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## Illiquidity and derivative valuation

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# Illiquidity and Derivative Valuation

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# Illiquidity and Derivative Valuation

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## Abstract

In illiquid markets, option traders may have an incentive to increase their portfolio value by using their impact on the dynamics of the underlying. We provide a mathematical framework within which to value derivatives under market impact in a multi-player framework by introducing strategic interactions into the model of Almgren and Chriss (2001). Specifically, we consider a financial market model with several strategically interacting players that hold European contingent claims and whose trading decisions have an impact on the price evolution of the underlying. We establish existence and uniqueness of equilibrium results for risk neutral and CARA investors and show that the equilibrium dynamics can be characterized in terms of a coupled system of possibly non-linear PDEs. For the linear cost function used in Almgren and Chriss (2001), we obtain a (semi) closed form solution. Analyzing this solution, we show how market manipulation can be reduced.

**AMS classification:** 91B28, 91B70, 60K10

**JEL classification:** C73, G12, G13

**Keywords:** Stochastic differential games, illiquidity, market impact, derivative valuation.

## 1 Introduction

Standard financial market models assume that asset prices follow an exogenous stochastic process and that all transactions can be settled at the prevailing price without any impact on market dynamics. The assumption that all trades can be carried out at exogenously given prices is

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appropriate for small investors that trade only a negligible proportion of the overall daily trading volume; it is not appropriate for institutional investors trading large blocks of shares over a short time span. Trading large amounts of shares is likely to move stock prices in an unfavorable direction and often carries significant trading costs. This is a particular challenge for traders that need to liquidate or acquire large portfolios. In derivative markets the situation is more ambiguous. A trader that is endowed with a large number of options may have an incentive to utilize her impact on the price dynamics of the underlying in order to move the option value in a favorable direction<sup>1</sup>. Pirrong (2001) write that “a trader with a large long position in a cash-settled contract can drive up its settlement value by buying excessive quantities [of the underlying]”. Kumar and Seppi (1992) call such trading behavior “punching the close”. This paper addresses the problem of derivative valuation in financial markets with strategically interacting investors that have an incentive “to punch the close.” We model the interaction between the investors as a stochastic differential game and establish existence and uniqueness of Markov equilibria for risk neutral and CARA investors. This allows us to discuss some ideas how manipulation in the sense of punching the close can possibly be avoided.

Our work builds on previous research in at least three different fields. The first is the mathematical modeling of illiquid financial markets. The last few years the role of liquidity as a source of financial risk has been extensively investigated in both the mathematical finance and financial economics literature. Much of the literature focusses on either optimal hedging and portfolio liquidation strategies for a single large investor under market impact (Çetin, Jarrow, and Protter (2004), Alfonsi, Fruth, and Schied (2010), Rogers and Singh (2008)), predatory trading (Brunnermeier and Pedersen (2005), Carlin, Lobo, and Viswanathan (2007), Schied and Schöneborn (2007)) or the role of derivative securities including the problem of market manipulation using options (Jarrow (1994), Kumar and Seppi (1992)). It has been shown by Jarrow (1994), for instance, that by introducing derivatives into an otherwise complete and arbitrage-free market, certain manipulation strategies for a large trader may appear, such as market corners and front runs. Schönbucher and Wilmott (2000) discuss an illiquid market model where a large trader can influence the stock price with vanishing costs and risk. They argue that the risk of manipulation on the part of the large trader makes the small traders unwilling to trade derivatives any more. In particular, they predict that the option market breaks down. Our analysis indicates that markets do not necessarily break down when stock price manipulation is costly as it is in our model. Kraft and Kühn (2009) analyze the behaviour of an investor in a Black Scholes type market, where trading has a linear permanent impact on the stock’s drift. They construct the hedging strategy and the indifference price of a European payoff for a CARA investor, and show that the optimal strategy is a combination of hedging and manipulation. In order to exploit her market impact, the investor over- or underhedges the option, depending on her endowment and the sign of the impact term.

The second line of research our paper is connected to is the strategic interaction between large investors and its implications for market microstructure, as discussed in Kyle (1985), Foster

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<sup>1</sup>Gallmeyer and Seppi (2000) provide some evidence that in illiquid markets option traders are in fact able to increase a derivative’s value by moving the price of the underlying.

and Viswanathan (1996), Back, Cao, and Willard (2000), and Chau and Vayanos (2008), for instance. Brunnermeier and Pedersen (2005), Carlin, Lobo, and Viswanathan (2007) and Schied and Schöneborn (2007) consider predatory trading, where liquidity providers try to benefit from the liquidity demand that comes from some “large” investor. Vanden (2005) considers a pricing game in continuous time where the option issuer controls the volatility of the underlying but does not incur liquidity or spread crossing costs. He derives a Nash equilibrium in the two player, risk neutral case and shows that “seemingly harmless derivatives, such as ordinary bull spreads, offer incentives for manipulation that are identical to those offered by digital options” (p. 1892, l. 36). Closest to our setup is the paper by Gallmeyer and Seppi (2000). They consider a binomial model with three periods and finitely many risk neutral agents holding call options on an illiquid underlying. Assuming a linear permanent price impact and linear transaction costs, and assuming that all agents are initially endowed with the same derivative they prove the existence of a Nash equilibrium trading strategy and indicate how market manipulation can be reduced.

A third line of research we build on is market manipulation. Different notions of market manipulations have been discussed in the literature including short squeezes, the use of private information or false rumours, cf. Kyle (1985), Back (1992), Jarrow (1994), Allen and Gale (1992), Pirrong (2001), Dutt and Harris (2005), Kyle and Viswanathan (2008). However, it seems to us that no generally accepted definition of “market manipulation” has yet been established. Kyle and Viswanathan (2008, p. 1) classify a trading strategy as “illegal price manipulation” if it “undermines economic efficiency *both* by making prices less accurate as signals for efficient resource allocation *and* by making markets less liquid for risk transfer”. Our notion of manipulation refers to the practice of “punching the close”, i.e. moving the stock price in order to increase the payoff of a given option. Kyle and Viswanathan (2008) call strategies such as punching the close “benign”, to emphasize that they are not necessarily illegal. Punching the close nonetheless reduces the accuracy of prices as signals about a stock’s fundamental value; we thus argue that it fulfills at least the first part of Kyle and Viswanathan’s definition of illegal manipulation.

Most of the aforementioned articles on manipulation are set up in discrete time. We suggest a general mathematical framework in continuous time within which to value derivative securities in illiquid markets under strategic interactions. Specifically, we consider a pricing game between a finite number of large investors (“players”) holding European claims written on an illiquid stock. Their goal is to maximize expected utility at maturity from trading the stock where their portfolio value at maturity depends on the trading strategies of all the other players through their impact on the dynamics of the underlying. Following Almgren and Chriss (2001) we assume that the players have a permanent impact on stock prices and that all trades are settled at the prevailing market price plus a liquidity premium. The liquidity premium can be viewed as an instantaneous price impact that affects transaction prices but not the value of the players’ inventory. This form of market impact modeling is analytically more tractable than that of Obizhaeva and Wang (2005) which also allows for temporary price impacts and resilience effects. It has also been adopted by, e.g. Carlin, Lobo, and Viswanathan (2007) and Schied and Schöneborn (2007) and some practitioners from the financial industry, as pointed out by Schied and Schöneborn (2008).

Our framework is flexible enough to allow for rather general liquidity costs including the linear

cost function of Almgren and Chriss (2001) and some form of spread crossing costs, cf. Example 2.4. We show that when the market participants are risk neutral or have CARA utility functions the pricing game has a unique Nash equilibrium in the class of absolutely continuous Markovian trading strategies; existence results for more general utility functions are given for the single player case. We solve the problem of equilibrium pricing using techniques from the theory of stochastic optimal control and stochastic differential games. Assuming that players' action sets are given by a class of absolutely continuous Markovian trading strategies we show that the family of the players' value functions can be characterized as the solution to a coupled system of non-linear PDEs. Coupled systems of non-linear PDEs arise naturally in differential stochastic games. Since general existence and uniqueness of solution results for systems of non-linear PDEs on unbounded state spaces are unavailable much of the literature on stochastic differential games is confined to bounded state spaces; see. e.g., the seminal paper of Friedman (1972). Without a priori estimates it is usually hard to prove the existence of a global solution to a non-linear PDE system. We prove an a priori estimate Nash equilibria. More precisely we prove that under rather mild conditions any equilibrium trading strategy is uniformly bounded. This allows us to prove that the PDE system that describes the equilibrium dynamics has a unique classical solution. The equilibrium problem can be solved in closed form for a specific market environment, namely the linear cost structure used in Almgren and Chriss (2001) and risk neutral agents.

It is important to know which measures may reduce market manipulation. For instance, Dutt and Harris (2005) propose position limits; Pirrong (2001) suggests efficient contract designs. We use the explicit solution for risk neutral investors to show when "punching the close" is not beneficial. For instance, no manipulation occurs in zero sum games, i.e., in a game between an option writer and an option issuer. In our model manipulation decreases with the number of informed liquidity providers and with the number of competitors, if the product is split between them. Furthermore, we find that the bid ask spread is important determinant of market manipulation. It turns out that the higher the spread, the less beneficial market manipulation: high spread crossing costs make trading more costly and hence discourage frequent re-balancing of portfolio positions.

This paper is organized as follows: We present the market model in section 2. In section 3, we formulate the optimization problem, derive a priori estimates for Nash equilibria and prove the existence of a solution for one player with general utility function. We solve the multi-player case in section 4 for risk neutral and CARA agents. We use these solutions in section 5 to show how market manipulation can be reduced. Section 6 concludes.

## 2 The Model

We adopt the market impact model of Schied and Schöneborn (2007) with a finite set of *agents*, or *players*, trading a single stock whose price process depends on the agents' trading strategies. Following Almgren and Chriss (2001) we shall assume that the players have a permanent impact on asset prices and that all trades are settled at prevailing market prices plus a liquidity premium which depends on the *change* in the players' portfolios. In order to be able to capture changes in portfolio positions in an analytically tractable way, we follow Almgren and Chriss (2001) and

Schied and Schöneborn (2007) and restrict ourselves to absolutely continuous Markov trading strategies. A Markov trading strategy for player  $j \in J \triangleq \{1, \dots, N\}$  is a family of mappings  $X_t^j : \mathbb{R} \rightarrow \mathbb{R}$  for any time  $t \in [0, T]$  where  $X_t^j(p)$  denotes the number of stock shares the player holds at time  $t$ , given the stock current price is  $p$ . We assume that the strategy  $X^j$  belongs to the class

$$\mathfrak{X} \triangleq \{X : [0, T] \times \mathbb{R} \mapsto \mathbb{R} \mid X \text{ absolutely continuous adapted and } X_0 = 0\},$$

write  $dX_t^j = \dot{X}_t^j dt$  and call  $\dot{X}^j$  the *trading speed* of player  $j$ .

**Remark 2.1.** *We appreciate that it may be considered undesirable to allow for absolutely continuous strategies only. Jumps are naturally captured in discrete time models, such as Kumar and Seppi (1992) or Gallmeyer and Seppi (2000). However, strategies with absolutely continuous and jump parts in continuous time would call for methods of singular or impulse control, viscosity solutions, and (systems of) quasi variational inequalities. This is beyond the scope of this article; we refer the interested reader to Ly Vath, Mnif, and Pham (2007) and Guo and Pham (2005) for a discussion of investment problems with more general trading strategies.  $\diamond$*

## 2.1 Price dynamics and the liquidity premium

Our focus is on valuation schemes for derivatives with short maturities under strategic market interactions. For short trading periods it is appropriate to model the *fundamental stock price*, i.e., the value of the stock in the absence of any market impact, as a Brownian Motion with volatility ( $\sigma B_t$ ). Market impact is accounted for by assuming that the investors' accumulated stock holdings  $\sum_{i=1}^N X^i$  have a linear impact on the stock process ( $P_t$ ) so that

$$P_t = P_0 + \sigma B_t + \lambda \sum_{i=1}^N X_t^i \tag{2.1}$$

with a permanent impact parameter  $\lambda > 0$ . The linear permanent impact is consistent with the work of Huberman and Stanzl (2004) who argue that linearity of the permanent price impact is important to exclude quasi-arbitrage.

A trade at time  $t \in [0, T]$  is settled at a *transaction price*  $\tilde{P}_t$  that includes an additional instantaneous price impact, or *liquidity premium*. Specifically,

$$\tilde{P}_t = P_t + g \left( \sum_{i=1}^N \dot{X}_t^i \right) \tag{2.2}$$

with a *cost function*  $g$  that depends on the instantaneous change  $\sum_{i=1}^N \dot{X}^i$  in the agents' position in a possibly non-linear manner. The liquidity premium accounts for limited available liquidity, transaction costs, fees or spread crossing costs, cf. Example 2.4. Spread-crossing costs are of particular importance and have not been considered in the previous literature on market impact.

**Remark 2.2.** *In our model the liquidity costs are the same for all traders and depend only on the aggregate demand throughout the entire set of agents. This captures situations where the agents trade through a market maker or clearing house that reduces the trading costs by collecting all orders and matching incoming demand and supply prior to settling the outstanding balance  $\sum_{i=1}^N \dot{X}_t^i$  at market prices.*  $\diamond$

We assume with no loss of generality that  $g$  is normalized,  $g(0) = 0$ , and that  $g$  is smooth. The following additional mild assumptions on  $g$  will guarantee that the equilibrium pricing problem has a solution for risk neutral and CARA investors.

**Assumption 2.3.**    • *The derivative  $g'$  is bounded away from zero, that is  $g' > \varepsilon > 0$ .*

• *The mapping  $z \mapsto g(z) + zg'(z)$  is strictly increasing.*

The first assumption is a technical condition needed in the proof of Proposition 3.2. It appears not too restrictive for a cost function. Since the liquidity costs associated with a net change in the overall position  $z$  is given by  $zg(z)$ , the second assumption states that the agents face increasing marginal costs of trading. Our assumptions on  $g$  are satisfied for the following important examples:

**Example 2.4.** *Among the cost functions which satisfy Assumption 2.3 are the linear cost function  $g(z) = \kappa z$  with  $\kappa > 0$ , used in Almgren and Chriss (2001) and cost functions of the form*

$$g(z) = \kappa z + s \frac{2}{\pi} \arctan(Cz) \quad \text{with } s, C > 0.$$

*The former is the cost function associated with a block-shaped limit order book. The latter can be viewed as a smooth approximation of the map  $z \mapsto \kappa z + s \cdot \text{sign}(z)$  which is the cost function associated with a block-shaped limit order book and bid ask spread  $s > 0$ .*

## 2.2 Preferences and endowments

Each agent is initially endowed with a contingent claim  $H^j = H^j(P_T)$ , whose payoff depends on the stock price  $P_T$  at maturity. Our focus is on optimal trading strategies in the stock, given an initial endowment. As in Gallmeyer and Seppi (2000) and Kraft and Kühn (2009), we assume that the agents do not trade the option in  $[0, T]$ . A consistent model for trading an illiquid option with illiquid underlying in a multiplayer framework in continuous time is not available, to the best of our knowledge. Our work might be considered a first step in this direction. Although it is not always necessary, we assume that the functions  $H^j$  are smooth and bounded with bounded derivatives  $H_p^j$ .

**Remark 2.5.** *We only consider options with cash settlement. This assumption is key. While cash settlement is susceptible to market manipulation, we show in Corollary 5.4 below that when deals are settled physically, i.e., when the option issuer delivers the underlying, market manipulation is not beneficial: Any price increase is outweighed by the liquidity costs of subsequent liquidation. We notice that this only applies to “punching the close”. There are other types of market manipulation, such as corners and short squeezes, which might be beneficial when deals are settled physically, but which are not captured by our model, cf. Jarrow (1994) or Kyle and Viswanathan (2008).*  $\diamond$



If the agent  $j$  follows an absolutely continuous trading strategy  $X^j$ , then the value of her portfolio at maturity is given by the option payoff  $H^j(P_T)$  less the costs of trading  $-\int_0^T \dot{X}_t^j \tilde{P}_t dt$  plus the liquidation value  $LV^j(X_T^j)$  of the stock position  $X_T^j$  at maturity.

**Remark 2.6.** *Defining and computing the liquidation value of an asset position in an illiquid market is currently an active field of research, cf. Almgren and Chriss (2001), Obizhaeva and Wang (2005), Alfonsi, Fruth, and Schied (2010) and Schied and Schöneborn (2008), to mention only a few. For the special case of a single risk neutral investor the expected liquidation value under infinitely slow liquidation equals the expected trading costs in the absence of market impact so the optimization problem is given by*

$$\sup_{X^j \in \mathfrak{X}} \mathbb{E} \left[ - \int_0^T \dot{X}_t^j g(\dot{X}_t^j) dt + H^j(P_T) \right].$$

*In particular, in optimizing her expected portfolio value the investor can focus entirely on the tradeoff between increased trading costs (market impact) and increased option payoffs.  $\diamond$*

Rigorously defining a form of liquidation value in a game-theoretic setting is challenging, even under risk neutrality. One reason is that all the agents optimize their trading rules against their beliefs about the other players' assessments of all the portfolio values at maturity. In order to simplify the analysis, and in order to focus on the interplay between the gain from manipulating option payoffs and the increase in trading costs that accompanies it, we shall assume that all agents value their portfolios according to its acquisition costs in the absence of market impact. Of course, this is just a first benchmark, but it nonetheless yields some insight into the structure of optimal trading under strategic interactions. Without Assumption 2.7, the equilibrium analysis becomes rather intractable as we illustrate in Appendix C.

**Assumption 2.7.** *All agents optimize their utility assuming that for all  $j = 1, \dots, N$*

$$LV^j(X_T^j) = \int_0^T \dot{X}_t^j P_t dt. \quad (2.3)$$

Under the preceding assumption the optimization problems of the individual market participants reduce to

$$\sup_{X^j \in \mathfrak{X}} \mathbb{E} \left[ u^j \left( - \int_0^T \dot{X}_t^j g \left( \sum_{i=1}^N \dot{X}_t^i \right) dt + H^j(P_T) \right) \right]. \quad (2.4)$$

This optimization problem reflects each investor's tradeoff between high liquidity costs and an increased option payoff.<sup>2</sup> If there is no market impact, agents do not trade, as we shall see in Remark 3.1.

**Definition 2.8.** We say that a vector of strategies  $(\dot{X}^1, \dots, \dot{X}^N)$  is a *Nash equilibrium* if for each agent  $j \in J$  her trading strategy  $\dot{X}^j$  is a best response against the behavior of all the other players, i.e., if  $\dot{X}^j$  solves (2.4), given the other players' aggregate trading  $\dot{X}^{-j} \triangleq \sum_{i \neq j} \dot{X}^i$ .

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<sup>2</sup>The only purpose of trading is an increased option payoff and not, for instance, hedging. For a study on the interplay of hedging and manipulation we refer the reader to Kraft and Kühn (2009).

In the following section we derive a priori estimates for equilibrium trading strategies and use standard results from the theory of stochastic optimization to show that Nash equilibria can be characterized in terms of a coupled system of partial differential equations (PDEs). For the special case of risk neutral and CARA investors we show that the system of PDEs has a solution so that a unique (in a certain class) equilibrium exists.

### 3 Equilibrium Dynamics and A-Priori Estimates

In this section we formulate the optimization problem (2.4) as a stochastic control problem, derive the associated Hamilton-Jacobi-Bellman-equations, HJB for short, and transform it into a system of coupled PDEs. To this end, we choose the stock price  $P$  and the trading costs  $R^j$  of the agent  $j \in J$  as state variables. They evolve according to:

$$\begin{cases} dP_t = \sigma dB_t + \lambda \sum_{i=1}^N \dot{X}_t^i dt, & P_0 = p_0 \\ dR_t^j = \dot{X}_t^j g \left( \sum_{i=1}^N \dot{X}_t^i \right) dt, & R_0^j = 0. \end{cases}$$

For a given time  $t < T$ , spot price  $p$  and a vector of trading costs  $r = (r^1, \dots, r^N)$  the *value function* of the player  $j$ , defined by

$$V^j(t, p, r) \triangleq \sup_{X^j \in \mathfrak{X}} \mathbb{E}_t \left[ u^j \left( -r^j - \int_t^T \dot{X}_s^j g \left( \sum_{i=1}^N \dot{X}_s^i \right) ds + H^j(P_T) \right) \mid P_t = p \right], \quad (3.1)$$

denotes the maximal expected portfolio value at maturity that the player can achieve by trading the underlying. The associated HJB-equation is (cf. Fleming and Soner (1993)):

$$\begin{cases} 0 = v_t^j + \frac{1}{2} \sigma^2 v_{pp}^j + \sup_{c^j \in \mathbb{R}} \left[ \lambda (c^j + \dot{X}^{-j}) v_p^j + g(c^j + \dot{X}^{-j}) \left( c^j v_{r^j}^j + \sum_{i \neq j} \dot{X}^i v_{r^i}^j \right) \right] \\ v^j(T, p, r) = u^j(-r^j + H^j(p)) \end{cases} \quad (3.2)$$

The HJB-equation is formulated in terms of the *candidate* value functions  $v^1, \dots, v^N$  instead of the actual value functions  $V^1, \dots, V^N$ . We first need to show existence and uniqueness of a smooth solution to (3.2) before we can identify  $v^i$  with  $V^i$ . Given the aggregate trading strategy  $\dot{X}^{-j}$  of all the other agents, a candidate for the maximizer  $c^j = \dot{X}^j$  in (3.2) should satisfy

$$0 = \lambda v_p^j + g(c^j + \dot{X}^{-j}) v_{r^j}^j + g'(c^j + \dot{X}^{-j}) \left( c^j v_{r^j}^j + \sum_{i \neq j} \dot{X}^i v_{r^i}^j \right). \quad (3.3)$$

Under some mild technical assumption which will all be satisfied for the case of risk neutral and CARA investors studied in Section 4 we are now going to turn the individual HJB equations into a coupled system of non-linear PDE. In terms of this PDE system we shall then characterize a class of equilibrium trading strategies.

### 3.1 The equilibrium PDE system

Let us assume that the matrix  $(v_{r^j}^i(t, p, r))_{i,j \in J}$  is invertible for any fixed argument  $(t, p, r)$ . We will show in Section 4 that this is true for risk neutral and CARA investors. In these cases,  $(v_{r^j}^i(t, p, r))_{i,j \in J}$  is a diagonal matrix. If  $(v_{r^j}^i(t, p, r))_{i,j \in J}$  is invertible, there exist  $\delta^1, \dots, \delta^N \in \mathbb{R}$  such that  $\sum_{i=1}^N \delta^i v_{r^j}^i = 1$  for each  $j \in J$ . Thus, multiplying equation (3.3) with  $\delta^j$  and summing up the equations for  $j \in J$  yields the following characterization of the aggregate trading speed  $\sum_{i=1}^N \dot{X}_t^i$ :

$$\begin{aligned} 0 &= \lambda \sum_{j=1}^N \delta^j v_p^j + g \left( \sum_{i=1}^N \dot{X}_t^i \right) \sum_{j=1}^N \delta^j v_{r^j}^j + g' \left( \sum_{i=1}^N \dot{X}_t^i \right) \sum_{j=1}^N \delta^j \sum_{i=1}^N \dot{X}_t^i v_{r^i}^j \\ &= \lambda \sum_{j=1}^N \delta^j v_p^j + g \left( \sum_{i=1}^N \dot{X}_t^i \right) \sum_{j=1}^N \delta^j v_{r^j}^j + g' \left( \sum_{i=1}^N \dot{X}_t^i \right) \sum_{i=1}^N \dot{X}_t^i. \end{aligned} \quad (3.4)$$

We further assume that  $K \triangleq \sum_{j=1}^N \delta^j v_{r^j}^j \geq 1$ . Again, this is true for risk neutral and CARA investors, where  $\delta^j v_{r^j}^j = 1$ . Due to Assumption 2.3,  $z \mapsto K \cdot g(z) + z \cdot g'(z)$  is strictly increasing. Hence, equation (3.4) admits a unique solution

$$\dot{X}^* \triangleq \sum_{i=1}^N \dot{X}_t^i$$

which depends on  $\sum_{j=1}^N \delta^j v_p^j$  and  $\sum_{j=1}^N \delta^j v_{r^j}^j$ . Plugging the solution  $\dot{X}^*$  back into (3.3) allows to compute the expression  $\sum_{i=1}^N \dot{X}_t^i v_{r^i}^j$  in terms of  $\dot{X}^*$  as

$$\sum_{i=1}^N \dot{X}_t^i v_{r^i}^j = -\frac{1}{g'(\dot{X}^*)} \left[ \lambda v_p^j + g(\dot{X}^*) v_{r^j}^j \right]. \quad (3.5)$$

This expression is well defined since  $g' > 0$ . To conclude, we have turned the family of individual HJB-equations (3.2) into the following system of coupled PDEs for  $j = 1, \dots, N$ :

$$\begin{cases} 0 = v_t^j + \frac{1}{2} \sigma^2 v_{pp}^j + \lambda \left( \dot{X}^* - \frac{g(\dot{X}^*)}{g'(\dot{X}^*)} \right) v_p^j - \frac{g(\dot{X}^*)^2}{g'(\dot{X}^*)} v_{r^j}^j \\ v^j(T, p, r) = u^j(-r^j + H^j(p)) \end{cases} \quad (3.6)$$

where the coupling stems from  $\dot{X}^*$ , which is uniquely defined via (3.4). It will become clear in Section 4 that the state variable  $r$  is redundant for risk neutral and CARA investors. In these cases, the value function is translation invariant (after a suitable transformation), which allows to drop the variable  $r$ .

**Remark 3.1.** *In a market without price impact ( $\lambda = 0$ ), manipulation is not beneficial. To see this, first note that  $\dot{X}^* = \sum_{i=1}^N \dot{X}_t^i = 0$  is the unique solution to (3.4). From (3.5), we get  $\sum_{i=1}^N \dot{X}_t^i v_{r^i}^j = 0$ . On the other hand, we have from (3.1) and (3.2) that  $v^j(t, p, r) = V^j(t, p, r) = \mathbb{E}_t u^j(-r^j + H^j(P_T))$ , and in particular  $v_{r^j}^j \neq 0$  and  $v_{r^i}^j = 0$  for  $i \neq j$ . This implies that  $0 = \sum_{i=1}^N \dot{X}_t^i v_{r^i}^j = \dot{X}_t^j v_{r^j}^j$  and finally  $\dot{X}_t^j = 0$  for each  $j$ .  $\diamond$*

### 3.2 A priori equilibrium estimate

Solving the system (3.6) is delicate, to say the least. The problem is the non-linearity coming from the expressions  $\frac{g(\dot{X}^*)}{g'(\dot{X}^*)}$  and  $\frac{g(\dot{X}^*)^2}{g'(\dot{X}^*)}$  along with the implicit dependence of  $\dot{X}^*$  on the derivatives  $v_p^i$  and  $v_{r,i}^i$ ,  $i \in J$ . In Section 4, we show that a unique classical solution to the system (3.6) exists for risk neutral and CARA investors. The proof uses the following a priori estimates for the optimal trading strategies. It states that, if an equilibrium exists, then each player's trading speed is bounded. In particular, there is no equilibrium with unbounded strategies. As a result, the agents' utilities from trading and the value function associated with their respective HJB equations along with (as we shall see) their derivatives are bounded.

**Proposition 3.2.** *Let  $(\dot{X}^1, \dots, \dot{X}^N)$  be a Nash equilibrium for problem (2.4). Then each strategy  $\dot{X}^j$  satisfies*

$$\left| \dot{X}_t^j \right| \leq N \frac{\lambda}{\varepsilon} \left( \max_i \|H_p^i\|_\infty + 1 \right),$$

where  $\varepsilon$  is taken from Assumption 2.3.

*Proof.* Let  $j \in J$ ,  $h \triangleq \max_i \|H_p^i\|_\infty$  and  $A \triangleq \{(t, \omega) : \sum_{i=1}^N \dot{X}_t^i(t, \omega) \geq 0\}$  be the set where the aggregate trading speed is nonnegative. Let us fix the sum of the competitors' strategies  $\dot{X}^{-j}$ . On the set  $A$  the best response  $\dot{X}_t^j$  is bounded from above by  $K \triangleq \frac{\lambda}{\varepsilon}(h+1)$ . Otherwise the truncated strategy  $\dot{Y}_t^j \triangleq \dot{X}_t^j \wedge K \mathbf{1}_A + \dot{X}_t^j \mathbf{1}_{A^c}$  would outperform  $\dot{X}_t^j$ . To see this, let us compare the payoffs associated with  $\dot{X}^j$  and  $\dot{Y}^j$ . We denote by  $P_T^{Y^j}$  and  $P_T^{\dot{X}^j}$  the stock price under the strategies  $\dot{Y}^j$  and  $\dot{X}^j$ , respectively. The payoff associated with  $\dot{Y}^j$  minus the payoff associated with  $\dot{X}^j$  can be estimated from below as

$$\begin{aligned} & - \int_0^T \dot{Y}_t^j g(\dot{Y}_t^j + \dot{X}_t^{-j}) dt + H^j(P_T^{Y^j}) \\ & + \int_0^T \dot{X}_t^j g(\dot{X}_t^j + \dot{X}_t^{-j}) dt - H^j(P_T^{\dot{X}^j}) \\ & \geq \int_0^T \dot{Y}_t^j \left( g(\dot{X}_t^j + \dot{X}_t^{-j}) - g(\dot{Y}_t^j + \dot{X}_t^{-j}) \right) dt \\ & + \int_0^T (\dot{X}_t^j - \dot{Y}_t^j) g(\dot{X}_t^j + \dot{X}_t^{-j}) dt - \lambda(X_T^j - Y_T^j) \|H_p\|_\infty. \end{aligned}$$

Note that  $\dot{X}_t^j + \dot{X}_t^{-j} \geq 0$  on  $A$  and thus  $g(\dot{X}_t^j + \dot{X}_t^{-j}) \geq 0$  due to Assumption 2.3. Furthermore,  $g(\dot{X}_t^j + \dot{X}_t^{-j}) - g(\dot{Y}_t^j + \dot{X}_t^{-j}) \geq \varepsilon(\dot{X}_t^j - \dot{Y}_t^j)$ , again by Assumption 2.3. The difference in the payoffs is therefore larger than

$$\begin{aligned} & \int_0^T \dot{Y}_t^j \varepsilon (\dot{X}_t^j - \dot{Y}_t^j) dt - \lambda h \int_0^T (\dot{X}_t^j - \dot{Y}_t^j) dt \\ & = \int_{\dot{X}_t^j > \dot{Y}_t^j} (\varepsilon \dot{Y}_t^j - \lambda h) (\dot{X}_t^j - \dot{Y}_t^j) dt \end{aligned}$$

On the set  $\{\dot{X}_t^j > \dot{Y}_t^j\}$  we have  $\dot{Y}_t^j = K = \frac{\lambda}{\varepsilon}(h+1)$  and the above expression is strictly positive, a contradiction. This shows that  $\dot{X}_t^j$  is bounded above by  $K$  on the set  $A$  for each  $j \in J$ . Still on the set  $A$ , we get the following lower bound:

$$\dot{X}_t^j = \sum_{i=1}^N \dot{X}_t^i + \sum_{i \neq j} -\dot{X}_t^i \geq 0 - (N-1)K. \quad (3.7)$$

A symmetric argument on the set  $B \triangleq \{(t, \omega) : \sum_{i=1}^N \dot{X}_t^i(t, \omega) \leq 0\}$  completes the proof.  $\square$

In the one player framework we can use a standard result from the theory of stochastic control to show that (3.6) admits a unique solution. We say that a function  $\psi$  satisfies a polynomial growth condition if there are constants  $p, C > 0$  such that  $|\psi(x)| \leq C(1 + |x|^p)$ .

**Proposition 3.3.** *Let  $N = 1$ . Let  $u \in \mathcal{C}^3$  be such that  $u$  and  $u'$  satisfy a polynomial growth condition. Then the HJB-equation (3.2) admits a unique classical solution in  $\mathcal{C}^{1,2}$ , which coincides with the value function  $V$ .*

*Proof.* Due to the a priori estimates in Proposition 3.2, it is enough to choose the control  $\dot{X}$  from a compact set. Thus, we can apply Theorem IV.4.3 in Fleming and Soner (1993), which yields that (3.2) admits a unique solution in  $\mathcal{C}^{1,2}$ , which is of polynomial growth. It remains to apply the Verification Theorem IV.3.1 from Fleming and Soner (1993) to see that this solution coincides with the agent's value function  $V$ .  $\square$

Under suitable smoothness conditions on the cost function the PDE system (3.6) always has a bounded solution with bounded derivatives for small time steps; see Taylor (1997), Proposition 15.1.1. The challenge is to establish a global solution, i.e., a solution on the whole time interval  $[0, T]$ . The main difficulty stems from the dependence of the players' optimal trading strategies on the derivative of the value function with respect to the trading costs. If the derivatives  $v_{r,i}^j$  are constant, then our a priori estimate together with the boundedness assumptions on the payoffs  $H^j$  and the characterization (3.5) of the equilibrium aggregate trading speed guarantees that the value function are bounded with bounded derivatives and hence that an equilibrium exists. This argument will be made more precise in the following section.

## 4 Solution for Risk Neutral and CARA Investors

In this section we establish existence and uniqueness of equilibrium results for risk neutral and CARA investors. For risk neutral investors and linear cost functions the equilibrium strategies can be given in closed form; if spread crossing costs are involved a closed form solution is not available and we report numerical results instead.

## 4.1 Risk Neutral Agents

Let us assume that all players are risk neutral, i.e.  $u^j(z) = z$ . In a first step we prove existence of a unique solution to the system (3.6) for general cost functions  $g$ . Subsequently we construct an explicit solution to (3.6) for the linear cost structure used in Almgren and Chriss (2001).

### 4.1.1 General Cost Structure

Let  $g$  be a general cost function which satisfies Assumption 2.3. In the risk neutral case the value function of player  $j$  turns into

$$V^j(t, p, r) = -r^j + \sup_{X^j \in \mathfrak{X}} \mathbb{E}_t \left[ - \int_t^T \dot{X}_s^j g \left( \sum_{i=1}^N \dot{X}_s^i \right) ds + H^j(P_T) | P_t = p \right].$$

In particular,  $V_{r^i}^j = -\mathbb{1}_{i=j}$  and the optimal strategies do not depend on the trading costs. In other words, the state variable  $r$  is redundant and we omit it in this section. We write

$$V^j(t, p) \triangleq V^j(t, p, 0),$$

and call this function *value function*, if there is no danger of confusion. The HJB-equation (3.2) turns into

$$0 = v_t^j + \frac{1}{2} \sigma^2 v_{pp}^j + \sup_{c^j \in \mathbb{R}} \left[ \lambda \left( c^j + \dot{X}^{-j} \right) v_p^j - c^j g \left( c^j + \dot{X}^{-j} \right) \right] \quad (4.1)$$

where we have used  $V_{r^i}^j = -\mathbb{1}_{i=j}$ . The optimal trading speed from (3.5) is given by

$$c^j = \dot{X}^j = - \frac{1}{g'(\dot{X}^*)} \left[ -\lambda v_p^j + g(\dot{X}^*) \right] \quad (4.2)$$

where the aggregate trading speed  $\dot{X}^* = \sum_{i=1}^N \dot{X}^i$  is the unique solution to

$$0 = \lambda \sum_{i=1}^N v_p^i - N g \left( \sum_{i=1}^N \dot{X}_t^i \right) - \left( \sum_{i=1}^N \dot{X}_t^i \right) g' \left( \sum_{i=1}^N \dot{X}_t^i \right). \quad (4.3)$$

This is (3.4) with  $\delta^j = -1$ . The system of PDEs (3.6) therefore takes the form

$$0 = v_t^j + \frac{1}{2} \sigma^2 v_{pp}^j + \lambda \left( \dot{X}^* - \frac{g(\dot{X}^*)}{g'(\dot{X}^*)} \right) v_p^j + \frac{g(\dot{X}^*)^2}{g'(\dot{X}^*)} \quad (4.4)$$

with terminal condition  $v^j(T, p) = H^j(p)$  for  $j \in J$ . Systems of the form (4.4) appear naturally in the theory of differential games, but we did not find a reference which covers this particular case. Theorem 1 of Friedman (1972) for instance is valid only on a bounded state space. However, in view of our a-priori estimates of Proposition 3.2 the proof follows from a general existence result for nonlinear systems of PDEs stated in Appendix A. The following theorem shows that a unique solution exists if  $H^j \in \mathcal{C}_b^2$ , i.e.  $H^j$  and its derivatives up to order 2 are bounded for each  $j$ .

**Theorem 4.1.** *Let  $H \in \mathcal{C}_b^2$ . Then the Cauchy problem (4.4) admits a unique classical solution in  $\mathcal{C}^{1,2}$ , which coincides with the vector of value functions.*

An alternative way of solving the system (4.4) is the following: If we sum up the  $N$  equations, we get a Cauchy problem for the aggregate value function  $v \triangleq \sum_{i=1}^N v^i$ , namely

$$0 = v_t + \frac{1}{2}\sigma^2 v_{pp} + \dot{X}^* \left[ \lambda v_p - g(\dot{X}^*) \right] \quad (4.5)$$

with terminal condition  $v(T, p) = \sum_{i=1}^N H^i(p)$ . Existence and uniqueness of a solution to this one-dimensional problem can be shown using Theorem IV.8.1 in Ladyzenskaja, Solonnikov, and Ural'ceva (1968). Once the solution is known, we can plug it back into (4.4) and get  $N$  decoupled equations. This technique is applied in the following section where we construct an explicit solution for linear cost functions.

#### 4.1.2 Linear Cost Structure

For the particular choice  $g(z) = \kappa z$  ( $\kappa > 0$ ) used in Almgren and Chriss (2001) and Schied and Schöneborn (2007), the solution to (4.4) can be given explicitly.

**Corollary 4.2.** *Let  $g(z) = \kappa z$ . Then the solution of (4.4) can be given in closed form as the solution to a nonhomogeneous heat equation.*

*Proof.* The optimal trading speed from (4.2) and the aggregate trading speed from (4.3) are

$$\dot{X}^j = \frac{\lambda}{\kappa} \left( v_p^j - \frac{1}{N+1} \sum_{i=1}^N v_p^i \right) \quad (4.6)$$

$$\dot{X}^* = \sum_{i=1}^N \dot{X}^i = \frac{\lambda}{\kappa(N+1)} \sum_{i=1}^N v_p^i = \frac{\lambda}{\kappa(N+1)} v_p. \quad (4.7)$$

Let us sum up the  $N$  equations from (4.4). This yields the following PDE for the aggregate value function  $v = \sum_{i=1}^N v^i$ :

$$0 = v_t + \frac{1}{2}\sigma^2 v_{pp} + \frac{\lambda^2 N}{\kappa(N+1)^2} v_p^2 \quad (4.8)$$

with terminal condition  $v(T, p) = \sum_{i=1}^N H^i(p)$ . This PDE is a variant of *Burgers' equation*, cf. Rosencrans (1972). It allows for an explicit solution, which we cite in Lemma 4.3. With this solution at hand, we can solve for each single investor's value function. We plug the solution  $v$  back into the equations (4.6) and (4.7) for the trading speeds, and those into the PDE (4.4). This yields

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j + \frac{\lambda^2}{\kappa(N+1)^2} v_p^2$$

with terminal condition  $v^j(T, p) = H^j(p)$ . This nonhomogeneous heat equation is solved by

$$v^j(T-t, p) = \int_{\mathbb{R}} H^j d\mathcal{N}(p, \sigma t) + \frac{\lambda^2}{\kappa(N+1)^2} \int_0^t \int_{\mathbb{R}} v_p^2(s, \cdot) d\mathcal{N}(p, \sigma(t-s))$$

where  $v$  is given in Lemma 4.3 and  $\mathcal{N}$  denotes the heat kernel.  $\square$

In the preceding proof and in Corollary 4.5 we need the solution to a variant of *Burgers' equation*. We cite it in the following Lemma.

**Lemma 4.3.** *Let  $A > 0, B \neq 0$  and  $G : \mathbb{R} \rightarrow \mathbb{R}$  be smooth and bounded. The PDE*

$$0 = 2v_t + Av_{pp} + Bv_p^2$$

*with terminal value*

$$v(T, p) = G(p)$$

*is solved by*

$$v(t, p) = \frac{A}{B} \log \left[ \int_{\mathbb{R}} \exp \left( \frac{B}{A} G(\sqrt{A}z) \right) d\mathcal{N} \left( \frac{p}{\sqrt{A}}, T - t \right) \right].$$

*Proof.* By means of a linear transformation we can reduce the problem to  $A = B = 1$ . This particular case is solved in Rosencrans (1972).  $\square$

### 4.1.3 Numerical Illustrations

In the risk neutral setting, we were able to reduce the system of PDEs from the multi-player setting to the one-dimensional PDE (4.8) for the aggregate value function. This can be interpreted as the value function of the representative agent. Such reduction to a representative agent is not always possible for more general utility functions. In the sequel we illustrate the optimal trading speed  $\dot{X}(t, p)$  and surplus of a representative agent as functions of time and spot prices for a European call option  $H(P_T) = (P_T - K)^+$  and digital option  $H(P_T) = \mathbb{1}_{\{P_T \geq K\}}$ , respectively.<sup>3</sup> By surplus, we mean the difference between the representative agent's optimal expected utility  $v(t, p)$  and the conditional expected payoff  $\mathbb{E}_t[H(P_T)|P_t = p]$  in the absence of any market impact. It represents the expected net benefit due to price manipulation.

We choose a linear cost function, strike  $K = 100$ , maturity  $T = 1$ , volatility  $\sigma = 1$  and liquidity parameters  $\lambda = \kappa = 0.01$ . We see from Figure 1 that for the case of a call option both the optimal trading speed and the surplus increases with the spot; the latter also increases with the time to maturity. Furthermore, the increase in the trading speed is maximal when the option is at the money. For digital options the trading speed is highest for at the money options close to maturity as the trader tries to push the spot above the strike. If the spot is far away from the strike, the trading speed is very small as it is unlikely that the trader can push the spot above the strike before expiry.

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<sup>3</sup>Note that the cost function in (4.9) is not smooth, and the Call and Digital options are not smooth and bounded, so Theorem 4.1 does not apply directly. There are two ways to overcome this difficulty: We could either approximate  $g$  and  $H$  by smooth and bounded functions. Or we could interpret  $v$  not as a classical, but only as a viscosity solution of (4.1), cf. Fleming and Soner (1993).



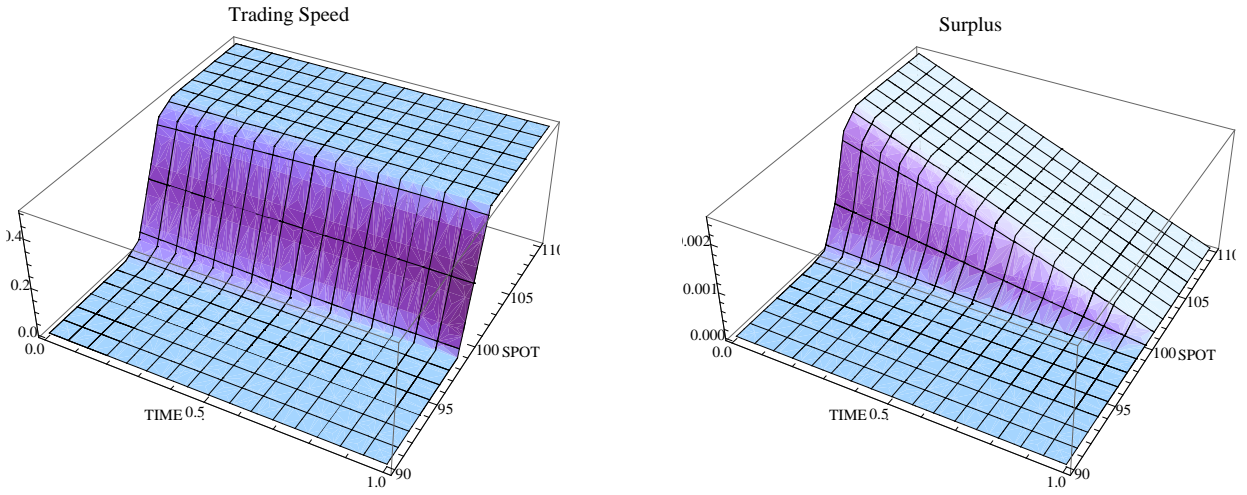


Figure 1: Trading speed and surplus for one risk neutral investor holding a European Call option.

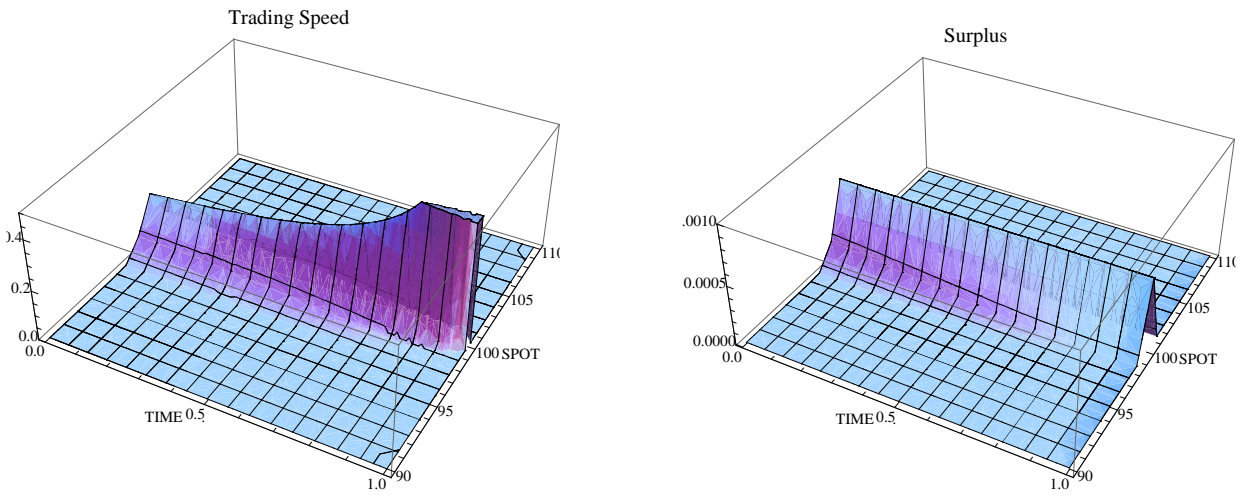


Figure 2: Trading speed and surplus for one risk neutral investor holding a Digital option.

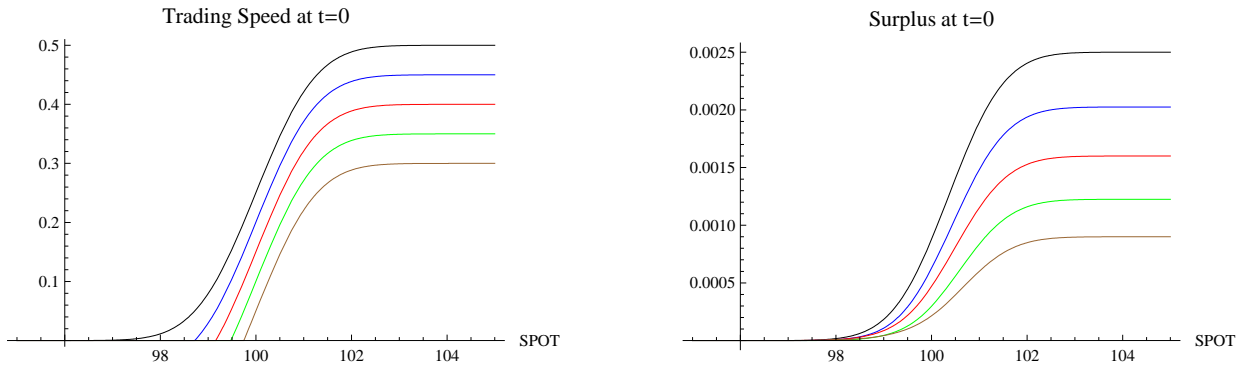


Figure 3: Trading speed and surplus for a risk neutral investor holding a European Call option for different spread sizes  $s = 0$  (black), 0.001 (blue), 0.002 (red), 0.003 (green), 0.004 (brown). The higher the spread, the smaller the trading speed and the surplus.

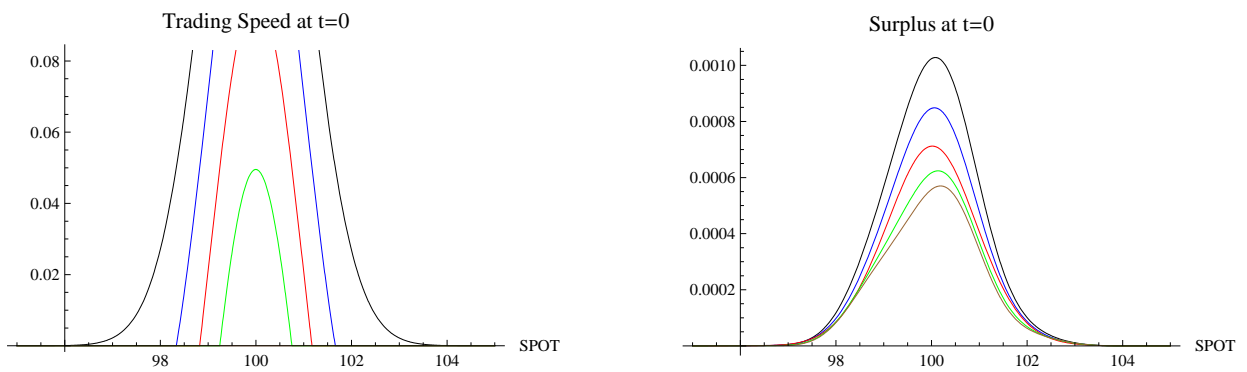


Figure 4: Trading speed and surplus for a risk neutral investor holding a Digital option for different spread sizes  $s = 0$  (black), 0.001 (blue), 0.002 (red), 0.003 (green), 0.004 (brown). The higher the spread, the smaller the trading speed and the surplus.

Figures 3 and 4 illustrate that a high spread makes manipulation unattractive. It shows the optimal trading speed and the surplus at time  $t = 0$  for the Call and Digital option in the one player framework. We used the cost function

$$g(z) = \kappa z + s \cdot \text{sign}(z) \quad \text{for different spreads } s \in \{0, 0.001, 0.002, 0.003, 0.004\} \quad (4.9)$$

with the remaining parameters as above. We see that the higher the spread, the smaller the trading speed and the surplus. This is intuitive as frequent trading, in particular, when the option is at the money, incurs high spread crossing costs. The same is true for fixed transaction costs which also discourage frequent trading.

## 4.2 Risk Averse Agents

A second class which yields explicit results is those of exponential utility functions  $u^j(z) = -\exp(-\alpha^j z)$  for  $j = 1, \dots, N$ , where  $\alpha^j > 0$  is the risk aversion coefficient. In this case the value functions satisfy

$$V^j(t, p, r) = \exp(\alpha^j r^j) \cdot V^j(t, p, 0)$$

and thus  $V_{r^i}^j = \alpha^j V^j \mathbf{1}_{i=j}$ . We suppress the state variable  $r$  and write  $V^j(t, p) \triangleq V^j(t, p, 0)$ . As above, we first show existence and uniqueness of a solution for a general cost structure. In a second step, we derive the closed form solution for the linear cost function in the single player framework.

### 4.2.1 General Cost Structure

The HJB-equation (3.2) turns into

$$0 = v_t^j + \frac{1}{2} \sigma^2 v_{pp}^j + \sup_{c^j \in \mathbb{R}} \left[ \lambda \left( c^j + \dot{X}^{-j} \right) v_p^j + c^j g \left( c^j + \dot{X}^{-j} \right) \alpha^j v^j \right] \quad (4.10)$$

with terminal condition  $v^j(T, p) = -\exp(-\alpha^j H^j(p))$ . We apply the logarithmic transformation  $\tilde{v}^j \triangleq -\frac{1}{\alpha^j} \log(-v^j)$  to turn the HJB equation into

$$0 = \tilde{v}_t^j + \frac{1}{2} \sigma^2 \tilde{v}_{pp}^j - \frac{1}{2} \sigma^2 \alpha^j \left( \tilde{v}_p^j \right)^2 + \sup_{c^j \in \mathbb{R}} \left[ \lambda \left( c^j + \dot{X}^{-j} \right) \tilde{v}_p^j - c^j g \left( c^j + \dot{X}^{-j} \right) \right] \quad (4.11)$$

with terminal condition  $\tilde{v}^j(T, p) = H^j(p)$ . Note that this equation equals the HJB-equation (4.1) in the risk neutral setting, up to the quadratic term  $-\frac{1}{2} \sigma^2 \alpha^j \left( \tilde{v}_p^j \right)^2$ . As in (4.2), the optimal trading speeds are

$$c^j = \dot{X}^j = -\frac{1}{g'(\dot{X}^*)} \left[ -\lambda \tilde{v}_p^j + g(\dot{X}^*) \right]$$

where the aggregate trading speed  $\dot{X}^*$  is the unique solution to

$$0 = \lambda \sum_{i=1}^N \tilde{v}_p^i - N g \left( \sum_{i=1}^N \dot{X}_t^i \right) - \left( \sum_{i=1}^N \dot{X}_t^i \right) g' \left( \sum_{i=1}^N \dot{X}_t^i \right). \quad (4.12)$$

If we plug  $\dot{X}^*$  and  $\dot{X}^j$  back into (4.11), we get

$$0 = \tilde{v}_t^j + \frac{1}{2}\sigma^2\tilde{v}_{pp}^j - \frac{1}{2}\sigma^2\alpha^j(\tilde{v}_p^j)^2 + \lambda\left(\dot{X}^* - \frac{g(\dot{X}^*)}{g'(\dot{X}^*)}\right)\tilde{v}_p^j + \frac{g(\dot{X}^*)^2}{g'(\dot{X}^*)}. \quad (4.13)$$

We can show existence and uniqueness of a solution.

**Theorem 4.4.** *Let  $H^j \in \mathcal{C}_b^2$  for each  $j \in J$ . The Cauchy problem (4.11) admits a unique solution, which coincides with the vector of value functions (up to an exponential transformation).*

*Proof.* See appendix A. □

### 4.2.2 Linear Cost Structure, Single Player

For the one player case with linear cost structure, we have an explicit solution:

**Corollary 4.5.** *Let  $N = 1$  and  $g(z) = \kappa z$ . Then the Cauchy problem (4.11) admits a unique solution, which can be given in closed form.*

*Proof.* The maximizer in (4.11) is

$$c = \dot{X} = \frac{\lambda}{2\kappa}\tilde{v}_p$$

and the Cauchy problem (4.13) turns into

$$0 = \tilde{v}_t + \frac{1}{2}\sigma^2\tilde{v}_{pp} + \left(\frac{\lambda^2}{4\kappa} - \frac{1}{2}\sigma^2\alpha\right)\tilde{v}_p^2$$

with terminal condition  $\tilde{v}(T, p) = H(p)$ . This is Burgers' equation. Its explicit solution is given in Lemma 4.3. □

### 4.2.3 Numerical Illustrations

Let us conclude this section with numerical illustrations. We simulated the system (4.10) for two players. Figure 5 shows the aggregate optimal trading speed and the surpluses  $v^j(0, p) - \mathbb{E}[u^j(H(P_T)) | P_0 = p]$  for time  $t = 0$  and different spot prices  $p \in [95, 105]$  for the European Call option  $H(P_T) = (P_T - K)^+$ ; we assume that Player 1 (blue) is the option writer and Player 2 (red) the option issuer. We chose the strike  $K = 100$ , maturity  $T = 1$ , volatility  $\sigma = 2$  and liquidity parameters  $\lambda = \kappa = 0.01$  and risk aversion parameters  $\alpha^1 = 0.01, \alpha^2 = 0.01$  (solid), respectively,  $\alpha^1 = 0.001, \alpha^2 = 0.1$  (dashed). Since Player 1 has a long position in the option, she has an incentive to buy the underlying; for the same reason Player 2 has an incentive to sell it (Panel (b)). Our simulations suggest that the dependence of the equilibrium trading speed on the agents' risk aversion is weak (Panels (b) and (c)) and that overall the option issuer is slightly more active than the option writer. Furthermore, we see from Panel (d) that the issuer benefits more from reducing her loss than the writer benefits from increasing her gains. This effect is due to the concavity of the utility function and increases with the risk aversion.

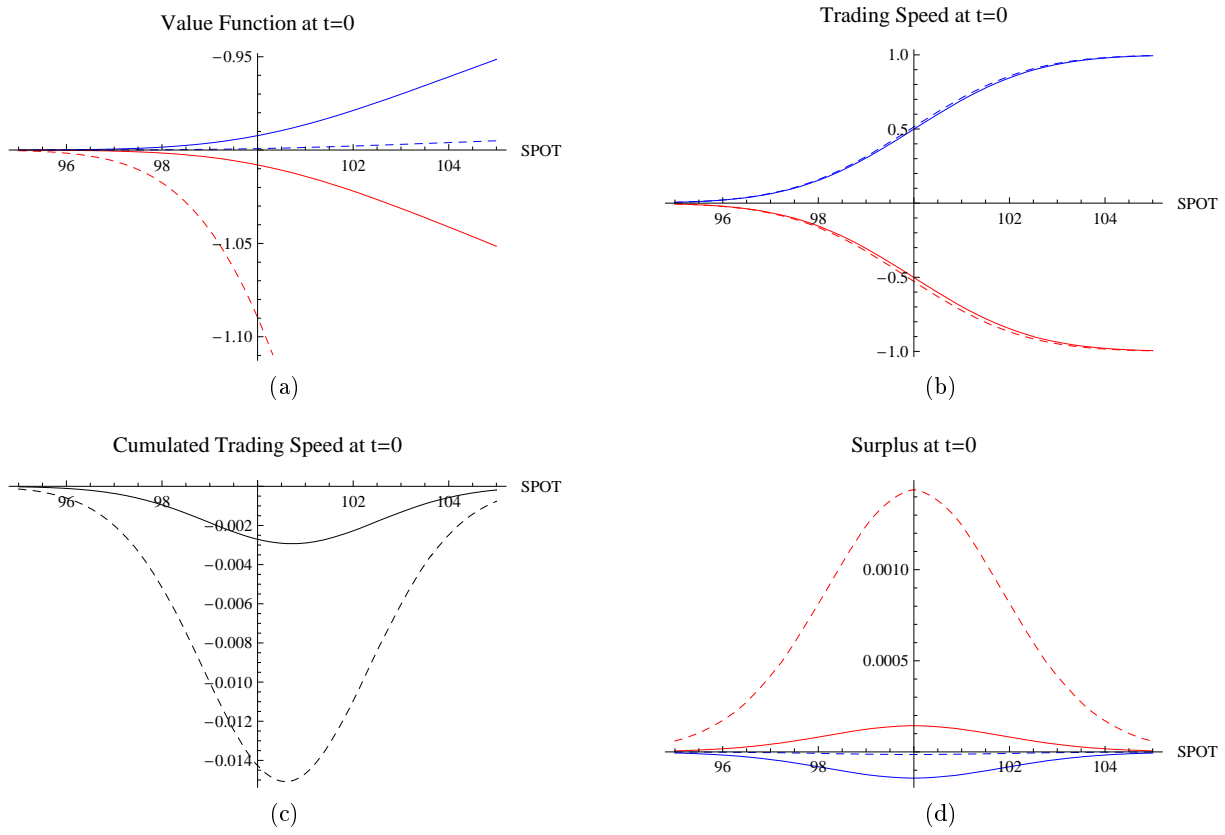


Figure 5: Value function, trading speed, aggregate trading speed and surplus for the writer (blue) and issuer (red) of a European Call option when both agents are risk averse. The solid (dashed) curves display the case where issuer is about as (more) risk averse than the option writer.

## 5 How to Avoid Manipulation

In the absence of market impact, manipulation does not occur, as we saw in Remark 3.1. In this section, we use the closed form solutions for risk neutral agents derived in subsection 4.1.2 to illustrate how an option issuer may prevent<sup>4</sup> other market participants from trading against her by using their impact on the dynamics of the underlying. Some of our observations were already made in Kumar and Seppi (1992) for Futures in a two period model and in Gallmeyer and Seppi (2000) for Call options in a three period binomial model. We start with the simplest case of a zero-sum game.

**Corollary 5.1.** *Let all players be risk neutral with offsetting payoffs  $\sum_{i=1}^N H^i = 0$ . Then the aggregate trading speed is  $\sum_{i=1}^N \dot{X}^i \equiv 0$ .*

*Proof.* Consider the PDE (4.5) for the aggregate value function with terminal condition zero and the characterization (4.3) of the aggregate trading speed.  $\dot{X}^* = \sum_{i=1}^N \dot{X}^i \equiv 0$  and  $v = \sum_{i=1}^N v^i \equiv 0$  is the unique solution to this coupled system.  $\square$

In a zero-sum game, if all option traders are risk neutral and willing to move the market in their favor, their combined effect cancels. We note that this is no longer true for general utility functions, as illustrated in figure 5 for the CARA case.

In reality, some (or all) of the investors might not want to manipulate, e.g. for legal reasons<sup>5</sup>. This is why we now look at the following asymmetric situation: The option issuer, Player 0, does not trade the underlying; her competitor, Player 1, owns the payoff  $H^1 \neq 0$  and intends to move the stock price to her favor. In addition, there are  $N - 1$  informed investors without option endowment in the market. They are “predators” that may supply liquidity and thus reduce the first player’s market impact, cf. Carlin, Lobo, and Viswanathan (2007) and Schied and Schöneborn (2007). Note that Corollaries 5.3 and 5.2 are only valid for the linear cost function, as the proofs hinge on the closed form solution obtained in Corollary 4.2. The following result states that the aggregate trading speed is decreasing in the number of players. The more liquidity suppliers, the less market manipulation. If the number of players goes to infinity, manipulation vanishes.

**Corollary 5.2.** *Let  $g(z) = \kappa z$ . Let  $H^1 \in \mathcal{C}_0^2$  be nondecreasing and  $H^i = 0$  for  $i = 2, \dots, N$ . Then the aggregate trading speed  $\sum_{i=1}^N \dot{X}_t^i$  is decreasing in  $N$  and*

$$\lim_{N \rightarrow \infty} \sum_{i=1}^N \dot{X}_t^i = 0.$$

*Proof.* See appendix B.  $\square$

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<sup>4</sup>Let us emphasize again that our results only apply to the practice of “punching the close”, i.e. manipulating the stock price in order to increase a given option payoff. There are other types of market manipulation not covered by our setup, such as market corners, short squeezes, the use of private information or false rumours. We refer the interested reader to Jarrow (1994) and Kyle and Viswanathan (2008).

<sup>5</sup>A discussion of legal issues is beyond the scope of this paper, but see the discussion in Kyle and Viswanathan (2008)

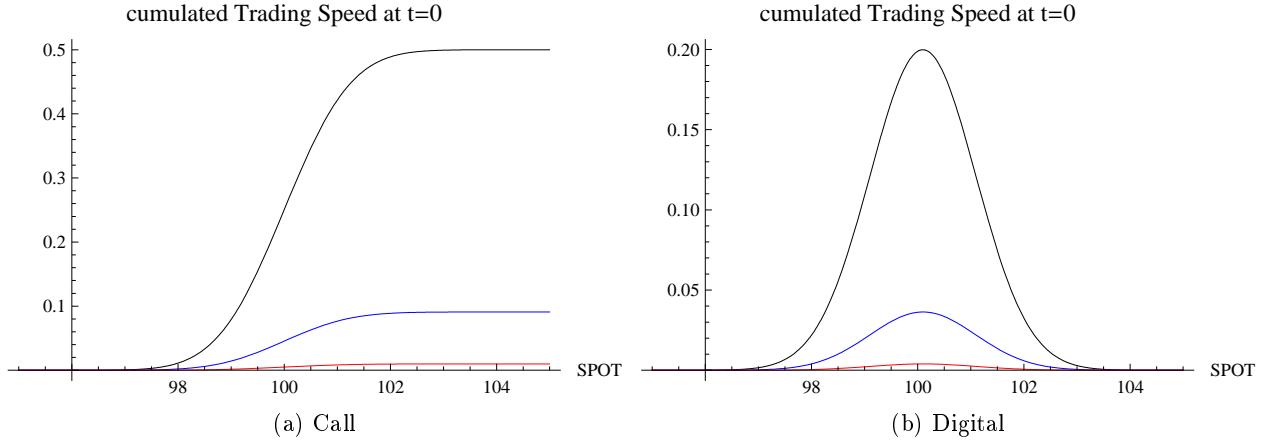


Figure 6: Aggregate trading speed  $\dot{X}^*$  at time  $t = 0$  for  $N = 1$  (black), 10 (blue), 100 (red) players each holding  $1/N$  shares of a Call (left) and Digital (right) option with strike  $K = 100$ . The more agents, the less aggregate manipulation.

Let us modify the preceding setting a little. Again, Player 0 issues a product  $H$  and does not intend to manipulate the underlying, while her competitors do. More precisely, assume that player 0 splits the product  $H$  into pieces and sells them to  $N$  risk neutral competitors, such that each of them gets  $\frac{1}{N}H$ . We find that their aggregate trading speed  $\sum_{i=1}^N \dot{X}^i$  is decreasing in the number of competitors  $N$ . Consequently, the option issuer should sell her product to as many investors as possible in order to avoid being outsmarted. We illustrate this result in figure 6, which shows the aggregate trading speed at time  $t = 0$  of  $N$  players each holding  $1/N$  option shares.

**Corollary 5.3.** *Let  $g(z) = \kappa z$ . Let  $H \in \mathcal{C}_b^2$  be nondecreasing and  $H^i = \frac{1}{N}H$  for  $i = 1, \dots, N$ . Then the aggregate trading speed  $\sum_{i=1}^N \dot{X}_t^i$  is decreasing in  $N$  and*

$$\lim_{N \rightarrow \infty} \sum_{i=1}^N \dot{X}_t^i = 0.$$

*Proof.* See appendix B. □

The preceding results indicate how an option issuer can prevent her competitors from manipulation. One strategy is public announcement of the transaction: the more informed liquidity suppliers on the market, the smaller the impact on the underlying. A second strategy is splitting the product into pieces - the more option writers, the less manipulation. Let us conclude this section with a surprisingly simple way to avoid manipulation: using options with physical settlement.

**Corollary 5.4.** *Calls, Puts and Forwards with physical settlement do not induce stock price manipulation.*

*Proof.* Consider a risk neutral agent who owns  $\Theta > 0$  Call options with physical settlement and strike  $K$ . As above, we denote by  $X$  her strategy in the underlying. At maturity, she exercises  $0 \leq \theta \leq \Theta$  of her Call options. The optimization problem is

$$\sup_{X \in \mathfrak{X}, 0 \leq \theta \leq \Theta} \mathbb{E} \left[ \int_0^T -\dot{X}_t \tilde{P}_t dt + (X_T + \theta) \left( P_T - \frac{1}{2} \lambda (X_T + \theta) \right) - \theta K \right]$$

where the first term represents the expected trading costs in  $[0, T]$ , the second term describes the liquidation value of  $\theta + X_T$  stock shares in  $[T, \infty)$  under infinitely slow liquidation (cf. Almgren and Chriss (2001)) and  $\theta K$  is the exercise price of  $\theta$  Call options. Using (2.1), (2.2) and  $X_0 = 0$ , it can be shown that this equals

$$\sup_{X \in \mathfrak{X}, 0 \leq \theta \leq \Theta} \mathbb{E} \left[ \int_0^T -\dot{X}_t g(\dot{X}_t) dt + \theta P_0 + \theta \sigma B_T - \frac{1}{2} \lambda \theta^2 - \theta K \right].$$

The cost term  $\int_0^T \dot{X}_t g(\dot{X}_t) dt$  is nonnegative, so the optimal trading strategy in the stock is clearly  $\dot{X} = 0$ . A similar argument holds true for Put options and Forward options with physical settlement.  $\square$

At first glance, Corollary 5.4 might contradict Pirrong (2001, p.1). He states that “replacement of delivery settlement of futures contracts with cash settlement is frequently proposed to reduce the frequency of market manipulation”. While his notion of market manipulation refers to market corners and short squeezes (see also Garbade and Silber (1983)), Corollary 5.4 shows that this is not always true for manipulation strategies in the sense of “punching the close”. The heuristic argument is the following: It is not beneficial to drive up the stock price at maturity if the option is settled physically and the investor needs to liquidate the stocks she receives at maturity. Any price increase is outweighed by subsequent liquidation and has no positive effect, but it is costly. This confirms a claim made in Kumar and Seppi (1992, p.1497), who argue that whether “futures contracts with a ‘physical delivery’ option [are] also susceptible to liquidity-driven manipulation [...] depends on whether ‘offsetting’ trades can be used to unwind a futures position with little price impact”.

## 6 Conclusion

We investigated the strategic behavior of option holders in illiquid markets. If trading the underlying has a permanent impact on the stock price, the possession of derivatives with cash settlement may induce market manipulation. We showed the existence and uniqueness of optimal trading strategies in continuous time and for a general cost function; in the one player framework for general utility functions, and in the multi-player case for risk neutral as well as CARA investors. Moreover, we showed how market manipulation can be reduced.

Our work may be extended in several directions. Foremost, we derived our results under Assumption 2.7. This assumption is only satisfied in the single-player risk-neutral case where the expected costs of buying a portfolio over a finite time interval under market impact equals its



expected liquidation value under infinitely slow liquidation and does not hold in general. The problem of defining a proper notion of liquidation value under strategic interaction is important but was not our focus and is left for future research. Furthermore, it would be interesting to consider American or path-dependent options, more sophisticated market impact models such as Obizhaeva and Wang (2005) that account for resilience effects and, thirdly, trading strategies which allow for jumps.

## A An Existence Result

In this section, we prove Theorems 4.1 and 4.4 where the PDE (4.4) in the risk neutral setting is a special case of the system (4.11) for risk averse agents, with  $\alpha^j = 0$  for each  $j$ . In order to establish our existence and uniqueness of equilibrium result, we adopt the proof of Proposition 15.1.1 in Taylor (1997) to our framework. After time inversion from  $t$  to  $T - t$  both systems of PDEs are of the form

$$v_t = Lv + F(v_p) \tag{A.1}$$

for  $v \triangleq (v^1, \dots, v^N)$ , where  $L$  is the Laplace-operator

$$L = \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial p^2}$$

and  $F = (F^1, \dots, F^N)$  is of the form

$$F^j(v_p) = -\frac{1}{2} \sigma^2 \alpha^j (v_p^j)^2 + \lambda \left( \dot{X}^* - \frac{g(\dot{X}^*)}{g'(\dot{X}^*)} \right) v_p^j + \frac{g(\dot{X}^*)^2}{g'(\dot{X}^*)}.$$

Here  $\dot{X}^* = \dot{X}^*(v_p)$  is given implicitly by (4.3). The initial condition is

$$v(0, p) = H(p) = (H^1, \dots, H^N). \tag{A.2}$$

We rewrite (A.1) in terms of an integral equation as

$$v(t) = e^{tL} + \int_0^t e^{(t-s)L} F(v_p(s)) ds \triangleq \Psi v(t). \tag{A.3}$$

and seek a fixed point of the operator  $\Psi$  on the following set of functions:

$$\mathbb{X} = \mathcal{C}_b^1(\mathbb{R}, \mathbb{R}^N) \triangleq \{v \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^N) \mid v, v_p \text{ bounded}\}$$

equipped with the norm

$$\|v\|_{\mathbb{X}} \triangleq \|v\|_{\infty} + \|v_p\|_{\infty}.$$

We set  $\mathbb{Y} \triangleq \mathcal{C}_b$ . Note that  $\mathbb{X}$  and  $\mathbb{Y}$  are Banach spaces and the semi-group  $e^{tL}$  associated with the Laplace operator is strongly continuous on  $\mathbb{X}$ , sends  $\mathbb{Y}$  on  $\mathbb{X}$  and satisfies

$$\|e^{tL}\|_{\mathcal{L}(\mathbb{Y}, \mathbb{X})} \leq Ct^{-\gamma}$$

for some  $C > 0$ ,  $\gamma < 1$  and  $t \leq 1$ . Furthermore, the nonlinearity  $F$  is locally Lipschitz and belongs to  $\mathcal{C}^\infty$ . Indeed, if we apply the implicit function theorem to  $\dot{X}^*$  given by (4.3), we see that the map  $a \mapsto \dot{X}^*(a)$  is  $\mathcal{C}^\infty$  with first derivative

$$\frac{\partial}{\partial v_p} \dot{X}^*(v_p) = \frac{\lambda}{(N+1)g'(\dot{X}^*(v_p)) + \dot{X}^*(v_p)g''(\dot{X}^*(v_p))}$$

where the denominator is positive due to Assumption 2.3. The cost function  $g$  is  $\mathcal{C}^\infty$  by assumption. In particular, the assumptions of Proposition 15.1.1 in Taylor (1997) are satisfied.

Before we proceed, we need the following lemma. It states that the value function<sup>6</sup> satisfies  $\|V^j\|_{\mathbb{X}} \leq K$  for each  $j \in J$  and some constant  $K$ , so it suffices to construct a solution in the following set:

$$\mathbb{X}_K \triangleq \{v \in \mathbb{X} \mid \|v\|_{\mathbb{X}} \leq K\}.$$

**Lemma A.1.** *There is a constant  $K$  such that  $\|V^j\|_{\mathbb{X}} \leq K$  for each  $j \in J$ .*

*Proof.* We prove the assertion for risk neutral agents, the CARA case follows by the same arguments. Our a priori estimates of Proposition 3.2 yield that  $\dot{X}^j$  is bounded for each  $j \in J$ . By the definition of  $V^j(t, p) = V^j(t, p, 0)$ , we get that  $V^j$  is bounded. Equation (3.5) implies that  $v_p^j$  is bounded, since  $v_{r_i}^j = -\mathbb{1}_{i=j}$  and  $\dot{X}^j$  as well as  $\dot{X}^*$  are bounded.  $\square$

We are now ready to prove existence and uniqueness of a solution to (A.3). In a nutshell, the argument is the following: Using Proposition 15.1.1 in Taylor (1997), we construct a solution to (A.1)-(A.2) for a small time horizon  $[0, \tau]$ , with  $\tau > 0$  specified below. The vector  $v$  coincides with the vector of value functions by Theorem IV.3.1 in Fleming and Soner (1993), so by Lemma A.1 the constructed solution is in  $\mathbb{X}_K$ . We apply his argument recursively to extend the solution to  $[0, T]$ .

**Proposition A.2.** *There is  $\tau > 0$  such that for each  $n \in \mathbb{N}_0$ , the PDE (A.3) with initial condition (A.2) admits a unique classical, bounded solution in  $\mathbb{X}_K$  on the time horizon  $[0, n\tau \wedge T]$ . This solution coincides with the value function.*

*Proof.* 1. For  $n = 0$ , there is nothing to prove. Pick  $n \in \mathbb{N}$  such that  $n\tau < T$ . By induction, we can assume that there is a solution  $v^{(n)} \in \mathbb{X}_K$  on the time horizon  $[0, n\tau]$ . In particular, the initial condition for the next recursion step  $h^{(n)} \triangleq v^{(n)}(n\tau)$  is in  $\mathbb{X}_K$ .

2. Fix  $\delta > 0$ . We construct a short time solution on the following set of functions:

$$Z^{(n+1)} \triangleq \left\{ v \in \mathcal{C}([n\tau, (n+1)\tau], \mathbb{X}) \mid v(n\tau) = h^{(n)}, \left\| v(t) - h^{(n)} \right\|_{\mathbb{X}} \leq \delta \forall t \in [n\tau, (n+1)\tau] \right\}.$$

We first show that  $\Psi : Z^{(n+1)} \rightarrow Z^{(n+1)}$  is a contraction, if  $\tau > 0$  is chosen small enough. For this, let  $\tau_1$  be small enough such that for  $t \leq \tau_1$  and any  $v \in \mathbb{X}_K$  we have

$$\left\| e^{tL}v - v \right\|_{\mathbb{X}} \leq \frac{1}{2}\delta.$$

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<sup>6</sup>By *value function*, we mean the function  $V^j(t, p) = V^j(t, p, 0)$ , where is state variable  $r$  is dropped.

Here we used that  $e^{tL}$  is a continuous semigroup and  $\|v\|_{\mathbb{X}} \leq K$ . In particular, for  $v = h^{(n)}$ :

$$\left\| e^{tL}h^{(n)} - h^{(n)} \right\|_{\mathbb{X}} \leq \frac{1}{2}\delta.$$

For  $v \in Z^{(n+1)}$ , the derivative  $v_p$  is uniformly bounded in the sense  $\|v_p\|_{\infty} \leq \|h^{(n)}\|_{\mathbb{X}} + \delta \leq K + \delta$ . Hence, we only evaluate  $F$  on compact sets. By assumption,  $F$  is locally Lipschitz. In particular,  $F$  is Lipschitz on compact sets. In other words, there is a constant  $K_1$  such that for any  $v, w \in Z^{(n+1)}$  we have

$$\|F(v_p) - F(w_p)\|_{\mathbb{Y}} \leq K_1 \|v - w\|_{\mathbb{X}}$$

This implies, for  $w = h^{(n)}$

$$\begin{aligned} \|F(v_p)\|_{\mathbb{Y}} &\leq \|F(h_p^{(n)})\|_{\mathbb{Y}} + K_1 \|v - h^{(n)}\|_{\mathbb{X}} \\ &\leq K + K_1\delta \triangleq K_2. \end{aligned}$$

This, together with the boundedness assumption on  $e^{tL}$ , yields

$$\begin{aligned} \left\| \int_{n\tau}^t e^{(t-y)L} F(v_p(y)) dy \right\|_{\mathbb{X}} &\leq t \|e^{tL}\| \sup_{n\tau \leq y \leq t} \|F(v_p(y))\|_{\mathbb{Y}} \\ &\leq t^{1-\gamma} CK_2. \end{aligned}$$

This quantity is  $\leq \frac{1}{2}\delta$  if  $t \leq \tau_2 \triangleq \left(\frac{\delta}{2CK_2}\right)^{\frac{1}{1-\gamma}}$ .

Finally, it follows that for  $v \in Z^{(n+1)}$  we have

$$\begin{aligned} \|\Psi v - h^{(n)}\|_{\mathbb{X}} &\leq \|e^{tL}h^{(n)} - h^{(n)}\|_{\mathbb{X}} + \left\| \int_{n\tau}^t e^{(t-y)L} F(v_p(y)) dy \right\|_{\mathbb{X}} \\ &\leq \frac{1}{2}\delta + \frac{1}{2}\delta = \delta. \end{aligned}$$

This shows that  $\Psi$  maps  $Z^{(n+1)}$  into itself.

It remains to show that  $\Psi$  is a contraction. Let  $v, w \in Z^{(n+1)}$ . Then

$$\begin{aligned} \|\Psi v(t) - \Psi w(t)\|_{\mathbb{X}} &= \left\| \int_{n\tau}^t e^{(t-y)L} [F(v_p(y)) - F(w_p(y))] dy \right\|_{\mathbb{X}} \\ &\leq t \|e^{tL}\| \sup_{n\tau \leq y \leq t} \|F(v_p(y)) - F(w_p(y))\|_{\mathbb{Y}} \\ &\leq t^{1-\gamma} CK_2 \sup_{n\tau \leq y \leq t} \|v(y) - w(y)\|_{\mathbb{X}} \end{aligned}$$

The quantity  $t^{1-\gamma} CK_2$  is  $\leq \frac{1}{2}$  if  $t \leq \tau_3 \triangleq \left(\frac{1}{2CK_2}\right)^{\frac{1}{1-\gamma}}$ . This proves that  $\Psi$  is a contraction in  $Z^{(n+1)}$ , if  $\tau$  is small in the sense

$$0 < \tau \triangleq \min\{\tau_1, \tau_2, \tau_3\}.$$

Note that the time step  $\tau$  does not depend on  $n$ . It is the same in every recursion step.

3. It follows that  $\Psi$  has a unique fix point  $v$  in  $Z^{(n+1)}$ . In other words, we constructed a function  $v \in \mathcal{C}([n\tau, (n+1)\tau], \mathbb{X}) = \mathcal{C}^{0,1}[n\tau, (n+1)\tau]$  which solves the PDE (A.3) with initial condition  $v(s) = h^{(n)} = v^{(n)}(n\tau)$  on the time interval  $[n\tau, (n+1)\tau]$ .

This solution is actually in  $\mathcal{C}^{1,2}((n\tau, (n+1)\tau) \times \mathbb{R}, \mathbb{R}^N)$ , due to Proposition 15.1.2 in Taylor (1997). Furthermore,  $v$  is bounded by construction. Indeed,  $\|v\|_\infty \leq \|h^{(n)}\|_\mathbb{X} + \delta \leq K + \delta$ . We define the new solution as

$$v^{(n+1)} \triangleq v^{(n)} \mathbb{1}_{\{0 \leq t \leq n\tau\}} + v \mathbb{1}_{\{n\tau < t \leq (n+1)\tau\}}.$$

By construction,  $v^{(n+1)}$  solves (A.3) on the time horizon  $[0, (n+1)\tau]$  and is bounded and in  $\mathcal{C}^{1,2}$ . Hence, we can apply the Verification Theorem IV.3.1 from Fleming and Soner (1993), which yields that  $v^{(n+1)}$  coincides with the vector of value functions (up to time reversal, reintroducing  $r^j$  and an exponential transformation, if  $\alpha^j > 0$ ). Due to Lemma A.1 we have  $v^{(n+1)} \in \mathbb{X}_K$ . In particular,  $\|v^{(n+1)}((n+1)\tau)\|_\mathbb{X} \leq K$ , which is necessary for the next recursion step.

This completes the proof.  $\square$

## B Proof of Corollary 5.2 and Corollary 5.3

The argument is the same for both corollaries. Fix  $N \in \mathbb{N}$ . The aggregate trading speed for  $N$  players is given from equation (4.7) as

$$\dot{X}^* = \sum_{i=1}^N \dot{X}^i = \frac{\lambda}{\kappa} \frac{1}{N+1} v_p,$$

where the aggregate value function  $v = \sum_{i=1}^N v_i$  from (4.8) solves Burgers' equation

$$0 = v_t + \frac{1}{2} \sigma^2 v_{pp} + \frac{\lambda^2}{\kappa} \frac{N}{(N+1)^2} v_p^2 \quad (\text{B.1})$$

with terminal condition  $v(T, p) = \sum_{i=1}^N H^i(p) = H^1(p) \triangleq H(p)$ . On the other hand, the aggregate trading speed for  $N+1$  players is

$$\dot{Y}^* = \sum_{i=1}^{N+1} \dot{Y}^i = \frac{\lambda}{\kappa} \frac{1}{N+2} w_p,$$

where the aggregate value function  $w = \sum_{i=1}^{N+1} w_i$  solves

$$0 = w_t + \frac{1}{2} \sigma^2 w_{pp} + \frac{\lambda^2}{\kappa} \frac{N+1}{(N+2)^2} w_p^2$$

with terminal condition  $w(T, p) = H(p)$ . We have to show that  $\dot{X}^* \geq \dot{Y}^*$ . To this end, let us define  $\tilde{w} \triangleq \frac{N+1}{(N+2)^2} \frac{(N+1)^2}{N} w$ . It is enough to show that  $v_p \geq \tilde{w}_p$ , since then  $\frac{1}{N+1} v_p \geq \frac{1}{N+1} \tilde{w}_p$  and, by definition,  $\frac{1}{N+1} \tilde{w}_p \geq \frac{1}{N+2} w_p$ . This implies  $\dot{X}^* \geq \dot{Y}^*$ .

To show  $v_p \geq \tilde{w}_p$ , first note that  $\tilde{w}$  is chosen such that it satisfies the same PDE (B.1) as  $v$ , namely

$$0 = \tilde{w}_t + \frac{1}{2}\sigma^2\tilde{w}_{pp} + \frac{\lambda^2}{\kappa} \frac{N}{(N+1)^2}\tilde{w}_p^2 \quad (\text{B.2})$$

with a smaller terminal condition:  $\tilde{w}(T, p) = \frac{N+1}{(N+2)^2} \frac{(N+1)^2}{N} H(p) \triangleq (1-\delta)H(p)$ . The solutions to (B.1) and (B.2) are given in Lemma 4.3 as

$$v(t, p) = c_1 \log \int_{\mathbb{R}} \exp(c_2 H(c_3 z)) d_{\mathcal{N}}(c_4 p, T-t)$$

and

$$\tilde{w}(t, p) = c_1 \log \int_{\mathbb{R}} \exp(c_2(1-\delta)H(c_3 z)) d_{\mathcal{N}}(c_4 p, T-t)$$

with constants  $c_1, c_2, c_3, c_4 \in \mathbb{R}$  and  $\delta \in (0, 1)$ . To verify  $v_p \geq \tilde{w}_p$ , it is enough to show

$$\frac{\partial}{\partial p} \log \int \exp(G) d_{\mathcal{N}}(p, 1) \geq \frac{\partial}{\partial p} \log \int \exp((1-\delta)G) d_{\mathcal{N}}(p, 1)$$

for an increasing function  $G \in \mathcal{C}_b^2$ . This is equivalent to

$$\frac{\int (z-p)e^G d_{\mathcal{N}}}{\int e^G d_{\mathcal{N}}} \geq \frac{\int (z-p)e^{(1-\delta)G} d_{\mathcal{N}}}{\int e^{(1-\delta)G} d_{\mathcal{N}}}$$

or

$$\int z e^{\delta G} \frac{e^{(1-\delta)G} d_{\mathcal{N}}}{\int e^{(1-\delta)G} d_{\mathcal{N}}} \geq \int z \frac{e^{(1-\delta)G} d_{\mathcal{N}}}{\int e^{(1-\delta)G} d_{\mathcal{N}}} \int e^{\delta G} \frac{e^{(1-\delta)G} d_{\mathcal{N}}}{\int e^{(1-\delta)G} d_{\mathcal{N}}}$$

or

$$\text{cov}_{\mathbb{Q}}(id, e^{\delta G}) \geq 0$$

under the measure  $\mathbb{Q}$  with  $d\mathbb{Q} \triangleq \frac{e^{(1-\delta)G} d_{\mathcal{N}}}{\int e^{(1-\delta)G} d_{\mathcal{N}}}$ . The covariance of two increasing functions is surely nonnegative. This finally proofs the assertion  $\dot{X}^* \geq \dot{Y}^*$ .

It remains to show  $\lim_{N \rightarrow \infty} \sum_{i=1}^N \dot{X}_t^i = 0$ . We have

$$\begin{aligned} \dot{X}^*(t, p) &= \sum_{i=1}^N \dot{X}_t^i = \frac{\lambda}{\kappa} \frac{1}{N+1} v_p(t, p) \\ &= \frac{\partial}{\partial p} \frac{\lambda}{\kappa} \frac{1}{N+1} \frac{\sigma^2 \kappa (N+1)^2}{2\lambda^2 N} \log \int \exp\left(\frac{2\lambda^2 N}{\sigma^2 \kappa (N+1)^2} H(\sigma z)\right) d_{\mathcal{N}}\left(\frac{p}{\sigma}, T-t\right) \\ &= \frac{\partial}{\partial p} \frac{\lambda}{\kappa} \frac{1}{N+1} \frac{\sigma^2 \kappa (N+1)^2}{2\lambda^2 N} \log \int \exp\left(\frac{2\lambda^2 N}{\sigma^2 \kappa (N+1)^2} H\left(\sigma z + \frac{p}{\sigma}\right)\right) d_{\mathcal{N}}(0, T-t) \\ &= \frac{\lambda}{\kappa} \frac{1}{N+1} \frac{1}{\sigma} \frac{\int H_p\left(\sigma z + \frac{p}{\sigma}\right) \exp\left(\frac{2\lambda^2 N}{\sigma^2 \kappa (N+1)^2} H\left(\sigma z + \frac{p}{\sigma}\right)\right) d_{\mathcal{N}}(0, T-t)}{\int \exp\left(\frac{2\lambda^2 N}{\sigma^2 \kappa (N+1)^2} H\left(\sigma z + \frac{p}{\sigma}\right)\right) d_{\mathcal{N}}(0, T-t)}, \end{aligned}$$

where we used Lemma 4.3 in the second line. This expression is nonnegative, since  $H_p \geq 0$ . Furthermore, we have  $\|H_p\|_\infty < \infty$  by assumption. It follows that

$$0 \leq \sum_{i=1}^N \dot{X}_t^i \leq \frac{\lambda}{\kappa} \frac{1}{N+1} \frac{1}{\sigma} \|H_p\|_\infty \xrightarrow{N \rightarrow \infty} 0.$$

This completes the proof.

## C The system of HJB PDEs without Assumption 2.7

In this section, we illustrate that the optimization problem under consideration becomes rather intractable without Assumption 2.7. In this case, problem (2.4) takes the form

$$\sup_{X^j \in \mathfrak{X}} \mathbb{E} \left[ u^j \left( - \int_0^T \dot{X}_t^j \left( P_t + g \left( \sum_{i=1}^N \dot{X}_t^i \right) \right) dt + H^j(P_T) + LV^j(X_T^j) \right) \right],$$

where the first term captures the trading costs (fundamental price plus liquidity costs) in  $[0, T]$ , the second term represents the option payoff and the third term the liquidation value of  $X_T^j$  stock shares at maturity. The dynamics of the state variables are

$$\begin{cases} dP_t = \sigma dB_t + \lambda \sum_{i=1}^N \dot{X}_t^i dt, & P_0 = p_0 \\ dR_t^j = \dot{X}_t^j \left( P_t + g \left( \sum_{i=1}^N \dot{X}_t^i \right) \right) dt, & R_0^j = 0 \\ dX_t^j = \dot{X}_t^j dt, & X_0^j = 0. \end{cases}$$

The associated HJB equation for agent  $j \in J$  is

$$0 = v_t^j + \frac{1}{2} \sigma^2 v_{pp}^j + \sup_{c^j \in \mathbb{R}} \left\{ \lambda (c^j + \dot{X}^{-j}) v_p^j + c^j v_{x^j}^j + \right. \\ \left. + \sum_{i \neq j} \dot{X}^i v_{x^i}^j + c^j \left[ p + g(c^j + \dot{X}^{-j}) \right] v_{r^j}^j + \sum_{i \neq j} \dot{X}^i \left[ p + g(c^j + \dot{X}^{-j}) \right] v_{r^i}^j \right\} \quad (\text{C.1})$$

with terminal condition  $v^j(T, p, r, x) = u^j(-r^j + H^j(p) + LV^j(x^j))$ . As above  $\dot{X}^{-j} = \sum_{i \neq j} \dot{X}^i$  denotes the opponents' aggregate trading speed. In this general formulation it is not clear that the a priori estimates from Proposition 3.2 hold. Moreover, the PDE (C.1) is not uniformly parabolic, so the proof of Theorems 4.1 and 4.4 given in Appendix A is not valid. In particular, we cannot apply Proposition 15.1.2 of Taylor (1997).

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