

Söhl, Jakob

Working Paper

Polar sets of anisotropic Gaussian random fields

SFB 649 Discussion Paper, No. 2009-058

Provided in Cooperation with:

Collaborative Research Center 649: Economic Risk, Humboldt University Berlin

Suggested Citation: Söhl, Jakob (2009) : Polar sets of anisotropic Gaussian random fields, SFB 649 Discussion Paper, No. 2009-058, Humboldt University of Berlin, Collaborative Research Center 649 - Economic Risk, Berlin

This Version is available at:

<https://hdl.handle.net/10419/39281>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

SFB 649 Discussion Paper 2009-058

Polar sets of anisotropic Gaussian random fields

Jakob Söhl*



* Humboldt-Universität zu Berlin, Germany

This research was supported by the Deutsche
Forschungsgemeinschaft through the SFB 649 "Economic Risk".

<http://sfb649.wiwi.hu-berlin.de>
ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin
Spandauer Straße 1, D-10178 Berlin



SFB 649 ECONOMIC RISK BERLIN

Polar sets of anisotropic Gaussian random fields*

Jakob Söhl[†]
Humboldt-Universität zu Berlin

November 13, 2009

Abstract

This paper studies polar sets of anisotropic Gaussian random fields, i.e. sets which a Gaussian random field does not hit almost surely. The main assumptions are that the eigenvalues of the covariance matrix are bounded from below and that the canonical metric associated with the Gaussian random field is dominated by an anisotropic metric. We deduce an upper bound for the hitting probabilities and conclude that sets with small Hausdorff dimension are polar. Moreover, the results allow for a translation of the Gaussian random field by a random field, that is independent of the Gaussian random field and whose sample functions are of bounded Hölder norm.

Keywords: Anisotropic Gaussian fields · Hitting probabilities · Polar sets · Hausdorff dimension · European option · Jump diffusion · Calibration

MSC (2000): 60G60 · 60G15 · 60G17 · 28A80

JEL Classification: G13 · C14

1 Introduction

Anisotropic Gaussian random fields arise naturally in stochastic partial differential equations, image processing, mathematical finance and other areas. Let $X = \{X(t) | t \in I \subset \mathbb{R}^N\}$ be a centered Gaussian random field with values in \mathbb{R}^d , where I is bounded. We will call X an (N, d) -Gaussian random field. The distance in the *canonical metric* associated with the Gaussian random field is $\sqrt{\mathbb{E}[\|X(s) - X(t)\|^2]}$, where $\|\cdot\|$ denotes the Euclidean metric. Polar sets of Gaussian random fields are investigated in [5] under the assumptions that the components are independent copies of the same random field,

*This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk". The author expresses his appreciation of the guidance he has received from his thesis advisor, Markus Reiß.

[†]E-mail address: soehl@math.hu-berlin.de

that the variance is constant and that $\sqrt{\mathbb{E}[\|X(s) - X(t)\|^2]} \leq c\|s - t\|^\beta$ holds with constants $c, \beta > 0$. The recent works [7] and [2] consider the anisotropic metric

$$\rho(s, t) := \sum_{j=1}^N |s_j - t_j|^{H_j} \quad (1)$$

with $H \in]0, 1]^N$ and assume $\sqrt{\mathbb{E}[\|X(s) - X(t)\|^2]} \leq c\rho(s, t)$. In addition they require for the variance only to be bounded from below. In this paper the assumptions on the variance and on the independent copies in the components are substituted by the milder assumption that the eigenvalues of the covariance matrix are bounded from below. The random fields in the components neither need to be identical distributed nor independent. Hence, we require weaker assumptions on the dependency structure of the components of the Gaussian random field than [5], [7] and [2]. It follows from an upper bound on the hitting probabilities of X that sets with Hausdorff dimension smaller than $d - \sum_{j=1}^N 1/H_j$ are polar. Our results allow for a translation of the Gaussian random field X by a random field, that is independent of X and whose sample functions are Lipschitz continuous with respect to the metric ρ .

As an application we show that an estimator in [1], which calibrates an exponential Lévy model by option data, is almost surely well-defined.

2 Main results

Let X be an (N, d) -Gaussian random field. Recall that we suppose the index set I to be bounded. We will assume the following two conditions.

Condition 1. There is a constant $c > 0$ such that for all $s, t \in I$ we have $\sqrt{\mathbb{E}[\|X(s) - X(t)\|^2]} \leq c\rho(s, t)$.

Condition 2. There is a constant $\lambda > 0$ such that for all $t \in I$ and for all $e \in \mathbb{R}^d$ with $\|e\| = 1$ we have $\mathbb{E}[(\sum_{j=1}^d e_j X_j(t))^2] \geq \lambda$.

Condition 1 bounds the canonical metric in terms of the anisotropic metric ρ . Condition 2 bounds the eigenvalues of the covariance matrix from below. It excludes, for example, cases where X takes values only in some vector subspace.

We will use a uniform modulus of continuity, see (69) in [7, p. 167]. We restate this result in the next inequality. A weaker formulation suffices for our purpose and is proved in the Appendix. Let X be an (N, d) -Gaussian random field, that satisfies Condition 1. Then there is a version X' of X and a constant $\tilde{c} > 0$ such that almost surely the following inequality holds:

$$\limsup_{\epsilon \downarrow 0} \sup_{s, t \in I, \rho(s, t) \leq \epsilon} \frac{\|X'(s) - X'(t)\|}{\epsilon \sqrt{\log(\epsilon^{-1})}} \leq \tilde{c}. \quad (2)$$

We will always assume that X is a version, which satisfies (2). We define by $\text{Lip}_\rho(L) := \{f : I \rightarrow \mathbb{R}^d \mid \|f(s) - f(t)\| \leq L\rho(s,t) \forall s, t \in I\}$ the L -Lipschitz functions with respect to the metric ρ . In each direction j the functions in $\text{Lip}_\rho(L)$ are Hölder continuous with exponent H_j . We denote by $B_\rho(t, r) := \{s \in \mathbb{R}^N \mid \rho(s, t) \leq r\}$ the closed ball of radius r around t .

Lemma 1. *Let X be an (N, d) -Gaussian random field, that satisfies Conditions 1 and 2. Then for each $L \geq 0$ there is a constant $C > 0$ such that for all $t \in I$, for all $r > 0$ and for all functions $f \in \text{Lip}_\rho(L)$ we have*

$$\mathbb{P} \left(\inf_{s \in B_\rho(t, r) \cap I} \|X(s) - f(s)\| \leq r \right) \leq Cr^d. \quad (3)$$

Proof. For all integers $n \geq 1$ we define $\epsilon_n := r \exp(-2^{n+1})$ and denote by $N_n := N_\rho(B_\rho(t, r) \cap I, \epsilon_n)$ the covering number, that is the minimum number of ρ -balls with radii ϵ_n and centers in $B_\rho(t, r) \cap I$ that are needed to cover $B_\rho(t, r) \cap I$. We have the inclusion $B_\rho(t, r) \subseteq \prod_{j=1}^N [t_j - r^{1/H_j}, t_j + r^{1/H_j}]$. On the other hand each set $\prod_{j=1}^N [s_j, s_j + (\epsilon_n/N)^{1/H_j}]$ can be covered by a single ball with radius ϵ_n . Hence there is a constant $c_1 > 0$ independent of n such that $N_n \leq \prod_{j=1}^N ((2rN/\epsilon_n)^{(1/H_j)} + 1) \leq c_1 \exp(Q2^{n+1})$ where $Q = \sum_{j=1}^N 1/H_j$.

We denote by $\{t_i^{(n)} \in B_\rho(t, r) \cap I \mid 1 \leq i \leq N_n\}$ a set of points such that the balls with the centers $\{t_i^{(n)}\}$ and radii ϵ_n cover $B_\rho(t, r) \cap I$. We define

$$r_n := \beta \epsilon_n 2^{\frac{n+1}{2}},$$

where $\beta > \tilde{c}$ is some constant to be determined later. For all integers $n, k \geq 1$ and $1 \leq i \leq N_k$, we define the following events

$$A_i^{(k)} := \left\{ \|X(t_i^{(k)}) - f(t_i^{(k)})\| \leq r + \sum_{l=k}^{\infty} r_l \right\}, \quad (4)$$

$$A^{(n)} := \bigcup_{k=1}^n \bigcup_{i=1}^{N_k} A_i^{(k)} = A^{(n-1)} \cup \bigcup_{i=1}^{N_n} A_i^{(n)}, \quad (5)$$

where the last equality only holds for $n \geq 2$. We will show that the probability in (3) can be dominated by the limit of the probabilities of the sets $A^{(n)}$

$$\mathbb{P} \left(\inf_{s \in B_\rho(t, r) \cap I} \|X(s) - f(s)\| \leq r \right) \leq \lim_{n \rightarrow \infty} \mathbb{P}(A^{(n)}). \quad (6)$$

For all $s \in B_\rho(t, r) \cap I$ and all $n \geq 1$ there exists i_n such that $\rho(s, t_{i_n}^{(n)}) \leq \epsilon_n$. By (2) we obtain almost surely

$$\limsup_{n \rightarrow \infty} \sup_{s \in I} \frac{\|X(s) - X(t_{i_n}^{(n)})\|}{r_n} \leq \frac{\tilde{c}}{\beta} < 1,$$

where the supremum over s is to be understood such that i_n varies according to s . Let $\kappa \in]\tilde{c}/\beta, 1[$. Especially there is N such that for all $n \geq N$ we have

$$\sup_{s \in I} \frac{\|X(s) - X(t_{i_n}^{(n)})\|}{r_n} \leq \kappa. \quad (7)$$

By going over to a possibly greater constant N we ensure that $(1-\kappa)\tilde{c}2^{\frac{N+1}{2}} \geq L$. On the event $\inf_{s \in B_\rho(t,r) \cap I} \|X(s) - f(s)\| \leq r$ there exists $s_0 \in B_\rho(t,r) \cap I$ such that

$$\|X(s_0) - f(s_0)\| \leq r + \sum_{l=N+1}^{\infty} r_l. \quad (8)$$

Choose i_N such that $\rho(s_0, t_{i_N}^{(N)}) \leq \epsilon_N$. Using (7), (8) and the Lipschitz continuity of f we obtain

$$\begin{aligned} & \|X(t_{i_N}^{(N)}) - f(t_{i_N}^{(N)})\| \\ & \leq \|X(t_{i_N}^{(N)}) - X(s_0)\| + \|X(s_0) - f(s_0)\| + \|f(s_0) - f(t_{i_N}^{(N)})\| \\ & \leq \kappa r_N + r + \sum_{l=N+1}^{\infty} r_l + L\rho(s_0, t_{i_N}^{(N)}) \\ & \leq \kappa r_N + r + \sum_{l=N+1}^{\infty} r_l + (1-\kappa)\tilde{c}2^{\frac{N+1}{2}}\epsilon_N \leq r + \sum_{l=N}^{\infty} r_l \end{aligned}$$

and (6) is established.

Trivially we have for $n \geq 2$

$$\mathbb{P}(A^{(n)}) \leq \mathbb{P}(A^{(n-1)}) + \mathbb{P}(A^{(n)} \setminus A^{(n-1)})$$

and by (5) we have

$$\mathbb{P}(A^{(n)} \setminus A^{(n-1)}) \leq \sum_{i=1}^{N_n} \mathbb{P}(A_i^{(n)} \setminus A_{i'}^{(n-1)}),$$

where i' is chosen such that $\rho(t_i^{(n)}, t_{i'}^{(n-1)}) < \epsilon_{n-1}$. We note that for $n \geq 2$

$$\begin{aligned} & \mathbb{P}(A_i^{(n)} \setminus A_{i'}^{(n-1)}) \quad (9) \\ & = \mathbb{P} \left(\|X(t_i^{(n)}) - f(t_i^{(n)})\| \leq r + \sum_{l=n}^{\infty} r_l, \|X(t_{i'}^{(n-1)}) - f(t_{i'}^{(n-1)})\| > r + \sum_{l=n-1}^{\infty} r_l \right) \\ & \leq \mathbb{P} \left(\|X(t_i^{(n)}) - f(t_i^{(n)})\| \leq c_2 r, \|X(t_i^{(n)}) - X(t_{i'}^{(n-1)})\| > r_{n-1} - L\epsilon_{n-1} \right) \\ & \leq \mathbb{P} \left(\|X(t_i^{(n)}) - f(t_i^{(n)})\| \leq c_2 r, \|X(t_i^{(n)}) - X(t_{i'}^{(n-1)})\| > (\beta 2^{\frac{n}{2}} - L)\epsilon_{n-1} \right), \end{aligned}$$

where $c_2 = 1 + \beta \sum_{l=1}^{\infty} 2^{\frac{l+1}{2}} \exp(-2^{l+1})$. We ensure $(\beta 2^{\frac{n}{2}} - L) > 0$ by choosing $\beta > L/2$. The idea is to rewrite $X(t_i^{(n)}) - X(t_{i'}^{(n-1)})$ as a sum of two terms, one expressed by $X(t_i^{(n)})$ and the other independent of $X(t_i^{(n)})$.

$\text{Lip}_\rho(L)$ is invariant under orthogonal transformations. By the spectral theorem we may choose new coordinates such that the covariance matrix at $t_i^{(n)}$ is diagonal. Then the components of $X(t_i^{(n)})$ are independent. By assumption $\sigma_j(s) := \sqrt{\mathbb{E}[X_j(s)^2]} > 0$. We define the standard normal random variables

$$Y_j(s) := \frac{X_j(s)}{\sigma_j(s)}.$$

Note that $\mathbb{E}[Y(t_i^{(n)})Y(t_i^{(n)})^\top] = \text{Id}$ holds. If $\mathbb{E}[(X_j(s) - X_j(t))^2] > 0$ we define

$$Y_j(s, t) := \frac{X_j(s) - X_j(t)}{\sqrt{\mathbb{E}[(X_j(s) - X_j(t))^2]}}$$

and $Y_j(s, t) := 0$ otherwise. We further define a matrix η and a random vector Z by

$$\begin{aligned} \eta &:= \mathbb{E} \left[Y(t_i^{(n)}, t_{i'}^{(n-1)}) Y(t_i^{(n)})^\top \right], \\ Z(t_i^{(n)}, t_{i'}^{(n-1)}) &:= Y(t_i^{(n)}, t_{i'}^{(n-1)}) - \eta Y(t_i^{(n)}). \end{aligned}$$

We observe that $|\eta_{jk}| \leq 1$ and hence in the operator norm $\|\eta\| \leq d$. The random vectors $Z(t_i^{(n)}, t_{i'}^{(n-1)})$ and $Y(t_i^{(n)})$ are independent because the covariance matrix is the zero matrix. By the definition of $Y(t_i^{(n)})$ we see that $Z(t_i^{(n)}, t_{i'}^{(n-1)})$ and $X(t_i^{(n)})$ are independent, too. We want to bound $\mathbb{P}(A_i^{(n)} \setminus A_{i'}^{(n-1)})$. If $t_i^{(n)} = t_{i'}^{(n-1)}$ then $\mathbb{P}(A_i^{(n)} \setminus A_{i'}^{(n-1)}) = 0$ holds. Thus we may assume that $\rho(t_i^{(n)}, t_{i'}^{(n-1)}) > 0$. (9) is bounded by

$$\begin{aligned} &\mathbb{P} \left(\|X(t_i^{(n)}) - f(t_i^{(n)})\| \leq c_2 r, \|Y(t_i^{(n)}, t_{i'}^{(n-1)})\| > \frac{(\beta 2^{\frac{n}{2}} - L) \epsilon_{n-1}}{c \rho(t_i^{(n)}, t_{i'}^{(n-1)})} \right) \\ &\leq \mathbb{P} \left(\|X(t_i^{(n)}) - f(t_i^{(n)})\| \leq c_2 r, \|Z(t_i^{(n)}, t_{i'}^{(n-1)})\| + \|\eta Y(t_i^{(n)})\| > \frac{\beta 2^{n/2} - L}{c} \right) \\ &\leq \mathbb{P} \left(\|X(t_i^{(n)}) - f(t_i^{(n)})\| \leq c_2 r, \|Z(t_i^{(n)}, t_{i'}^{(n-1)})\| > \frac{\beta 2^{n/2} - L}{2c} \right) \\ &+ \mathbb{P} \left(\|X(t_i^{(n)}) - f(t_i^{(n)})\| \leq c_2 r, d \|Y(t_i^{(n)})\| > \frac{\beta 2^{n/2} - L}{2c} \right) \\ &:= I_1 + I_2. \end{aligned}$$

Each component of $Z(t_i^{(n)}, t_{i'}^{(n-1)})$ is a weighted sum of at most $d+1$ standard normal random variables with weights in $[-1, 1]$. Hence the variance of each

component is at most $(d+1)^2$. In the following c_l with $l \in \mathbb{N}$ will denote positive constants. By the independence of $X(t_i^{(n)})$ and $Z(t_i^{(n)}, t_{i'}^{(n-1)})$ we have

$$\begin{aligned}
I_1 &= \mathbb{P} \left(\|X(t_i^{(n)}) - f(t_i^{(n)})\| \leq c_2 r \right) \mathbb{P} \left(\|Z(t_i^{(n)}, t_{i'}^{(n-1)})\| > \frac{\beta 2^{n/2} - L}{2c} \right) \\
&\leq c_3 r^d \mathbb{P} \left(\|Z(t_i^{(n)}, t_{i'}^{(n-1)})\| > \frac{\beta 2^{n/2} - L}{2c} \right) \\
&\leq c_3 r^d \frac{2d}{\sqrt{2\pi}} \frac{2\sqrt{d}(d+1)c}{\beta 2^{\frac{n}{2}} - L} \exp \left(-\frac{(\beta 2^{\frac{n}{2}} - L)^2}{8c^2 d(d+1)^2} \right) \\
&\leq c_4 r^d \exp \left(-\frac{(\beta 2^{\frac{n}{2}} - L)^2}{8c^2 d(d+1)^2} \right).
\end{aligned}$$

By the definition of $Y(t_i^{(n)})$ we have with the abbreviation $\sigma_j = \sigma_j(t_i^{(n)})$

$$\begin{aligned}
I_2 &\leq \int_{\{\|u - f(t_i^{(n)})\| \leq c_2 r, \|(\frac{u_k}{\sigma_k})_k\| > \frac{\beta 2^{n/2} - L}{2dc}\}} \left(\frac{1}{2\pi} \right)^{\frac{d}{2}} \frac{1}{\sigma_1 \cdots \sigma_d} e^{-\frac{1}{2} \left(\frac{u_1^2}{\sigma_1^2} + \cdots + \frac{u_d^2}{\sigma_d^2} \right)} du \\
&\leq \int_{\{\|u - f(t_i^{(n)})\| \leq c_2 r\}} \left(\frac{1}{2\pi} \right)^{\frac{d}{2}} \frac{1}{\sigma_1 \cdots \sigma_d} e^{-\frac{1}{4} \left(\frac{u_1^2}{\sigma_1^2} + \cdots + \frac{u_d^2}{\sigma_d^2} \right)} du e^{-\frac{1}{4} \left(\frac{(\beta 2^{\frac{n}{2}} - L)^2}{4d^2 c^2} \right)} \\
&\leq c_5 r^d \exp \left(-\frac{(\beta 2^{\frac{n}{2}} - L)^2}{16d^2 c^2} \right).
\end{aligned}$$

To sum it up

$$\begin{aligned}
\mathbb{P}(A^{(n)}) &\leq \mathbb{P}(A^{(n-1)}) + c_6 r^d N_n \exp \left(-\frac{(\beta 2^{\frac{n}{2}} - L)^2}{16d(d+1)^2 c^2} \right) \\
&\leq \mathbb{P}(A^{(1)}) + c_6 r^d \sum_{k=2}^{\infty} N_k \exp \left(-\frac{(\beta 2^{\frac{k}{2}} - L)^2}{16d(d+1)^2 c^2} \right) \\
&\leq c_7 r^d + c_6 r^d \sum_{k=2}^{\infty} c_1 \exp \left(Q 2^{k+1} - \frac{(\beta 2^{\frac{k}{2}} - L)^2}{16d(d+1)^2 c^2} \right).
\end{aligned}$$

We choose $\beta > \max(\tilde{c}, L/2)$ such that $\frac{\beta^2}{16d(d+1)^2 c^2} > 2Q$. Then the sum converges and the lemma follows by (6). \square

Let \mathcal{H}_α denote the α -dimensional *Hausdorff measure*, see the definition, for instance, in [4, p. 129]. Recall that $Q = \sum_{j=1}^N 1/H_j$ with H_j as in the definition of the metric ρ .

Theorem 1. *Let X be an (N, d) -Gaussian random field that satisfies Conditions 1 and 2. If $Q < d$, then for each $L \geq 0$ there is a constant $C > 0$ such that all Borel sets $F \subseteq \mathbb{R}^d$ and all random fields Y which are independent of X and whose sample functions are all in $\text{Lip}_\rho(L)$ satisfy*

$$\mathbb{P}(\exists s \in I : X(s) + Y(s) \in F) \leq C\mathcal{H}_{d-Q}(F). \quad (10)$$

Proof. By Fubini's theorem it suffices to show for all functions $f \in \text{Lip}_\rho(L)$

$$\mathbb{P}(\exists s \in I : X(s) + f(s) \in F) \leq C\mathcal{H}_{d-Q}(F).$$

We choose some constant $\gamma > \mathcal{H}_{d-Q}(F)$. By definition of the Hausdorff measure there is a set of balls $\{B(x_l, r_l) : l = 0, 1, 2, \dots\}$ such that

$$F \subseteq \bigcup_{l=0}^{\infty} B(x_l, r_l) \quad \text{and} \quad \sum_{l=0}^{\infty} (2r_l)^{d-Q} \leq \gamma. \quad (11)$$

For all j we cut the bounded index set I orthogonal to the j -axis with distance $(r_l/N)^{1/H_j}$ between the cuts. Each piece of I can be covered by a single ball of radius r_l in the metric ρ . Hence there is a constant $c_8 > 0$ such that I can be covered by at most $c_8 r_l^{-Q}$ balls. We apply Lemma 1 to these balls. By summing up we obtain

$$\mathbb{P}(\exists s \in I : X(s) + f(s) \in B(x_l, r_l)) \leq c_9 r_l^{d-Q}. \quad (12)$$

By (11) and (12) we have

$$\begin{aligned} & \mathbb{P}(\exists s \in I : X(s) + f(s) \in F) \\ & \leq \sum_{l=0}^{\infty} \mathbb{P}(\exists s \in I : X(s) + f(s) \in B(x_l, r_l)) \leq c_{10}\gamma. \end{aligned}$$

We have $\mathbb{P}(\exists s \in I : X(s) + f(s) \in F) \leq c_{10}\mathcal{H}_{d-Q}(F)$, since $\gamma > \mathcal{H}_{d-Q}(F)$ was chosen arbitrarily. \square

3 Application

In this section we apply our results to nonparametric statistics. Observations in the Gaussian white noise model lead to a Gaussian process to which then Theorem 1 is applied. We conclude that a certain estimator is almost surely well-defined. Existing results on Gaussian random fields have too restrictive assumptions for the application considered here.

The common approach to the estimation of Lévy processes is to take advantage of the Lévy-Khintchine representation. This often involves taking a complex logarithm of the empirical characteristic function, see e.g. [1]

and [3]. The branch of the logarithm needs to be taken such that the log-characteristic function is continuous. This is referred to as the *distinguished logarithm*. To this end the empirical characteristic function has to be distinct from zero everywhere which means that zero needs to be a polar set. For instance in the set-up of [1] the following function is estimated:

$$\psi : \mathbb{R} \rightarrow \mathbb{C}, v \mapsto \frac{1}{T} \log(1 + iv(1 + iv)\mathcal{F}\mathcal{O}(v)),$$

where $T > 0$, \mathcal{O} is some $L^1(\mathbb{R})$ function modeling option prices and \mathcal{F} denotes the Fourier transform. The argument of the logarithm is distinct from zero for all $v \in \mathbb{R}$ and the logarithm is taken such that ψ is continuous with $\psi(0) = 0$. The function $\mathcal{O}(x)$ is dominated by an exponentially decaying function for $|x| \rightarrow \infty$. Especially $x\mathcal{O}(x)$ is integrable. To simplify matters we suppose that \mathcal{O} is continuously observed as in the Gaussian white noise model, cf. [1]. For $\epsilon \in L^2(\mathbb{R})$ we observe

$$d\tilde{\mathcal{O}}(x) = \mathcal{O}(x)dx + \epsilon(x) dW(x) \quad \text{for all } x \in \mathbb{R}.$$

Taking the Fourier transform yields

$$\mathcal{F}(d\tilde{\mathcal{O}})(v) = \mathcal{F}\mathcal{O}(v) + \int_{-\infty}^{\infty} e^{ivx} \epsilon(x) dW(x).$$

Thus the canonical estimator for ψ is

$$\tilde{\psi}(v) := \frac{1}{T} \log\left(1 + iv(1 + iv)\mathcal{F}(d\tilde{\mathcal{O}})(v)\right) \quad \text{for all } v \in \mathbb{R}.$$

In [1] a trimmed log-function is used to bound the real part. But this is of no importance for the question whether the estimator $\tilde{\psi}$ is well-defined. We require the following condition on ϵ .

Condition 3. There is a $p > 1$ such that $\int_{-\infty}^{\infty} (1 + |x|)^p \epsilon(x)^2 dx < \infty$.

For example, if $\epsilon \in L^2(\mathbb{R})$ and $\epsilon(x) = O(|x|^{-p})$ for $|x| \rightarrow \infty$ with $p > 1$, then the condition is satisfied. Condition 3 and Lemma 4 in the Appendix imply the uniform modulus of continuity (2) for a version of $X(v) := \int_{-\infty}^{\infty} e^{ivx} \epsilon(x) dW(x)$. We will assume that X is a version that satisfies (2). Thus in the definition of $\tilde{\psi}$ the argument of the logarithm is almost surely continuous.

Lemma 2. *Let ϵ fulfill Condition 3. Let $\mathcal{O} \in L^1(\mathbb{R})$ such that $x\mathcal{O}(x)$ is integrable. Then the estimator $\tilde{\psi}$ is almost surely well-defined.*

Proof. We have to show that almost surely the argument of the logarithm does not hit zero. The process $1 + iv(1 + iv)\mathcal{F}(d\tilde{\mathcal{O}})(v)$ equals 1 at $v = 0$. It suffices to consider the process on $\mathbb{R} \setminus \{0\}$. We rewrite the process as

$$iv(1 + iv) \left(\frac{1}{iv(1 + iv)} + \mathcal{F}\mathcal{O}(v) + \int_{-\infty}^{\infty} e^{ivx} \epsilon(x) dW(x) \right).$$

We define

$$f(v) := \frac{1}{iv(1+iv)} + \mathcal{FO}(v) \quad \text{and} \quad X(v) := \int_{-\infty}^{\infty} e^{ivx} \epsilon(x) dW(x).$$

On a bounded index set X is an $(1,2)$ -Gaussian random field. We will apply Theorem 1 to X , $Y = f$ and $F = \{0\}$. It is proved in the Appendix that under Condition 3 there is a constant $c > 0$ such that for all $u, v \in \mathbb{R}$ the inequality

$$\sqrt{\mathbb{E}[\|X(u) - X(v)\|^2]} \leq c|u - v|^{\min(p/2, 1)}. \quad (13)$$

holds. This gives reason to the definition $\rho(u, v) := |u - v|^H$ with $H = \min(p/2, 1) \in]1/2, 1]$. Thus Condition 1 is satisfied and we have $d - Q = 2 - 1/H > 0$.

It remains to show that Condition 2 is fulfilled and that f is Lipschitz continuous with respect to the metric ρ . For $\epsilon = 0 \in L^2(\mathbb{R})$ we have $\tilde{\psi} = \psi$ and thus $\tilde{\psi}$ is well-defined. We will now show that the covariance matrix of $X(v)$ is not degenerated if $\epsilon \neq 0 \in L^2(\mathbb{R})$ and $v \neq 0$. Let $e \in \mathbb{R}^2$ such that $e_1^2 + e_2^2 = 1$. Then there is $\varphi \in [0, 2\pi]$ such that $e_1 = \sin \varphi$ and $e_2 = \cos \varphi$. Consider X as a \mathbb{R}^2 -valued stochastic process. The Itô isometry yields

$$\begin{aligned} \mathbb{E}[(e_1 X_1(v) + e_2 X_2(v))^2] &= \mathbb{E} \left[\left(\int_{-\infty}^{\infty} (e_1 \cos(vx) + e_2 \sin(vx)) \epsilon(x) dW(x) \right)^2 \right] \\ &= \int_{-\infty}^{\infty} (e_1 \cos(vx) + e_2 \sin(vx))^2 \epsilon(x)^2 dx \\ &= \int_{-\infty}^{\infty} (\sin(\varphi + vx))^2 \epsilon(x)^2 dx > 0. \end{aligned}$$

The function

$$\mathbb{R} \times [0, 2\pi] \rightarrow \mathbb{R}, (v, \varphi) \mapsto \int_{-\infty}^{\infty} (\sin(\varphi + vx))^2 \epsilon(x)^2 dx$$

is continuous by dominated convergence. On $([-V, -1/V] \cup [1/V, V]) \times [0, 2\pi]$ it takes a minimum $\lambda_V > 0$ for $V > 0$. Hence Condition 2 is fulfilled on the index set $I_V = [-V, -1/V] \cup [1/V, V]$.

Since $x\mathcal{O}(x)$ is integrable we have that \mathcal{FO} is Lipschitz continuous on \mathbb{R} . $1/(iv(1+iv))$ is Lipschitz continuous on sets bounded away from zero. Hence f is Lipschitz continuous on I_V . Since I_V is bounded it follows that f is Lipschitz continuous with respect to the metric ρ on I_V .

Thus we may apply Theorem 1 to the index set $I_V = [-V, -1/V] \cup [1/V, V]$. Since $\mathcal{H}_{d-Q}(\{0\}) = 0$ we obtain $\mathbb{P}(\exists v \in I_V : X(v) + f(v) = 0) = 0$. Because $V > 0$ was chosen arbitrarily the Lemma follows. \square

Appendix

Lemma 3. *Let X be an (N, d) -Gaussian random field that fulfills Condition 1. Then there is a version X' of X and a constant $\tilde{c} > 0$ with the following property. If $\epsilon_n \downarrow 0$ satisfies $\sum_{n=1}^{\infty} \epsilon_n^\vartheta < \infty$ for each $\vartheta > 0$ then*

$$\limsup_{n \rightarrow \infty} \sup_{s, t \in I, \rho(s, t) \leq \epsilon_n} \frac{\|X'(s) - X'(t)\|}{\epsilon_n \sqrt{\log(\epsilon_n^{-1})}} \leq \tilde{c}$$

holds almost surely.

Proof. We define the Gaussian random field $Y = \{Y(t, s) | t, (t + s) \in I\}$ by $Y(t, s) := X(t + s) - X(t)$. By Condition 1 we have

$$\begin{aligned} d((t, s), (t', s')) &:= \sqrt{\mathbb{E}[\|Y(t, s) - Y(t', s')\|^2]} \\ &\leq c_1 \min(\rho(0, s) + \rho(0, s'), \rho(s, s') + \rho(t, t')). \end{aligned}$$

Let $\epsilon \in]0, 1/4[$. We denote by D_ϵ the diameter of $T_\epsilon := \{(t, s) \in I \times B_\rho(0, \epsilon) | t + s \in I\}$, where $B_\rho(0, \epsilon) := \{s \in \mathbb{R}^N | \rho(s, 0) \leq \epsilon\}$. By the first argument of the minimum $D_\epsilon \leq 2c_1\epsilon$ holds. For $\delta \in]0, \epsilon[$ denote by $N_d(T_\epsilon, \delta)$ the covering number of T_ϵ in the metric d , i.e. the minimum number of d -balls with radius δ that are needed to cover T_ϵ . By the second argument of the minimum the covering number satisfies $N_d(T_\epsilon, \delta) \leq c_2\epsilon^Q/\delta^{2Q}$. By the previous estimates we obtain with the substitution $y = \delta/(c_2^{1/2Q}\sqrt{\epsilon})$

$$\int_0^{D_\epsilon} \sqrt{\log N_d(T_\epsilon, \delta)} d\delta \leq \sqrt{2Q}c_2^{1/2Q} \sqrt{\epsilon} \int_0^{2c_1\sqrt{\epsilon}/c_2^{1/2Q}} \sqrt{\log y^{-1}} dy.$$

By choosing a possibly larger constant c_1 or c_2 we may assume $2c_1 = c_2^{1/2Q}$. The integral is solved by

$$\int_0^x \sqrt{\log y^{-1}} dy = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \text{Erf}(\sqrt{\log x^{-1}}) + x\sqrt{\log x^{-1}},$$

where $\text{Erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt$. Estimating $\text{Erf}(y)$ from below yields some constant $c_3 > 0$ such that for all $x \in]0, 1/2[$

$$\int_0^x \sqrt{\log y^{-1}} dy \leq c_3 x \sqrt{\log x^{-1}}.$$

Thus for $\epsilon \in]0, 1/4[$

$$\int_0^{D_\epsilon} \sqrt{\log N_d(T_\epsilon, \delta)} d\delta \leq c_4 \epsilon \sqrt{\log \epsilon^{-1/2}}$$

holds. By Lemma 2.2 in [6] there is a constant $K > 0$ such that for all $u \geq 2Kc_4\epsilon\sqrt{\log \epsilon^{-1/2}}$ we obtain

$$\mathbb{P}\left(\sup_{(t,s) \in T_\epsilon} \|X'(t+s) - X'(t)\| \geq u\right) \leq \exp\left(-\frac{u^2}{(2KD_\epsilon)^2}\right).$$

Using the Borel-Cantelli lemma the statement follows. \square

Lemma 4. *Let ϵ fulfill Condition 3. Then there exists a number $c > 0$ such that for all $u, v \in \mathbb{R}$ the stochastic process $X(v) = \int_{-\infty}^{\infty} e^{ivx}\epsilon(x) dW(x)$ satisfies $\sqrt{\mathbb{E}[\|X(u) - X(v)\|^2]} \leq c|u - v|^{\min(p/2, 1)}$.*

Proof. Condition 3 is satisfied for $q := \min(p, 2)$ as well. We conclude that:

$$\begin{aligned} & \mathbb{E}[\|X(u) - X(v)\|^2] \\ &= \mathbb{E}\left[\left|\int_{-\infty}^{\infty} (e^{iux} - e^{ivx})\epsilon(x) dW(x)\right|^2\right] \\ &= \int_{-\infty}^{\infty} |e^{iux} - e^{ivx}|^2 \epsilon(x)^2 dx \\ &\leq \int_{-\infty}^{\infty} \min(4, (u-v)^2 x^2) \epsilon(x)^2 dx \\ &= \int_{|x| \geq 2|u-v|^{-1}} 4\epsilon(x)^2 dx + \int_{|x| < 2|u-v|^{-1}} (u-v)^2 x^2 \epsilon(x)^2 dx \\ &\leq \int_{|x| \geq 2|u-v|^{-1}} 4\left(\frac{|x|}{2|u-v|^{-1}}\right)^q \epsilon(x)^2 dx \\ &\quad + \int_{|x| < 2|u-v|^{-1}} \left(\frac{2|u-v|^{-1}}{|x|}\right)^{2-q} (u-v)^2 x^2 \epsilon(x)^2 dx \\ &= 2^{2-q} |u-v|^q \int_{-\infty}^{\infty} |x|^q \epsilon(x)^2 dx \end{aligned}$$

This shows the lemma. \square

References

- [1] Denis Belomestny and Markus Reiß. Spectral calibration of exponential Lévy models. *Finance Stoch.*, 10(4):449–474, 2006.
- [2] Hermine Biermé, Céline Lacaux, and Yimin Xiao. Hitting probabilities and the Hausdorff dimension of the inverse images of anisotropic Gaussian random fields. *Bull. Lond. Math. Soc.*, 41(2):253–273, 2009.
- [3] S. Gugushvili. Nonparametric estimation of the characteristic triplet of a discretely observed Lévy process. *Journal of Nonparametric Statistics*, 21(3):321–343, 2009.

- [4] Jean-Pierre Kahane. *Some random series of functions*, volume 5 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1985.
- [5] Michel Weber. Polar sets of some Gaussian processes. In *Probability in Banach spaces, IV (Oberwolfach, 1982)*, volume 990 of *Lecture Notes in Math.*, pages 204–214. Springer, Berlin, 1983.
- [6] Yimin Xiao. Packing measure of the sample paths of fractional Brownian motion. *Trans. Amer. Math. Soc.*, 348(8):3193–3213, 1996.
- [7] Yimin Xiao. Sample path properties of anisotropic Gaussian random fields. Khoshnevisan, Davar (ed.) et al., *A minicourse on stochastic partial differential equations. Papers based on the presentations at the minicourse, Salt Lake City, UT, USA, May 8–19, 2006*. Berlin: Springer. *Lecture Notes in Mathematics* 1962, 145-212 (2009)., 2009.

SFB 649 Discussion Paper Series 2009

For a complete list of Discussion Papers published by the SFB 649, please visit <http://sfb649.wiwi.hu-berlin.de>.

- 001 "Implied Market Price of Weather Risk" by Wolfgang Härdle and Brenda López Cabrera, January 2009.
- 002 "On the Systemic Nature of Weather Risk" by Guenther Filler, Martin Odening, Ostap Okhrin and Wei Xu, January 2009.
- 003 "Localized Realized Volatility Modelling" by Ying Chen, Wolfgang Karl Härdle and Uta Pigorsch, January 2009.
- 004 "New recipes for estimating default intensities" by Alexander Baranovski, Carsten von Lieres and André Wilch, January 2009.
- 005 "Panel Cointegration Testing in the Presence of a Time Trend" by Bernd Droge and Deniz Dilan Karaman Örsal, January 2009.
- 006 "Regulatory Risk under Optimal Incentive Regulation" by Roland Strausz, January 2009.
- 007 "Combination of multivariate volatility forecasts" by Alessandra Amendola and Giuseppe Storti, January 2009.
- 008 "Mortality modeling: Lee-Carter and the macroeconomy" by Katja Hanewald, January 2009.
- 009 "Stochastic Population Forecast for Germany and its Consequence for the German Pension System" by Wolfgang Härdle and Alena Mysickova, February 2009.
- 010 "A Microeconomic Explanation of the EPK Paradox" by Wolfgang Härdle, Volker Krätschmer and Rouslan Moro, February 2009.
- 011 "Defending Against Speculative Attacks" by Tijmen Daniëls, Henk Jager and Franc Klaassen, February 2009.
- 012 "On the Existence of the Moments of the Asymptotic Trace Statistic" by Deniz Dilan Karaman Örsal and Bernd Droge, February 2009.
- 013 "CDO Pricing with Copulae" by Barbara Choros, Wolfgang Härdle and Ostap Okhrin, March 2009.
- 014 "Properties of Hierarchical Archimedean Copulas" by Ostap Okhrin, Yarema Okhrin and Wolfgang Schmid, March 2009.
- 015 "Stochastic Mortality, Macroeconomic Risks, and Life Insurer Solvency" by Katja Hanewald, Thomas Post and Helmut Gründl, March 2009.
- 016 "Men, Women, and the Ballot Woman Suffrage in the United States" by Sebastian Braun and Michael Kvasnicka, March 2009.
- 017 "The Importance of Two-Sided Heterogeneity for the Cyclicity of Labour Market Dynamics" by Ronald Bachmann and Peggy David, March 2009.
- 018 "Transparency through Financial Claims with Fingerprints – A Free Market Mechanism for Preventing Mortgage Securitization Induced Financial Crises" by Helmut Gründl and Thomas Post, March 2009.
- 019 "A Joint Analysis of the KOSPI 200 Option and ODAX Option Markets Dynamics" by Ji Cao, Wolfgang Härdle and Julius Mungo, March 2009.
- 020 "Putting Up a Good Fight: The Galí-Monacelli Model versus 'The Six Major Puzzles in International Macroeconomics'", by Stefan Ried, April 2009.
- 021 "Spectral estimation of the fractional order of a Lévy process" by Denis Belomestny, April 2009.
- 022 "Individual Welfare Gains from Deferred Life-Annuities under Stochastic Lee-Carter Mortality" by Thomas Post, April 2009.

SFB 649, Spandauer Straße 1, D-10178 Berlin
<http://sfb649.wiwi.hu-berlin.de>

This research was supported by the Deutsche
Forschungsgemeinschaft through the SFB 649 "Economic Risk".



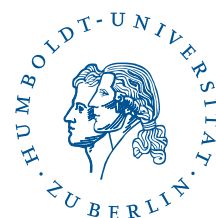
SFB 649 Discussion Paper Series 2009

For a complete list of Discussion Papers published by the SFB 649, please visit <http://sfb649.wiwi.hu-berlin.de>.

- 023 "Pricing Bermudan options using regression: optimal rates of convergence for lower estimates" by Denis Belomestny, April 2009.
- 024 "Incorporating the Dynamics of Leverage into Default Prediction" by Gunter Löffler and Alina Maurer, April 2009.
- 025 "Measuring the effects of geographical distance on stock market correlation" by Stefanie Eckel, Gunter Löffler, Alina Maurer and Volker Schmidt, April 2009.
- 026 "Regression methods for stochastic control problems and their convergence analysis" by Denis Belomestny, Anastasia Kolodko and John Schoenmakers, May 2009.
- 027 "Unionisation Structures, Productivity, and Firm Performance" by Sebastian Braun, May 2009.
- 028 "Optimal Smoothing for a Computationally and Statistically Efficient Single Index Estimator" by Yingcun Xia, Wolfgang Härdle and Oliver Linton, May 2009.
- 029 "Controllability and Persistence of Money Market Rates along the Yield Curve: Evidence from the Euro Area" by Ulrike Busch and Dieter Nautz, May 2009.
- 030 "Non-constant Hazard Function and Inflation Dynamics" by Fang Yao, May 2009.
- 031 "De copulis non est disputandum - Copulae: An Overview" by Wolfgang Härdle and Ostap Okhrin, May 2009.
- 032 "Weather-based estimation of wildfire risk" by Joanne Ho and Martin Odening, June 2009.
- 033 "TFP Growth in Old and New Europe" by Michael C. Burda and Battista Severgnini, June 2009.
- 034 "How does entry regulation influence entry into self-employment and occupational mobility?" by Susanne Prantl and Alexandra Spitz-Oener, June 2009.
- 035 "Trade-Off Between Consumption Growth and Inequality: Theory and Evidence for Germany" by Runli Xie, June 2009.
- 036 "Inflation and Growth: New Evidence From a Dynamic Panel Threshold Analysis" by Stephanie Kremer, Alexander Bick and Dieter Nautz, July 2009.
- 037 "The Impact of the European Monetary Union on Inflation Persistence in the Euro Area" by Barbara Meller and Dieter Nautz, July 2009.
- 038 "CDO and HAC" by Barbara Choroś, Wolfgang Härdle and Ostap Okhrin, July 2009.
- 039 "Regulation and Investment in Network Industries: Evidence from European Telecoms" by Michał Grajek and Lars-Hendrik Röller, July 2009.
- 040 "The Political Economy of Regulatory Risk" by Roland Strausz, August 2009.
- 041 "Shape invariant modelling pricing kernels and risk aversion" by Maria Grith, Wolfgang Härdle and Juhyun Park, August 2009.
- 042 "The Cost of Tractability and the Calvo Pricing Assumption" by Fang Yao, September 2009.

SFB 649, Spandauer Straße 1, D-10178 Berlin
<http://sfb649.wiwi.hu-berlin.de>

This research was supported by the Deutsche
Forschungsgemeinschaft through the SFB 649 "Economic Risk".



SFB 649 Discussion Paper Series 2009

For a complete list of Discussion Papers published by the SFB 649, please visit <http://sfb649.wiwi.hu-berlin.de>.

- 043 "Evidence on Unemployment, Market Work and Household Production" by Michael C. Burda and Daniel S. Hamermesh, September 2009.
- 044 "Modelling and Forecasting Liquidity Supply Using Semiparametric Factor Dynamics" by Wolfgang Karl Härdle, Nikolaus Hautsch and Andrija Mihoci, September 2009.
- 045 "Quantifizierbarkeit von Risiken auf Finanzmärkten" by Wolfgang Karl Härdle and Christian Wolfgang Friedrich Kirchner, October 2009.
- 046 "Pricing of Asian temperature risk" by Fred Benth, Wolfgang Karl Härdle and Brenda López Cabrera, October 2009.
- 047 "MM-Stat – MultiMedia-Statistik: Statistische Datenanalyse – webbasiert, interaktiv und multimedial" by Sigbert Klinke, Dina Kuhlee, Christian Theel, Cornelia Wagner and Christian Westermeier, October 2009.
- 048 "Migration of the Highly Skilled: Can Europe catch up with the US?" by Lydia Mechtenberg and Roland Strausz, October 2009.
- 049 "A blocking and regularization approach to high dimensional realized covariance estimation" by Nikolaus Hautsch, Lada M. Kyj and Roel C.A. Oomen, October 2009.
- 050 "Generalized single-index models: The EFM approach" by Xia Cui, Wolfgang Karl Härdle and Lixing Zhu, October 2009.
- 051 "The Market Impact of a Limit Order" by Nikolaus Hautsch and Ruihong Huang, October 2009.
- 052 "On economic evaluation of directional forecasts" by Oliver Blaskowitz and Helmut Herwartz, October 2009.
- 053 "Monetary Policy Implementation and Overnight Rate Persistence" by Dieter Nautz and Jan Scheithauer, November 2009.
- 054 "Depression Econometrics: A FAVAR Model of Monetary Policy During the Great Depression" by Pooyan Amir Ahmadi and Albrecht Ritschl, November 2009.
- 055 "Representations for optimal stopping under dynamic monetary utility functionals" by Volker Krätschmer and John Schoenmakers, November 2009.
- 056 "Product policy and the East-West productivity gap" by Bernd Görzig, Martin Gornig, Ramona Voshage and Axel Werwatz, November 2009.
- 057 "Real and Nominal Rigidities in Price Setting: A Bayesian Analysis Using Aggregate Data" by Fang Yao, November 2009.
- 058 "Polar sets of anisotropic Gaussian random fields" by Jakob Söhl, November 2009.

SFB 649, Spandauer Straße 1, D-10178 Berlin
<http://sfb649.wiwi.hu-berlin.de>

This research was supported by the Deutsche
Forschungsgemeinschaft through the SFB 649 "Economic Risk".

