

Rogerson, William P.

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## Inter-temporal cost allocation and investment decisions

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# THE CENTER FOR THE STUDY OF INDUSTRIAL ORGANIZATION

AT NORTHWESTERN UNIVERSITY

Working Paper #0084

## INTER-TEMPORAL COST ALLOCATION AND INVESTMENT DECISIONS

By

**William P. Rogerson<sup>\*</sup>**  
**Northwestern University**

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## **ABSTRACT**

This paper considers the profit maximization problem of a firm that must make sunk investments in long-lived assets to produce output. It is shown that if per period accounting income is calculated using a simple and natural allocation rule for investment called the relative replacement cost (RRC) rule, that, in a broad range of plausible circumstances, the firm can choose the fully optimal sequence of investments over time simply by choosing a level of investment each period to maximize next period's accounting income. Furthermore, in a model where shareholders delegate the investment decision to a better-informed manager, it is shown that if accounting income based on the RRC allocation rule is used as a performance measure for the manager, robust incentives are created for the manager to choose the profit maximizing sequence of investments regardless of the manager's own personal discount rate or other aspects of the manager's personal preferences.

## INTRODUCTION

In a variety of industries, firms must make sunk investments in long-lived assets to produce output. Calculation of profit maximizing investment levels and evaluation of the firm's performance in such a situation is inherently complicated because of the need to consider implications for cash flows over multiple future periods. One technique that firms routinely use to create simplified single period "snapshots" of their performance, is to calculate per-period accounting income using accounting measures of cost that allocate the costs of purchasing long-lived assets over the periods that the assets will be used. Firms use these single-period snapshots of performance both to directly guide their investment decisions and to evaluate the performance of managers who make investment decisions. Given their widespread use to both directly and indirectly guide investment decisions, it is perhaps surprising that there has been almost no formal analysis in the economics, finance, or accounting literature that attempts to investigate whether there is any basis for these accounting practices and, if so, how the choice of an allocation rule ought to be affected by factors such as the pattern of depreciation of the underlying asset, the firm's discount rate, the rate at which asset prices are changing over time, and the manager's own rate of time preference. This paper provides a theory which addresses these questions. It shows that, in a broad range of plausible circumstances, a natural and simple allocation rule, which will be called the relative replacement cost (RRC) rule, can be used both to simplify calculation of the optimal level of investment and to create robust incentives for managers to choose this level of investment when the decision is delegated to them.

In particular, two major results are proven. First, it is shown that, when accounting income is calculated using the RRC allocation rule, the firm can choose the fully optimal

sequence of investments simply by choosing a level of investment each period to maximize next period's accounting income. Second, in a model where shareholders delegate the investment decision to a better informed manager, it is shown that if shareholders base the manager's wage each period on current and past period's accounting income calculated using the RRC rule, that the manager will have the incentive to choose the fully optimal sequence of investments so long as each period's wage is weakly increasing in current and past period's accounting income. Furthermore, this result holds regardless of the manager's own personal discount rate or other aspects of the manager's personal preferences. Therefore, the investment incentive problem is solved in a robust way and the firm is left with considerable degrees of freedom to address any other incentive problems that may exist, such as providing incentives for the manager to exert effort each period, by choosing the precise functional form of the wage function each period.

In the formal model of this paper, it is assumed that assets have a known but arbitrary depreciation pattern and that the purchase price of new assets changes at a known constant rate over time. The RRC allocation rule is defined to be the unique allocation rule that satisfies the two properties that: (i) the cost of purchasing an asset is allocated across periods of its lifetime in proportion to the relative cost of replacing the surviving amount of the asset with new assets; and (ii) the present discounted value of the cost allocations using the firm's discount rate is equal to the initial purchase price of the asset.

Property (i) can be interpreted as a version of the "matching principal" from accrual accounting that states that investment costs should be allocated across periods so as to match costs with benefits where the "benefit" that an asset contributes to any period is interpreted to be the avoided cost of purchasing new capacity in that period. Property (ii) can be viewed as stating

that the investment should be fully allocated taking the time value of money into account. Most traditional accounting systems ignore the time value of money when allocating investment costs over time. The term “residual income” is generally used in the accounting literature to describe income measures that are calculated using an allocation rule for investment that takes the time value of money into account (Horngren and Foster 1987, pp. 873-74). Recently there has been an explosion of applied interest in using residual income both to directly guide capital budgeting decisions and as a performance measure for managers who make capital budgeting decisions. Management consulting companies have renamed this income measure “economic value added” (EVA) and very successfully marketed it as an important new technique for maximizing firm value. *Fortune*, for example, has run a cover story on EVA, extolling its virtues and listing a long string of major companies that have adopted it (Tully 1993).<sup>1</sup> This paper provides an explicit formal model which justifies the use of residual income in the capital budgeting process and also specifically identifies the particular allocation rule that should be used to calculate residual income and how it depends on the depreciation pattern of the underlying assets .

This paper’s results are based on Arrow’s (1964) analysis of the optimal investment problem under certainty for general patterns of depreciation.<sup>2</sup> Arrow shows that for any given

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<sup>1</sup>See the roundtable discussion in the *Continental Journal of Applied Corporate Finance* (Stern and Stewart 1994) and the associated articles (Sheehan 1994, Stewart 1994).

<sup>2</sup>Most of the literature on the optimal investment problem under certainty restricts itself to considering the case of exponential depreciation, where a constant share of the capital stock is assumed to depreciate each year regardless of the age profile of the capital stock. See Jorgensen (1963) for an early analysis of this case and see Abel(1990) for a more extensive discussion of the optimal investment literature and further references. The assumption of exponential depreciation dramatically simplifies the analysis because the age profile of the existing capital stock can be ignored. However, for the purposes of this paper’s study of cost allocation rules, it is important to allow for general patterns of depreciation because one of the most interesting

depreciation pattern of assets, a vector of “user costs”<sup>3</sup> can be calculated with the property that, under a broad range of plausible circumstances, the seemingly complex optimal investment problem collapses into a series of additively separable single period problems, where the firm can be viewed as choosing the amount of capital to “rent” each period with rental rates given by the vector of user costs. This paper’s basic insight is that a simple cost allocation rule (namely the RRC rule) can be defined with the property that the cost it allocates to any period of an asset’s lifetime is equal to the surviving amount of the asset multiplied by that period’s user cost. The desirable characteristics of the RRC allocation rule are then shown to follow from this property. Intuitively, Arrow’s results establish that, for purposes of calculating the optimal sequence of investments, a planner can assume that the true marginal cost of providing an additional unit of capital in any period is equal to that period’s user cost, and the RRC allocation rule provides correct information to guide investment decisions by reflecting this fact.

As part of the proof that the RRC rule has the property that the cost it allocates to any period of an asset’s lifetime is equal to the surviving amount of the asset multiplied by that period’s user cost, this paper derives a different and much simpler formula for calculating user cost than the formula derived by Arrow. In particular, Arrow’s formula for calculating user cost

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questions to investigate regarding cost allocation rules is how the nature of the appropriate cost allocation rule should change as the depreciation pattern of the underlying assets changes. Obviously the pattern of depreciation must be a factor which can be exogenously varied in order to investigate this question. Furthermore, the case of exponential depreciation is not a particularly natural case to consider for most real applications. In most real applications a much more natural case to consider is the so-called case of one-hoss shay depreciation, where assets are assumed to have finite lifetimes and to remain equally productive over their lifetimes. This paper’s analysis of the general case, will, in particular, apply to the case of one-hoss shay depreciation.

<sup>3</sup>This term was coined by Jorgensen(1963).

depends on the vector of so-called “replacement rates” from renewal theory, which describe the series of replacements of an original asset as it depreciates, replacements of the replacements as they in turn depreciate, etc. that would be required to generate a permanent increase in the stock of capital of one unit. For the case of general depreciation patterns, the formula for the vector of replacement rates is complicated and difficult to calculate and is defined by an infinite series of recursively defined functions. This paper shows that it is possible to derive an alternate and much simpler formula for user cost which does NOT depend on replacement rates.<sup>4</sup> In particular, it shows that a very simple formula exists to calculate hypothetical “perfectly competitive” rental prices for assets, and then proves that these hypothetical perfectly competitive rental prices must be equal to user costs. The fact that user cost can be calculated by a very simple formula that does not depend on replacement rates is an interesting result independent of its application to cost allocation rules. Furthermore, the fact that user costs can be interpreted as hypothetical perfectly competitive rental prices provides some extra economic intuition to explain their role.

There are two recent groups of papers in the literature that have investigated the question of whether or not allocation rules for investment can be identified such that the resulting measures of per period accounting income can play a useful role in the capital budgeting process. One approach is due to Anctil (1996) and Anctil, Jordan and Mukherji (1998). They assume that depreciation is exponential, that there are adjustment costs to changing the size of the capital

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<sup>4</sup> For the special case of exponential depreciation, the formula determining replacement rates is very simple - a constant share of the asset is replaced each period - and Arrow observes that his formula for user cost collapses into the simple formula directly derived by Jorgensen (1963) for this case. The incremental contribution of this paper is to show that a similarly simple formula to calculate user costs exists for general depreciation patterns, even when there is no simple formula to calculate the vector of replacement rates.



stock, and that the environment is completely stationary so that a stationary equilibrium level of capital stock exists. The papers show that the time path of capital stock when fully optimal investments are chosen and the time path of capital stock when the firm simply attempts each period to maximize that period's residual income both converge to the stationary capital stock and thus converge to one another. This means that the policy of attempting to maximize residual income on a period-by-period basis yields a policy that converges to the optimal policy. While this is an interesting result, it only shows convergence to the optimal policy in the limit and only applies to the case of a stationary environment with exponential depreciation.

The other approach is due to Rogerson (1997).<sup>5</sup> It derives similar sorts of results to the results of this paper for the simpler case where it is assumed that the firm engages in a one-time investment at the beginning of the first period, instead of investing in every period, as in this paper. An allocation rule called the relative benefits (RB) rule is shown to have the same sorts of desirable properties in the model of Rogerson(1997) that the RRC allocation rule is shown to have in the model of this paper. While the allocation rules identified by both papers can be interpreted as allocating investment costs across periods in proportion to the relative benefit that the investment creates across periods, the relevant notion of "benefit" turns out to be very different in each case. In particular, in the one-time investment model of Rogerson(1997), the optimal allocation rule is determined solely by the demand-side factor of how the level of demand varies across periods. In contrast, the optimal allocation rule in the model of this paper

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<sup>5</sup>See Rogerson (1992) for an earlier, related, result. Papers that have generalized Rogerson's(1997) result and applied it in a number of different settings include Baldenius and Reichelstein (2005), Baldenius and Ziv (2003), Dutta and Reichelstein (1999, 2002), and Reichelstein (1997, 2000).

is determined solely by supply side factor of how investments made in different periods can substitute for one another in creating capital stock to be used in a given period. In particular the optimal allocation rule does not depend on how demand varies over time. Therefore the economic factors that determine the optimal allocation rule are quite different depending upon whether investments in different periods substitute for one another or not.

The results of this paper are also related to the larger economics and accounting literature that studies the allocation of joint costs of any sort, that are not necessarily inter-temporal in nature.<sup>6</sup> This paper's theory of cost allocation exhibits a striking difference from most of these existing theories, which stems from a fundamental difference in the manner in which joint costs affect the nature of the underlying cost function. In the model of this paper, the firm can be thought of as a multi-product firm producing joint products, where the level of capital stock available in each period is a separate product. The function giving the present discounted cost of providing any vector of capital stocks can then be viewed as the firm's cost function. From this perspective, the essential content of the user cost result is that the firm's cost function is linear and additively separable in each period's capital stock over the relevant range of capital stocks. In contrast, the literature on allocation of joint costs typically considers models where the presence of joint costs implies that the resulting cost function is not additively separable. The explanation for this difference, is that the existing literature typically considers the case where there is a *single* joint cost that applies to multiple products. In such a model, the only way that the firm can increase the output of any product is by increasing its investment in the single joint cost, and this results in increased output of all of the products. However in the model of this

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<sup>6</sup>See for example, Demski(1981), Thomas(1978) and Young(1985).

paper, there are *multiple overlapping* joint costs. Namely, if assets have a lifetime of  $T$  years, then investment in period 0 is a joint cost of producing stocks of capital in periods 1 through  $T$ , investment in period 1 is a joint cost of producing stocks of capital in periods 2 through  $T$ , etc. In this case, it is possible to adjust the entire vector of planned investments so as to increase output in the next period while holding output in all subsequent periods constant, and the marginal cost of increasing output in the next period is simply equal to the present discounted value of this entire series of adjustments. Given the linear cost of purchasing assets, the result of this property is that the underlying cost function turns out to be linear and additively separable in each period's capital stock over the relevant range of capital stocks, even though there are clearly joint costs of production.

Most existing theories of cost allocation begin with models where the presence of joint costs implies that it is impossible to both fully allocate costs and set the unit accounting cost of each product equal to marginal cost. This means that theories of (full) cost allocation necessarily involve departures of unit accounting cost from marginal cost. In contrast, this paper's theory considers a model where the underlying cost function turns out to be linear and additively separable in each product even though there are clearly joint costs of production. Therefore setting the unit accounting cost of each product equal to marginal cost will fully allocate costs. However, in this paper's model, the value of marginal cost in any given period is not readily apparent because it is equal to the present discounted value of a potentially very complicated series of adjustments in future investments. The result of this paper is essentially that a simple cost allocation rule can be used to calculate the true value of the marginal cost.

In a companion paper to this paper (Rogerson 2007), it is shown that the approach of this

paper can also be applied to the issue of calculating welfare maximizing prices for a regulated firm. In particular it is shown that when there are constant returns to scale within each period (i.e., when output in each period is proportional to the capital stock in each period), that the accounting cost of output calculated using the RRC allocation rule is equal to long run marginal cost. Therefore prices set equal to accounting cost calculated using the RRC allocation rule are first best in that they both induce efficient consumption decisions and allow the firm to break even.

The paper is organized as follows. Section I presents the model and section II presents the results on user cost. Section III introduces notation to define cost allocation rules and shows that the RRC allocation rule has the property that it sets the accounting cost of using a unit of capital in any period of its lifetime equal to Arrow's user cost of capital in that period. Section IV shows that the optimal level of investment in any period can be calculated by maximizing next period's accounting income when accounting income is calculated using the RRC rule. Section V considers an extension to the basic model in which it is assumed that shareholders delegate the investment decision to a better-informed manager, and shows that robust incentives for the manager to choose the fully optimal sequence of investments can be created by using accounting income based on the RRC allocation rule as a performance measure for the manager. Section VI briefly explains how the results generalize to the case where future asset prices do not change at a constant rate. Finally section VII draws a brief conclusion. All proofs are contained in Appendix A. Appendix B describes and derives Arrow's original formula for user cost that depends on replacement rates and discusses its connection to the formula derived by this paper.

## I. THE MODEL

Suppose that there are an infinite number of periods indexed by  $t \in \{0, 1, 2, \dots\}$  where period 0 is the current period.<sup>7</sup> Let  $I_t \in [0, \infty)$  denote the number of assets that the firm purchases in period  $t \in \{0, 1, \dots\}$ . This will also be referred to as the level of investment in period  $t$ . Let  $\mathbf{I} = (I_0, I_1, \dots)$  denote the entire vector of asset purchases.

Assume that a certain fraction of the asset “depreciates” or becomes permanently unavailable for further use in each period of its life. It will be convenient to use notation that directly defines the share of the asset that survives, and is thus available for use in each period, rather than the share that depreciates. Let  $s_t$  denote the share of an asset that survives until at least the  $t^{\text{th}}$  period of the asset’s lifetime and let  $\mathbf{s} = (s_1, s_2, \dots)$  denote the entire vector of survival shares. It will be assumed that  $s_t \in [0, 1]$  for every  $t$ ,  $s_1 = 1$ , and that  $s_t$  is weakly decreasing in  $t$ .

Two natural and simple examples of depreciation patterns are the cases of exponential depreciation and one-hoss shay depreciation. The case of exponential depreciation occurs when a constant share of the asset depreciates each period. Formally, this means that  $s_t$  is given by

$$(1) \quad s_t = \beta^{t-1}$$

for some  $\beta \in (0, 1)$ . The case of one-hoss shay depreciation occurs when assets have a finite lifetime of  $T$  years and remain equally productive over their entire lifetime. Formally, this corresponds to the case where  $s_t$  is of the form

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<sup>7</sup>It is possible to conduct the entire analysis in continuous time without any significant changes.

$$(2) \quad s_t = \begin{cases} 1, & t \in \{1, 2, \dots, T\} \\ 0, & \text{otherwise} \end{cases}$$

where  $T$  is a positive integer.

Assume that an asset becomes available for use one period after it is purchased.<sup>8</sup> Let  $K_t$  denote the number of assets the firm has available for use in period  $t \in \{1, 2, \dots\}$  *due to assets purchased in period 0 or later*.<sup>9</sup> This will also be referred to as the capital stock in period  $t$ . Let  $\mathbf{K} = (K_1, K_2, \dots)$  denote the entire vector of capital stocks. Let  $\psi_t(I_0, \dots, I_{t-1})$  denote the function from  $R^t$  to  $R$  giving the capital stock in period  $t$  resulting from investments in periods 0 through  $t-1$ . It is defined by

$$(3) \quad \psi_t(I_0, \dots, I_{t-1}) = \sum_{i=1}^t I_{t-i} s_i$$

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<sup>8</sup> i.e., an asset purchased during period 0 becomes available for use in period 1, etc. The analysis generalizes to allow for any lag length between the period when an asset is purchased and the period in which it first becomes useful. The assumption of a single period of lag minimizes notational clutter.

<sup>9</sup>Note that the above formulation does not exclude the possibility that the firm enters period 0 with existing assets that were purchased in earlier periods. However, the capital stock variable,  $K_t$ , is defined to only include assets purchased in period 0 or thereafter. It could therefore be thought of as the “incremental” capital stock created by the firm’s investment decisions beginning in the current period. As will be seen, there will be no need in this paper to formally introduce notation to describe the capital stock created by assets that the firm enters period 0 with. That is, the only notion of capital stock that it will be necessary to formally model will be the “incremental” capital stock as defined by equation (3). Therefore, for purposes of this paper, unless otherwise indicated, the terms “capital stock” and “incremental capital stock” will be used as interchangeable synonyms. In the occasional instance when it is useful to refer to assets that the firm already owns at the beginning of period 0, these will be referred to as “legacy assets.”

Let  $\psi(\mathbf{I}) = (\psi_1(I_0), \psi_2(I_0, I_1), \dots)$  denote the function from  $\mathbb{R}^\infty$  to  $\mathbb{R}^\infty$  giving the entire vector of capital stocks generated by an entire vector of investments. The function  $\psi(\mathbf{I})$  will be called the capital accumulation function.

Note that the capital accumulation function is linear and that  $K_t$  is determined only by  $I_0$  through  $I_{t-1}$ . A simple induction argument establishes that the capital accumulation function is invertible and that the required level of investment in period  $t$  is determined as a linear function of  $K_1$  through  $K_{t+1}$ . For future reference, this will be stated as a lemma.

**Lemma 1:**

The inverse of the capital accumulation function exists. Furthermore, for any  $t \in \{0, 1, \dots\}$ , the required level of investment in period  $t$  is determined as a linear function of  $K_1$  through  $K_{t+1}$ .

**Proof:**

See Appendix A.

QED

Let  $\varphi_t(K_1, \dots, K_{t+1})$  be the function from  $\mathbb{R}^{t+1}$  to  $\mathbb{R}$  giving the required level of investment in period  $t$  necessary to generate the vector of capital stocks  $(K_1, \dots, K_{t+1})$  and let  $\varphi(\mathbf{K}) = (\varphi_0(K_1), \varphi_1(K_1, K_2), \dots)$  be the entire inverse capital accumulation function.

Let  $\delta \in (0, 1)$  denote the firm's discount rate. Let  $z_t$  denote the price of purchasing a new unit of the asset in period  $t$ . Assume that  $z_t$  is determined by

$$(4) \quad z_t = \alpha^t z_0$$

where  $z_0 \in (0, \infty)$  and  $\alpha \in (0, 1/\delta)$ . That is, the price of a new unit of the asset in period 0 is a

known positive number and the price of purchasing a unit of the asset either stays constant over time or changes at a known constant percentage rate.<sup>10</sup> The assumption that  $\alpha < 1/\delta$  guarantees that the firm does not find it profitable to “stockpile” assets ahead of time. The present discounted cost of any vector of investments is given by

$$(5) \quad \sum_{t=0}^{\infty} z_t I_t \delta^t$$

It will be convenient to view the firm as directly choosing a vector of capital stocks rather than as directly choosing a vector of investments. Define  $C(\mathbf{K})$  to be the function from  $\mathbb{R}^{\infty}$  to  $\mathbb{R}$  giving the present discounted cost of generating any vector of capital stocks. It is formally defined by substituting the inverse capital accumulation function into (5) to yield

$$(6) \quad C(\mathbf{K}) = \sum_{t=0}^{\infty} \varphi_t(K_1, \dots, K_{t+1}) z_t \delta^t$$

The function  $C(\mathbf{K})$  will sometimes be referred to as the “cost function” of the firm.

To complete the description of the model, let  $B(K, t)$  be the function determining the firm’s operating profit or “benefit” in period  $t \in \{1, 2, \dots\}$  if it has the capital stock  $K \in [0, \infty)$ . Let  $B_K(K, t)$  denote the marginal benefit function. It will be assumed, that for every  $t$ ,  $B_K(K, t)$  exists, is continuous, is strictly decreasing when it is strictly positive, and that it either converges

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<sup>10</sup>Section VI will describe the manner in which the results generalize to the case where asset prices do not change at a constant rate.



to zero or becomes negative for large enough values of  $K$ .<sup>11</sup> Let  $B(\mathbf{K})$  denote the present discounted value of the benefits created by the vector of capital stocks,  $\mathbf{K}$ .

$$(7) \quad B(\mathbf{K}) = \sum_{t=1}^{\infty} B(K_t, t) \delta^t$$

The firm's optimization problem is to choose a vector capital stocks to maximize the present discounted value of its cash flows subject to the constraint that the levels of investment required to generate this vector of capital stocks are all non-negative.

$$(8) \quad \underset{\mathbf{K}}{\text{maximize}} \quad B(\mathbf{K}) - C(\mathbf{K})$$

Subject to:

$$(9) \quad \varphi_t(K_1, \dots, K_t) \geq 0 \quad \text{for } t \in \{0, 1, \dots\}$$

It will be necessary to assume that one additional condition with real economic content is satisfied in order to guarantee that the non-negativity constraints on investment do not bind in the above problem. In particular a condition will be assumed to hold that is sufficient to guarantee that the vector of capital stocks which is the unconstrained maximizer of (8) is weakly increasing in  $t$ . It is, of course, clear that a vector of capital stocks that is weakly increasing in  $t$  will automatically satisfy the non-negativity of investment constraints in (9). The condition that will

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<sup>11</sup>If there is no operating cost associated with letting capital stock sit idle for a period, then  $B_K(K, t)$  will never be negative but will instead be equal to zero once the level of capital stock is reached such that production and sale of further output would reduce operating profit.

be assumed to hold is that the marginal benefit function is weakly increasing over time when marginal benefit is calculated in “real” dollars using the cost of purchasing one new unit of the asset as a numeraire. This will be referred to as the monotonicity (M) condition since its role is to guarantee monotonicity of the capital stock. It is formally stated below.

**The Monotonicity (M) Condition:**

$[B_K(K, t)/\alpha^t]$  is weakly increasing in  $t$  for every  $K$ .

This condition is essentially a generalized version of the condition that marginal revenue is weakly increasing over time and therefore we would generally expect to see it satisfied in situations where demand is weakly increasing over time. For the remainder of this paper, it will be assumed that the monotonicity condition is satisfied.

## **II. USER COSTS AND HYPOTHETICAL PERFECTLY COMPETITIVE RENTAL PRICES**

Recall that the present discounted cost to the firm of providing itself with a vector of capital stocks, given that it must purchase its own assets, is given by the cost function,  $C(\mathbf{K})$ , defined by equation (6). Inspection of equation (6) reveals that the present discounted cost is a linear function of the required investments in each period which are given by the  $\phi_t(K_1, \dots, K_{t+1})$  functions. However, Lemma 1 has already established that the  $\phi_t(K_1, \dots, K_{t+1})$  functions are linear in each period’s capital stock. It therefore follows immediately that the entire cost function  $C(\mathbf{K})$  is linear in each period’s capital stock and can therefore be written in the form

$$(10) \quad C(\mathbf{K}) = \sum_{t=1}^{\infty} c_t K_t \delta^t$$

for some vector of constants,  $\mathbf{c} = (c_1, c_2, \dots) \in \mathbb{R}^{\infty}$ . Substitution of (7) and (10) into (8) then shows that firm's objective function in its optimization problem becomes

$$(11) \quad \sum_{t=1}^{\infty} \delta^t [B(K_t, t) - c_t K_t]$$

In particular, the objective function is the sum of a sequence of terms where the  $t^{\text{th}}$  term only depends on  $K_t$  and can be interpreted as the discounted profit the firm would earn in period  $t$  if it rented  $K_t$  units of capital stock for that period at a rental rate of  $c_t$ . Therefore, so long as the non-negativity constraints on investment given by (9) do not bind, the optimization problem collapses into a series of very simple completely separate single period optimization problems where the firm can view itself as being able to rent capital each period at rental rates given by the vector  $\mathbf{c}$ . The method of solution is therefore to determine a formula for calculating the vector of constants  $\mathbf{c}$  and then to show that the monotonicity condition is sufficient to guarantee that the vector of capital stocks that is the unconstrained maximizer of (11) automatically satisfies the non-negativity of investment constraints given by (9).

The vector of constants  $\mathbf{c} = (c_1, c_2, \dots)$  such that the cost function is given by (10) will be called the vector of user costs. Arrow(1964) determines a formula for calculating the vector of user costs by directly calculating the coefficients of the inverse capital accumulation function and then substituting the results into equation (6). The “problem” with this approach is that the

formulas for calculating the coefficients of the inverse capital accumulation function are quite complex and difficult to work with, and that this complexity carries over to the ultimate formula for the vector of user costs that is derived. In particular, the resulting formula depends on the vector of so-called “replacement rates” from renewal theory which describe the sequence of replacements of an original asset as it depreciates, replacements of the replacements as they in turn depreciate, etc. that would be required to permanently increase the capital stock by one unit. The replacement rates are determined by a relatively complex infinite series of recursively defined equations. This paper will show that an alternate approach can be used to derive a different (but equivalent) formula for the vector of user costs which is dramatically simpler and which, in particular, does not depend on replacement rates.<sup>12</sup>

More specifically, this paper will consider a hypothetical scenario where it is assumed that a rental market for assets exists. It will shown that an extremely simple formula exists to calculate a vector of zero-profit, perfectly competitive rental prices for this hypothetical rental market and that these hypothetical perfectly competitive rental prices must necessarily be equal to the vector of user costs. Thus, the very simple formula for calculating hypothetical perfectly competitive rental prices provides an alternate (but equivalent) formula for calculating user costs.

The main use that this paper will make of the simple alternate formula for calculating user costs will be to prove the result that the RRC allocation rule has the property that the cost it

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<sup>12</sup>Arrow’s original formula is derived and compared to this paper’s formula in Appendix B. As already mentioned in the introduction, for the special case of exponential depreciation, the formula determining replacement rates is very simple - a constant share of the asset is replaced each period - and Arrow observes that his formula for user cost collapses into the simple formula directly derived by Jorgensen (1963) for this case. The incremental contribution of this paper is to show that a similarly simple formula to calculate user costs exists for general depreciation patterns, even when there is no simple formula to calculate the vector of replacement rates.

allocates to any period of an asset's lifetime is equal to that period's user cost multiplied by the surviving amount of the asset. However, the fact that user costs can be calculated by a very simple formula that does not depend on replacement rates is an interesting result independent of its application to cost allocation rules. The assets involved in many real investment problems obviously do not exhibit an exponential pattern of depreciation, and the alternate formula for user costs provides a new simple method of characterizing the optimal investment path for such cases. Furthermore, the fact that user costs can be interpreted as hypothetical perfectly competitive rental prices provides some extra economic intuition to explain their role.

This section is organized as follows. Subsection A defines hypothetical perfectly competitive rental prices, determines the simple formula which can be used to calculate them, and shows that these rental prices must be equal to user costs. Subsection B provides some extra intuition for the result that the cost function is linear in each period's capital stock by directly calculating marginal cost for the case of one-hoss shay depreciation. Finally subsection C verifies that the monotonicity condition implies that the vector of capital stocks that maximizes (8) satisfies the constraints in (9).

### **A. Perfectly Competitive Rental Prices**

Let  $c_t$  denote the price of renting one unit of capital stock in period  $t$  and let  $\mathbf{c} = (c_1, c_2, \dots)$  denote an entire vector of rental prices. Suppose that a hypothetical supplier of rental services can enter the market in any period by purchasing one unit of the asset and then renting out the available capital stock over the asset's life. Then under the assumptions that suppliers incur no extra costs besides the cost of purchasing the asset, that they can rent the full

remaining amount of the asset to a customer every period, and that their discount rate is equal to  $\delta$ , the zero profit condition that must be satisfied by a perfectly competitive equilibrium is

$$(12) \quad z_t = \sum_{i=1}^{\infty} c_{t+i} s_i \delta^i \quad \text{for every } t \in \{0, 1, 2, \dots\}$$

Let  $c_t^*$  denote the period  $t$  rental price given by

$$(13) \quad c_t^* = k^* z_t$$

where the constant  $k^*$  is defined by<sup>13</sup>

$$(14) \quad k^* = 1 / \left[ \sum_{i=1}^{\infty} s_i (\delta \alpha)^i \right]$$

and let  $\mathbf{c}^* = (c_1^*, \dots)$  denote the entire vector of these values for rental prices. It is straightforward to verify that these rental prices satisfy the zero profit condition (12). This will be stated as Lemma 2.

### **Lemma 2:**

The vector of rental prices  $\mathbf{c}^*$  satisfies

$$(15) \quad z_t = \sum_{i=1}^{\infty} c_{t+i}^* s_i \delta^i \quad \text{for every } t \in \{0, 1, 2, \dots\}$$

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<sup>13</sup>For the case of exponential depreciation defined by equation (2),  $k^* = [1 - \beta \alpha \delta] / \alpha \delta$ . For the case of one-hoss shay depreciation defined by equation (3),  $k^* = [1 - \alpha \delta] / [(\delta \alpha)^T - (\delta \alpha)^{T+1}]$ .

**Proof:**

Straightforward algebra.

QED

Proposition 1 now states the main result of this section, which is that a vector of rental prices satisfies the zero profit condition if and only if it is the vector of user costs.

**Proposition 1:**

A vector of rental prices  $\mathbf{c} = (c_1, c_2, \dots)$  satisfies the zero profit condition (12) if and only if the cost function, as defined by equation (6), satisfies condition (10).

**Proof:**

See Appendix A.

QED

It now follows immediately from Lemma 2 and Proposition 1 that the vector of rental costs  $\mathbf{c}^*$  defined by equations (13)-(14) is also the vector of user costs.<sup>14</sup>

**Corollary 1:**

The vector of rental costs  $\mathbf{c}^*$  defined by equation (13)-(14) is the vector of user costs.

That is, the cost function  $C(\mathbf{K})$  can be written as

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<sup>14</sup>It also follows from Proposition 1 that  $\mathbf{c}^*$  is the unique vector of rental costs that satisfies the zero profit condition, (12). The vector of user costs is, by definition, unique. (i.e. since the cost function  $C(\mathbf{K})$  is a well-defined function, it cannot simultaneously satisfy equation (10) for two different vectors of constants). Since Proposition 1 states that a vector of rental costs satisfying the zero profit condition must be equal to the vector of user costs, it therefore follows that there can be at most one vector of rental costs that satisfies the zero profit condition.

$$(16) \quad C(\mathbf{K}) = \sum_{t=1}^{\infty} c_t^* K_t \delta^t$$

**Proof:**

As above.

QED

The vector of rental costs  $\mathbf{c}^*$  will be interchangeably referred to as either the vector of user costs or the vector of perfectly competitive rental prices. According to equations (13)-(14), the user cost/perfectly competitive rental price in any period is equal to the constant  $k^*$  multiplied by the cost of purchasing assets in that period. A relatively simple formula directly defined in terms of the depreciation pattern of the assets determines the constant  $k^*$ .<sup>15</sup>

## **B. Directly Calculating the Marginal Cost of Increasing Capital Stock in a Single Period**

Since each asset represents a joint cost of producing stocks of capital across multiple periods, it may seem counter-intuitive that the resulting cost function defined over vectors of capital stocks is linear and additively separable in each period's capital stock. The explanation for this result is that the case where a firm engages in ongoing investment in sunk assets creates an unusual structure of "multiple overlapping joint costs" which essentially allows the firm to separately control the capital stock in any given period by simultaneously making adjustments to the entire vector of planned current and future investments.

For example, consider the case of one-hoss shay depreciation where each asset lasts with

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<sup>15</sup>Arrow's formula for  $k^*$  in terms of the replacement rates is given by equation (B.11) in Appendix B, where the replacement rates are defined by (B.1)-(B.4).



undiminished productivity for  $T$  years. Then the firm can increase its capital stock in period 1 by one unit while holding the capital stock in all other periods constant by implementing the following series of adjustments to its investment plans. The firm must purchase an additional unit of the asset in period 0 to increase the capital stock by one unit in period 1. However, it will now be able to reduce its asset purchases by one unit in period 1. Now when period  $T$  arrives, the extra asset that the firm purchased in period 0 will no longer be available in the next period, so the firm will have to purchase an extra unit of the asset in period  $T$  to maintain its level of capital stock at the previously planned level. However, as before, it will now be able to reduce its asset purchases by one unit in period  $T+1$ . This process continues indefinitely. That is, the firm can increase its capital stock in period 1 by exactly one unit and hold the capital stock in all other periods constant by shifting the purchase of one unit of the asset forward in time from period 1 to 0,  $T+1$  to  $T$ ,  $2T+1$  to  $2T$ , etc. The present discounted value of the cost of these adjustments calculated in period 1 dollars is, by definition, the marginal cost of increasing the capital stock by one unit in period 1. It is straightforward to directly calculate this value and show that it is equal to  $k \cdot z_1$ . A similar calculation shows that the marginal cost of increasing the capital stock in any period  $t$  calculated in period  $t$  dollars is equal to  $k \cdot z_t$ .

Note that for the simple case of one-hoss shay depreciation, it is easy to directly calculate the series of required incremental changes in investment and therefore directly calculate their present discounted value which is equal to marginal cost. However, for more complex patterns of depreciation, the pattern of shifts in asset purchases over all future periods required to increase the capital stock in the next period by one unit while holding the capital stock in all future periods constant can become very complicated and difficult to directly calculate. Arrow's

original calculation of user costs is based on directly calculating the entire time series of shifts in asset purchases and then calculating their present discounted value. Proposition 1 and Lemma 2 essentially show that there is a very simple formula which can be used to directly calculate the present discounted value of the required shifts in asset purchases even when the formula for calculating the entire series of shifts in assets purchases becomes very complicated.

### C. The Solution to the Constrained Optimization Problem

Let  $\mathbf{K}^* = (K_1^*, K_2^*, \dots)$  denote the unique vector of capital stocks that maximizes (8) subject to no constraint.<sup>16</sup> It is the unique solution to the first order conditions

$$(17) \quad \{B_K(K_t, t) = c_t^* \text{ and } K_t^* \geq 0\} \text{ or } \{B_K(K_t^*, 0) < 0 \text{ and } K_t^* = 0\}$$

The monotonicity condition obviously implies that  $K_t^*$  is weakly increasing in  $t$  which in turn implies that  $\mathbf{K}^*$  satisfies the non-negativity on investment constraints given by (9). Since  $\mathbf{K}^*$  is the unconstrained maximum to (8) and  $\mathbf{K}^*$  satisfies (9), it is obviously the solution to the constrained problem (8)-(9). For future reference this will be stated as a lemma.

#### Lemma 3:

The vector of capital stocks  $\mathbf{K}^*$  is the unique solution to the optimization problem (8)-(9).

#### Proof:

As above.

QED

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<sup>16</sup>Since  $B(K, t)$  is only defined for non-negative values of  $K$ , the “unconstrained” problem still requires imposition of the constraint that each capital stock be non-negative.

### III. COST ALLOCATION RULES

This section will introduce notation to define cost allocation rules, depreciation rules, and accounting income. It will also define the RRC allocation rule, and show that RRC allocation rule has the property that the cost of purchasing an asset allocated to any period of the asset's lifetime is equal to the surviving amount of the asset multiplied by that period's user cost.

#### A. Allocation and Depreciation Rules

Define a depreciation rule to be a vector  $\mathbf{d} = (d_1, d_1, \dots)$  such that  $d_i \geq 0$  for every  $i$  and

$$(18) \quad \sum_{i=1}^{\infty} d_i = 1$$

where  $d_i$  is interpreted as the share of depreciation allocated to the  $i^{\text{th}}$  period of the asset's life. Let  $D$  denote the set of all depreciation rules. Define an allocation rule to be a vector  $\mathbf{a} = (a_1, a_2, \dots)$  that satisfies  $a_i \geq 0$  where  $a_i$  is interpreted as the share of the asset's purchase cost that is allocated to the  $i^{\text{th}}$  period of the asset's life. Let  $A$  denote the set of all allocation rules. For any discount rate  $\gamma \in [0, \infty)$  an allocation rule will be said to be "complete given  $\gamma$ " if the discounted sum of the allocation shares using  $\gamma$  is equal to 1, i.e., if

$$(19) \quad \sum_{i=1}^{\infty} a_i \gamma^i = 1$$

Let  $\Gamma(\mathbf{a})$  denote the unique value of  $\gamma$  such that  $\mathbf{a}$  is complete with respect to  $\gamma$ .

Firms generally think of themselves as directly choosing a depreciation rule and a discount rate instead of as directly choosing an allocation rule. The cost allocated to each period

is then calculated as the sum of the depreciation allocated to that period plus imputed interest on the remaining (non-depreciated) book value of the asset. Formally, for any depreciation rule,  $\mathbf{d}$ , and discount rate  $\gamma$  the corresponding allocation rule is given by

$$(20) \quad a_i = d_i + \{(1-\gamma)/\gamma\} \sum_{j=i}^{\infty} d_j.$$

It is straightforward to verify that the resulting allocation rule determined by (20) is complete given  $\gamma$ . It is also straightforward to verify that for any  $\mathbf{a} \in \mathbf{A}$ , there is a unique  $(\mathbf{d}, \gamma) \in \mathbf{D} \times [0, \infty)$  such that (20) maps  $(\mathbf{d}, \gamma)$  into  $\mathbf{a}$ . It is defined by  $\gamma = \Gamma(\mathbf{a})$  and

$$(21) \quad d_i = \sum_{j=i+1}^{\infty} \gamma^{j-i} a_j - \sum_{j=i+2}^{\infty} \gamma^{j-i-1} a_j$$

Therefore one can equivalently think of the firm as choosing either a depreciation rule and a discount rate or as choosing an allocation rule. For the purposes of this paper, it will be more convenient to view the firm as directly choosing an allocation rule.<sup>17</sup>

## B. Accounting Cost and Accounting Income

The accounting cost of using assets in any period is by definition equal to the sum of the costs allocated to that period because of investments in previous periods. Let  $A_t(K_1, \dots, K_t, \mathbf{a})$

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<sup>17</sup>See Rogerson (1992) for a fuller discussion of the relationship between depreciation and allocation rules and their properties.

denote the accounting cost in period  $t$  conditional on the firm's choice of capital stocks up until that point and the allocation rule it uses. It is formally defined by

$$(22) \quad A_t(K_1, \dots, K_t, \mathbf{a}) = \sum_{i=1}^t \phi_{t-i}(K_1, \dots, K_{t+1-i}) z_{t-i} a_i$$

The  $i$ th term of equation (22) is the accounting cost of using assets that are in the  $i$ th period of their lifetime, which is equal to the number of such assets,  $\phi_{t-i}(K_1, \dots, K_{t+1-i})$ , multiplied by the purchase price of assets in that period,  $z_{t-i}$ , multiplied by the share of the purchase cost allocated to period  $t$ ,  $a_i$ . The total accounting cost is the sum the accounting costs due to investment in all previous periods.<sup>18</sup> Let  $Y_t(K_1, \dots, K_t, \mathbf{a})$  denote the accounting income in period  $t$ , which is equal to the operating profit in period  $t$  minus the accounting cost allocated to period  $t$ .

$$(23) \quad Y_t(K_1, \dots, K_t, \mathbf{a}) = B(K_t, t) - A_t(K_1, \dots, K_t, \mathbf{a}).$$

### C. The RRC Allocation Rule

If one unit of an asset is purchased in some period  $t$ , the asset will be in the  $i$ th period of its lifetime during period  $t+i$ , so the price of purchasing new assets in the  $i$ th period of its lifetime is equal to  $z_t \alpha^i$ . Since  $s_i$  units of the original asset will survive in the  $i$ th period of its

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<sup>18</sup>The accounting cost in period  $t$  should also include whatever costs of legacy assets are also allocated to that period. However, these are completely fixed costs and are thus irrelevant to determining which vector of capital stocks maximizes accounting income in any given period. Since this will be the only question regarding accounting income that will be investigated, these legacy accounting costs can thus be ignored.

lifetime, the total cost of purchasing new assets to replace the surviving amount of the asset in the  $i$ th period of its lifetime is equal to  $s_i z_i \alpha^i$ . Therefore an allocation rule  $\mathbf{a} = (a_1, a_2, \dots)$  can be said to allocate costs in proportion to the cost of replacing the surviving amount of the asset with new assets if it satisfies

$$(24) \quad a_i = k s_i \alpha^i$$

for some positive real number  $k$ . It is easy to verify that an allocation rule of the form in (24) is complete with respect to  $\delta$  if and only if the constant  $k$  is equal to the value  $k^*$  defined by (14). Let  $\mathbf{a}^*$  denote the allocation rule determined by setting  $k$  equal to  $k^*$ .

$$(25) \quad a_i^* = k^* s_i \alpha^i$$

This will be called the relative replacement cost (RRC) allocation rule. It is the unique allocation rule that satisfies the two properties that: (i) it allocates costs in proportion to replacing the surviving amount of the asset with new assets, and (ii) it is complete with respect to  $\delta$ .

The RRC allocation rule takes a particularly simple form for the case where assets follow the one-hoss shay depreciation pattern defined by equation (3) where assets last  $T$  years and remain equally productive over their entire lifetimes. In this case the RRC rule is given by

$$(26) \quad a_i^* = \begin{cases} k^* \alpha^i, & i \in \{1, \dots, T\} \\ 0, & i \in \{T+1, \dots\} \end{cases}$$

When the purchase price of replacement assets stays constant over time (i.e., when  $\alpha = 1$ ), the RRC allocation rule allocates a constant share of cost to each period. If the purchase price of replacement assets is changing over time, then the allocation shares change at the same rate that the price of replacement assets is changing at.

#### **D. The Relationship Between Accounting Cost Under the RRC Rule and User Cost**

This section will show that there is a very close connection between accounting cost calculated under the RRC allocation rule and user cost. In particular, it will be shown that the RRC allocation rule has the property that the cost allocated to any period of an investment's lifetime is simply equal to that period's user cost multiplied by the surviving amount of the asset. All of the results of the following sections will be shown to follow from this property.

More formally, suppose that the firm purchases one unit of the asset in period  $t$ . Then the total cost that an allocation rule  $\mathbf{a}$  assigns to the  $i$ th period of the asset's life is given by

$$(27) \quad z_t a_i$$

since the total cost of the asset is  $z_t$  and the allocation rule  $\mathbf{a}$  allocates the share  $a_i$  of this cost to the  $i$ th period of the asset's lifetime. Since the asset is purchased in period  $t$ , it will be in the  $i$ th period of its lifetime in period  $t+i$ . Therefore the user cost in the  $i$ th period of the asset's lifetime multiplied by the surviving number of assets in that period is given by

$$(28) \quad c_{t+i} * s_i.$$

Therefore an allocation rule  $\mathbf{a}$  has the property that the cost of period  $t$  investment allocated to the  $i$ th period of the investment's life is equal to that period's user cost multiplied by the surviving amount of the asset if and only if the expression in (27) is equal to the expression in (28).

**Definition:**

An allocation rule  $\mathbf{a} = (a_1, a_2, \dots)$  will be said to satisfy the “user cost property for allocations of period  $t$  investment to the  $i$ th period of the investment's lifetime,” or the “user cost for  $(t, i)$  property” if

$$(29) \quad z_t a_i = c_{t+i} * s_i$$

Of course equation (29) can be rewritten as

$$(30) \quad a_i = c_{t+i} * s_i / z_t$$

Therefore, whether or not an allocation rule  $\mathbf{a}$  satisfies the user cost for  $(t, i)$  property depends only the  $i$ th period allocation share,  $a_i$ . In particular, for any  $(t, i)$ , an allocation rule satisfies the user cost for  $(t, i)$  property if and only if its  $i$ th period allocation share is equal to the value given by the RHS of equation (30). Substitution of equations (4), (13), and (25) into (30) shows that (30) can be rewritten as

$$(31) \quad a_i = a_i^*$$



Note that the RHS of (31) does NOT depend on  $t$ . This means setting  $a_i = a_i^*$  simultaneously guarantees that an allocation rule satisfies the user cost for  $(t, i)$  property for every  $t$ . Conversely, an allocation rule that satisfies the user cost for  $(t, i)$  property for any  $t \in \{0, 1, \dots\}$  must exhibit the property that  $a_i = a_i^*$ . Proposition 2 summarizes these results.

**Proposition 2:**

Let  $\mathbf{a} = (a_1, a_2, \dots)$  denote an allocation rule. Then for any  $i \in \{1, 2, \dots\}$  the following three statements are equivalent.

- (i)  $\mathbf{a}$  satisfies the user cost for  $(t, i)$  property for some  $t \in \{0, 1, \dots\}$
- (ii)  $\mathbf{a}$  satisfies the user cost for  $(t, i)$  property for every  $t \in \{0, 1, \dots\}$
- (iii)  $a_i = a_i^*$

**Proof:**

As above.

QED

In particular, then, Proposition 2 implies that the RRC allocation rule satisfies the user cost property for  $(t, i)$  for every value of  $t$  and  $i$ . It follows from this that the total accounting cost of using capital in any period must simply be equal to the existing number of units of capital in that period multiplied by that period's user cost. This is stated as Corollary 2.

**Corollary 2:**

Suppose that the RRC allocation rule is used to calculate accounting cost. Then the accounting cost in any period is equal to the number of units of capital in that period multiplied by that

period's user cost. Formally,

$$(32) \quad A_t(K_1, \dots, K_t, \mathbf{a}^*) = c_t^* K_t$$

**Proof:**

See Appendix A.

QED

#### IV. A SIMPLE RULE FOR CALCULATING OPTIMAL INVESTMENT

This section will consider the procedure where the firm chooses a level of investment every period to maximize next period's accounting income, which will be called period-by-period maximization of accounting income. It will be shown that a sufficient condition for period-by-period maximization of accounting income to yield the fully optimal vector of capital stocks is that the allocation rule used by the firm to calculate accounting income satisfy  $a_1 = a_1^*$ , i.e., that the first period allocation share used by the firm be the set according to the RRC rule.

To see this, suppose that the firm is in period  $t \in \{0, 1, \dots\}$  and that it will therefore choose  $K_{t+1}$  by its current-period investment decision. According to Lemma 3, the optimal capital stock in period  $t+1$  maximizes

$$(33) \quad B(K_{t+1}, t+1) - c_{t+1}^* K_{t+1}$$

Now suppose that the firm simply chooses a level of investment in period  $t$  to maximize period  $t+1$  accounting income. Then, so long as the non-negativity constraint on investment does not bind,  $K_{t+1}$  will be chosen to maximize

$$(34) \quad B(K_{t+1}, t+1) - z_t a_1 K_{t+1}$$

since  $z_t a_1$  is the accounting cost allocated to period  $t+1$  if one unit of the asset is purchased in period  $t$ . By comparing equations (33) and (34), it obviously follows that, if the firm chooses investment every period to maximize next period's accounting income, then a sufficient condition for the firm to choose the fully optimal vector of investments is that the coefficients multiplying  $K_{t+1}$  in equations (33) and (34) be equal for every  $t$ , i.e, that

$$(35) \quad z_t a_1 = c_{t+1}^*$$

for every  $t$ . However, equation (35) is simply the user cost for  $(t, 1)$  property and Proposition 3 has already established that (35) is true if and only if  $a_1 = a_1^*$ . Proposition 4 states the result.

**Proposition 4:**

Suppose that the firm calculates accounting income using the allocation rule  $\mathbf{a} = (a_1, a_2, \dots)$ .

Then a sufficient condition for period-by-period maximization of accounting income to yield the fully optimal sequence of investments is that  $a_1 = a_1^*$ .

**Proof:**

As above.

QED

Of course the sufficient condition in Proposition 4 only specifies the first period allocation share of the allocation rule used by the firm. Therefore, while the RRC allocation rule

satisfies this sufficient condition, there are obviously many other allocation rules that also satisfy it. However, the RRC allocation rule is a particularly simple and natural allocation rule and it is not clear that it would be possible to identify some other equally simple and natural allocation rule that sets  $a_1$  equal to  $a_1^*$  but sets  $a_i$  unequal to  $a_i^*$  for other values of  $i$ . Furthermore, the next section will consider a more complex model where shareholders delegate the investment decision to management and accounting income is used as a managerial performance measure, and will show that a sufficient condition for an allocation rule to create good investment incentives in this model is that the allocation share in every period be set according the RRC allocation rule.

## **V. MANAGERIAL INVESTMENT INCENTIVES**

This section will consider an extension to the basic model in which it is assumed that shareholders delegate the investment decision to a better-informed manager and show that there is a sense in which robust incentives for the manager to choose the fully optimal sequence of investments can be created by using accounting income based on the RRC allocation rule as a performance measure for the manager. Subsection A will describe the basic model. Then Subsection B will present the main result and Subsection C will discuss it.

### **A. The Model With Delegation of the Investment Decision**

Assume that the production/demand environment is as described in the previous sections. Assume that shareholders know the parameters of the model necessary to calculate the RRC allocation rule, i.e., they know the depreciation pattern of assets,  $\mathbf{s}$ , and they know the future rate of change of asset prices,  $\alpha$ . However, assume that shareholders do not know current or future

demand functions and therefore do not know the benefit function,  $B(K_t, t)$ . This means that they do not have sufficient information to calculate the optimal investment level. Assume that the manager knows all of the functions and parameters in the model so that the manager is able to calculate the optimal investment level.

Suppose that shareholders delegate the production decision to the manager in order to take advantage of his private information and that they create a compensation scheme for the manager by choosing an allocation rule and wage function. The allocation rule is used to calculate each period's accounting income. The wage function determines the wage the manager receives each period as a function of the accounting income in the current period and past periods. Assume that the manager has a utility function defined over vectors of wage payments over time. For any given wage function, one can therefore define an indirect utility function of the manager over vectors of accounting income. It will turn out to be useful to employ notation that suppresses the manager's direct utility function over wage payments and the wage function chosen by shareholders and instead focuses on the indirect utility function over accounting income created by the composition of these two functions. Let  $y_t$  denote period  $t$  accounting income and let  $\mathbf{y} = (y_1, y_2, \dots)$  denote an entire vector of accounting incomes. Let  $U(\mathbf{y})$  denote the manager's indirect utility function over vectors of accounting incomes that is created by the choice of a wage function and the manager's own direct utility function over wage payments.

For any allocation rule  $\mathbf{a}$  the manager will choose the vector of capital stocks  $\mathbf{K}$  to maximize

$$(36) \quad U(Y_1(K_1, \mathbf{a}), Y_2(K_1, K_2, \mathbf{a}), \dots).$$

The question which will be investigated in this section is whether or not a wage function and allocation rule can be identified such that the manager has the incentive to choose the efficient vector of capital stocks,  $\mathbf{K}^*$ . In general this would appear to be a very complex problem whose solution depends on the precise functional form of the manager's underlying direct utility function over wage payments, including the manager's own personal discount rate. Therefore, it appears that even if it was possible to identify a wage function and allocation rule that induced the manager to choose the efficient level of investment each period, that it would be necessary to have detailed information about the manager's preferences, including his own personal discount rate, in order to determine such a wage function and allocation rule.

The key idea of this section is that almost all of this apparent complexity can be avoided by the appropriate choice of an allocation rule. In particular, most of the apparent complexity is created by the fact that, in general, the manager faces trade-offs between maximizing accounting income in different periods. That is, an investment plan that would maximize accounting income in any given period is unlikely to also maximize accounting income in all other periods. Therefore, selecting the optimal investment plan requires the manager to make complex trade-offs between periods. The wage function and the manager's own underlying direct utility function over wage payments both will have significant and potentially complex effects on how the manager trades off accounting income between periods. However, suppose that there was a vector of capital stocks that simultaneously maximized the accounting income in *every* period. Then, so long as the indirect utility function  $U$  was weakly increasing in all of its arguments, this vector of capital stocks would also obviously maximize the managers's utility regardless of the precise functional form of  $U$ . The fairly modest condition that each period's wage is weakly

increasing in current and past periods' accounting income is clearly sufficient to guarantee that  $U$  is weakly increasing in each of its arguments.<sup>19</sup> Therefore if there is a vector of capital stocks that simultaneously maximizes accounting income in every period, this vector of capital stocks would maximize the manager's utility for the entire class of wage functions satisfying the modest condition that each period's wage is a weakly increasing function of current and past periods' accounting income.

Formally, an allocation  $\mathbf{a}$  will be said to create robust incentives for the manager to choose the vector  $\mathbf{K}$  if  $\mathbf{K}$  maximizes each period's accounting income calculated using  $\mathbf{a}$ .

**Definition:**

An allocation rule  $\mathbf{a}$  will be said to create robust incentives for the manager to choose the capital stock vector  $\mathbf{K}' = (K_1', K_2', \dots)$  if

$$(37) \quad (K_1', K_2', \dots, K_t') \in \underset{(K_1, K_2, \dots, K_t)}{\operatorname{argmax}} Y_t(K_1, \dots, K_t, \mathbf{a}) \text{ for every } t \in \{1, 2, \dots\}$$

**B. The Result**

The result that the RRC allocation rule creates robust incentives for the manager to choose the optimal vector of capital stocks,  $\mathbf{K}^*$ , follows immediately from Corollary 2. Substitution of (32) into (23) shows that accounting income under the RRC allocation rule is given by

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<sup>19</sup>Of course, it must also be assumed that the agent's direct utility function over wages is weakly increasing in each period's wage.

$$(38) \quad Y_t(K_1, \dots, K_t, \mathbf{a}^*) = B(K_t, t) - c_t^* K_t.$$

Note that accounting income in period  $t$  only depends on  $K_t$  and that it is maximized at  $K_t^*$ .

This, of course, implies that the vector of capital stocks  $\mathbf{K}^* = (K_1^*, K_2^*, \dots)$  simultaneously maximizes accounting income for every time period, which establishes the result of interest.

**Proposition 4:**

The RRC allocation rule,  $\mathbf{a}^*$ , creates robust incentives for the manager to choose the fully optimal vector of capital stocks,  $\mathbf{K}^*$ .

**Proof:**

As above.

QED

**C. Discussion**

The above result does not formally show that a contract using the RRC allocation rule is the optimal solution to a completely specified principal agent problem. It is clear that such a result would be straightforward to prove in a model where it was assumed that the only incentive/information problem was that the manager is better informed than shareholders about some information necessary to calculate the fully optimal investment plan. However, there would be no need in such a model to base the manager's wage on any measure of the firm's performance. This is because one fully optimal contract would be for shareholders to simply pay the manager a constant wage each period sufficient to induce the manager to accept the job. Then the manager would be (weakly) willing to choose the profit maximizing investment plan.



Therefore, in reality, the result of this paper will only be useful in situations where there is some additional incentive problem which requires shareholders to base the manager's wage on some measure of the firm's performance. A natural candidate would be to assume that there is a moral hazard problem within each period, i.e., that each period the manager can exert unobservable effort which affects the firm's cash flow that period. This would create a multi-period moral hazard problem with asymmetric information.<sup>20</sup> The modeling problem this creates is that solutions to such problems are extremely complex and the nature of the solution generally depends on particular aspects of the environment (such as the agent's preferences), that the principal is unlikely to have reliable information about. Thus, it is not clear that such contracts would be suitable for use in the real world where robustness to small changes in the environment is likely to be important.

In light of these difficulties, the result of this paper can be interpreted as offering a useful alternative approach. In particular, this paper shows that, by restricting themselves to choosing a compensation scheme where accounting income is calculated using the RRC allocation rule and where each period's wage is a weakly increasing function of current and past periods' accounting income, shareholders can guarantee in a robust way that the investment incentive problem will be completely solved and still leave themselves considerable degrees of freedom to address remaining incentive issues. For example, by using accounting income based on the RRC allocation rule as a performance measure, shareholders could thereby guarantee that the investment incentive problem was completely solved and then use a "trial and error" process over

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<sup>20</sup>It must be assumed that the manager has private information in order to create the need for shareholders to delegate the investment decision to the manager in the first place.

time to identify a wage function that appeared to create the appropriate level of effort incentives.

Note that in cases where it is possible to calculate a fully optimal contract, it may well be that the fully optimal contract will not necessarily induce the agent to choose the profit maximizing level of investment. (A general lesson from the incentives literature is that when one calculates the fully optimal contractual solution to a situation involving multiple interacting incentive problems, it is often the case that it is optimal to purposely distort the solution to one problem away from the first best in order to get extra leverage on the other problem.) However, it precisely these sorts of calculations that are exceedingly complex and that are unlikely to be robust to small changes in the contracting environment. Therefore, if shareholders have an opportunity to guarantee that a first best solution is created to one of the two incentive problems in a simple robust way while still leaving themselves considerable degrees of freedom to address the second incentive problem through a trial and error process, this may be a very attractive alternative in the real world.

This paper's approach of showing that a certain allocation rule induces the manager to make first-best investment decisions so long as each period's wage is weakly increasing in current and past periods' accounting income was first used in Rogerson (1997) to analyze a model where it was assumed that investment only occurs once, in the initial period. As discussed in the introduction, the current paper essentially provides a similar sort of result for the more complex case where investment occurs every period. Dutta and Riechelstein (2002) have shown that Rogerson's (1997) allocation rule can be made part of a fully optimal contractual solution to a fully specified principal agent model that includes a moral hazard component in what they refer to as a LEN model - which means that contracts are assumed to be linear, utility

is assumed to be exponential, and noise is assumed to be normal. An interesting project for future research would be to determine if the Dutta/Reichelstein(2002) approach could be adapted to the model of this paper.

## **VI. GENERALIZATIONS**

The results of this paper were proven under the assumption that the purchase price of new assets changes at a constant rate. This section will describe the extent to which the results generalize to the case where the vector of purchase prices for new assets is allowed to be any vector of non-negative prices. The findings of this section can be summarized as follows. So long as the non-negativity constraints on investment do not bind, cost allocation rules can still be identified that have the same sorts of desirable properties that the RRC allocation rule was shown to have in previous sections. However, there is no longer necessarily any simple or natural way to describe these allocation rules in terms of the underlying parameters of the model. Therefore, while the generalization is of analytic interest, because it helps clarify precisely why the cost allocation result is true and what it depends on, it may be of more limited practical interest. However, a firm's information about future prices is likely to be somewhat imprecise in any event, so that it may be very natural and reasonable in many applied cases to project future prices by simply specifying a likely average future growth rate.

The remainder of this section will briefly describe the manner in which the results generalize. Because the generalization is a relatively natural and simple extension of the results of previous sections, detailed formal statements of results and proofs will not be provided.

The same model will be considered except it will no longer be assumed that the purchase

price of new assets changes at a constant rate. Rather,  $\mathbf{z} = (z_0, z_1, \dots)$  will be allowed to be any vector of non-negative numbers.<sup>21</sup> It is straightforward to verify that the initial observation of this paper - that the capital accumulation function is linear and invertible and that, as a consequence, the cost function  $C(\mathbf{K})$  must also be linear - is still true. That is, there still must be a vector of constants  $\mathbf{c}^* = (c_1^*, c_2^*, \dots)$  such that the cost function can be written as in equation (16). Arrow(1964) provides a formula for calculating this vector of constants for the general case. See Appendix B of this paper for a statement of the general formula and a sketch of its derivation. The main difference in the general case is that it is no longer possible to use this paper's approach to derive a simpler version of the formula that does not depend on the vector of replacement rates.<sup>22</sup> The user cost in any period is still equal to the present discounted value of the entire vector of adjustments to investments that would increase capital stock in that period by one unit while holding the capital stock in all other periods constant. However, without the regularity created by the assumption that asset prices change at a constant rate over time, the formula does not collapse to any simple form. Therefore, in the general case, each period's user cost is potentially a very complex function of the vector of all current and future asset prices and the vector of survival shares.

For the general case it will be necessary to potentially allow the firm to choose a different allocation rule to allocate each period's investment. Let  $\mathbf{a}_t = (a_{t1}, a_{t2}, \dots)$  denote the allocation

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<sup>21</sup>The monotonicity condition must also be replaced by a generalized condition. This is that  $B_K(\mathbf{K}, t) - c_t^*$  is weakly increasing in  $t$  where  $c_t^*$  denotes period  $t$  user cost as defined below.

<sup>22</sup>Proposition 1, which states that the vector of perfectly competitive rental prices must be equal to the vector of user costs, still holds for the general case. However, there is no longer any simple method to calculate the vector of perfectly competitive rental prices for the general case.

rule used to allocate assets purchased in period  $t$  for  $t \in \{0, 1, 2, \dots\}$  and let  $\mathbf{v} = (\mathbf{a}_0, \mathbf{a}_1, \dots)$  denote the entire vector of allocation rules chosen by firm. Let  $A^t(K_1, \dots, K_t, \mathbf{a}_0, \dots, \mathbf{a}_{t-1})$  denote the accounting cost in period  $t$  given that the firm chooses investments to produce the vector of capital stocks  $(K_1, \dots, K_t)$  and the allocation rules  $(\mathbf{a}_0, \dots, \mathbf{a}_{t-1})$  are used to allocate the investments. It is formally defined by

$$(39) \quad A_t(K_1, \dots, K_t, \mathbf{a}_0, \dots, \mathbf{a}_{t-1}) = \sum_{i=1}^t \phi_{t-i}(K_1, \dots, K_{t+1-i}) z_{t-i} a_{t-i, i}$$

As before, accounting income in any period is defined to be operating profit minus accounting cost.

Now, two potentially different vectors of allocation rules will be defined. (As will be seen, these vectors of allocation rules turn out to be identical under the assumption that asset prices change at a constant rate over time. However, in the general case, it will turn out that they are not identical.) First, the vector of user cost allocation rules is defined as follows. For any period  $t$ , define the user cost allocation rule for period  $t$ , denoted by  $\mathbf{a}_t^U = (a_{t1}^U, a_{t2}^U, \dots)$ , to be the allocation rule such that the cost allocated to the  $i$ th period of the asset's lifetime is equal to that period's user cost multiplied by the surviving amount of the asset. It is formally defined by

$$(40) \quad a_{ti}^U = c_t * s_i / z_t.$$

Let  $\mathbf{v}^U = (\mathbf{a}_0^U, \mathbf{a}_1^U, \dots)$  denote the entire vector of user cost allocation rules. The second, potentially different, vector of allocation rules that will be defined will be called the vector of

relative replacement cost (RRC) allocation rules. For any period  $t$ , define the RRC allocation rule for period  $t$ , denoted by  $\mathbf{a}_t^R = (a_{t1}^R, a_{t2}^R, \dots)$ , to be the unique allocation rule that satisfies the following two properties for any investment made in period  $t$ : (i) the cost allocated to any period of the asset's lifetime is proportional to the cost of replacing the surviving amount of the investment with new assets; and (ii) the present discounted value of the allocations using the firm's discount rate is equal to the original purchase price of the asset. It is formally defined by

$$(41) \quad a_{ti}^R = \frac{z_{t+i}s_i}{\sum_{j=1}^{\infty} z_{t+j}s_j \delta^j}$$

Recall that sections IV and V showed that using the vector of allocation rules  $(\mathbf{a}^*, \mathbf{a}^*, \dots)$  had various desirable properties in the special case where asset prices change at a constant rate. A review of the proofs of these sections will show that the only property of the vector  $(\mathbf{a}^*, \mathbf{a}^*, \dots)$  that was used to prove any of the propositions in these sections was that the cost of purchasing an asset allocated to any period of its lifetime was equal to that period's user cost multiplied by the surviving amount of the asset. However, by equation (40), the vector of user cost allocation rules is constructed to have this property. Therefore, for the general case, the same arguments can be used to show that the vector of user cost allocation rules has these same desirable properties. Namely, the arguments of Section IV show that if the firm calculates accounting income using this vector of allocation rules, then period-by-period maximization of accounting income will yield the fully optimal vector of investments. The arguments of Section V show that the vector of allocation rules  $\mathbf{v}^U$  creates robust incentives for the manager to choose the fully efficient vector of investments in the sense that this vector of investments simultaneously maximizes each

period's accounting income. It is also straightforward to verify that the proof of Proposition does not depend on the assumption that asset prices change at a constant rate and is thus still true for the general case. It follows from this that each of the user cost allocation rules is complete with respect to the firm's discount rate,  $\delta$ . Therefore the result of previous sections that holds true for the general case is that a vector of allocation rules - namely the vector of user cost allocation rules - can always be defined which has desirable properties for guiding investment decisions as described above, and such that the allocation rule used every period is complete with respect to the firm's discount rate.

For the special case where asset prices change at a constant rate over time, it is easy to see that the period  $t$  RRC allocation rule, as defined by (41), is the same for every period and is equal to the allocation rule  $\mathbf{a}^*$  defined by (24)-(25) which, in previous sections, was simply referred to as *the* RRC allocation rule. Furthermore, given this paper's formula for the vector of user costs for the special case, as given by (13)-(14), it is also straightforward to observe that the period  $t$  user cost rule defined by (40) is the same for every period and is also equal to the allocation rule  $\mathbf{a}^*$ . Therefore, for the special case where asset prices change at a constant rate, it turns out that the vector of user cost allocation rules is simply equal to the vector of RRC allocation rules.

This result does NOT generalize. That is, in the general case it is no longer true that the vector of user cost allocation rules is equal to the vector of RRC allocation rules.<sup>23</sup> One can still determine

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<sup>23</sup>The following observation establishes that this result will not generalize. In the general case, the vector of user cost allocation rules is still constructed to have the property that the cost of using any vintage of asset in any period will be equal to that period's user cost multiplied by the surviving number of units of the asset. That is, the user cost allocation rule has the property that, in any given period, the unit cost of using any vintage of asset is constant. However, it is easy to see that the cost of using different vintages of assets will NOT generally be constant under the vector of RRC allocation rules so long as asset prices do not change at a constant rate.

a formula for calculating the vector of user cost allocation rules in terms of the underlying parameters of the model by substituting the formula expressing user cost as a function of these underlying parameters into equation (41). However, in the general case, the resulting formula is NOT the formula for calculating the vector of RRC allocation rules, nor does it appear to collapse into any other sort of simple or natural form.

## **VI. CONCLUSION**

Firms that make sunk investments in long-lived assets create simplified single-period snapshots of their performance by allocating the cost of purchasing long-lived assets over the periods the assets will be used to calculate period-by-period accounting income. This paper has shown that, in broad range of plausible circumstances, a very simple and natural allocation rule - the RRC rule - exists such that measures of period-by-period accounting income calculated using this rule can play a useful role in guiding investment decisions. In particular two results are shown. First, it is shown that if per period accounting income is calculated using the RRC rule, the firm can choose the fully optimal sequence of investments over time simply by choosing a level of investment each period to maximize next period's accounting income. Second, in a model where shareholders delegate the investment decision to a better-informed manager, it is shown that if accounting income based on the RRC allocation rule is used as a performance measure for the manager, robust incentives are created for the manager to choose the profit maximizing sequence of investments regardless of the manager's own personal discount rate or other aspects of the manager's personal preferences.



## APPENDIX A: PROOFS

### Lemma 1:

Let  $\mathbf{K} = (K_1, K_2, \dots)$  be the vector of capital stocks that the firm would like to generate. Obviously,  $I_0$  must be set equal to  $K_1$ . This establishes that period 0 satisfies the induction hypothesis that  $I_t$  is a linear function of  $K_1$  through  $K_{t+1}$ . Now suppose that  $I_0$  through  $I_{t-1}$  have been determined and  $I_t$  is a linear function of  $K_1$  through  $K_{t+1}$  for every  $i \in \{0, \dots, t-1\}$ . Let  $K_{t+1}^A$  denote the amount of capital that will be available in period  $t+1$  if the firm chooses zero investment in period  $t$ . The variable  $K_{t+1}^A$  is a linear function of  $I_0$  through  $I_{t-1}$  determined by the capital accumulation function. Therefore by the induction hypothesis,  $K_{t+1}^A$  is a linear function of  $K_1$  through  $K_t$ . Obviously,  $I_t$  must be set equal to the difference between  $K_{t+1}$  (the required amount of capital in period  $t$ ) and  $K_{t+1}^A$  (the available amount of capital in period  $t$ ). Since  $K_{t+1}^A$  is a linear function of  $K_1$  through  $K_t$ , this means that  $I_t$  is a linear function of  $K_1$  through  $K_{t+1}$ . QED

### Proposition 1:

First suppose that  $\mathbf{c}$  satisfies equation (12). If  $\mathbf{I}$  generates  $\mathbf{K}$  then,  $C(\mathbf{K})$  is given by

$$(A.1) \quad C(\mathbf{K}) = \sum_{t=0}^{\infty} z_t I_t \delta^t$$

Substitution of (12) into (A.1) yields

$$(A.2) \quad C(\mathbf{K}) = \sum_{t=0}^{\infty} \sum_{i=1}^{\infty} c_{t+i} s_i \delta^i I_t \delta^t.$$

Reorganize the summation to yield

$$(A.3) \quad C(\mathbf{K}) = \sum_{t=1}^{\infty} c_t \delta^t \sum_{i=1}^t s_i I_{t-i}$$

Substitution of  $K_t$  for the inner summation in (A.3) yields equation (10).

Now suppose that the cost function satisfies equation (10). For any  $t \in \{0, 1, \dots\}$  suppose that firm purchases one unit of the asset in period  $t$  and no units of the asset in any other period.

Let  $\mathbf{K}^t = (K_1^t, K_2^t, \dots)$  denote the resulting vector of capital stocks given by

$$(A.4) \quad K_i^t = \begin{cases} 0, & i \leq t \\ s_{i-t}, & i \geq t+1 \end{cases}$$

From the definition of  $C(\mathbf{K})$  in equation (6), we know that

$$(A.5) \quad C(\mathbf{K}^t) = \delta^t z_t.$$

However, since  $C(\mathbf{K})$  satisfies equation (10), substitution of (A.4) into (10) shows that

$$(A.6) \quad C(\mathbf{K}^t) = \sum_{i=1}^{\infty} c_{t+i} s_i \delta^{t+i}.$$

Setting the RHS of (A.5) equal to the RHS of (A.6) then yields equation (12). QED

### Corollary 2:

Since  $\mathbf{a}^*$  satisfies the user cost for  $(j, i)$  property for every  $j \in \{0, 1, 2, \dots\}$  and  $i \in \{1, 2, \dots\}$ ,

$$(A.7) \quad z_j a_i^* = c_{j+i}^* s_i \quad \text{for every } j \in \{0, 1, 2, \dots\} \text{ and } i \in \{1, 2, \dots\}.$$

Substitute  $t = j+i$  into (A.7) to rewrite it as

$$(A.8) \quad z_{t-i} a_i^* = c_t^* s_i \quad \text{for every } t \in \{1, 2, \dots\} \text{ and } i \in \{1, \dots, t\}.$$

Substitution of  $\mathbf{a} = \mathbf{a}^*$  and (A.8) into (22) and reorganization yields

$$(A.9) \quad A_t(K_1, \dots, K_t, \mathbf{a}^*) = \sum_{i=1}^t c_t^* \{ \sum_{i=1}^t \phi_{t-i}(K_1, \dots, K_{t+1-i}) s_i \}$$

Substitute  $K_t$  for the term in brackets in (A.9) to yield (32). QED

## APPENDIX B: ARROW'S(1964) DERIVATION OF THE USER COST FORMULA

Arrow's original derivation of a formula for calculating the vector of user costs was based on directly calculating the coefficients of the inverse capital accumulation function. For the case where the purchases price of new assets changes at a constant rate over time, this paper presents an alternate derivation that relies on the showing that the vector of user costs can also be interpreted as being a vector of perfectly competitive prices, which produces a much simpler (but equivalent) formula for calculating the vector of user costs. However, Arrow's original formula is still needed to calculate the vector of user costs for the general case considered in section VI. Furthermore, the reader may be interested to compare Arrow's original formula for the case where the purchase price of new assets changes at a constant rate over time, to the formula derived by this paper. Since Arrow uses a somewhat different model than the model used by this paper,<sup>24</sup> a reader would likely have to invest a considerable amount of time in reading Arrow's paper in order to translate Arrow's results to the model presented in this paper. Therefore, this Appendix sketches Arrow's original derivation of the formula for user cost in the context of this paper's model.

Define the vector of replacement rates, denoted by  $\mathbf{r} = (r_0, r_1, \dots)$ , by the following

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<sup>24</sup>In Arrow's formulation, time is modeled as being continuous and the function that this paper calls the "cost function" is never explicitly defined or calculated. Furthermore, the capital stock variable that is explicitly modeled includes both incremental capital stock (i.e., capital stock resulting from investments made in period 0 and after) and legacy capital stock (i.e., capital stock resulting from investments made before period 0). Finally, Arrow denominates all cash flows using the purchase price of new assets as numeraire, so that he can assume without loss of generality that the purchase price of new assets is simply equal to one in every period. While Arrow is correct that this can be done without loss of generality, this makes it difficult to disentangle effects on the vector of user costs due to the way the firm values cash flows from effects due to the way in which the purchase price of assets changes over time.

thought experiment. Suppose that a firm purchases one unit of the asset in period 0 and then in every subsequent period purchases just enough new assets to replace assets as they wear out, so that the firm maintains a stock of capital equal to 1 forever. Let  $r_t$  denote the number of assets that the firm purchases in period  $t$  for  $t \in \{0, 1, 2, \dots\}$ . By construction,  $r_0$  is equal to 1. The formula for calculating the replacement rate in any subsequent period  $t \in \{1, 2, \dots\}$  is calculated by summing up the entire series of replacements of original assets, replacements of replacements, etc. that will occur in any period. Formally,

$$(B.1) \quad r_t = \sum_{n=1}^t m^n_t \quad \text{for } t \in \{1, 2, \dots\}$$

where  $m^n_t$  is the number of asset's that experience their "nth replacement" in period  $t$  and is defined recursively by

$$(B.2) \quad m^1_i = 1 - s_i$$

and

$$(B.3) \quad m^{n+1}_i = \begin{cases} 0, & \text{for } i \leq n \\ \sum_{j=1}^{i-1} m^n_j m_{i-j} & \text{for } i \geq n+1 \end{cases}$$

Arrow reports<sup>25</sup> that a formula for  $\phi_t(K_1, \dots, K_t)$  in terms of the vector of replacement rates is given by

$$(B.4) \quad \phi_t(K_1, \dots, K_{t+1}) = \sum_{i=0}^t [K_{t+1-i} - K_{t-i}] r_i$$

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<sup>25</sup>Arrow credits Feller(1941) with the original derivation of this result. Feller(1941) considers the case of continuous time which likely involves subtleties related to existence and/or uniqueness that do not arise in the discrete case.

where  $K_0$  is interpreted to be 0. This result is very intuitive. One can think of generating a vector of capital stocks  $\mathbf{K} = (K_1, K_2, \dots)$  by creating a successive series of “permanent” increments in the capital stock. In period 1 the capital stock is “permanently” changed from 0 to  $K_1$ , in period 2 the capital stock is “permanently” changed from  $K_1$  to  $K_2$ , etc. The effect of each permanent increment on current and future periods’ investment is determined by the replacement shares. Equation (B.5) presents the results of this calculation. For the purposes of this paper it will be useful to rewrite (B.5) as

$$(B.5) \quad \varphi_t(K_1, \dots, K_{t+1}) = K_{t+1} + \sum_{i=1}^t K_{t+1-i} [r_i - r_{i-1}]$$

The cost function,  $C(\mathbf{K})$ , can now be calculated by substituting (B.5) into (6) which yields

$$(B.6) \quad C(\mathbf{K}) = \sum_{t=1}^{\infty} c_t^* K_t \delta^t$$

where  $c_t^*$  is given by

$$(B.7) \quad c_t^* = r_0 z_{t-1} / \delta + \sum_{i=0}^{\infty} [r_{i+1} - r_i] \delta^i z_{t+i}$$

Equation (B.7) provides the formula for period  $t$  user cost for the general case.

Some extra intuition for why the functional form determining user cost in equation (B.7) takes the precise form that it does, can be had by directly calculating the incremental cost of increasing the stock of capital in a given period  $t$  while holding the stock of capital in all subsequent periods fixed. The present discounted cost of this change can be directly calculated by first determining the changes in investment that are required to generate this change and then

calculating the present discounted cost of these changes in investment. This calculation is illustrated in Table 1.

**Table 1**  
**Cost of Changes in Investment Necessary to Increase Capital Stock by 1 Unit in Period t**  
**While Holding Capital Constant In All Other Periods**

| <b>Period</b> | <b>(1)*</b> | <b>(2)*</b> | <b>(3)*</b>     | <b>(4)*</b>                            |
|---------------|-------------|-------------|-----------------|--|
| t-1           | $+r_0$      | 0           | $+r_0$          | $r_0 Z_{t-1} / \delta$                 |
| t             | $+r_1$      | $-r_0$      | $r_1 - r_0$     | $(r_1 - r_0) Z_t$                      |
| t+1           | $+r_2$      | $-r_1$      | $r_2 - r_1$     | $(r_2 - r_1) Z_{t+1} \delta$           |
| .             | .           | .           | .               | .                                      |
| .             | .           | .           | .               | .                                      |
| t+i           | $+r_{i+1}$  | $-r_i$      | $r_{i+1} - r_i$ | $(r_{i+1} - r_i) Z_{t+i} \delta^{i-1}$ |

- \* Explanation of Column Headings:
- Column (1) = change in investment to permanently increase capital stock by one unit beginning in period t.
- Column (2) = change in investment to permanently decrease capital stock by one unit beginning in period t+1.
- Column (3) = sum of columns (1) and (2)
- Column (4) = cost of change in column (3) in period t dollars

The key idea is that an increase in the capital stock of one unit in period t, while holding the capital stock all other periods fixed, can equally well be thought of as the net result of two different changes - a permanent increase in the capital stock by one unit in period t followed by a permanent decrease in the capital stock of one unit in period t+1. Column (1) lists the changes in investment necessary in each period to produce a permanent increase in the stock of capital beginning in period t. Column (2) lists the changes in investment necessary in each period to produce a permanent decrease in the capital stock of one unit beginning in period t+1. Column (3) then lists the sum of these changes and column (4) multiplies the period j entry in column (3)

by  $z_t \delta^{i-t}$  to calculate the present discounted value of the cost of these investment changes in period  $t$  dollars. By construction, the sum of the terms in column (4) is equal to the incremental cost of increasing capital by one unit in period  $t$ . It is easy to verify that the sum of the terms in column (4) yields the expression for  $c_{t*}$  in equation (B.7).

The summation in (B.7) can be reorganized to produce a formula for  $c_t^*$  that depends on replacement shares instead of first differences of replacement shares. This reorganization yields

$$(B.8) \quad c_t^* = \sum_{i=0}^{\infty} r_i \delta^{i-1} (z_{t-1+i} - \delta z_{t+i})$$

and corresponds to Arrow's (1964) formula for user cost given by the LHS of equation (14).

Now suppose that the purchase price of new units of the asset changes at a constant rate over time as described by equation (4). Substitution of equation (4) into (B.8) yields

$$(B.9) \quad c_t^* = k^* z_t$$

where

$$(B.10) \quad k^* = [(1-\delta\alpha)/\delta\alpha] \sum_{i=0}^{\infty} r_i (\delta\alpha)^i.$$

Equations (B.9)-(B.10) present Arrow's formula for user cost for the case when asset prices change at a constant rate over time. Recall that this paper's formula for user cost is given by equations (13)-(14). A comparison of (B.10) and (14) shows that Arrow's formula for  $k^*$  depends on the vector of replacement rates which are in turn relatively complicated functions of the vector of survival shares. In contrast, this paper's formula for  $k^*$  is expressed directly as a relatively simple function of the vector of survival shares.

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