*) < 0$ and $\tilde{\beta}^+ > 1$ the second term is positive. It can be easily verified that $\partial_{\bar{\eta}} \tilde{\lambda}^- < 0$ and $\tilde{\lambda}^- < 0$ by definition. Therefore $\partial_{\bar{\eta}} \tilde{y}_* > 0$.

Finally, we show that $\partial_{\bar{\mu}} \tilde{y}_* > 0$. Taking the implicit derivative with respect to $\bar{\mu}$ and using the same equation (21) to replace $\frac{e^{\tilde{y}_*}\left(\lambda^-_{(\eta)} - 1\right)(1+\varphi)}{\lambda^-_{(\eta)}}$ we can write the derivative of $\tilde{x}^*$ with respect to $\bar{\mu}$

as:

$$\left(\frac{\left(\tilde{\lambda}^{-}-\tilde{\beta}^{+}\right)e^{\tilde{\beta}^{+}(\tilde{y}_{*}-\tilde{x}^{*})}}{\tilde{\lambda}^{-}}-1\right)\partial_{\underline{\mu}}\tilde{y}_{*}=$$
$$\left[\left(\tilde{\beta}^{+}\partial_{\underline{\mu}}\tilde{x}^{*}-\partial_{\underline{\mu}}\tilde{\beta}^{+}\left(\tilde{y}_{*}-\tilde{x}^{*}\right)\right)\left(\tilde{\lambda}^{-}-\tilde{\beta}^{+}\right)+\frac{\left(\tilde{\lambda}^{-}-1\right)\partial_{\underline{\mu}}\tilde{\beta}^{+}}{\left(\tilde{\beta}^{+}-1\right)}\right]\frac{e^{\tilde{\beta}^{+}(\tilde{y}_{*}-\tilde{x}^{*})}}{\left(\tilde{\beta}^{+}-1\right)\tilde{\lambda}^{-}} \quad (35)$$

The term in brackets can be further reduce to $\partial_{\underline{\mu}}\tilde{\beta}^{+}(1-(\tilde{\lambda}^{-}-\tilde{\beta}^{+})(\tilde{y}_{*}-\tilde{x}^{*}))$ by plugging in $\partial_{\underline{\mu}}\tilde{x}^{*}$ as obtained in Claim A.2. $\partial_{\underline{\mu}}\tilde{y}_{*}>0$ if $(1-(\tilde{\lambda}^{-}-\tilde{\beta}^{+})(\tilde{y}_{*}-\tilde{x}^{*}))<0$, since $\partial_{\underline{\mu}\tilde{\beta}^{+}}<0$, or equivalently if:

$$(\tilde{y}_{*}-\tilde{x}^{*})<\frac{1}{\tilde{\lambda}^{-}-\tilde{\beta}^{+}}.$$

From Claim A.4 we know that $\frac{\tilde{\lambda}^{-}-\tilde{\beta}^{+}}{\tilde{\lambda}^{-}}e^{\tilde{\beta}^{+}(\tilde{y}_{*}-\tilde{x}^{*})}<1$ which implies that

$$(\tilde{y}_{*}-\tilde{x}^{*})<\frac{1}{\tilde{\beta}^{+}}\log\left(\frac{\tilde{\lambda}^{-}}{\tilde{\lambda}^{-}-\tilde{\beta}^{+}}\right)<\frac{1}{\tilde{\lambda}^{-}-\tilde{\beta}^{+}}$$

as shown in the proof of Claim A.4 ■

Proof of Propositon 3.7

Taking the derivative of equation (22) with respect to $\underline{\mu}$ we obtain:

$$\partial_{\underline{\mu}}\mathscr{V}_{1}^{*}(x)=-e^{\tilde{\lambda}^{-}(x-\tilde{y}_{*})}\tilde{\lambda}^{-}\left(U-e^{\tilde{y}_{*}}(1+\varphi)+\frac{e^{(\tilde{y}_{*}-\tilde{x}^{*})\tilde{\beta}^{+}}U}{\tilde{\beta}^{+}-1}\right)\partial_{\underline{\mu}}\tilde{y}_{*}$$
$$+e^{\tilde{\lambda}^{-}(x-\tilde{y}_{*})}\left(-e^{\tilde{y}_{*}}(1+\varphi)\partial_{\underline{\mu}}\tilde{y}_{*}-\frac{U\partial_{\underline{\mu}}\tilde{\beta}^{+}e^{(\tilde{y}_{*}-\tilde{x}^{*})\tilde{\beta}^{+}}}{\left(\tilde{\beta}^{+}-1\right)^{2}}+\frac{Ue^{(\tilde{y}_{*}-\tilde{x}^{*})\tilde{\beta}^{+}}\left(\tilde{\beta}^{+}\left(\partial_{\underline{\mu}}\tilde{y}_{*}-\partial_{\underline{\mu}}\tilde{x}^{*}\right)+\left(\tilde{y}_{*}-\tilde{x}^{*}\right)\partial_{\underline{\mu}}\tilde{\beta}^{+}\right)}{\tilde{\beta}^{+}-1}\right) \quad (36)$$

Taking the implicit derivative of equation (21) with respect to $\underline{\mu}$ we can get

$$\frac{Ue^{(\tilde{y}_{*}-\tilde{x}^{*})\tilde{\beta}^{+}}\left(\tilde{\beta}^{+}\left(\partial_{\underline{\mu}}\tilde{y}_{*}-\partial_{\underline{\mu}}\tilde{x}^{*}\right)+\left(\tilde{y}_{*}-\tilde{x}^{*}\right)\partial_{\underline{\mu}}\tilde{\beta}^{+}\right)}{\tilde{\beta}^{+}-1}=$$
$$\frac{e^{\tilde{y}_{*}}\left(\tilde{\lambda}^{-}-1\right)(1+\varphi)\partial_{\underline{\mu}}\tilde{y}_{*}}{\left(\tilde{\lambda}^{-}-\tilde{\beta}^{+}\right)}+\frac{U\ \partial_{\underline{\mu}}\tilde{\beta}^{+}e^{\tilde{\beta}^{+}(\tilde{y}_{*}-\tilde{x}^{*})}}{\left(\tilde{\beta}^{+}-1\right)}\left(\frac{1}{\left(\tilde{\beta}^{+}-1\right)}+\frac{1}{\left(\tilde{\lambda}^{-}-\tilde{\beta}^{+}\right)}\right)$$

Substituting the previous expression in equation (36) and using equation (21) to replace

$$\frac{U\left(\tilde{\lambda}^{-}-\tilde{\beta}^{+}\right)}{\left(\tilde{\beta}^{+}-1\right)\tilde{\lambda}^{-}}e^{\tilde{\beta}^{+}(\tilde{y}_{*}-\tilde{x}^{*})}=-U+\frac{(1+\varphi)\left(\tilde{\lambda}^{-}-1\right)}{\tilde{\lambda}^{-}}e^{\tilde{y}_{*}}.$$

we can write $\partial_{\underline{\mu}} \mathscr{V}_1^*(x)$ as

$$\partial_{\underline{\mu}} \mathscr{V}_1^*(x) = \left(\partial_{\underline{\mu}} \tilde{y}_* \left(\tilde{\lambda}^- - \left(\tilde{\lambda}^- - \tilde{\beta}^+ \right) e^{\tilde{\beta}^+ (\tilde{y}_* - \tilde{x}^*)} \right) + \frac{\partial_{\underline{\mu}} \tilde{\beta}^+}{\left(\tilde{\beta}^+ - 1 \right)} e^{\tilde{\beta}^+ (\tilde{y}_* - \tilde{x}^*)} \right) \frac{U e^{\tilde{\lambda}^- (x - \tilde{y}_*)}}{\left(\tilde{\lambda}^- - \tilde{\beta}^+ \right)} \tag{37}$$

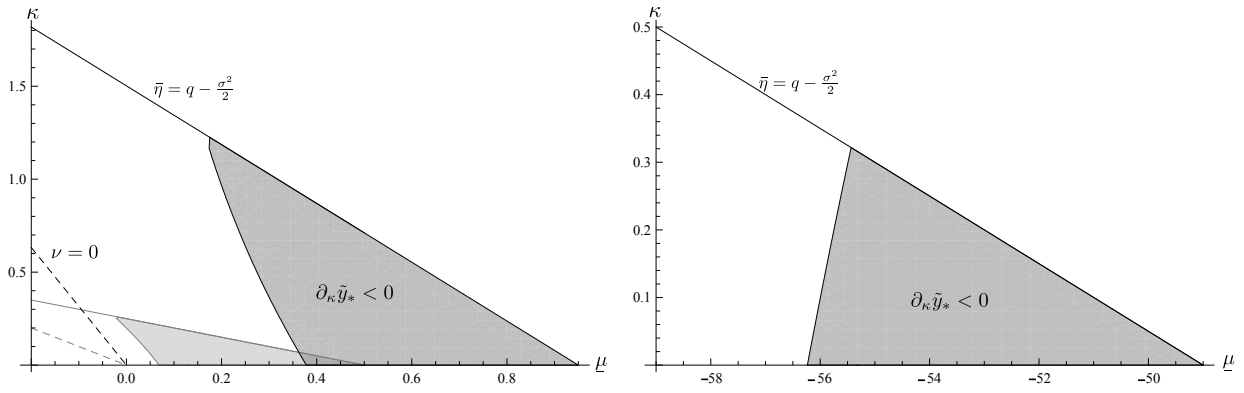
From Proposition 3.6, and Claim A.4 therein, $\left(\tilde{\lambda}^- - \left(\tilde{\lambda}^- - \tilde{\beta}^+ \right) e^{\tilde{\beta}^+ (\tilde{y}_* - \tilde{x}^*)} \right) < 0$ and $\partial_{\underline{\mu}} \tilde{y}_* > 0$. It is easy to verify that $\partial_{\underline{\mu}} \tilde{\beta}^+ < 0$, thus $\partial_{\underline{\mu}} \mathscr{V}_1^*(x) > 0$ ■

B Change in the buyer's threshold for extreme parameter values

In this appendix we use numerical methods to evaluate the effect that a change in κ has on the optimal buying threshold, $\tilde{y}_*$. From our Result 3.4 we have that the *ex ante* drift used to evaluate the option to resell the good is given by $\underline{\mu} = \nu - \sigma\kappa$ while the drift used to compute the value of the option to buy the good is $\bar{\eta} = \nu + \sigma\kappa$. An increase in κ will reduce $\underline{\mu}$ and increase $\bar{\eta}$. Formally, $\partial_\kappa \bar{\mu} = -\sigma$ and $\partial_\kappa \eta = \sigma$. The total change of $\tilde{y}_*$ with respect to κ is

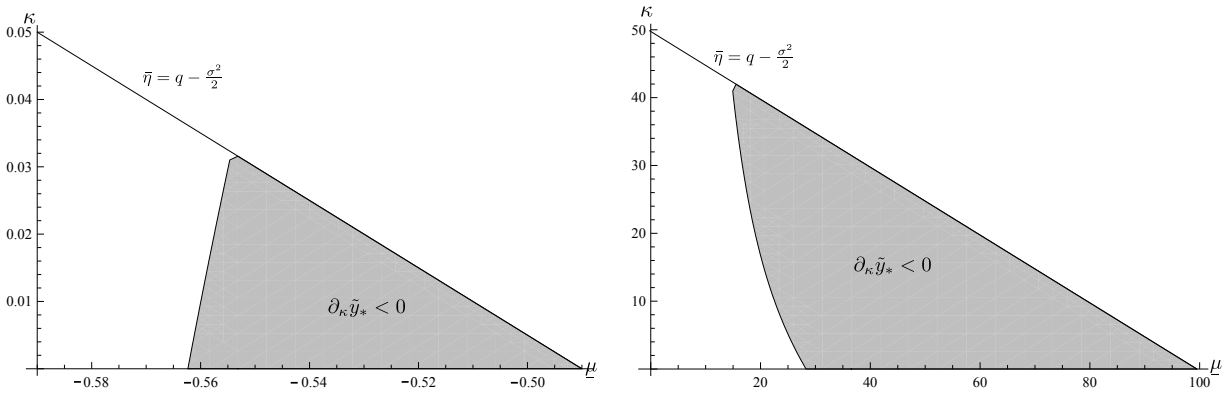
$$\frac{dy_*}{d\kappa} = \sigma \partial_{\bar{\eta}} \tilde{y}_* - \sigma \partial_{\underline{\mu}} \tilde{y}_*. \tag{38}$$

Using equations (34) and (35) we can evaluate the previous expression for a given set of parameters. To facilitate the exposition of the results we generate region plots highlighting the set of parameters for which the $\frac{dy_*}{d\kappa} < 0$. We vary the values of σ, q and φ to examine the result at "extreme" values of these parameters. In all the computations U was normalized to 1. Whenever possible the region depicted in Figure 3.2 is included in the figures in this appendix for comparison.



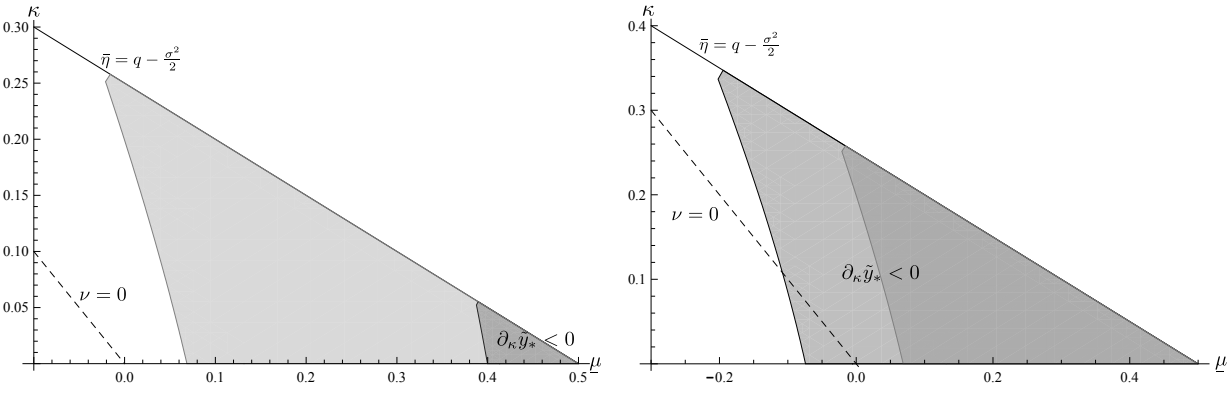
(a) $\sigma = \sqrt{0.1}$ (b) $\sigma = \sqrt{100}$

Figure B.1: *Region Plot for $\partial_\kappa \tilde{y}^* < 0$ for extreme values of σ ($q = 1$, $\varphi = 0.5$)*



(a) $q = 0.1$ (b) $q = 100$

Figure B.2: *Region Plot for $\partial_\kappa \tilde{y}^* < 0$ for extreme values of q ($\sigma = 1$, $\varphi = 0.5$)*



(a) $\varphi = 0.1$ (b) $\varphi = 0.9$

Figure B.3: *Region Plot for $\partial_\kappa \tilde{y}^* < 0$ for extreme values of φ ($\sigma = 1$, $q = 1$)*