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## Leverage and Feedback Effects on Multifactor Wishart Stochastic Volatility for Option Pricing

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# Leverage and Feedback Effects on Multifactor Wishart Stochastic Volatility for Option Pricing

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# Leverage and Feedback Effects on Multifactor Wishart Stochastic Volatility for Option Pricing\*

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## **Abstract**

The paper proposes a general asymmetric multifactor Wishart stochastic volatility (AMWSV) diffusion process which accommodates leverage, feedback effects and multifactor for the covariance process. The paper gives the closed-form solution for the conditional and unconditional Laplace transform of the AMWSV models. The paper also suggests estimating the AMWSV model by the generalized method of moments using information not only of stock prices but also of realized volatilities and covolatilities. The empirical results for the bivariate data of the NASDAQ 100 and S&P 500 indices show that the general AMWSV model is preferred among several nested models.

**Keywords:** Multivariate Stochastic Volatility; Wishart Process; Leverage Effects; Feedback Effects; Multifactor Model; Option Pricing.

**JEL classifications:** C32, C51, G13

# 1 Introduction

In the framework of discrete time models, a wide range of multivariate GARCH and stochastic volatility (SV) models has been developed, analyzed, and applied extensively in recent years to characterize the volatility that is inherent in high frequency financial time series data. Bauwens, Laurent and Rombouts (2006) provided a survey of multivariate GARCH models, while Asai, McAleer and Yu (2006) and Chib, Omori and Asai (2009) evaluated various kinds of multivariate SV models.

There has been growing interest in the continuous-time multivariate stochastic volatility processes, especially in (i) estimating integrated volatility and co-volatility, (ii) deriving the diffusion limit of discrete multivariate volatility models, and (iii) developing option pricing models. With respect to the first point, several authors, including Hayashi and Yoshida (2005), Voev and Lunde (2007), Zhang (2008), Malliavin and Mancino (2009), Wang and Zhou (2010) and Barndorff-Nielsen et al. (2011), have developed volatility matrix estimators in the context of realized volatility and co-volatility. Regarding diffusion limits in the univariate class, Nelson (1990) derived SV models as the diffusion limit of the GARCH and exponential GARCH (EGARCH) models. Nelson and Foster (1994), Drost and Werker (1996), Duan (1997) and Fornari and Mele (1997) examined various GARCH-type diffusions. While Nelson (1996) developed the continuous record asymptotic theory for several multivariate GARCH models, Hafner, Laurent and Violante (2010) derived the diffusion limit of the consistent DCC model of Aielli (2006) that modified the DCC model of Engle (2002) in such a way that the unstandardized conditional correlation matrix has martingale difference innovations.

For pricing options, Hull and White (1987), Scott (1987), Melino and Turnbull (1990) and Heston (1993) proposed option pricing SV models, while Duan (1997) suggested the option pricing

GARCH model. With respect to the multivariate derivative pricing models, Gouriéroux (2006) and Gouriéroux and Sufana (2010) extended Wishart Autoregressive (WAR) diffusion processes, as originally considered by Bru (1991). Recently, Muhle-Karbe, Pfaffel and Stelzer (2012) developed a multivariate Ornstein-Uhlenbeck-type stochastic volatility model based on a Lévy process.

The first purpose of the paper is to extend the WAR models of Gouriéroux (2006) and Gouriéroux and Sufana (2010) regarding the asymmetric effects and multifactor for the covariance process. There are two ways to introduce asymmetric effects in continuous SV models: one way is to accommodate feedback effects, while another is to have leverage effects. As Gouriéroux and Sufana (2010) suggested a WAR model with feedback effects in log-price processes, this paper considers leverage effects in addition to feedback effects. Turning to multifactors, Chernov et al. (2003) found evidence for the superiority of two-factor specifications over one-factor models for S&P 500 returns, and showed that introducing additional factors contributes to breaking the link between tail thickness and volatility. In this paper, we develop the new general asymmetric multifactor WAR stochastic volatility models.

The second purpose of the paper is to present a closed-form expression of the conditional Laplace transform of the new model as it is the basis for pricing derivatives, as shown in Duffie, Pan and Singleton (2000). The third purpose of the paper is to suggest estimation of the new models using generalized method of moments (GMM), based on the information not only of prices but also of realized volatility and co-volatilities.

The remainder of the paper is organized as follows. Section 2 proposes the new asymmetric multifactor Wishart stochastic volatility models, and derives the conditional and unconditional Laplace transforms. Section 2 also explains the method to price derivatives using Laplace transforms and the Fourier inversion technique. Section 3 suggests GMM estimation via the information

on stock prices and realized covariances, and conducts Monte Carlo simulations to investigate the finite sample properties of the estimator. Section 4 gives an empirical example for the pair of NASDAQ 100 and S&P 500 indices, and Section 5 gives the conclusions.

In the following,  $W(A, p)$  denotes the Wishart distribution with the scale matrix  $A$  and the degrees-of-freedom parameter  $p$ . For any symmetric matrix  $X$ ,  $X^{1/2}$  is defined by the spectral decomposition of  $X$  so that  $X^{1/2}X^{1/2} = X$ . For any square matrix  $X$ , the matrix-exponential operator is defined by  $\text{Exp}(X) = \sum_{i=0}^{\infty} (1/i!)X^i$ , with  $X^0 = I$ .

## 2 Asymmetric Multifactor Wishart Stochastic Volatility

### 2.1 The Model

In this section we develop new continuous-time multivariate stochastic volatility (MSV) models, based on the Wishart Autoregressive (WAR) process, as suggested by Bru (1991) and studied by Gouriéroux (2006). The  $\text{WAR}(\nu, \Phi, C)$  process is defined by

$$dW_t = (\nu C' C + W_t \Phi' + \Phi W_t) dt + W_t^{1/2} dB_t^M (C' C)^{1/2} + (C' C)^{1/2} dB_t^M W_t^{1/2}, \quad (1)$$

where the  $B_t^M$  are  $n$  dimensional and symmetric matrix-variate standard Brownian motions,  $\nu$  is a scalar such that  $\nu > n$ ,  $\Phi$  is an  $n \times n$  matrix, and  $C$  is an  $n$  dimensional upper triangular matrix.

Instead of specifying  $W_t$  as an MSV process, we consider a multifactor specification of WAR process as follows. Let  $\Omega_t$  be an  $n$  dimensional square matrix defined by

$$\Omega_t = \Theta' \left( \sum_{k=1}^K V_{kt} \right) \Theta, \quad (2)$$

$$dV_{kt} = (\nu_k I_n + V_{kt} \Phi_k' + \Phi_k V_{kt}) dt + V_{kt}^{1/2} dB_{kt}^M + dB_{kt}^M V_{kt}^{1/2}, \quad (3)$$

where the  $B_{kt}$ 's are symmetric matrix-variate standard Brownian motions, and are mutually independent. In this specification, each  $V_{kt}$  follows a  $\text{WAR}(\nu_k, \Phi_k, I_n)$  process independently. In the following, we work with the multifactor specification defined by  $\Omega_t$ .



Let  $p_t$  be an  $m$  dimensional vector of financial log-prices evolving in continuous time. In addition to MSV, the price process accommodates the feedback and leverage effects as follows.

We consider

$$dp_t = \mu^*(\Omega_t)dt + du_t, \quad (4)$$

with feedback effect,  $\mu^*(\Omega_t)$ , defined by

$$\mu^*(\Omega_t) = (\mu_1 + \text{tr}(M_1\Omega_t), \dots, \mu_n + \text{tr}(M_n\Omega_t))', \quad (5)$$

where  $\mu_i$  and  $M_i$  ( $i = 1, \dots, n$ ) are scalar parameters and  $n$ -dimensional symmetric matrices of parameters, respectively, and the second term in (4) accommodates leverage effects as

$$du_t = \sum_{k=1}^K \Theta' V_{kt}^{1/2} R_k \text{vec}(dB_{kt}^M) + \left( \sum_{k=1}^K \Theta' V_{kt}^{1/2} (I_n - R_k R_k') V_{kt}^{1/2} \Theta \right)^{1/2} dB_t, \quad (6)$$

where  $B_t$  is an  $n$ -vector of standard Brownian motions, and  $R_k$  ( $k = 1, \dots, K$ ) are  $n \times n^2$  matrices of parameters. We assume that  $(I_n - R_k R_k')$  are positive definite matrices. It is straightforward to show that  $E(du_t du_t' | \Omega_t) = \Omega_t$ . We impose restrictions on  $R_k$  to reduce the number of parameters, such that  $R_k = (I_n \otimes \rho_k')$ , where  $\rho_k$  is an  $n$ -vector of parameters. Instead of equation (6), we may also consider a specification given by

$$du_t = \Omega_t^{1/2} \left[ \sum_{k=1}^K R_k \text{vec}(dB_{kt}^M) + \left( I_n - \sum_{k=1}^K R_k R_k' \right) dB_t \right].$$

However, we will work with (6) for the computational convenience in calculating the Laplace transform. We will refer to the new model in (2)-(6) as the ‘general asymmetric multifactor Wishart stochastic volatility’ (AMWSV) model.

The new AMWSV model contains seven special cases. If  $k = 1$ ,  $R_k = O$  and  $M_i = O$ , the AMWSV model reduces to a simple Wishart stochastic volatility (WSV) model with no asymmetric effects. When  $k = 1$  and  $R_k = O$ , we have a Wishart stochastic volatility model with

feedback effects (WSV-F). For the case  $k = 1$  and  $M_i = O$ , we obtain a WSV model with leverage effects (WSV-L). Setting  $k = 1$  yields a general asymmetric WSV (AWSV) model. Similarly, we have the multifactor WSV model (MWSV), the MWSV model with feedback effects (MWSV-F), and the MWSV model with leverage effects (MWSV-L). Note that the model suggested by Gouriéroux and Sufana (2010) is essentially the same as the one-factor WSV-F model.

## 2.2 Laplace Transforms

As explained in Duffie, Pan and Singleton (2000), the risk-neutral Laplace transform is the basis for derivative pricing because it can be used to obtain explicit or quasi-explicit prices for various derivatives. In this subsection, we derive the conditional and unconditional Laplace transforms of  $p_t$  and  $\Omega_t$ . We discuss their risk-neutral Laplace transforms in the next subsection.

Applying the approach of Gouriéroux and Sufana (2010), we can show that the joint process  $(p_t, \Omega_t)$  is an affine process. In order to derive the conditional Laplace transform of  $p_t$ , we use the matrix Riccati linearization technique suggested by Fonseca, Grasselli and Tebaldi (2008), instead the approach of Gouriéroux and Sufana (2010). Proposition 1 shows the conditional Laplace transform of the log-price process,  $p_t$ .

**Proposition 1** *The conditional Laplace transform of the affine process (4) with equations (2), (3), (5) and (6) is*

$$\Psi_{\gamma,t}(h) \equiv E_t [\exp(\gamma' p_{t+h})] = \exp \left[ \sum_{k=1}^K \text{tr}(A_k(h) V_{kt}) + b(h)' p_t + c(h) \right] \quad (7)$$

where the symmetric matrices  $A_k(h)$ , vector  $b(h)$  and scalar  $c(h)$  satisfy the system of Riccati

equations:

$$\begin{aligned}
\frac{d}{dh}A_k(h) &= \Psi'_k A_k(h) + A_k(h)\Psi_k + 2[A_k(h)]^2 + 2\Theta(\gamma\rho'_k)A_k(h) + \Gamma^* \quad (k = 1, \dots, n), \\
\frac{d}{dh}b(h) &= 0, \\
\frac{d}{dh}c(h) &= \sum_{k=1}^K \text{tr}(\nu_k A_k(h)) + \gamma'\mu,
\end{aligned} \tag{8}$$

with  $\Gamma^* = \frac{1}{2}\Theta\gamma\gamma'\Theta' + \sum_{i=1}^n \gamma_i\Theta M_i\Theta'$ ,  $\mu = (\mu_1, \dots, \mu_n)'$  and initial conditions  $A_k(0) = O$ ,  $b(0) = \gamma$  and  $c(0) = 0$ . Their solutions are given by

$$\begin{aligned}
A_k(h) &= [N_{22,k}(h)]^{-1}N_{21,k}(h), \quad (k = 1, \dots, K), \\
b(h) &= \gamma, \\
c(h) &= -\frac{1}{2} \sum_{k=1}^K \nu_k [\log \det(N_{22,k}(h)) + h\text{tr}(\Psi'_k + 2\Theta\gamma\rho'_k)],
\end{aligned} \tag{9}$$

where

$$\begin{pmatrix} N_{11,k}(h) & N_{12,k}(h) \\ N_{21,k}(h) & N_{22,k}(h) \end{pmatrix} = \text{Exp } h \begin{pmatrix} \Psi_k & -2I_n \\ \Gamma^* & -(\Psi'_k + 2\Theta\gamma\rho'_k) \end{pmatrix},$$

if  $[N_{22,k}(h)]^{-1}$  exists.

**Proof.** See Appendix A.1.

Proposition 2 shows the conditional Laplace transform of the multifactor WAR process,  $\Omega_t$ .

**Proposition 2** *The conditional Laplace transform of the multifactor WAR process (2) with equation (3) is*

$$\begin{aligned}
\Psi_{\Xi,t}^\dagger(h) &\equiv E_t[\exp(\text{tr}\{\Xi\Omega_{t+h}\})] \\
&= \prod_{k=1}^K \frac{\exp(\text{tr}\{(I_n - 2\Theta\Xi\Theta'\Sigma_k(h))^{-1}\Theta\Xi\Theta'P_k(h)V_{kt}P_k(h)'\})}{[\det(I_n - 2\Theta\Xi\Theta'\Sigma_k(h))]^{\nu_k/2}},
\end{aligned} \tag{10}$$

where  $\Xi$  is a square matrix,

$$\begin{aligned}
P_k(h) &= \text{Exp}(\Phi_k h), \\
\text{vec}(\Sigma_k(h)) &= [(I_n \otimes \Phi_k) + (\Phi_k \otimes I_n)]^{-1} \text{vec}(\text{Exp}(\Phi_k h)\text{Exp}(\Phi'_k h) - I_n),
\end{aligned} \tag{11}$$

if  $\Phi_k^{-1}$  exists.

**Proof.** See Appendix A.2.

Applying the same arguments to the original WAR process, we have a similar result.

**Corollary 1** *The conditional Laplace transform of the  $WAR(\nu, \Phi, C)$  process (1) is*

$$\Psi_{\Xi, t}^*(h) \equiv E_t [\exp(\text{tr}\{\Xi W_{t+h}\})] = \frac{\exp(\text{tr}\{(I_n - 2\Xi\Sigma(h))^{-1}\Xi P(h)W_t P(h)'\})}{[\det(I_n - 2\Sigma(h)\Xi)]^{\nu/2}}, \quad (12)$$

where  $\Xi$  is a square matrix,

$$P(h) = \text{Exp}(\Psi h), \quad (13)$$

$$\text{vec}(\Sigma(h)) = [(I_n \otimes \Phi) + (\Phi \otimes I_n)]^{-1} \text{vec}(\text{Exp}(\Phi h)C' C \text{Exp}(\Phi' h) - C' C),$$

if  $\Phi^{-1}$  exists. Furthermore, an alternative form of  $\Sigma(h)$  is given by

$$\Sigma(h) = \int_0^h \text{Exp}(\Phi_k s)C' C \text{Exp}(\Phi_k' s)ds \quad (14)$$

**Proof.** See Appendix A.3.

With respect to Corollary 1, Proposition 5 of Gourieroux (2006) also showed the conditional Laplace transform (12) based on (14), using a different approach. However, Corollary 1 shows that we can obtain  $\Sigma(h)$  without any numerical integration.

By setting  $h \rightarrow \infty$ , we have the unconditional Laplace transform of the WAR process, as in the following corollary.

**Corollary 2** *The unconditional Laplace transform of the  $WAR(\nu, \Phi, C)$  process (1) is*

$$E [\exp(\text{tr}\{\Xi W_{t+h}\})] = \lim_{h \rightarrow \infty} \Psi_{\Xi, t}^*(h) = [\det(I_n - 2\Sigma(\infty)\Xi)]^{-\nu/2}, \quad (15)$$

where  $\Xi$  is a square matrix and

$$\text{vec}(\Sigma(\infty)) = -[(I_n \otimes \Phi) + (\Phi \otimes I_n)]^{-1} \text{vec}(C' C), \quad (16)$$

if all eigenvalues of  $\Phi$  are negative. In other words, if  $W_t \sim \text{WAR}(\nu, \Phi, C)$ , the invariant distribution of  $W_t$  is the Wishart distribution with the scale matrix  $\Sigma(\infty)$  and the degrees-of-freedom parameter  $\nu$ .

**Proof.** If all the eigenvalues of  $\Phi$  are negative,  $P(h) = \text{Exp}(\Psi h) \rightarrow O$  as  $h \rightarrow \infty$ , by the spectral decomposition shown in Appendix A.3. The invariant distribution, which is characterized by the moment generating function (16), is the Wishart distribution. ■

Define the vector of asset returns as  $y_{t+h} = p_{t+h} - p_t$ . The next corollary shows the moment generating functions of  $y_{t+h}$  and  $\Omega_{t+h}$ , which will be used for estimating the AMWSV models.

**Corollary 3** For the AMWSV process (2)-(6), the moment generating functions of  $y_{t+h}$  and  $\Omega_{t+h}$  are given by

$$E [\exp(\gamma' y_{t+h})] = \exp(c(h)) \prod_{k=1}^K [I_n - 2A_k(h)\Sigma_k(\infty)]^{\nu_k/2} \quad (17)$$

$$E [\exp(\text{tr}\{\Xi\Omega_{t+h}\})] = \prod_{k=1}^K [I_n - 2\Theta\Xi\Theta'\Sigma_k(\infty)]^{\nu_k/2}, \quad (18)$$

where  $A_k(h)$  and  $c(h)$  are defined by Proposition 1, and

$$\text{vec}(\Sigma_k(\infty)) = -[(I_n \otimes \Phi_k) + (\Phi_k \otimes I_n)]^{-1} \text{vec}(I_n).$$

**Proof.** By a simple application of Corollary 2 to equation (10), we have equation (18). For each  $V_{kt}$ , we also have  $E [\exp(\text{tr}\{\Xi V_{k,t+h}\})] = [I_n - 2\Xi\Sigma_k(\infty)]^{\nu_k/2}$  from Corollary 2. Then we obtain

$$\begin{aligned} E [\exp(\gamma'(p_{t+h} - p_t))] &= EE_t [\exp(\gamma'(p_{t+h} - p_t))] \\ &= E \exp(-\gamma' p_t) \Psi_{\gamma,t}(h) \\ &= \exp(c(h)) \exp \left[ \sum_{k=1}^K \text{tr}(A_k(h)V_{kt}) \right] \\ &= \exp(c(h)) \prod_{k=1}^K [I_n - 2A_k(h)\Sigma_k(\infty)]^{\nu_k/2}. \blacksquare \end{aligned}$$

### 2.3 Option Pricing

Using the result of Duffie, Pan and Singleton (2000), we can use the risk-neutral Laplace transform to solve various option pricing problems. Under the risk-neutral measure  $Q$ , we want to determine the price  $E^Q(e^{-rT} f(p_T))$  of a European option with payoff  $f(p_T)$  at maturity  $T$ , where  $r$  is an instantaneous interest rate and  $f$  is a measurable function such that  $f : \mathfrak{R}^n \rightarrow \mathfrak{R}_+$ .

In order to obtain the risk-neutral Laplace transform, we can apply the results of Proposition 6 derived by Gouriéroux and Sufana (2010) to our case. We need to adjust the drift terms of  $(p_t, \Omega_t)$  under the risk-neutral distribution, such that

$$\begin{aligned} E_t^Q(dp_t) &= r\iota - \frac{1}{2}(e_1'\Omega_t e_1, \dots, e_n'\Omega_t e_n)' dt \\ E_t^Q(d\Omega_t) &= E_t(d\Omega_t) + \text{Cov}[\text{tr}(C_t d\Omega_t), d\Omega_t] \\ &= E_t(d\Omega_t) + 2(\Omega_t \Xi_t \Theta' \Theta + \Theta' \Theta \Xi_t' \Omega_t) dt, \end{aligned}$$

where  $E_t^Q$  denotes the conditional expectation under the risk-neutral measure  $Q$ ,  $\iota$  is an  $n$ -vector of ones, and  $e_i$  is the canonical vector with zero components, except for the  $i$ th element which is equal to one. The risk premium on the Brownian motion of the return equation is fixed by the martingale condition, whereas the risk premia corresponding to the covariance can be fixed arbitrarily. As in Gouriéroux and Sufana (2010), we consider the case that the volatility risk premia are constant, such that  $\Xi_t = \bar{\Xi}$ . Then we may have the alternative form the above equations:

$$\begin{aligned} E_t^Q(dp_t) &= (\bar{\mu}_1 + \text{tr}(\bar{M}_1 \Omega_t), \dots, \bar{\mu}_n + \text{tr}(\bar{M}_n \Omega_t))' dt \\ E_t^Q(d\Omega_t) &= E_t(d\Omega_t) + \text{Cov}[\text{tr}(C_t d\Omega_t), d\Omega_t] \\ &= \sum_{k=1}^K \Theta' (\nu_k I_n + V_{kt} \bar{\Phi}'_k + \bar{\Phi}_k V_{kt}) \Theta dt, \end{aligned}$$

where  $\bar{\mu}_i = r$ ,  $\bar{M}_i = -\frac{1}{2}e_i e_i'$  and  $\bar{\Phi}_k = \Phi + 2\Theta\bar{\Xi}\Theta'$ . We can obtain the risk-neutral conditional Laplace transform by replacing  $E_t$  with  $E_t^Q$  in Propositions 1 and 2. We can also have the risk-neutral conditional Laplace transform of the pair  $(p_t, \Omega_t)$  by applying the same approach. Compared with the MWSV-F model, the option pricing AMWSV model can accommodate the asymmetric effects flexibly through the leverage effects.

For purposes of calculating option prices, we can explain the approach based on the Fourier inversion technique introduced by Carr and Madan (1999) and extended by Eberlein, Glau and Papapantoleon (2010) to the multivariate case. Let  $\hat{f}$  be the Fourier transform of  $f$ . Eberlein, Glau and Papapantoleon (2010) showed that

$$E^Q(e^{-rT} f(p_T)) = \frac{e^{-z'p_0 - rT}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-iu'p_0} \Psi_{iz-u,0}(T) \hat{f}(iz-u) du, \quad (19)$$

where  $\Psi_{\gamma,t}(h)$  is the conditional Laplace transform of  $p_{t+h}$ , and the vector  $z$  has to lie in the intersection of the domain of  $\Psi$  and  $\hat{f}$ .

Muhle-Karbe, Pfaffel and Stelzer (2012) give the following formulae for basket and spread options. The payoff function of a basket put option is defined by  $f(p) = (K - \sum_{j=1}^n e^{p_j})^+$ ,  $K > 0$ , and its Fourier transform is given by

$$\hat{f}(u) = K^{1+i\sum_{j=1}^n u_j} \frac{\prod_{j=1}^n \Gamma(iu_j)}{\Gamma(2+i\sum_{j=1}^n u_j)},$$

for all  $u \in \mathcal{C}^n$  with  $\text{Im}(u_j) < 0$  ( $j = 1, \dots, n$ ). We can derive the price of the corresponding call option using put-call-parity  $(K-x)^+ = (x-K)^+ - x + K$ . With respect to a spread call option, the payoff function is defined by  $f(p) = (e^{p_1} - e^{p_2} - K)^+$ ,  $K > 0$ , and its Fourier transform is given by

$$\hat{f}(u) = \frac{K^{1+iu_1+iu_2}}{iz_1(1+iz_1)} \frac{\Gamma(iz_2)\Gamma(-iz_1-iz_2-1)}{\Gamma(-iz_1-1)},$$

for all  $u \in \mathcal{C}^2$ , with  $\text{Im}(u_1) > 1$ ,  $\text{Im}(u_2) < 0$  and  $\text{Im}(u_1 + u_2) > 1$ . We can use Fast Fourier Transform (FFT) methods to compute these prices.

### 3 Estimation

For estimating the AMWSV models, we use a generalized method of moments (GMM) based on the moment generating function of  $y_{t+h}$  and  $\Omega_{t+h}$ . With respect to the one factor WSV-F model with  $\Theta = \theta I_n$ , Gouriéroux and Sufana (2010) also use GMM, and their Monte Carlo simulation results indicate that it is hard to estimate  $\theta$ , even for  $T = 5000$ . In order to cope with the problem, we suggest estimation using realized volatilities and co-volatilities. By the methods of Voev and Lunde (2007), Malliavin and Mancino (2009), Wang and Zhou (2010), Barndorff-Nielsen et al. (2011), and Zhang (2011), we can obtain a consistent estimator of  $\Omega_t$  (without assuming its structure).

We may call the difference between a consistent estimator of  $\Omega_t$  and  $\Omega_t$  itself as the ‘realized covariance error’. In univariate realized volatility models, Barndorff-Nielsen and Shephard (2002), Bollerslev and Zhou (2002), and Asai, McAleer, and Medeiros (2012a,b) showed it is useful to use an ad hoc approach that accommodates an error term with constant variance. Here, we consider a Wishart-like disturbance.

Consider a Wishart distribution,  $W_t^* \sim W((1/\nu_*)\Omega_t, \nu_*)$ , with the mean and variance of  $\text{vec}(W_t^*)$  given by  $E(\text{vec}(\Omega_t))$  and  $E((1/\nu_*)(I_n + P)(\Omega_t \otimes \Omega_t))$ , respectively, where  $P$  is a matrix which satisfies  $\text{vec}(A') = P\text{vec}(A)$  for any square matrix  $A$ . Let  $X_t$  be a consistent estimator of  $\Omega_t$ . We assume that the first two moments of  $x_t = \text{vec}(X_t)$  correspond to the moments of  $\text{vec}(W_t^*)$ . Although this specification is ad hoc, this structure can create disturbances of  $\Omega_t$  with a single nuisance parameter,  $\nu_*$ . By construction, we need to guarantee the consistency and positive semi-definiteness of  $X_t$ . We will use the moments of  $x_{ij,t}$  ( $1 \leq i \leq j \leq n$ ) and  $x_{ij,t}x_{kl,t}$



$(1 \leq i \leq j \leq n) (1 \leq k \leq l \leq n)$ , resulting in  $n^*(n^*+3)/2$  conditions for  $x_t$ , where  $n^* = n(n+1)/2$ .

Following Gourieroux and Sufana (2010), we work with the moments of  $\exp(u'_2 y_{t+2h} + u'_1 y_{t+h})$  for  $y_t$ . For the case  $n = 2$ , we consider 23 moment conditions of  $(u'_2, u'_1)$ :  $(0, 0, 1, 0)$ ,  $(0, 0, 0, 1)$ ,  $(0, 0, 2, 0)$ ,  $(0, 0, 0, 2)$ ,  $(0, 0, 1, 1)$ ,  $(1, 0, 1, 0)$ ,  $(0, 1, 0, 1)$ ,  $(1, 0, 0, 1)$ ,  $(0, 1, 1, 0)$ ,  $(1, 0, 2, 0)$ ,  $(0, 1, 0, 2)$ ,  $(1, 0, 0, 2)$ ,  $(0, 1, 2, 0)$ ,  $(2, 0, 2, 0)$ ,  $(0, 2, 0, 2)$ ,  $(2, 0, 0, 2)$ ,  $(0, 2, 2, 0)$ ,  $(0, 0, 3, 0)$ ,  $(0, 0, 0, 3)$ ,  $(0, 0, 3, 3)$ ,  $(0, 0, 4, 0)$ ,  $(0, 0, 0, 4)$ ,  $(0, 0, 4, 4)$ .

For  $n = 2$ , we use GMM with these 32 moment conditions to estimate 26 parameters of the AMWSV model. We estimate the weighting matrix according to Newey and West (1987), with a lag length of 50.

We present the results of a Monte Carlo study to investigate the finite sample performance of GMM estimation for  $n = 2$ . We generate  $R$  simulated time series of  $\{y_t, x_t\}_{t=1}^T$  for the AMWSV model in equations (2)–(6), and for some given “true” parameter vector. Subsequently, we treat the parameter vector as unknown and estimate it for each series using GMM. We compute the sample mean, standard deviation, and root mean squared error (RMSE), and compare these with the true parameter value. In order to obtain the simulated path of  $V_{kt}$ , we use the following approach. Noting that  $dV_{kt} = V_{kt}^{1/2} dV_{kt}^{1/2} + (dV_{kt}^{1/2})V_{kt}^{1/2}$ , we have

$$\text{vec}(dV_{kt}^{1/2}) = [(V_{kt}^{1/2} \otimes I_n) + (I_n \otimes V_{kt}^{1/2})]^{-1} \text{vec}(\nu I_n + V_{kt} \Phi'_k + \Phi_k V_{kt}) dt + \text{vec}(dB_{kt}^M),$$

where  $B_{kt}^M$  is the same symmetric matrix-variate Brownian motion process used in equation (3). Then we can apply the local linearization technique of Shoji and Ozaki (1998) to generate  $V_{kt}^{1/2}$  ( $V_{kt}$  and  $\Omega_t$ ) efficiently.

The true parameter values for generating Monte Carlo samples are given in the first column of Table 1, which is obtained from our empirical analysis in Section 4. The results given in Table 1 are for sample size  $T = 2500$ , with the number of iterations set to  $R = 2000$ . Table 1 shows that

most values of the standard deviations are close to their RMSE counterparts, indicating that the biases in finite samples are negligible.

## 4 Empirical Analysis

The empirical analysis focuses on the pair of indexes given by NASDAQ 100 and S&P 500. In calculating daily realized volatilities and co-volatilities, we use the averaging realized volatility matrix estimator suggested by Wang and Zou (2010). We clean the data for outliers and discard transactions outside trading hours, considering transactions between 9.30 A.M. and 4.00 P.M. Following Tao et al. (2011), we use grids selected in accord with 5 minute returns to take averages. The sample period is July 1, 1996 to September 28, 2012, giving  $T= 3961$  observations for the return vector,  $y_t$ , and realized covariance,  $X_t$ .

We estimate one-factor and two-factor WSV models including the simple WSV, WSV-F, WSV-L and general AWSV models. For purposes of a diagnostic statistic, we use the conventional GMM  $J$ -test of Hansen (1982). Under standard regularity conditions, the minimized value of the objective function multiplied by the sample size is asymptotically distributed as chi-square, which allows for an omnibus test of the overidentifying restrictions. We also examine the estimates of  $E(\text{tr}(M_i\Omega_t))$  and  $E(d\omega_{ii,t}du_{1t}) + E(d\omega_{ii,t}du_{2t})$  ( $i = 1, 2$ ) to check the feedback and leverage effects, respectively. Note that  $E[\text{vec}(d\Omega_t)du'_i|V_{1t}, V_{2t}] = (\Theta' \otimes \Theta') \sum_{k=1}^2 [(I_n \otimes V_{kt}^{1/2}) + (V_{kt}^{1/2} \otimes I_n)] R'_k V_{kt}^{1/2} \Theta dt$ . We expect  $E(\text{tr}(M_i\Omega_t)) > 0$  for the feedback effect, while we expect  $E(d\omega_{ii,t}du_{1t}) + E(d\omega_{ii,t}du_{2t}) < 0$  for the leverage effect.

Table 2 shows the estimates of the one-factor WSV models. For estimating the one factor models, we excluded the last 6 moment conditions for  $y_t$ . With respect to the simple one-factor WSV model, the estimate of  $\nu_1$  is 17.0, which is significant at the 5% level. The estimates of  $\phi_{11,1}$  and  $\phi_{22,1}$  are negative and significant, while those of  $\phi_{12,1}$  and  $\phi_{21,1}$  are insignificant. The

eigenvalues of the estimate of  $\Phi_1$  are negative, indicating that the invariant distribution of  $V_{1t}$  exists by Corollary 2. The estimate of  $\nu_*$  is 0.17 and is significant. We can interpret the variance of  $x_t$  conditional on  $\Omega_t$ , that is,  $(1/\nu_*)(I_n + P)(\Omega_t \otimes \Omega_t)$ , such that, as  $\nu$  decreases, the variance of  $x_t$  increases. The GMM omnibus test for the overidentifying restrictions rejects the simple one-factor WSV model.

Table 2 also shows the estimates of the one-factor WSV-F, WSV-L and Asymmetric WSV models. Compared with the simple WSV model, the estimates of  $\nu_1$  became smaller and are close to 2.1 for all three models. The estimates of  $\phi_{11,1}$  and  $\phi_{22,1}$  are negative and significant, while those of  $\phi_{12,1}$  and  $\phi_{21,1}$  are insignificant. The estimate of  $\nu_*$  is close to 0.3, which is greater than the value for the simple WSV model. With respect to asymmetry, Table 3 summarizes the estimated asymmetric effects. For the WSV-F model, the feedback effect to NASDAQ returns is positive, while the effect to S&P returns is negative. Hence, the specification (especially for the second variable) has room to improve.

Regarding the WSV-L model, the leverage effects can be examined by the estimates of  $E(d\omega_{ii,t}du_{1t}) + E(d\omega_{ii,t}du_{2t})$ , as noted above. The leverage effects to the NASDAQ returns volatility is negative, while the effect to the S&P returns volatility is positive. Again, the sign conditions are not satisfied. For the general AWSV model, the feedback effects are positive, while the leverage effects are negative for NASDAQ and S&P. This time the conditions for asymmetric effects are satisfied. However, the GMM omnibus tests for the overidentifying restrictions reject the WSV-F, WSV-L and AWSV models.

Table 4 reports the estimates of the two-factor WSV models. With respect to the simple two-factor WSV model without asymmetric effects, all the estimated parameters are significant. The estimate of  $\nu_1$  is 5.8, while the estimate of  $\nu_2$  is 5.8. The estimates of  $\phi_{11,i}$  and  $\phi_{22,i}$  are

negative for  $i = 1, 2$ , while those of  $\phi_{12,i}$  and  $\phi_{21,i}$  are positive. The estimate of  $\nu_*$  is 1.5, which is greater than the values given in Table 2. This result suggests that the unexplained component of  $x_t$  has become smaller than those of the one-factor models. The GMM omnibus test for the overidentifying restrictions does not reject the simple two-factor WSV model. Except for the parameters of asymmetric effects, all the estimates are significant for the two-factor WSV-F, WSV-L and AWSV models. Compared to the estimates of the simple two-factor WSV model, the WSV-F model has similar results. The WSV-L and AWSV models have smaller  $\nu_*$  values compared with the simple WSV model. The GMM omnibus tests for the overidentifying restrictions do not reject these three asymmetric two-factor WSV models.

Table 5 presents the estimated asymmetric effects. For the WSV-F model, the feedback effects to NASDAQ and S&P returns are positive, indicating that the sign condition is satisfied. Regarding the WSV-L model, the leverage effects to the NASDAQ and S&P returns volatilities are negative. For the general AWSV model, the feedback effects are positive, while the leverage effects are negative for NASDAQ and S&P. The results also show that the conditions are satisfied.

These empirical results show that the data favor the two-factor WSV models. Among the two-factor WSV models, the general asymmetric specification is preferred by the  $t$  tests. The two-factor AWSV model has another merit in that its option pricing variation can accommodate asymmetric effects flexibly via the leverage effect.

## 5 Concluding Remarks

In this paper, we suggested a general asymmetric multifactor Wishart stochastic volatility (AMWSV) model which accommodates leverage and feedback effects and multifactor for the covariance process. We also derived the closed-form expression of the conditional and unconditional Laplace transform of the AMWSV models, the results of which can be used for pricing derivatives. Fur-

thermore, we proposed to estimate the AMWSV model using GMM based on information not only of stock prices but also of realized volatilities and co-volatilities. The empirical results for the pair of NASDAQ 100 and S&P 500 indices show the usefulness of the new specification compared with other specifications.

## Appendix

### A.1 Proof of Proposition 1

By applying the Feynman-Kac argument, we have

$$\frac{d}{dh} \Psi_{\gamma,t}(h) = \mathcal{L}_{p,V_1,\dots,V_k} \Psi_{\gamma,t}(h), \quad \Psi_{\gamma,t}(0) = \exp(\gamma' p_t).$$

with the infinitesimal generator for  $(p_t, V_{1t}, \dots, V_{kt})$ . By equation (5.12) of Bru (1991), we can obtain the infinitesimal generator for the WAR process,  $V_{kt}$ , as follows:

$$\mathcal{L}_{V_k} = \text{tr} [(\nu_k I_n + V_k \Phi'_k + \Phi_k V_k) D_k + 2V_k D_k^2],$$

where  $D_k$  is a matrix differential operator with elements  $D_{ij,k} = (\partial/\partial V_{ij,k})$ . We define  $\delta = (\delta_1, \dots, \delta_n)'$  with  $\delta_i = (\partial/\partial p_i)$ . Then the infinitesimal generator for  $(p_t, V_{1t}, \dots, V_{kt})$  is given by

$$\begin{aligned} \mathcal{L}_{p,V_1,\dots,V_k} = & \delta' \mu + \sum_{i=1}^n \delta_i \text{tr} \left[ M_i \Theta' \left( \sum_{k=1}^K V_k \right) \Theta \right] + \frac{1}{2} \text{tr} \left[ \Theta' \left( \sum_{k=1}^K V_k \right) \Theta \delta \delta' \right] \\ & + \sum_{k=1}^K \text{tr} [(\nu_k I_n + V_k \Phi'_k + \Phi_k V_k) D_k + 2V_k D_k^2] + 2 \sum_{k=1}^K \text{tr} [V_k \Theta \gamma \rho_k D_k]. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} 0 = & - \sum_{k=1}^K \text{tr} \left[ \frac{d}{dh} A_k(h) V_k \right] - \frac{d}{dh} b(h)' p - \frac{d}{dh} c(h) \\ & + \sum_{k=1}^K \text{tr} [(\nu_k I_n + V_k \Phi'_k + \Phi_k V_k) A_k(h) + 2V_k A_k^2(h) + V_k \Theta \gamma \rho_k A_k(h)] \\ & + \mu' b(h) + \sum_{i=1}^n b_i(h) \text{tr} \left[ \Theta M_i \Theta' \sum_{k=1}^K V_k \right] + \frac{1}{2} \sum_{k=1}^K \text{tr} [V_k \Theta b(h) b(h)' \Theta'], \end{aligned}$$

with boundary conditions  $A_k(0) = O$  for all  $k$ ,  $b(0) = \gamma$  and  $c(0) = 0$ . By identifying the coefficients of  $(p, V_1, \dots, V_k)$ , we have the ordinary differential equations for  $b(h)$  and  $A_k(h)$  given by equation (8). As usual, the function  $c(h)$  can be obtained by direct integration.

Following Fonseca, Grasselli and Tebaldi (2008), we linearize the Matrix Riccati equation for  $A_k(h)$  by doubling the dimension of the problem. Consider a decomposition which satisfies

$$A_k(h) = [F_k(h)]^{-1}G_k(h).$$

Then we have

$$\frac{d}{dh}[F_k(h)A_k(h)] - \frac{d}{dh}[F_k(h)]A_k(h) = F_k(h)\frac{d}{dh}[A_k(h)].$$

Substituting the Matrix Riccati equation for  $A_k(h)$  in equation (8) into the above equation, we obtain

$$\frac{d}{dh}G_k(h) - \frac{d}{dh}[F_k(h)]A_k(h) = G_k(h)\Phi_k F_k(h)\Gamma^* + [F_k(h)(\Phi'_k + 2\Theta\gamma\rho'_k) + 2G_k(h)]A_k(h).$$

Hence, we can construct a system of  $(2n)$  linear equations given by

$$\begin{aligned}\frac{d}{dh}G_k(h) &= F_k(h)\Gamma^* + G_k(h)\Phi_k, \\ \frac{d}{dh}F_k(h) &= -F_k(h)(\Phi'_k + 2\Theta\gamma\rho'_k) - 2G_k(h),\end{aligned}$$

with initial conditions  $G_k(0) = O$  and  $F_k(0) = I_n$ . The above equations can be written as follows:

$$\frac{d}{dh} \begin{pmatrix} G_k(h) & F_k(h) \end{pmatrix} = \begin{pmatrix} G_k(h) & F_k(h) \end{pmatrix} \begin{pmatrix} \Phi_k & -2I_n \\ \Gamma^* & -(\Phi'_k + 2\Theta\gamma\rho'_k) \end{pmatrix}.$$

Hence, we obtain the solution by matrix-exponentiation:

$$\begin{pmatrix} G_k(h) & F_k(h) \end{pmatrix} = \begin{pmatrix} O & I_n \end{pmatrix} \text{Exp } h \begin{pmatrix} \Phi_k & -2I_n \\ \Gamma^* & -(\Phi'_k + 2\Theta\gamma\rho'_k) \end{pmatrix}.$$

Using the definition of  $N_{ij,k}(h)$   $i, j = 1, 2$  in equation (9), we have  $\begin{pmatrix} G_k(h) & F_k(h) \end{pmatrix} = \begin{pmatrix} N_{21,k}(h) & N_{22,k}(h) \end{pmatrix}$ , and  $A_k(h) = [N_{22,k}(h)]^{-1}N_{21,k}(h)$ . Now we turn to the ordinary differential equation for  $c(h)$ .

Multiplying  $F_k(h)$  before  $\frac{d}{dh}F_k(h)$  above, we have

$$[F_k(h)]^{-1}\frac{d}{dh}F_k(h) = -(\Phi'_k + 2\Theta\gamma\rho'_k) - 2A_k(h).$$

We can remove  $A_k(h)$  from the ODE of  $c(h)$ , in order to give

$$\frac{d}{dh}c(h) = -\frac{1}{2}\sum_{k=1}^K \nu_k \text{tr} \left( [F_k(h)]^{-1} \frac{d}{dh}F_k(h) + \Phi'_k + 2\Theta\gamma\rho'_k \right) + \gamma'\mu.$$

Now we can integrate the last equation and obtain the solution of  $c(h)$  given by equation (9).

## A.2 Proof of Proposition 2

We can handle the conditional Laplace transform of the covariance process:

$$\Psi_{\Xi,t}^\dagger(h) = E_t [\exp(\text{tr}\{\Xi\Omega_{t+h}\})] = \exp \left[ \sum_{k=1}^K \text{tr} \left( A_k^\dagger(h)V_{kt} \right) + b^\dagger(h)'p_t + c^\dagger(h) \right],$$

by replacing the corresponding boundary conditions and repeating the procedure in Appendix

A.1. Hence, we have

$$A_k^\dagger(h) = [\Theta\Xi\Theta'N_{12}(h) + N_{22,k}(h)]^{-1}\Theta\Xi\Theta'N_{11,k}(h), \quad (k = 1, \dots, K)$$

$$b^\dagger(h) = \mathbf{0},$$

$$c^\dagger(h) = -\frac{1}{2}\sum_{k=1}^K \nu_k [\log \det(N_{22,k}(h)) + h\text{tr}(\Psi'_k + 2\Theta\gamma\rho_k)],$$

where

$$\begin{pmatrix} N_{11,k}(h) & N_{12,k}(h) \\ N_{21,k}(h) & N_{22,k}(h) \end{pmatrix} = \text{Exp} h \begin{pmatrix} \Psi_k & -2I_n \\ O & -\Psi'_k \end{pmatrix}.$$

It is straightforward to show that  $N_{11,k}(h) = \text{Exp}(\Psi_k h)$ ,  $N_{22,k}(h) = \text{Exp}(-\Psi'_k h)$ ,  $N_{21,k}(h) = O$  and  $N_{12,k}(h) = -2\Sigma(h)\text{Exp}(-\Psi'_k h)$ , where  $\Sigma(h)$  is defined by (11). By a property of matrix-exponential transformation (for instance, see Chiu, Leonard and Tsui (1996)), we have

$\log \det(N_{22,k}(h)) = \log \det(\text{Exp}(-\Psi'_k h)) = \text{tr}(-\Psi'_k h)$ . Thus, we obtain

$$\begin{aligned} A_k^\dagger(h) &= \text{Exp}(\Psi'_k h) [I_n - 2\Theta\Xi\Theta'\Sigma(h)]^{-1} \Theta\Xi\Theta' \text{Exp}(\Psi_k h), \\ c^\dagger(h) &= -\frac{1}{2}\sum_{k=1}^K \nu_k \log \det (I_n - 2\Theta\Xi\Theta'\Sigma(h)). \end{aligned}$$

Therefore, we have the results in (10).



### A.3 Proof of Corollary 1

In order to prove Corollary 1, we need to show that equation (13) is equivalent to the result obtained by Proposition 5 of Gourieroux (2006). We also need to show some properties of the matrix-exponential operator. Note that for any  $n \times n$  matrix  $A$ , the spectral decomposition is given by  $A = ULU^{-1}$ , where  $L$  is the diagonal matrix of eigenvalues such that  $L = \text{diag}\{\lambda_1, \dots, \lambda_m\}$ , and  $U$  is the matrix of corresponding eigenvectors. Hence, we have  $A^i = UL^iU^{-1}$ , and we have the decomposition:

$$\text{Exp}(A) = U \begin{pmatrix} \exp(\lambda_1) & & O \\ & \ddots & \\ O & & \exp(\lambda_m) \end{pmatrix} U^{-1}.$$

(see Bellman (1970) and Chiu, Leonard and Tsui (1996)).

**Lemma 1** For any square matrix  $A$  ( $m \times m$ ),

$$\frac{d}{ds} \text{Exp}(As) = A \text{Exp}(As).$$

**Proof.** It is straightforward to show this from the definition as:

$$\begin{aligned} \frac{d}{ds} \text{Exp}(As) &= \frac{d}{ds} \left\{ I + \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{i!} A^i s^i \right\} = \lim_{n \rightarrow \infty} \frac{d}{ds} \sum_{i=1}^n \frac{1}{i!} A^i s^i \\ &= \lim_{n \rightarrow \infty} A \sum_{i=1}^n \frac{1}{(i-1)!} A^{i-1} s^{i-1} = A \text{Exp}(As). \quad \blacksquare \end{aligned}$$

Denote  $G(h) = \text{Exp}(\Phi h) C C' [\text{Exp}(\Phi h)]'$ . Then,  $\Omega(h) = \int_0^h G(s) ds$ . Noting that  $\frac{dG(s)}{ds} = \Phi G(s) + G(s) \Phi'$  by Lemma 1, we have

$$\text{vec}(\Omega(h)) = [(I_m \otimes \Phi) + (\Phi \otimes I_m)]^{-1} \text{vec}(\text{Exp}(\Phi h) C C' [\text{Exp}(\Phi h)]' - C C'),$$

if  $\Phi^{-1}$  exists.

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Table 1: Monte Carlo Results for the GMM Estimator for Two-Factor Asymmetric WSV Model with Noise

Para.	True	Mean	Std.Dev.	RMSE
$\mu_1$	-0.0047	-0.0212	0.00315	0.0169
$\mu_2$	-0.0013	-0.0066	0.00130	0.0055
$m_{11,1}$	0.8448	0.8432	0.00544	0.0057
$m_{21,1}$	-0.1867	-0.1839	0.00586	0.0065
$m_{22,1}$	0.1646	0.1710	0.00574	0.0085
$m_{11,2}$	0.1805	0.1803	0.00389	0.0039
$m_{21,2}$	-0.0778	-0.0774	0.00437	0.0044
$m_{22,2}$	0.2429	0.2482	0.00618	0.0081
$\theta_{11}$	0.0394	0.0388	0.00532	0.0054
$\theta_{12}$	0.0070	-0.0008	0.00549	0.0095
$\theta_{22}$	0.0051	-0.0073	0.00234	0.0126
$\nu_1$	7.7153	7.7161	0.00375	0.0038
$\phi_{11,1}$	-7.1421	-7.1417	0.00313	0.0032
$\phi_{21,1}$	-0.1148	-0.1151	0.00299	0.0030
$\phi_{12,1}$	-0.0394	-0.0395	0.00420	0.0042
$\phi_{22,1}$	-4.7369	-4.7365	0.00650	0.0065
$\rho_{1,1}$	-0.3159	-0.3158	0.00409	0.0041
$\rho_{2,1}$	-0.2855	-0.2857	0.00415	0.0042
$\nu_2$	2.8146	2.8151	0.01215	0.0122
$\phi_{11,2}$	-1.2305	-1.2371	0.01705	0.0183
$\phi_{21,2}$	0.8658	0.8648	0.00657	0.0066
$\phi_{12,2}$	-0.0967	-0.1138	0.00975	0.0197
$\phi_{22,2}$	-0.0076	-0.0269	0.01123	0.0223
$\rho_{1,2}$	-0.1095	-0.1108	0.00612	0.0063
$\rho_{2,2}$	0.1587	0.1604	0.00907	0.0092
$\nu_*$	0.6882	2.4928	0.01888	0.0202

Table 2: GMM Estimates for One-Factor WSV Processes

Para.	WSV		WSV-F		WSV-L		Asym. WSV	
$\mu_1$	-0.0005	(0.0002)	-0.0005	(0.0001)	-0.0004	(0.0002)	-0.0006	(0.0112)
$\mu_2$	$6.17 \times 10^{-5}$	$(5.06 \times 10^{-5})$	0.0004	(0.0001)	$7.14 \times 10^{-5}$	(0.0001)	$-3.68 \times 10^{-5}$	(0.0019)
$m_{11,1}$			0.0186	(0.0039)			0.2307	(0.0495)
$m_{21,1}$			0.0694	(0.0064)			-0.0270	(0.0283)
$m_{22,1}$			0.1131	(0.0372)			-0.0975	(0.0140)
$m_{11,2}$			-0.1258	(0.0524)			-0.0277	(0.0168)
$m_{21,2}$			-0.1646	(0.0039)			0.2807	(0.0275)
$m_{22,2}$			-0.2862	(0.0097)			0.1422	(0.0765)
$\theta_{11}$	0.0234	(0.0044)	0.0341	(0.0057)	0.0324	(0.0053)	0.0415	(0.0070)
$\theta_{12}$	0.0013	(0.0006)	-0.0025	(0.0054)	$6.57 \times 10^{-5}$	(0.0010)	0.0022	(0.0071)
$\theta_{22}$	-0.0114	(0.0021)	-0.0312	(0.0052)	-0.0303	(0.0051)	-0.0297	(0.0050)
$\nu_1$	17.004	(5.9687)	2.0619	(0.3323)	2.1981	(0.3206)	2.1327	(0.3540)
$\phi_{11,1}$	-3.0189	(1.1218)	-0.8373	(0.2688)	-0.7917	(0.2565)	-1.2414	(0.4102)
$\phi_{21,1}$	-0.1031	(0.2372)	-0.3504	(0.5046)	-0.1273	(0.3385)	-0.0218	(0.6447)
$\phi_{12,1}$	-0.0726	(0.0648)	-0.2729	(0.9217)	-0.2422	(0.9407)	-0.0869	(1.0963)
$\phi_{22,1}$	-2.5587	(0.9563)	-2.1070	(0.7059)	-2.1170	(0.7084)	-1.9826	(0.6713)
$\rho_{1,1}$					-0.4711	(0.2250)	-0.5232	(0.1502)
$\rho_{2,1}$					-0.0461	(0.5941)	-0.0114	(0.0813)
$\nu_*$	0.1712	(0.0123)	0.3139	(0.0354)	0.2974	(0.0274)	0.3052	(0.0329)
$J$ -test	25.015		24.578		24.378		24.294	
d.o.f.	(15)		(9)		(13)		(7)	
$p$ -value	0.0497		0.0035		0.0278		0.0010	

Note: Standard errors are in parentheses. ' $J$ -test' is the GMM test of overidentifying restrictions.

Table 3: Estimated Asymmetric Effects for One-Factor WSV Models

Para.	WSV	WSV-F	WSV-L	Asym. WSV
$E(\omega_{11,t})$	0.001537	0.001506	0.001489	0.001485
$E(\omega_{21,t})$	0.000117	0.000105	0.000107	0.000104
$E(\omega_{22,t})$	0.000443	0.000486	0.000483	0.000481
Feedback Effect				
$E(\text{tr}(M_1\Omega_t))$	0	$9.746 \times 10^{-5}$	0	0.00029003
$E(\text{tr}(M_2\Omega_t))$	0	-0.000363	0	$8.575 \times 10^{-5}$
Leverage Effects from $u_{1t}$				
$E(d\omega_{11,t}du_{1t})$	0	0	$-4.529 \times 10^{-5}$	$-6.453 \times 10^{-5}$
$E(d\omega_{21,t}du_{1t})$	0	0	$-0.039 \times 10^{-5}$	$-0.712 \times 10^{-5}$
$E(d\omega_{22,t}du_{1t})$	0	0	$0.016 \times 10^{-5}$	$-0.019 \times 10^{-5}$
Leverage Effects from $u_{2t}$				
$E(d\omega_{11,t}du_{2t})$	0	0	$-0.266 \times 10^{-5}$	$-0.429 \times 10^{-5}$
$E(d\omega_{21,t}du_{2t})$	0	0	$-1.918 \times 10^{-5}$	$-2.378 \times 10^{-5}$
$E(d\omega_{22,t}du_{2t})$	0	0	$0.078 \times 10^{-5}$	$-0.106 \times 10^{-5}$

Note: We omitted  $dt$  for leverage effects.



Table 4: GMM Estimates for Two-Factor WSV Processes

Para.	WSV		WSV-F		WSV-L		Asym. WSV	
$\mu_1$	-0.0007	(0.0002)	-0.0013	(0.0002)	-0.0022	(0.0002)	-0.0047	(0.0006)
$\mu_2$	0.0012	(0.0005)	0.0005	(0.0001)	-0.0006	(0.0001)	-0.0013	(0.0003)
$m_{11,1}$			0.0749	(0.0256)			0.8448	(0.1870)
$m_{21,1}$			-0.1373	(0.0464)			-0.1867	(0.0734)
$m_{22,1}$			0.0333	(0.0273)			0.1646	(0.0470)
$m_{11,2}$			0.0951	(0.0532)			0.1805	(0.0503)
$m_{21,2}$			-0.0998	(0.0270)			-0.0778	(0.0066)
$m_{22,2}$			0.0462	(0.0263)			0.2429	(0.2668)
$\theta_{11}$	0.0125	(0.0013)	0.0123	(0.0005)	0.0173	(0.0004)	0.0394	(0.0011)
$\theta_{12}$	0.0091	(0.0007)	0.0087	(0.0003)	0.0089	(0.0004)	0.0070	(0.0004)
$\theta_{22}$	-0.0131	(0.0008)	-0.0128	(0.0004)	-0.0118	(0.0003)	0.0051	(0.0002)
$\nu_1$	5.7615	(1.7031)	5.7753	(1.2572)	5.6577	(2.4992)	7.7153	(1.2193)
$\phi_{11,1}$	-5.2620	(0.6143)	-5.2497	(0.7034)	-5.3224	(1.1645)	-7.1421	(0.6422)
$\phi_{21,1}$	0.0428	(0.0034)	0.0360	(0.0111)	-0.0686	(0.0111)	-0.1148	(0.0231)
$\phi_{12,1}$	0.0437	(0.0234)	0.0365	(0.0069)	-0.0718	(0.0093)	-0.0394	(0.0073)
$\phi_{22,1}$	-5.0030	(0.7916)	-4.9987	(0.4301)	-5.0214	(1.2123)	-4.7369	(0.3698)
$\rho_{1,1}$					-0.2293	(0.0209)	-0.3159	(0.0236)
$\rho_{2,1}$					-0.2591	(0.0622)	-0.2855	(0.0207)
$\nu_2$	4.4075	(0.3416)	4.4533	(0.3251)	3.9378	(0.3432)	2.8146	(0.2341)
$\phi_{11,2}$	-0.7955	(0.0425)	-0.7348	(0.0137)	-0.5003	(0.0141)	-1.2305	(0.0741)
$\phi_{21,2}$	0.7869	(0.0630)	0.7314	(0.0203)	0.1403	(0.0179)	0.8658	(0.1434)
$\phi_{12,2}$	0.9960	(0.0609)	0.9404	(0.0197)	0.4003	(0.0135)	-0.0967	(0.0067)
$\phi_{22,2}$	-1.2469	(0.0927)	-1.2034	(0.0308)	-0.3147	(0.0188)	-0.0076	(0.0050)
$\rho_{1,2}$					-0.4146	(0.0264)	-0.1095	(0.0166)
$\rho_{2,2}$					0.1919	(0.0137)	0.1587	(0.0767)
$\nu_*$	1.5227	(0.0161)	1.5359	(0.0162)	0.6993	(0.0123)	0.6882	(0.0124)
$J$ -test	9.8073		9.8073		9.8071		9.8062	
d.o.f.	(16)		(10)		(12)		(6)	
$p$ -value	0.8765		0.4576		0.6328		0.1331	

Note: Standard errors are in parentheses. ‘ $J$ -test’ is the GMM test of overidentifying restrictions.

Table 5: Estimated Asymmetric Effects for Two-Factor WSV Models

Para.	WSV	WSV-F	WSV-L	Asym. WSV
$E(\omega_{11,t})$	0.002332	0.002338	0.002718	0.002776
$E(\omega_{21,t})$	0.000182	0.000185	0.000208	0.000218
$E(\omega_{22,t})$	0.000434	0.000426	0.000815	0.000763
Feedback Effect				
$E(\text{tr}(M_1\Omega_t))$	0	0.000138	0	0.002389
$E(\text{tr}(M_2\Omega_t))$	0	0.000205	0	0.000653
Leverage Effects from $u_{1t}$				
$E(d\omega_{11,t}du_{1t})$	0	0	$-0.845 \times 10^{-5}$	$-3.9434 \times 10^{-5}$
$E(d\omega_{21,t}du_{1t})$	0	0	$-2.712 \times 10^{-5}$	$-0.614 \times 10^{-5}$
$E(d\omega_{22,t}du_{1t})$	0	0	$-0.344 \times 10^{-5}$	$-0.025 \times 10^{-5}$
Leverage Effects from $u_{2t}$				
$E(d\omega_{11,t}du_{2t})$	0	0	$-0.008 \times 10^{-5}$	$-0.557 \times 10^{-5}$
$E(d\omega_{21,t}du_{2t})$	0	0	$-0.537 \times 10^{-5}$	$-0.556 \times 10^{-5}$
$E(d\omega_{22,t}du_{2t})$	0	0	$-0.944 \times 10^{-5}$	$0.029 \times 10^{-5}$

Note: We omitted  $dt$  for leverage effects.