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Debt-Sensitive Majority Rules

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Debt-Sensitive Majority Rules

Johannes Gerd Becker* Hans Gersbach* Oliver Grimm*

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We examine debt-sensitive majority rules. According to such a rule, the higher a planned public debt, the higher the necessary parliamentary majority to approve it. In a two-period model, we compare debt-sensitive majority rules with the simple majority rule when individuals differ regarding their benefits from public good provision. We establish the existence of Condorcet winners under debt-sensitive majority rules and derive their properties. We find that equilibrium debt-levels are lower under the debt-sensitive majority rule if preferences regarding public goods are sufficiently heterogeneous and if the impact of debt on future public good provision is sufficiently strong. We illustrate how debt-sensitive majority rules act as political stabilizers of negative macroeconomic shocks.

Keywords: fiscal policy, public debt, public goods, voting, simple majority rule, debt-sensitive majority rule, debt restriction

JEL Classification: H63, D72, E61, H41

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1. Introduction

Most western countries have experienced rising debt levels over the last decades. This will be accelerated by the recent economic and financial crisis, as indicated by Figure 1. The issue of debt restriction has become a great concern of policy-makers and economists.

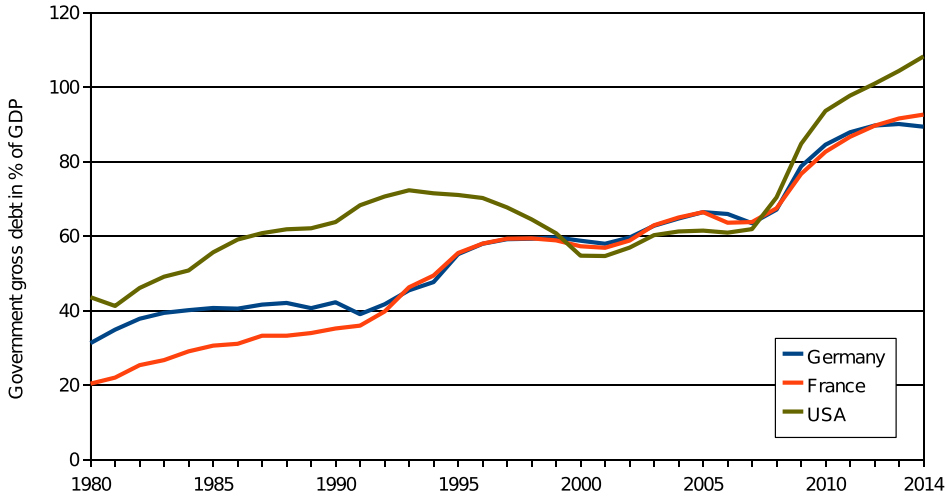

Figure 1: Government debt for selected countries (years 2009–2014 estimated). Source: IMF World Economic Outlook Database (2009)

In this paper, we examine new rules, called debt-sensitive majority rules, which can provide tighter debt brakes than the standard voting procedures. Under such rules, the required majority to pass a budget increases with the size of the budget deficit.

We consider a two-period model with private and public consumption. Individuals differ in their preference for public goods. Debt has a negative impact on the government's ability to provide public goods in the second period. In the first period, the society votes on a policy package including taxes, a level of public-good provision, and debt. We establish the existence of Condorcet winners under debt-sensitive majority rules and derive their properties. We compare the outcome of a standard simple majority voting rule to that of a debt-sensitive majority rule. We show that a suitable debt-sensitive majority rule can restrict debt accumulation if the individuals are sufficiently heterogeneous, if the society comprises individuals with a high valuation for public goods, and if the impact of debt on future public good provision is sufficiently strong. In contrast to fixed debt limits, debt-sensitive majority rules act as stabilizers when negative

macroeconomic shocks occur. Debt-sensitive voting rules allow higher debt-financing in downturns. This can be regarded as *political* stabilization, since the stabilization is induced by the voting rule.

While government debt can be justified on normative grounds – for instance, by the famous tax-smoothing argument of Barro (1979) –, the literature has highlighted that political-economic forces tend to push public debt beyond socially desirable levels. Fragmented governments are prone to excessive spending when few groups benefit from public goods, but the costs are distributed over the society as a whole (Weingast et al., 1981, Baron and Ferejohn, 1989, Baron, 1991). In a dynamic framework, Battaglini and Coate (2008) show that a government's incentive towards pork-barrel spending accounts for underprovision of public goods and accumulation of debt over time. Yared (2009) shows that a rent-seeking government might raise debts and taxes above a society's optimal level. The accumulation of debts allows governments to shift fiscal burdens to future generations, as these generations are underrepresented or not represented at all in today's elections (Song et al., 2009). Within a monetary union, public debt might even be raised further due to free-riding behavior: the debt burden of a single country has to be borne by all member countries (Beetsma and Bovenberg, 1999; Neck and Sturm, 2008).

Several proposals have been made to limit excessive public-debt accumulation. For instance, fixed budget-limits, debt-brakes, rainy-day funds and balanced-budget rules have been implemented. These rules, however, face credibility and flexibility problems. On the one hand, tight budget rules and debt-brakes have proved not to be very credible, as they were either repealed temporarily, or as violations were not sanctioned. On the other hand, when rules allowed for room to maneuver, policy makers exploited their flexibility even in quiet times, and debt accumulation was not slowed down. Canova and Pappa (2006), as well as Auerbach (2008), have assessed the success of fiscal rules in most U.S.-states, and their findings show that these rules had little success in limiting debt accumulation.

Debt-sensitive majority rules are a new concept. According to such a rule, the necessary majority in parliament depends on the new debt associated with a proposal. Other variants of proposal-dependent majority rules in the context of cake division and public-good provision have been discussed in Gersbach (2004) and Gersbach (2009).

Our paper is organized as follows. In Section 2, we outline the basic model and introduce the voting rules. In Section 3, we determine optimal policies. In Section 4, we prove

the existence of Condorcet winners and derive the unique equilibrium under the simple majority rule. In Section 5, we prove the existence of a unique equilibrium under a debt-sensitive majority rule. Section 6 contains a number of examples. We develop conditions under which the debt-sensitive rule restricts excessive debt-accumulation. In Section 7 we discuss how debt-sensitive majority rules can act as political stabilizers. In Section 8 we outline procedures to implement debt-sensitive majority rules and discuss extensions of the model. Section 9 concludes.

2. Model

We consider an economy populated by agents who live for two periods. Agents work when they are young, but do not work when they are old. We consider a competitive labor market and assume that each agent inelastically supplies one unit of labor when he is young. All workers receive the same wage w. We focus on the collective decision of the agents when they are young.

2.1. Utility Function

We follow Song et al. (2009), and assume that the utility of an agent in period 1 is given by the Cobb-Douglas utility function

$$U_\theta = \log c + \theta \log g + \beta\theta \log h, \tag{1}$$

where c is the consumption level of the private good and g and h denote public-good consumption in period 1 and 2, respectively. The discount factor is denoted by β. In order to simplify the presentation, $\log c$ captures the utility from consuming private goods over both periods. Note that this is not a restricting assumption, as it is straightforward to separate private consumption into period 1 and period 2 consumption by using

$$c_2 = (1 + r)(c - c_1),$$

where r is the real interest rate and c_1 and c_2 are the consumption levels in period 1 and period 2, respectively.

The agents differ in the relative weight $\theta > 0$ of utility from public goods. The distribution of types θ over the society is given by a probability measure μ, which means that

the mass of agents is normalized to unity. We assume that the median type is unique, i. e. that there is a unique type θ^* such that the intervals $(0;\theta^*]$ and $[\theta^*;\infty)$ both comprise a mass of at least $1/2$ of agents. For a very large society, neighboring agents are likely to differ only slightly in their types, so that we can introduce an additional agent if the median is not unique, without changing the outcome notably.

The amount of private consumption of an agent is given by his budget constraint

$$c = w(1-\tau), \tag{2}$$

with τ being the income tax rate.

2.2. Fiscal Policy

Fiscal policy is determined by a parliament that represents the electorate. The parliament decides upon the tax rate τ and the amount g of public good provided in the first period, and faces an inherited debt level a. The debt level at the beginning of the second period is denoted by b. The evolution of debt is given by

$$b = Ra + g - \tau w. \tag{3}$$

We assume that debt is issued on the international capital market and government bonds are held by foreigners. R denotes the interest rate factor for public debt. $R-1$ is the interest rate.

There exists a maximum level $\bar{b}$ of government debt that can be issued. This upper limit of debt is reached when the interest payment exhausts the maximum tax revenue. Thus,

$$\bar{b} = \frac{\bar{\tau} w}{R-1}\,, \tag{4}$$

where $\bar{\tau}$ is the revenue-maximizing tax rate. For ease of presentation, we assume that labor is supplied inelastically and the entire income can be taxed away, i. e. $\bar{\tau} = 1$.[1] We exclude subsidies and thus only consider tax rates $\tau \geq 0$. We assume that initial debt is below the level for which a state-bankruptcy would be inevitable, i. e. $Ra - w < \bar{b}$.

[1] There are various ways to relax this assumption. For instance, one can assume that households supply inelastically one unit of labor as long as the tax rate does not exceed a threshold $\bar{\bar{\tau}} < \bar{\tau} = 1$, and do not work if the tax level exceeds this threshold. Our results fully apply to this case.

Consumption of the public good in the second period depends on the government debt b inherited from the first period. This dependence is described by a function $h(b)$. We assume that for $b < \bar{b}$,

$$(\log h)'(b) < 0 \quad \text{and} \quad (\log h)''(b) < 0, \tag{5}$$

i. e. $\log h(b)$ is strictly decreasing and strictly concave. If debt inherited from period 1 reaches $\bar{b}$, the total tax income $\bar{\tau}w$ has to be used to finance interest payments on government debt, and no public goods can be provided, which implies $h(\bar{b}) = 0$.

For the purpose of this paper, we do not specify the function h exactly, but allow for different reactions of future electorates to higher debt. For the theoretical investigations in the next sections we only require that h fulfills the above assumptions. Specific examples will be discussed in Section 6.

We summarize the sequence of events in Figure 2.

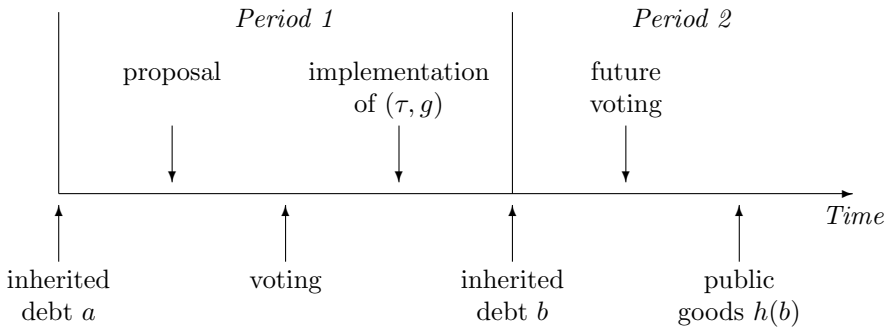

Figure 2: Sequence of events.

It is useful to make the following definitions.

Definition 1 *We denote the set of policies by* $\mathcal{P} := [0;1] \times \mathbb{R}_{\geq 0}$. *A* policy *is a pair* $(\tau, g) \in \mathcal{P}$, *consisting of a tax rate* τ *and a level of government goods* g.

A policy determines the associated debt level

$$B(\tau, g) := Ra + g - \tau w. \tag{6}$$

For $b \leq \bar{b}$ *define*

$$\mathcal{P}_{=b} := \big\{(\tau, g) \in \mathcal{P} \;\big|\; B(\tau, g) = b\big\},$$

and

$$\mathcal{P}_{\leq b} := \big\{(\tau, g) \in \mathcal{P} \;\big|\; B(\tau, g) \leq b\big\}.$$

In words, $\mathcal{P}_{=b}$ is the set of all policies that imply a debt level of exactly b left to the next period, and $\mathcal{P}_{\leq b}$ is the set of all policies that imply a debt level of at most b.

By Equations (1), (2), and (3), the utility of an agent of type θ from a policy $(\tau, g) \in \mathcal{P}$ with $B(\tau, g) < \bar{b}$ is

$$V_\theta(\tau, g) = \log(1 - \tau) + \theta \log g + \beta\theta \log h\big(B(\tau, g)\big). \tag{7}$$

To simplify the exposition, we have made a monotonic transformation and dropped the constant term $\log w$ here. For convenience of notation, we set

$$V_\theta(\tau, g) := -\infty$$

for $(\tau, g) \in \mathcal{P}$ with $B(\tau, g) \geq \bar{b}$. V_θ can be viewed as the indirect utility function of an agent of type θ.

2.3. Voting Schemes

We will compare the outcome under the simple majority rule and debt-sensitive majority rules. The rules are defined in the following.

2.3.1. Simple Majority Rule

We assume that each member of parliament can make a proposal $(\tau, g) \in \mathcal{P}$ on tax rate and level of public-good provision. The legislature decides between the proposals by pairwise voting.

Definition 2 (Simple majority rule) *The voting process under the simple majority rule is described as follows:*

- *The proposals are sorted arbitrarily and pairwise voting occurs sequentially. First, the parliament decides between the first two proposals. The winning proposal is pitted against the third proposal, etc. The proposal that survives this process is implemented.*

- *In a pairwise voting, a proposal wins against another proposal if it receives a majority of strictly more than* 0.5*. If both proposals receive an equal vote share of 0.5, the winning proposal is chosen by fair randomization.*

In general, the outcome of such a voting game may depend on the order in which the proposals appear in the voting process. In Section 4, however, we prove the existence of a unique Condorcet winner. As the Condorcet winner beats all other proposals in the voting process, the order in which the proposals enter the process does not play any role. Furthermore, the voting process does not necessarily have to be sequential in the sense that the winner of the previous voting competes against the next proposal. We could use any form of a tournament with pairwise votes to determine the winning proposal.

We assume there is no commitment problem, i. e. the government will always implement the winning proposal.

2.3.2. Debt-Sensitive Majority Rule

An exogenously given debt-sensitive majority rule is a rule that is sensitive as to the planned government debt level b. The rule is established by constitution and requires a majority of parliamentarians that grows with the level of planned public debt.

We define a status quo policy as any policy following a *balanced-budget rule*. That is, such a policy leaves the debt level unchanged. If a status quo policy has to be implemented, tax revenues have to cover the expenditures for public goods and the interest payments on the inherited debt.

Definition 3 *A* status quo policy *is a tax / public good plan* $(\tau, g) \in \mathcal{P}$ *that does not increase the government debt level, i. e.* $B(\tau, g) \leq a$.

We next introduce the concept of a debt-sensitive majority rule:

Definition 4 (Debt-sensitive Majority Rule) *A* debt-sensitive majority rule *is described by a left-continuous, weakly increasing function* $\phi\colon \mathbb{R}_{\geq 0} \to [0; 1]$ *with the following consequences in the voting process:*

Stage 1: Proposal-making

- *Each legislator makes*[2] *a proposal* (τ, g)*. The winning proposal* (τ^F, g^F) *is determined by the simple majority rule.*

Stage 2: Winning Proposal Versus Status Quo

The legislators decide between the winning proposal (τ^F, g^F) *of stage 1 and implementing a status quo policy. The winning proposal* (τ^F, g^F) *needs a majority of* $\phi(b)$ *to be passed, with* $b = B(\tau^F, g^F)$*. Specifically,*

- *if* (τ^F, g^F) *receives a vote share greater than or equal to* $\phi(b)$*, the winning proposal of stage 1 has passed and will be implemented;*
- *if* (τ^F, g^F) *receives a vote share smaller than* $\phi(b)$*, a status quo policy has to be implemented. This status quo policy is chosen by simple majority voting, where only proposals from the set* $\mathcal{P}_{\leq a}$ *are allowed.*

As we shall see in Section 5, the left-continuity of ϕ will ensure uniqueness of the voting outcome. To illustrate our definition, we provide a simple example. Let $\delta_1 = 0.5\,\%$ and $\delta_2 = 2\,\%$. Consider the rule

$$\phi(b) := \begin{cases} 0.50 & \text{for } b \leq a + \delta_1 \cdot \text{GDP}, \\ 0.55 & \text{for } a + \delta_1 \cdot \text{GDP} < b \leq a + \delta_2 \cdot \text{GDP}, \\ 0.67 & \text{for } b > a + \delta_2 \cdot \text{GDP}, \end{cases} \tag{8}$$

where a denotes debt inherited from the previous period. According to this rule, government budgets with new debts below $\delta_1 = 0.5\,\%$ of GDP can be passed with simple majority. New debts between $\delta_1 = 0.5\,\%$ and $\delta_2 = 2\,\%$ of GDP require the support of $55\,\%$ of legislators, while higher new debts require a 2/3-majority in parliament.

2.4. Equilibrium Concept

In Sections 4 and 5, we determine the equilibrium outcomes under the two voting regimes. We will employ the subgame-perfect strong Nash equilibrium for the voting games of Definitions 2 and 4. The concept of a strong Nash equilibrium was proposed and introduced by Aumann (1959). A strong equilibrium is defined to be a strategy profile such that no subset of players can jointly deviate in a way which is beneficial for all of them.

[2] Of course, a legislator can abstain from proposal-making. As can be easily seen, such a deviation can never be profitable.

Consequently, in a strong Nash equilibrium players cannot do better even if they are allowed to communicate and collaborate before the game. This property is especially desirable for the analysis of a political decision process.

3. Optimal Policies

To prepare our analysis of voting outcomes in Sections 4 and 5, we now examine which policies agents consider to be optimal. For this purpose, we show that the indirect utility function is strictly concave. The concavity guarantees the existence and the uniqueness of a maximum. We characterize this maximum explicitly.

3.1. Properties of the Indirect Utility Function

Under debt-sensitive majority rules, policies that imply a high debt level may not survive the second stage of the voting process and thus will be excluded from the set of policies legislators will consider in the first stage. Moreover, the status quo policy is chosen from the set $\mathcal{P}_{\leq a}$, which is a proper subset of $\mathcal{P}$. In the following, we prove the existence and the uniqueness of optimal policies within $\mathcal{P}$, as well as within the proper subsets of $\mathcal{P}$.

Denote

$$\mathcal{P}^{\mathrm{f}} := \big\{(\tau, g) \in \mathcal{P} \;\big|\; V_\theta(\tau, g) > -\infty\big\}.$$

Obviously, this definition does not depend on the choice of θ. We now state:

Proposition 1

(i) The set $\mathcal{P}^{\mathrm{f}}$ is convex, and for each $\theta > 0$ the restriction of V_θ to $\mathcal{P}^{\mathrm{f}}$ is strictly concave.

(ii) For any closed and convex set $A \subseteq \mathcal{P}$, the restriction of V_θ to A attains a maximum; the point of maximum is unique if $\mathcal{P}^{\mathrm{f}} \cap A \neq \emptyset$.

The Proposition can be proved by examining the partial derivatives of V_θ. The first-order partial derivatives are given by

$$\partial_\tau V_\theta(\tau, g) = -\frac{1}{1-\tau} - \beta\theta w(\log h)'(Ra + g - \tau w) \tag{9}$$

and

$$\partial_g V_\theta(\tau, g) = \frac{\theta}{g} + \beta\theta(\log h)'(Ra + g - \tau w). \tag{10}$$

The second-order derivatives are

$$\partial_\tau \partial_\tau V_\theta(\tau, g) = -\frac{1}{(1-\tau)^2} + \beta\theta w^2(\log h)''(Ra + g - \tau w), \tag{11}$$

$$\partial_g \partial_\tau V_\theta(\tau, g) = -\beta\theta w(\log h)''(Ra + g - \tau w), \tag{12}$$

$$\partial_g \partial_g V_\theta(\tau, g) = -\frac{\theta}{g^2} + \beta\theta(\log h)''(Ra + g - \tau w). \tag{13}$$

By inspecting the Hessian, we find that V_θ is concave. For details, we refer to the proof of Proposition 1 in the Appendix.

An immediate consequence of Proposition 1 is the following corollary:

Corollary 1 *V_θ possesses a unique point of maximum $\big(\tau^0(\theta), g^0(\theta)\big)$ on $\mathcal{P}$. Furthermore, for each debt level $b \in (Ra - w; \bar{b})$, there is a unique optimal policy $\big(\tau^0_{=b}(\theta), g^0_{=b}(\theta)\big)$ among all policies in $\mathcal{P}_{=b}$, as well as a unique optimal policy $\big(\tau^0_{\leq b}(\theta), g^0_{\leq b}(\theta)\big)$ in $\mathcal{P}_{\leq b}$.*

The optimal policy $(\tau^0(\theta), g^0(\theta))$ within $\mathcal{P}$ is characterized by the first-order conditions

$$\partial_\tau V_\theta\big(\tau^0(\theta), g^0(\theta)\big) \leq 0 \text{ and } \partial_g V_\theta\big(\tau^0(\theta), g^0(\theta)\big) = 0,$$

where strict inequality is only permitted if $\tau^0(\theta) = 0$. Note that the $\leq$-sign results from the strict concavity of V_θ, which implies that a corner solution at $\tau = 0$ occurs if and only if $\partial_\tau V_\theta(0, g) \leq 0$. We do not need to consider the other possible corner solution $\tau = 1$, as $\lim_{\tau \to 1} V_\theta(\tau, g) = -\infty$ and thus a tax rate of $\tau = 1$ cannot be optimal. For the same reason, we can neglect boundary solutions with $g = 0$. Due to (9) and (10), the first-order conditions read

$$\frac{1}{1-\tau} + \beta\theta w \cdot (\log h)'(Ra + g - \tau w) \geq 0, \tag{14}$$

$$\frac{\theta}{g} + \beta\theta \cdot (\log h)'(Ra + g - \tau w) = 0. \tag{15}$$

For the case of an interior solution with $\tau > 0$, Equation (14) holds with equality, and the combination of both equations yields

$$\tau = 1 - \frac{g}{\theta w}\,. \tag{16}$$

Inserting this expression into in the budget restriction leads to

$$g = \frac{b - Ra + w}{1 + \theta^{-1}}\,. \tag{17}$$

Inserting this result into (15) yields

$$\frac{1 + \theta^{-1}}{b - Ra + w} + \beta(\log h)'(b) = 0.$$

Thus, in the case of an interior solution, the optimal debt level $b^0(\theta) := B\big(\tau^0(\theta), g^0(\theta)\big)$ for type θ satisfies the equation

$$b^0(\theta) = Ra - w + \left(1 + \frac{1}{\theta}\right) \frac{-1}{\beta(\log h)'(b^0(\theta))}\,. \tag{18}$$

With $b^0(\theta)$ being given, the optimal values $g^0(\theta)$ and $\tau^0(\theta)$ are determined by the Equations (17) and (16). A corner solution occurs if and only if this procedure leads to a negative value of τ. Then, $\tau^0(\theta) = 0$, and $g^0(\theta)$ and $b^0(\theta)$ are determined by Equation (15) and the budget constraint (3).

In the appendix we show:

Remark 1 The optimal debt level $b^0(\theta)$ depends negatively on the taste parameter θ for public goods.

3.2. Optimal Policy with an Upper Debt Limit

In the following, we determine the optimal policy for an agent of type θ among all policies associated with a debt level below some upper limit $\tilde{b}$, i. e. the optimal policy in the set $\mathcal{P}_{\leq \tilde{b}}$. Recall that we denote this optimal policy by $\big(g^0_{\leq \tilde{b}}(\theta), \tau^0_{\leq \tilde{b}}(\theta)\big)$; we define

$$b^0_{\leq \tilde{b}}(\theta) := B\big(g^0_{\leq \tilde{b}}(\theta), \tau^0_{\leq \tilde{b}}(\theta)\big)$$

to be the corresponding debt level.

We first derive the optimal policy $\big(\tau_{=b}(\theta), g_{=b}(\theta)\big)$ within $\mathcal{P}_{=b}$ for a given b. Since $\tau \leq 1$ and $g \geq 0$, and due to Equation (4), we can restrict ourselves to b satisfying

$$Ra - w < b < \bar{b}. \tag{19}$$

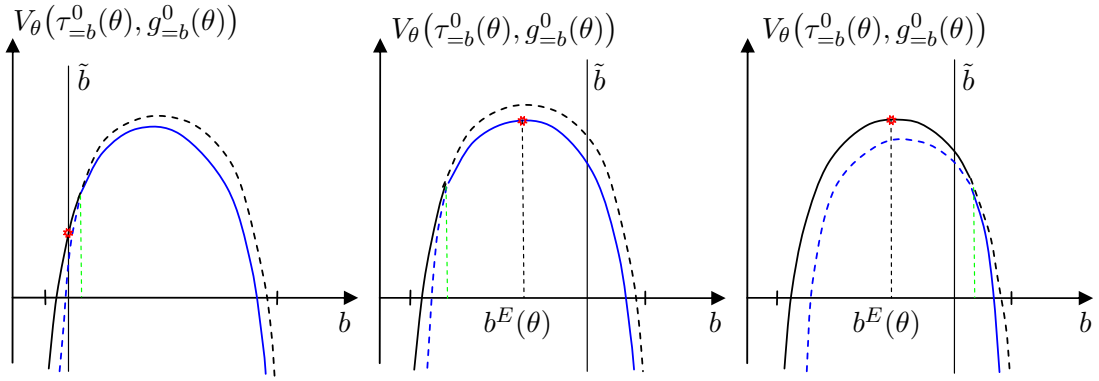

Figure 3: The three candidates for the optimal debt level from Equation (27). The solid curve represents Equation (24) on the interval $\big(Ra - w; \tilde{b}\big)$. The green vertical lines indicate a debt level of $b = Ra + \theta w$, right of which $\tau^0_{=b}(\theta) = 0$.

Lemma 1 *For b satisfying (19), we have*

$$\tau^0_{=b}(\theta) = \max\left\{0; \frac{\theta w - b + Ra}{w(1+\theta)}\right\}, \tag{20}$$

$$g^0_{=b}(\theta) = b - Ra + \tau^0_{=b}(\theta) w. \tag{21}$$

The proof of this Lemma is given in the Appendix. The expressions from the Lemma can be rewritten as

$$1 - \tau^0_{=b}(\theta) = \min\left\{1; \frac{w + b - Ra}{w(1+\theta)}\right\}, \tag{22}$$

$$g^0_{=b}(\theta) = \max\left\{b - Ra; \frac{\theta(w + b - Ra)}{1+\theta}\right\}. \tag{23}$$

By these equations, the indirect utility of type θ as a function of the debt level b amounts to

$$\begin{aligned} &V_\theta\big(\tau^0_{=b}(\theta), g^0_{=b}(\theta)\big) \\ &= \log\min\left\{1; \frac{w + b - Ra}{w(1+\theta)}\right\} + \theta \log\max\left\{b - Ra; \frac{\theta(w + b - Ra)}{1+\theta}\right\} + \theta\beta \log h(b) \\ &= \begin{cases} C + (1+\theta)\log(w + b - Ra) + \theta\beta \log h(b) & \text{if } b < Ra + \theta w, \\ \log(b - Ra) + \theta\beta \log h(b) & \text{if } b \geq Ra + \theta w. \end{cases} \end{aligned} \tag{24}$$

with the constant C reading

$$C := -\log w - (1+\theta)\log(1+\theta) + \theta \log \theta.$$

For any $\tilde{b} \in (Ra - w; \bar{b})$, under the constraint that debt is restricted to $\tilde{b}$, the optimal debt level is

$$b^0_{\leq \tilde{b}}(\theta) = \arg\max_{b \leq \tilde{b}} V_\theta\big(\tau^0_{=b}(\theta), g^0_{=b}(\theta)\big).$$

We denote by $b^E(\theta)$ the maximum of the function $b \mapsto V_\theta\big(\tau^0_{=b}(\theta), g^0_{=b}(\theta)\big)$. This function is depicted in Figure 3. The first case of Equation (24) corresponds to the outer curve, and the second case to the inner curve. The graph of the function is given by the outer curve for debt levels below $Ra + \theta w$, which marks the osculation point of the two curves, and by the inner curve for higher debt levels. The point of maximum $b^E(\theta)$ is given by the maximum of the inner curve if the slope is positive at the osculation point; it is given by the maximum of the outer curve if the slope is negative.

The figure illustrates three cases for $b^0_{\leq \tilde{b}}(\theta)$. If the debt constraint $\tilde{b}$ is binding, the utility-maximizing debt level $b^0_{\leq \tilde{b}}(\theta)$ is given by $\tilde{b}$ (left diagram). If the debt constraint is not binding, $b^0_{\leq \tilde{b}}(\theta)$ is given by the maximum $b^E(\theta)$, which is either the maximum of the inner curve (middle diagram) or the outer curve (right diagram). The following proposition gives a formal characterization.

Proposition 2 *Suppose $\tilde{b} \in \big(Ra - w; \bar{b}\big]$. If $-(\log h)'(Ra + \theta w) > 1/(\beta\theta w)$, let $b^E(\theta)$ being the unique solution of*

$$\frac{1+\theta}{w + b - Ra} = -\theta\beta(\log h)'(b). \tag{25}$$

If $-(\log h)'(Ra + \theta w) \leq 1/(\beta\theta w)$, let $b^E(\theta)$ be the unique solution of

$$\frac{\theta}{b - Ra} = -\theta\beta(\log h)'(b). \tag{26}$$

With this definition, $b^E(\theta)$ is the minimum of the solution of (25) *and the solution of* (26)*. The optimal debt level b under the constraint $b \leq \tilde{b}$ is given by*

$$b^0_{\leq \tilde{b}}(\theta) = \min\big\{\tilde{b}; b^E(\theta)\big\}. \tag{27}$$

The optimal tax rate and the optimal level of government expenditure are given by

$$\tau^0_{\leq\tilde{b}}(\theta) = \max\left\{\frac{\theta w - \tilde{b} + Ra}{w(1+\theta)}; \frac{\theta w - b^E(\theta) + Ra}{w(1+\theta)}; 0\right\}, \tag{28}$$

$$g^0_{\leq\tilde{b}}(\theta) = b^0_{\leq\tilde{b}}(\theta) - Ra + \tau^0_{\leq\tilde{b}}(\theta)w. \tag{29}$$

We prove this proposition in the appendix. For future use we emphasize:

Remark 2 For $\tilde{b} \leq Ra + \theta w$, we have

$$1 - \tau^0_{\leq\tilde{b}}(\theta) = \frac{1}{\theta w} \cdot g^0_{\leq\tilde{b}}(\theta) \tag{30}$$

and

$$b^0_{\leq\tilde{b}}(\theta) = Ra - w + \frac{1+\theta}{\theta} \cdot g^0_{\leq\tilde{b}}(\theta). \tag{31}$$

Proof. For $\tilde{b} \leq Ra + \theta w$, the first term on the right-hand side of Equation (28) is non-negative. Hence, we can neglect the third term (the zero) in this case, and we achieve

$$\tau^0_{\leq\tilde{b}}(\theta) = \frac{1}{w(1+\theta)}\big(\theta w - b^0_{\leq\tilde{b}}(\theta) + Ra\big).$$

Using Equations (27) and (29), we obtain (30). The budget constraint (3) then yields Equation (31). □

For the subsequent analysis, the median type θ^* is particularly important. To simplify the notation, we will henceforth use the abbreviations $\tau^* := \tau^0(\theta^*)$, $g^* := g^0(\theta^*)$, $b^* := b^0(\theta^*)$, $\tau^*_{\leq b} := \tau^0_{\leq b}(\theta^*)$ etc. to describe the optimal values for the median type.

Let us give a short graphical illustration of the agents' optimization problem which we have analyzed in this section. Figure 4 exhibits the indifference curves of the median voter θ^* and his desired policy (τ^*, g^*), with the parameters that will also be used in Example 3 on page 27. If debt is restricted to some $\tilde{b}$, his best policy within $\mathcal{P}_{\leq\tilde{b}}$ is given by $(\tau^*_{\leq\tilde{b}}, g^*_{\leq\tilde{b}})$; it is located on the highest indifference curve that touches $\mathcal{P}_{\leq\tilde{b}}$. The status quo $(\tau^*_{\leq a}, g^*_{\leq a})$ is determined in the same way, with debt being restricted to a. If $\tau^* > 0$, the points $(\tau^*_{\leq\tilde{b}}, g^*_{\leq\tilde{b}})$ for varying values of $\tilde{b}$ lie on the straight line that connects the points (τ^*, g^*) and $(1, 0)$; the line is characterized by Equation (30). If $\tau^* = 0$ (i. e. if we have a corner solution), the straight line hits the vertical axis given by $\tau = 0$. The point (τ^*, g^*) is now located on the vertical axis somewhere above the intersection point.

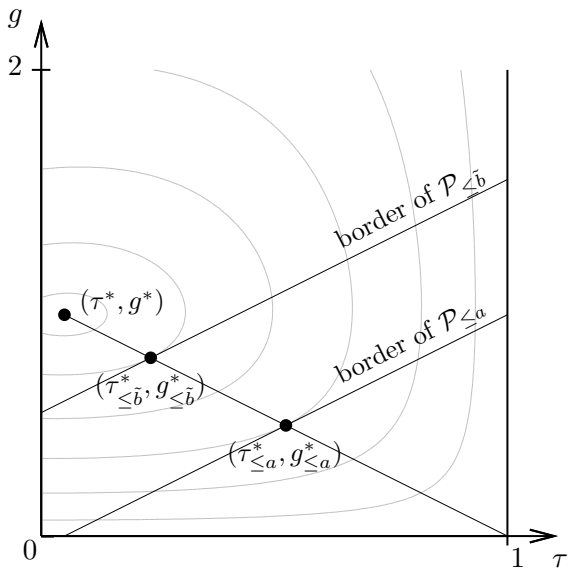

Figure 4: Indifference curves. The parameter values are taken from Example 3 (see page 27).

The set of points $(\tau^*_{\leq \tilde{b}}, g^*_{\leq \tilde{b}})$ then has a kink; it contains the complete straight line as well as the points between the intersection point and the point (τ^*, g^*).

4. Condorcet Winners and the Outcomes Under the Simple Majority Rule

In this section, we turn our attention to the equilibrium of the voting process under the simple majority rule, which we specified in Section 2.3.1. We will establish the existence of a unique Condorcet winner, which is the winner against all other policy proposals under pairwise voting. We show that the Condorcet winner is determined by the median voter's preferences. We further show that it is the equilibrium policy under the simple majority rule.

We use the notation of McKelvey (1974): For any two proposals (τ, g) and (τ', g'), we write $(\tau, g) \succ_\theta (\tau', g')$ if and only if a type-θ agent strictly prefers (τ, g) to (τ', g'), i. e.

$$(\tau, g) \succ_\theta (\tau', g') \;:\Leftrightarrow\; V_\theta(\tau, g) > V_\theta(\tau', g').$$

We introduce the abbreviation $\big\{(\tau, g) \succ_\theta (\tau', g')\big\} := \big\{\theta > 0 \;\big|\; (\tau, g) \succ_\theta (\tau', g')\big\}$.

For any measurable subset $M \subseteq (0;\infty)$, we denote by $|M|$ the measure of M, which means that a fraction $|M| := \mu(M)$ of all agents belongs to M. We use the shorthand

$$\big|(\tau,g) \succ_\theta (\tau',g')\big| := \Big|\big\{(\tau,g) \succ_\theta (\tau',g')\big\}\Big|.$$

We shall use a similar notation for $\succsim_\theta$, $\precsim_\theta$, $\prec_\theta$, etc. instead of $\succ_\theta$.

Definition 5 *A proposal (τ,g) is called a* Condorcet winner *within a set $A \subseteq \mathcal{P}$ of policies if no alternative (τ',g') is strictly preferred to (τ,g) by a simple majority of voters. Formally, (τ,g) is called a Condorcet winner if*

$$\big|(\tau',g') \succ_\theta (\tau,g)\big| \leq \frac{1}{2} \quad \textit{for all } (\tau',g') \in A.$$

The concept of a Condorcet winner within a proper subset A of $\mathcal{P}$ will be particularly relevant for the analysis of the debt-sensitive majority rules, while only the case $A = \mathcal{P}$ will matter for the simple majority rule.

In the following proposition, we describe the set of agents preferring a particular policy to another:

Proposition 3 *Consider any two policies $(\tau,g),(\tau',g') \in \mathcal{P}$. Let $M := \big\{(\tau,g) \succ_\theta (\tau',g')\big\}$. Then, if $\tau = \tau'$, either $M = \emptyset$ or $M = \big(0;\infty\big)$. If $\tau < \tau'$, either $M = \emptyset$ or $M = \big(0;\infty\big)$, or $\tilde\theta \in \big(0;\infty\big)$ exists such that $M = \big(0;\tilde\theta\big)$. If $\tau > \tau'$, either $M = \emptyset$ or $M = \big(0;\infty\big)$, or $\tilde\theta \in \big(0;\infty\big)$ exists such that $M = \big(\tilde\theta;\infty\big)$. In particular, $\big\{(\tau,g) \succ_\theta (\tau',g')\big\}$ and $\big\{(\tau,g) \precsim (\tau',g')\big\}$ are (possibly empty) intervals.*

Proof. If $V_\theta(\tau,g) = -\infty$ or $V_\theta(\tau',g') = -\infty$, the statement is obvious. Otherwise, define the function $d\colon (0;\infty) \to \mathbb{R}$ by

$$d(\theta) := \frac{1}{\theta} \cdot \big(V_\theta(\tau,g) - V_\theta(\tau',g')\big).$$

By equation (7), we have

$$d(\theta) = \frac{1}{\theta}\big(\log(1-\tau) - \log(1-\tau')\big) + \tilde C,$$

with $\tilde C$ being a term that does not depend on θ. We now observe that d is constant if $\tau = \tau'$, strictly decreasing if $\tau < \tau'$, and strictly increasing if $\tau > \tau'$. Furthermore, d is continuous. Since $M = \big\{\theta \;\big|\; d(\theta) > 0\big\}$, the assertion follows. □

By inspecting the proof, we observe:

Remark 3 If $\tilde{\theta}$ in Proposition 3 exists, an individual of type $\tilde{\theta}$ is indifferent between the proposals (τ, g) and (τ', g').

Proposition 3 together with the uniqueness of the median yield the following technical result, which we prove in the appendix.

Corollary 2 *For any two policies $(\tau, g), (\tau', g') \in \mathcal{P}$, let $M := \{(\tau, g) \succ_\theta (\tau', g')\}$. If the median type θ^* is contained in M, then $|M| > 1/2$.*

As a consequence of Proposition 3 and Corollary 2, we obtain a characterization of Condorcet winners:

Corollary 3 *Consider any set $A \subseteq \mathcal{P}$. A policy proposal is a Condorcet winner in A if and only if it is the median voter's optimal policy in A.*

Proof. Consider any proposal (τ, g) that is not a Condorcet winner. Then a proposal (τ', g') exists with

$$\left|(\tau', g') \succ_\theta (\tau, g)\right| > \frac{1}{2}\,. \tag{32}$$

According to Proposition 3, the set $\{(\tau', g') \succ_\theta (\tau, g)\}$ is an interval; thus inequality (32) implies that the set contains the median voter's type. Hence the median voter strictly prefers (τ', g') to (τ, g), so that he does not consider (τ, g) to be optimal in A.

To prove the reverse, take any proposal (τ, g) that is not considered to be optimal by the median voter. Let (τ', g') be any proposal which he strictly prefers to (τ, g). From Corollary 2 we know that $\left|(\tau', g') \succ_\theta (\tau, g)\right| > 1/2$. Hence (τ, g) is not a Condorcet winner. □

From Corollary 3 we observe that Condorcet winners coincide with optimal policies of the median voter. This property allows to characterize voting outcomes below. Recall that θ^* denotes the median voter's type and that we use the abbreviations $\tau^* := \tau^0(\theta^*)$, $g^* := g^0(\theta^*)$, $b^* := b^0(\theta^*)$, $\tau^*_{\leq b} := \tau^0_{\leq b}(\theta^*)$ etc.

Corollary 4 *For each b, a unique Condorcet winner in $\mathcal{P}_{\leq b}$ exists. It is given by $(\tau^*_{\leq b}, g^*_{\leq b})$. In particular, (τ^*, g^*) is the unique Condorcet winner in $\mathcal{P}$.*

Proof. The Corollary follows from Proposition 1, applied to V_{θ^*}, and Corollary 3. □

Corollary 5 *Consider any set $A \subseteq \mathcal{P}$. If (τ, g) is a unique Condorcet winner within A, then*

$$\left|(\tau', g') \succ_\theta (\tau, g)\right| < \frac{1}{2} \quad \textit{for any proposal } (\tau', g') \in A.$$

The proof can be found in the appendix.

As the following proposition states, the Condorcet winner is the equilibrium outcome when a simple majority rule is applied:

Proposition 4 *Let $A \subseteq \mathcal{P}$ be a set of policies, and suppose a unique Condorcet winner (τ, g) within A exists. If the agents decide among the policies of A using sequential pairwise voting with a simple majority rule, then a subgame-perfect strong Nash equilibrium exists, and (τ, g) will be the equilibrium outcome in any subgame-perfect strong Nash equilibrium.*

Proof. Let (τ, g) be a Condorcet winner within A. Consider the strategy profile in which everybody proposes (τ, g) and the proposal is accepted unanimously. This is a subgame-perfect strong Nash equilibrium, which we clarify as follows: Since (τ, g) is the unique Condorcet winner within A, Corollary 5 yields

$$\left|(\tau', g') \succ_\theta (\tau, g)\right| < \frac{1}{2} \quad \text{for any proposal } (\tau', g') \in A.$$

Hence, only a minority of agents would ponder on following a strategy leading to an outcome different from (τ, g), but such a minority does not have the power to alter the outcome. Consequently, no coalition has an incentive to deviate from the beforementioned strategy profile.

It remains to be shown that the outcome of any subgame-perfect strong Nash equilibrium is a Condorcet winner. Suppose there would be a subgame-perfect strong Nash equilibrium with an outcome (τ', g') which is not a Condorcet winner. Then a proposal (τ'', g'') would exist such that

$$\left|(\tau'', g'') \succ_\theta (\tau', g')\right| > \frac{1}{2}.$$

Thus, a majority of agents would deviate by proposing and voting for (τ'', g''). This would contradict the fact that (τ', g') is the outcome of a subgame-perfect strong Nash equilibrium. □

An important remark is in order. A unanimous voting result, as presented in the proof, is unlikely to be observed in actual parliamentary vote. Minorities will fight for their ideas, even if they know that their proposals will not pass. As we have just shown, no coalition of parliamentarians has the power to secure themselves an outcome that is better than (τ, g). Hence, in our situation, parliamentarians would have other motives to deviate than influencing the voting outcome. For the purpose of the proof, it was sufficient to consider the simplest possible strategy profile, in which all agents vote for the winning proposal.

We conclude this section and combine our findings in the first main result:

Theorem 1 *The median voter's desired policy (τ^*, g^*) is the unique voting outcome under the simple majority rule.*

5. Debt-Sensitive Majority Rule

In this section, we analyze the outcome under a debt-sensitive majority rule. Again, we employ the concept of subgame-perfect strong Nash equilibrium. We show that the equilibrium policy is unique and characterize the equilibrium.

5.1. Equilibrium: Definition and Existence

Recall that by Definition 4, a debt-sensitive majority rule is described by a function ϕ, which is assumed to be left-continuous and weakly increasing.

For any policy $(\tau, g) \in \mathcal{P}$, we denote by

$$\alpha(\tau, g) := \Big|(\tau, g) \succsim_\theta \big(\tau^*_{\leq a}, g^*_{\leq a}\big)\Big| \tag{33}$$

the fraction of voters who at least weakly prefer (τ, g) to the status quo.

The following equilibrium definition will turn out to be appropriate.

Definition 6 *Let ϕ be a debt-sensitive majority rule. A policy (τ, g) is called a* semi-equilibrium under ϕ *if it satisfies the following requirements:*

(i) (τ, g) *is a Condorcet winner in* $\mathcal{P}_{\leq B(\tau,g)}$,

(ii) $\alpha(\tau, g) \geq \phi\big(B(\tau, g)\big)$.

A semi-equilibrium (τ, g) *under* ϕ *is called an* equilibrium under ϕ *if no semi-equilibrium* (τ', g') *with* $B(\tau', g') > B(\tau, g)$ *exists.*

To illustrate this definition, we refer to Figure 4 again. The semi-equilibria are exactly those points between (τ^*, g^*) and the status quo $(\tau^*_{\leq a}, g^*_{\leq a})$ for which condition (ii) is fulfilled. The semi-equilibrium with the highest associated debt-level, that is, the highest of these points, is the equilibrium.

In the subsequent proposition, we show that the equilibria in the sense of Definition 6 are the equilibrium outcomes of the voting game described in Section 2.3.2.

Proposition 5 *A policy* (τ, g) *is an equilibrium (in the sense of Definition 6) if and only if it is the outcome of a subgame-perfect strong Nash equilibrium of the voting process under the debt-sensitive majority rule (in the sense of Definition 4).*

Proof. We proceed as in the proof of Proposition 4. Since now voting is two-staged, the arguments are more involved.

Let (τ^ϕ, g^ϕ) be an equilibrium in the sense of Definition 6 and let $b^\phi = B(\tau^\phi, g^\phi)$ be the associated debt level. We show that a subgame-perfect strong Nash equilibrium with the outcome (τ^ϕ, g^ϕ) exists (i. e. the "only if" part of Proposition 5).

Consider the following strategy profile: Everybody proposes (τ^ϕ, g^ϕ); the proposal is accepted unanimously in both stages. This is a subgame-perfect strong Nash equilibrium, which we clarify as follows: No proposal (τ, g) with $B(\tau, g) > b^\phi$ will survive stage 2. Since (τ^ϕ, g^ϕ) is the unique Condorcet winner within the set of policies $\mathcal{P}_{\leq b^\phi}$, Corollary 5 yields

$$\left|(\tau, g) \succ_\theta (\tau^\phi, g^\phi)\right| < \frac{1}{2} \quad \text{for any proposal } (\tau, g) \text{ with } B(\tau, g) \leq b^\phi.$$

Hence, only a minority of agents would ponder on following a strategy leading to an outcome different from (τ^ϕ, g^ϕ), but such a minority does not have the power to alter the outcome. Consequently, no coalition has an incentive to deviate from the above-mentioned strategy profile.

It remains to be shown that the outcome of any subgame-perfect strong Nash equilibrium is an equilibrium in the sense of Definition 6 (i. e. the "if" part of Proposition 5).

Let (τ, g) be the outcome of a subgame-perfect strong Nash equilibrium. Then (τ, g) is the status quo or it is a policy that survives the two stages of the voting game. In any

case, (τ, g) is the unique Condorcet winner within $\mathcal{P}_{\leq B(\tau,g)}$, and $\alpha(\tau, g) \geq \phi\big(B(\tau, g)\big)$. Hence, (τ, g) is a semi-equilibrium.

Suppose that another semi-equilibrium (τ', g') with $B(\tau', g') > B(\tau, g)$ exists. As (τ', g') is the unique Condorcet winner within $\mathcal{P}_{\leq B(\tau',g')}$ and (τ, g) is contained in the set, (τ', g') is strictly preferred to (τ, g) by the median voter, due to Corollary 3. Hence, by Corollary 2,

$$\big|(\tau', g') \succ_\theta (\tau, g)\big| > \frac{1}{2}.$$

It follows that a majority of agents has an incentive to deviate by proposing and voting for (τ', g'), and hence (τ, g) would not be the outcome of a subgame-perfect strong Nash equilibrium. This is a contradiction, and thus (τ, g) is an equilibrium. □

Again, as we have argued after the proof of Proposition 4, a unanimous voting outcome is unlikely to occur in reality, but is sufficient for the purpose of the proof. We next state our second main result.

Theorem 2 *For each debt-sensitive majority rule ϕ, a unique equilibrium under ϕ exists.*

The proof of the theorem can be found in the appendix.

5.2. Characterization of the Equilibrium Policy

We now characterize the equilibrium policy in more detail. The fraction of people who prefer the proposal $(\tau^*_{\leq b}, g^*_{\leq b})$ over the status quo $(\tau^*_{\leq a}, g^*_{\leq a})$ is given by $\alpha(\tau^*_{\leq b}, g^*_{\leq b})$, as defined in Equation (33). As shown in the last section, the equilibrium debt level under the debt-sensitive majority rule $\phi(\cdot)$ is the largest b such that $\alpha(\tau^*_{\leq b}, g^*_{\leq b}) \geq \phi(b)$. The following proposition allows to determine $\alpha(\tau^*_{\leq b}, g^*_{\leq b})$.

Proposition 6 *Define the function $T\colon [a; \bar{b}) \to \mathbb{R}$ by*

$$T(b) := \begin{cases} \dfrac{\log g^*_{=b} - \log g^*_{=a} + \beta\big(\log h(b) - \log h(a)\big)}{\log(1 - \tau^*_{=a}) - \log(1 - \tau^*_{=b})} & \text{for } b \in (a; \bar{b}), \\ -\infty & \text{for } b = a. \end{cases} \tag{34}$$

(i) For b with $a < b \leq Ra + \theta^* w$, we have

$$T(b) = -1 + \beta \cdot \frac{\log h(a) - \log h(b)}{\log(w + b - Ra) - \log(w + a - Ra)}. \tag{35}$$

(ii) $T(b)$ *is strictly increasing for* $a \leq b \leq Ra + \theta^* w$.

(iii) Consider b *with* $a \leq b \leq b^*$. *An agent of type* θ *prefers the policy proposal* $(\tau^*_{\leq b}, g^*_{\leq b})$ *to the status quo* $(\tau^*_{\leq a}, g^*_{\leq a})$ *if and only if*

$$\frac{1}{\theta} \geq T(b). \tag{36}$$

The proof of this proposition can be found in the appendix. The proposition allows us to characterize the equilibrium proposal under a particular debt-sensitive majority rule in a direct way.

Corollary 6 *If* $b^* \geq a$, *the equilibrium debt level under a debt-sensitive majority rule* ϕ *is the largest* $b \in [a; b^*]$ *satisfying*

$$\alpha(\tau^*_{\leq b}, g^*_{\leq b}) \equiv \mu\big\{\theta \;\big|\; 1/\theta \geq T(b)\big\} \geq \phi(b).$$

The equilibrium under the debt-sensitive majority rule is the Condorcet winner within a set of policies that honor some upper debt limit. This upper debt limit is chosen such that the supermajority required by the rule prefers the Condorcet winner within the restricted set to the Condorcet winner when no new debt is allowed (status quo).

We further note that for $b^* < a$ even the median voter favors a policy that implies debt reduction. Hence, any debt-sensitive majority rule will yield the same outcome as a simple majority rule in such circumstances.

6. Examples

6.1. The Setup

In the following, we will compare the simple and the debt-sensitive majority rule in several examples. In all these examples, the relationship between debt and government expenditure is given by

$$h(b) = \gamma(\bar{b} - b)^{\zeta} \quad \text{for } b < \bar{b} \tag{37}$$

with some $\gamma, \zeta > 0$.

Equation (25) then reads

$$\frac{1+\theta}{w + b - Ra} = \frac{\beta\theta\zeta}{\bar{b} - b}, \tag{38}$$

and Equation (26) reads

$$\frac{\theta}{b - Ra} = \frac{\beta\theta\zeta}{\bar{b} - b}. \tag{39}$$

$b^E(\theta)$ is the solution of Equation (38) or (39), depending on whether $-(\log h)'(Ra + \theta w)$ is above or below $1/(\beta\theta w)$; hence,

$$b^E(\theta) = \begin{cases} \dfrac{\beta\theta\zeta(Ra - w) + (1+\theta)\bar{b}}{1 + \theta + \beta\zeta\theta} & \text{for } \dfrac{\beta\theta\zeta w}{\bar{b} - Ra - \theta w} \leq 1, \\ \dfrac{\beta\zeta Ra + \bar{b}}{1 + \beta\zeta} & \text{for } \dfrac{\beta\theta\zeta w}{\bar{b} - Ra - \theta w} > 1. \end{cases} \tag{40}$$

We obtain the optimal debt level from Equation (27). It is given by

$$b^0_{\leq\tilde{b}}(\theta) = \min\left\{\tilde{b},\ \frac{\beta\theta\zeta(Ra - w) + (1+\theta)\bar{b}}{1 + \theta + \beta\zeta\theta},\ \frac{\beta\zeta Ra + \bar{b}}{1 + \beta\zeta}\right\}. \tag{41}$$

Using Equations (28) and (29), we achieve

$$1 - \tau^0_{\leq\tilde{b}}(\theta) = \min\left\{\frac{w + \tilde{b} - Ra}{w(1+\theta)},\ \frac{w + \bar{b} - Ra}{w(1 + \theta + \beta\zeta\theta)},\ 1\right\}, \tag{42}$$

$$g^0_{\leq\tilde{b}}(\theta) = b^0_{\leq\tilde{b}}(\theta) - Ra + \tau^0_{\leq\tilde{b}}(\theta)w. \tag{43}$$

In Section 4, we showed that under the simple majority rule, the policy (τ^*, g^*) will be chosen. This policy corresponds to the optimum of the median voter if there is no restriction on the allowed debt level, which means $(\tau^*, g^*) = \left(\tau^0_{\leq\bar{b}}(\theta^*), g^0_{\leq\bar{b}}(\theta^*)\right)$ and $b^* = b^0_{\leq\bar{b}}$. With the Equations (41), (42) and (43) and taking into account that $1+\theta^*+\beta\theta^* > 1$, we attain the outcome under the simple majority rule:

$$b^* = \min\left\{\frac{\beta\zeta\theta^*(Ra - w) + (1+\theta^*)\bar{b}}{1 + \theta^* + \beta\zeta\theta^*},\ \frac{\beta\zeta Ra + \bar{b}}{1 + \beta\zeta}\right\}, \tag{44}$$

$$\tau^* = \max\left\{1 - \frac{w + \bar{b} - Ra}{w(1 + \theta^* + \beta\zeta\theta^*)},\ 0\right\}, \tag{45}$$

$$g^* = b^* - Ra + \tau^* w\,. \tag{46}$$

For analyzing debt-sensitive majority rules, we use the results from Section 5.2. For h given by (37), we determine the fraction of agents who prefer the policy proposal $\left(\tau^*_{\leq b}, g^*_{\leq b}\right)$ to the status quo $\left(\tau^*_{\leq a}, g^*_{\leq a}\right)$, i. e. the types for which Condition (36) is fulfilled. The function T is given by Equation (34). For $a < b \leq Ra + \theta^* w$, we can use

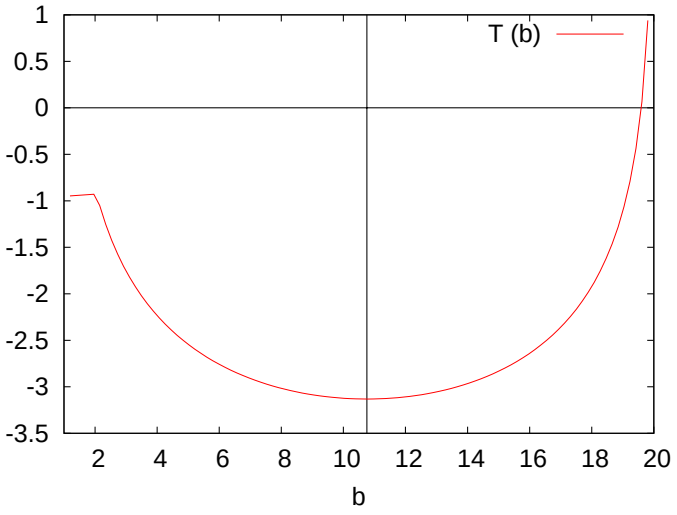

Figure 5: Function T for Example 1.

Equation (35), and obtain

$$T(b) = -1 + \beta\zeta \cdot \frac{\log\big(\bar{b} - a\big) - \log\big(\bar{b} - b\big)}{\log(w + b - Ra) - \log(w + a - Ra)} \ . \tag{47}$$

6.2. Parametrized Examples

We illustrate the impact of debt-sensitive majority rules for specific parameter values.

Example 1 (Linear $h(b)$ with low initial debt) We first consider the parameter values $a = 1$, $w = 1$, $\bar{b} = 20$, $R = 1.05$, $\beta = 1/R$, $\zeta = 1$. Let $1/\theta$, which is the relative weight of consumption in the utility function, be uniformly distributed on the interval $(0; 2]$. Then the median type is $\theta^* = 1$.

For the simple majority rule, Equations (44), (45) and (46) yield

$$b^* = 10.76, \qquad \tau^* = 0, \qquad g^* = 9.71,$$

which is a corner solution – a majority of agents prefers to finance government expenditure completely through debt. This result is mainly driven by the very low initial debt level, which leads to a relatively low utility loss of debt-financing compared to tax-financing. The status quo is given by the Equations (41) to (43)

$$b^*_{\leq a} = a = 1, \qquad \tau^*_{\leq a} = 0.525, \qquad g^*_{\leq a} = 0.475.$$

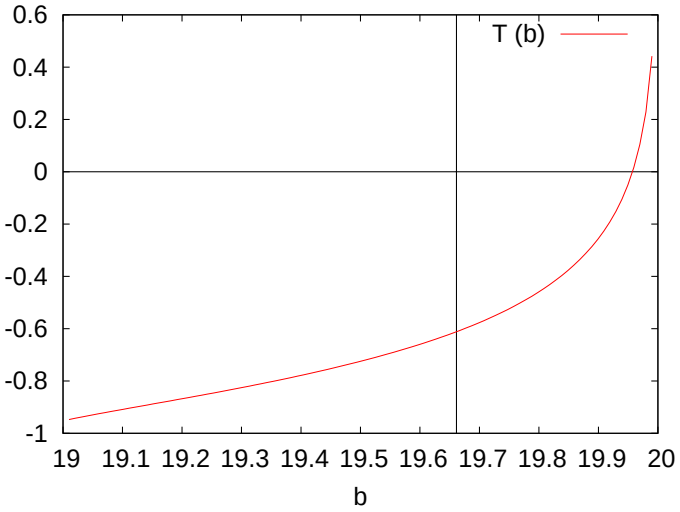

Figure 6: Function T for Example 2.

Figure 5 shows the graph of the function T, as defined in Equation (34). The curve has a kink at a debt level of $b = Ra + \theta^* w = 2.05$, above which the median type's desired tax rate is zero. The vertical line depicts the equilibrium debt level b^* under the simple majority rule. The function T is negative on the interval $(a; b^*)$. This means that all agents prefer the optimal policy (τ^*, g^*) of the median voter to the status quo. Hence, a debt-sensitive majority rule as introduced in Section 5 will have no effect. ♢

Example 2 (Linear $h(b)$ with high initial debt) Now, we consider a higher initial debt level given by $a = 19$, keeping the other parameters unchanged.

Here, we obtain the outcome

$$b^* = 19.66, \qquad \tau^* = 0.64, \qquad g^* = 0.36.$$

In this example, the status quo is given by

$$b^*_{\leq a} = a = 19, \qquad \tau^*_{\leq a} = 0.975, \qquad g^*_{\leq a} = 0.025.$$

The function T is depicted in Figure 6. Again, a debt-sensitive majority rule will have no effect, as T is negative on the relevant interval (a, b^*). ♢

In Examples 1 and 2, debt accumulation cannot be prevented by a debt-sensitive majority rule. The reason is two-fold. First, the marginal disutility of debt at any level

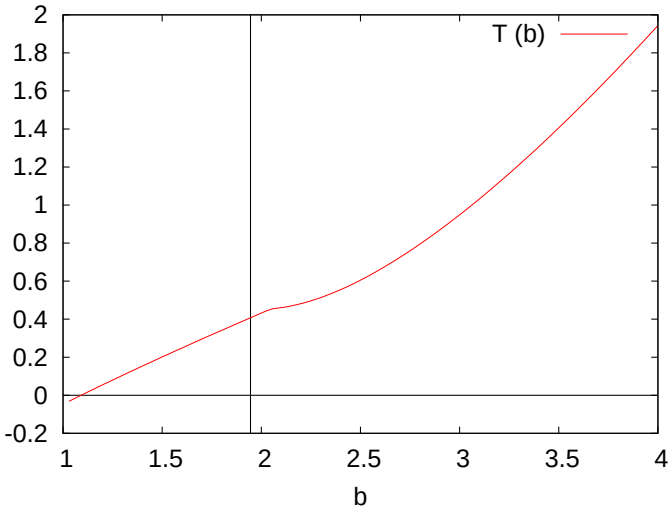

Figure 7: Function T for Example 3.

is small, as future government expenditures depend on debt linearly. Second, the status quo is extremely bad compared to the median voter's optimal policy, even for large values of θ. This is true even if initial debt is very high, because severe taxation is necessary to maintain the status quo in such circumstances, which results in a sharp cut of private consumption and thus, high marginal disutility. This effect outweighs the negative impact of debt-making on future public good provision.

We consider a third example where higher debt causes a disproportionately high decrease in h. Such a relationship seems reasonable. For instance, an increasing debt level may coincide with increasing interest rates, which additionally triggers the government budget, or agents may have altruistic motives and do not want to shift a high debt burden to future generations. The role of altruism will be discussed in Section 8.

Example 3 (Stronger impact of debt on future utility) In this example, we calibrate the model with $a = 1$, $w = 1$, $\bar{b} = 20$, $R = 1.05$, $\beta = 1/R$, $\zeta = 20$. These parameter values yield

$$b^* = 1.95, \qquad \tau^* = 0.05, \qquad g^* = 0.95.$$

The status quo is the same as in Example 1. $T(b)$ is depicted in Figure 7. One observes that $T(b)$ is no longer negative on the whole interval (a, b^*), but positive for $b > 1.1$. Thus, proposals that imply a debt level near b^* are not accepted unanimously. The fraction $\alpha(\tau^*_{\leq b}, g^*_{\leq b}) = \mu\{\theta \mid 1/\theta \geq T(b)\}$ of parliamentarians who accept the policy proposal $(\tau^*_{\leq b}, g^*_{\leq b})$ is decreasing in b and is strictly below 1 if b is larger than 1.1.

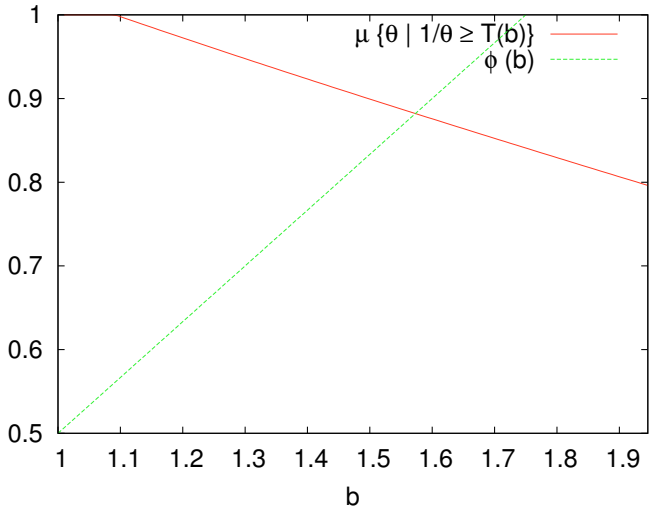

Figure 8: Function $\mu\{\theta \mid 1/\theta \geq T(b)\}$ (solid line) and $\phi(b)$ (dotted line) for Example 3.

Hence, we can prevent debt accumulation beyond this level by employing a suitable debt-sensitive majority rule. As we have proved in Section 5, the resulting debt level is the highest b such that $\mu\{\theta \mid 1/\theta \geq T(b)\} \leq \phi(b)$.

Consider, for instance, the debt-sensitive majority rule $\phi(b) = \frac{1}{2} + \frac{2}{3}(b - a)$, which is depicted in Figure 8. The intersection of the graphs of $\mu\{\theta \mid 1/\theta \geq T(b)\}$ and $\phi(b)$ gives the equilibrium value of b. In this example, the future debt level is reduced from 1.95 to around 1.55 by the introduction of the debt-sensitive majority rule $\phi(b)$. ♢

6.3. A Sufficient Condition for the Effectiveness of Debt-Sensitive Majority Rules

Examples 1–3 indicate that a debt-sensitive majority rule will have an effect only if the graph of T hits the horizontal axis below b^*. We now provide a sufficient condition for debt-sensitive majority rules to be effective.

A situation similar to the ones shown in the Examples 1 and 2 will not arise if T starts above the horizontal axis, i.e. if $\lim_{b \downarrow a} T(b) \geq 0$. By L'Hospital's rule, we have

$$\lim_{b \downarrow a} T(b) = -1 + \beta \lim_{b \downarrow a} \frac{-(\log h)'(b)}{(w + b - Ra)^{-1}} = -1 - \beta(w + a - Ra)(\log h)'(a).$$

Thus, we obtain

Proposition 7 *Suppose that $b^* > a$, and agents with arbitrarily large θ exist, i.e. 0 is a boundary point for the support of the distribution of $1/\theta$. Then, a sufficient condition for a debt-restricting debt-sensitive majority rule to exist is*

$$-\beta(w + a - Ra)(\log h)'(a) \geq 1.$$

Agents with high values of θ reflect those people of the society who strongly rely on public services and public goods provided by the government. Examples are older people who rely heavily on public transportation or on public health services, or people particularly concerned about security.

6.4. Risky Public-Good Provision

As a further example, we consider risky public-good provision. There are many different reasons for such risks. For instance, a high level of future debt increases the risk of a substantial cut in future public-good provision, since a future downturn might reduce the ability to pay interest, interest rates might rise, or the country might even default. It could also happen that future electorates will re-allocate government funds to activities that do not benefit the current electorate. In order to capture such uncertainties in our model, let $\log h(b)$ be a mixture of a first term reflecting a normal economic situation and a second term reflecting a situation in which public-good provision is severely encumbered. The weight of the second term increases with the debt level. We use the following specification:

$$\log h(b) = \frac{b}{\bar{b}} \log\bigl(\gamma_1(\bar{b} - b)^\zeta\bigr) + \left(1 - \frac{b}{\bar{b}}\right) \log\bigl(\gamma_2(\bar{b} - b)^\zeta\bigr), \tag{48}$$

with γ_2 considerably larger than γ_1. The value $1 - b/\bar{b}$ describes the probability of normal economic times. The fraction $b/\bar{b}$ reflects the probability of a drastic reduction of public-good provision. This reduction is modelled by a low value of γ_1. The probability $b/\bar{b}$ that the amount of public goods will be severely shortened increases with the level of public debt.

Again the median voter's desired debt level follows from Equation (27). It is given by $b^* = b^E(\theta^*)$, with $b^E(\theta^*)$ being the solution of Equation (25), which now reads

$$\frac{1 + \theta}{w + b - Ra} = -\beta\theta \left(\frac{\log \gamma_1 - \log \gamma_2}{\bar{b}} - \frac{\zeta}{\bar{b} - b}\right),$$

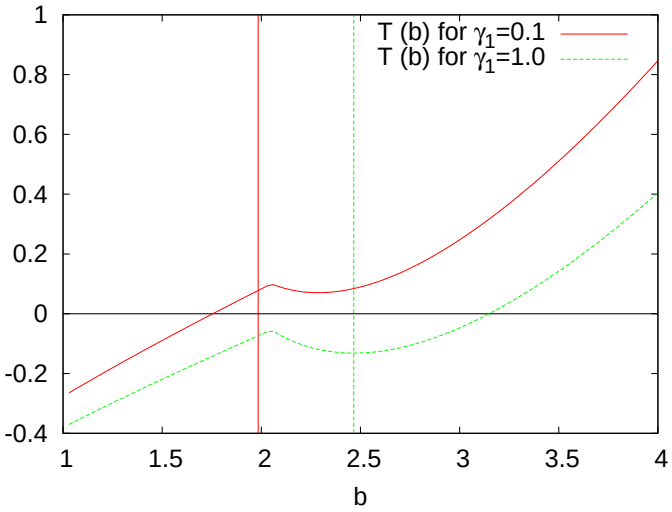

Figure 9: Functions T for Section 6.4.

or the solution of Equation (26), which reads

$$\frac{\theta}{b - Ra} = -\beta\theta \left(\frac{\log \gamma_1 - \log \gamma_2}{\bar{b}} - \frac{\zeta}{\bar{b} - b} \right).$$

From Equation (35), we obtain

$$T(b) = \frac{\log g^*_{=b} - \log g^*_{=a} + \beta\big[(b-a)\bar{b}^{-1}(\log \gamma_2 - \log \gamma_1) + \zeta\big(\log(\bar{b} - a) - \log(\bar{b} - b)\big)\big]}{\log(1 - \tau^*_{=a}) - \log(1 - \tau^*_{=b})}.$$

We use the distribution of θ from Example 1 and the parameter values $a = 1$, $w = 1$, $\bar{b} = 20$, $R = 1.05$, $\beta = 1/R$, $\zeta = 13$, $\gamma_1 = 0.1$, and $\gamma_2 = 1$. These yield

$$b^* = 1.98, \qquad \tau^* = 0.03, \qquad g^* = 0.97.$$

Again, the status quo is that of Example 1. In Figure 9, the function $T(b)$ is plotted for this example and for a scenario without the risk of a severe decline of public-good provision (i. e. $\gamma_1 = \gamma_2 = 1$). In the latter scenario, the optimal values for the median type are

$$b^* = 2.47, \qquad \tau^* = 0, \qquad g^* = 1.42.$$

The median voter's optimal debt b^* is lower in the presence of risk, which results in lower debt-making also under the simple majority rule. This is illustrated by the left-shift of the vertical line. The figure illustrates that a debt-sensitive majority rule can be

effective, as the solid $T(b)$-curve is partly above the horizontal axis. In the benchmark scenario without risky public-good provision, the $T(b)$-curve (dashed) hits the horizontal line at a point to the right of b^*, and hence, the introduction of a debt-sensitive majority rule would have no effect. The upward-shift of the T-curve can be explained as follows:

In the scenario with risky public-good provision, the expected utility $\log h(b)$ from future public good consumption is lower relative to the status quo utility $\log h(a)$, for any debt level b. Agents with a high value of θ, therefore, now tend to favor the status quo over the median voter's desired policy, if this policy is connected with a high debt level. This leads to a larger fraction of agents opposing debt accumulation compared to the case where $\gamma_1 = \gamma_2 = 1$.

7. Debt-Sensitive Majority Rules as Political Stabilizers

Debt-sensitive majority rules can limit debt accumulation. The same goal, however, could be achieved by fixed debt limits. Indeed, the following proposition holds as a direct consequence of Proposition 5:

Proposition 8 *Consider some debt-sensitive majority rule with an associated policy outcome (τ^ϕ, g^ϕ) and an associated debt level $b^\phi = B(\tau^\phi, g^\phi)$. If a debt limit of $b^{\text{fix}} = b^\phi$ were fixed by constitution, the simple majority rule would yield the same outcome (τ^ϕ, g^ϕ).*

We now illustrate that a debt-sensitive majority rule, in contrast to fixed debt limits, can stabilize macroeconomic shocks.

A crucial advantage of debt-sensitive majority rules over fixed debt limits is that debt can increase when negative macroeconomic shocks occur. Thus, a debt-sensitive majority rule allows for budget deficits in downturns, and hence acts as a macroeconomic stabilizer. As stabilization is the result of the voting procedure, we call debt-sensitive majority rules *political* stabilizers. We illustrate this property for a negative income shock.

Consider a temporary shock which implies a decline of w to $w_r < w$. Since the shock is not persistent, $\bar{b}$ is not affected. From Equation (35), we observe that with the reduction in wage, the function $T(b)$ increases more slowly than before, as long as we do not have a corner solution at $\tau = 0$. The median voter's desired debt level also increases from b^*

to b_r^* due to the recession (see Equation 25). According to Corollary 6, the wage shock will increase the debt level chosen under the debt-sensitive majority rule, i. e. $b_r^\phi > b^\phi$.

With a fixed debt limit, however, the debt level would remain unchanged at $b^{\text{fix}} < b_r^\phi$. The debt-sensitive majority rule thus guarantees more flexibility in policy-making by allowing for and resulting in a higher debt level, compared to a fixed debt limit, when negative shocks occur. As indicated by Equations (31) and (30), the higher debt level under the debt-sensitive majority rule is accompanied by a higher public good provision and a lower tax rate. Thus, debt-sensitive majority rules act as political stabilizers.

8. Implementation and Extensions

Our analysis has indicated that debt-sensitive majority rules may act as a debt-brake. Nevertheless, their introduction may turn out to be difficult, as such rules may not be in the interest of the current electorate. In Section 8.1 we assess ways to overcome these difficulties. Several extensions of our model are discussed in Section 8.2.

8.1. Implementation

A debt-sensitive majority rule allows a minority to block debt accumulation. Right in the cases in which the rule is effective, it leads to an outcome being utility inferior for a majority of agents, which is implies that the introduction of a debt-sensitive majority rule will not find a majority in parliament. In this section we outline two ways how debt-sensitive majority rules could be introduced nevertheless.

Prompt introduction, but future effectiveness. Consider a situation in which the present debt is moderate, so that the proposed debt-sensitive majority rule would not bind immediately. If in such a situation, agents expect a future political environment to be less disciplined with respect to debt accumulation, a debt-restricting policy-rule could find a majority.

Delayed application and altruism. A promising way to introduce debt-sensitive majority rules is immediate introduction, but delayed application. Suppose that agents exhibit some degree of altruism, which can be small. An agent who cares for his descendants to some extent will fear excessive debt accumulation, as it limits the government's ability to provide public goods. Hence, an altruistic agent may want a debt-sensitive rule to be used in the future. In the first generation for which the debt-sensitive majority rule is

effective, a majority of agents will, however, experience a utility loss – these agents will inherit a relatively high debt level from their parents, while they will be limited in their debt-financing potential. A rule may be advantageous for the second and all subsequent generations. If these utility gains outweigh the utility loss of the first generation, a majority of the current generation may be willing to adopt a debt-sensitive majority rule today if it is applied from the next generation onwards. The introduction has to be coupled with the rule that the abolishment requires a supermajority. Otherwise, the next generation would immediately abolish the debt-sensitive majority rule.[3]

Another reason that favors delayed application is common in a parliamentary democracy, in which politicians are elected only for a limited period of time. Politicians may be willing to introduce unpopular policy measures for some future point in time, as they will be no longer in office when negative side-effects appear.

An example for delayed application of a long-term, welfare-improving policy is the recent debt-brake in Germany *(Schuldenbremse)*. It states that the government's budget has to be nearly balanced every year – "nearly" meaning that the budget deficit is restricted to 0.35 % of GDP. Debt that is accumulated in severe economic crises has to be repaid as soon as the economy recovers. The rule was introduced in 2009 and will be fully enforced by the year 2020 (see Mody and Stehn, 2009, and Article 115 of the German Constitution).

8.2. Further Extensions

So far, we have assumed that agents differ in their taste for public goods. In practice, there are other sources of heterogeneities among citizens that may give debt-sensitive majority rules further foundations in their role to limit government debt-accumulation. We sketch three sources of heterogeneity in this section.

Initial old generation. We could add an old generation in the first period. Such a generation would be in favor of high debt. If the old generation were the majority, debt-sensitive majority rules would be particularly effective, as the simple majority rule would lead to very high debt levels.

Different degrees of altruism. Suppose that agents are altruistic and differ in the weight they attach to the utility of future generations. In our model, the simplest way to analyze

[3] The idea of a delayed application of constitutional rules profitable for future generations is discussed in Gersbach and Kleinschmidt (2009).

the effects of altruism is to interpret $h(b)$ as the overall utility of future periods, which comprises the agent's own utility at old age and the discounted total utility of future generations. Then, different levels of β would reflect different degrees of altruism. Since the utility $h(b)$ of future periods is strictly decreasing in the debt level b left to the next generation, a higher value of β exhibits higher altruistic motives and implies a higher distaste for public debt. If there is heterogeneity regarding β, highly altruistic agents will block the tendency towards higher public debt under a debt-sensitive majority rule.

Income heterogeneity. Assume that agents differ in their labor income. How income heterogeneities affect the effectiveness of debt-sensitive majority rules depends on the tax scheme. As long as the assumption of a flat tax, i. e. a constant tax rate for all individuals, is maintained, our results remain unaltered: The reason is that individual wage enters the indirect utility function as a constant summand, and thus does not influence the agents' policy preferences.

In addition, we consider now a two-level tax system where individuals with low incomes are exempted from taxation. In our model, such a tax system has two implications.

First, non-taxed individuals desire high tax rates to finance a high level of public goods, as they face the utility function

$$\begin{aligned} V_\theta &= \log(1-\tau)\big|_{\tau=0} + \theta \log g + \theta\beta \log h(b) \\ &= \theta \log g + \theta\beta \log h(b). \end{aligned}$$

This expression indicates that individuals exempted from taxation behave exactly like taxed individuals having an infinite weight $\theta \to \infty$ on public-good consumption. Hence, to study the impact of a group of non-taxed individuals, we can replace a certain fraction of the society by individuals with an extremely high value of θ.

Second, for any given tax rate, the per-capita tax-revenue will be smaller, since only a fraction of the society can be taxed. As the wage rate enters the optimization problem only through the government budget constraint, the assumption of non-taxed individuals formally implies a decrease in the wage rate w.

The effect on the outcome depends on the interrelation between the change in the individuals' types and the change in the wage rate. We analyze the effects under the precondition that we have an interior optimum with $\tau^* > 0$.

The existence of individuals with high θ will typically lead to an upward-shift of the median type θ^*. The median voter's desired debt level will decrease, as we have seen

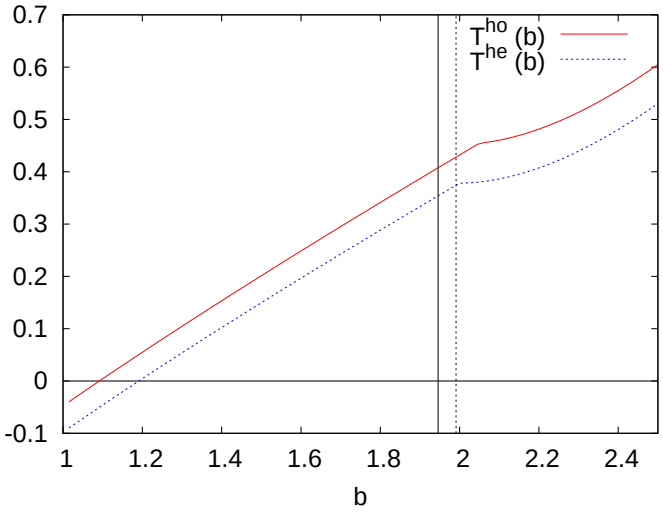

Figure 10: Function T without and with income heterogeneity (for the parameter values given in the text). The vertical lines indicate the resulting debt levels under the simple majority rule without (left, solid line) and with (right, dotted line) income heterogeneity.

at the end of Section 3.1. Hence, with income heterogeneity and discriminating taxes, one will already observe a more disciplined policy under the simple majority rule. As θ^* does not occur in Equation (35), the change of the median does not have a *direct* influence on the equilibrium debt level under a debt-sensitive majority rule.

To capture the remaining effects, we modify Example 3, and suppose that 15 % of the individuals are exempted from taxation, who earn 5 % of total wage. Assume that the decision not to tax low incomes can be revoked and, therefore, does not affect the maximum debt level $\bar{b}$. Further, to prescind from the effects of a changing median type, we assume that the non-taxed individuals are those with the highest valuation of public-good consumption anyway (i. e. individuals with $0 < 1/\theta < 0.3$ are exempted from taxation). Our assumptions imply that the taxed wage decreases from $w^{\text{ho}} = 1.0$ to $w^{\text{he}} = 0.95$, while all other parameter values, including $\theta^* = 1$, remain unchanged.

The decrease of w leads to a downward shift of the function T, which can easily be seen

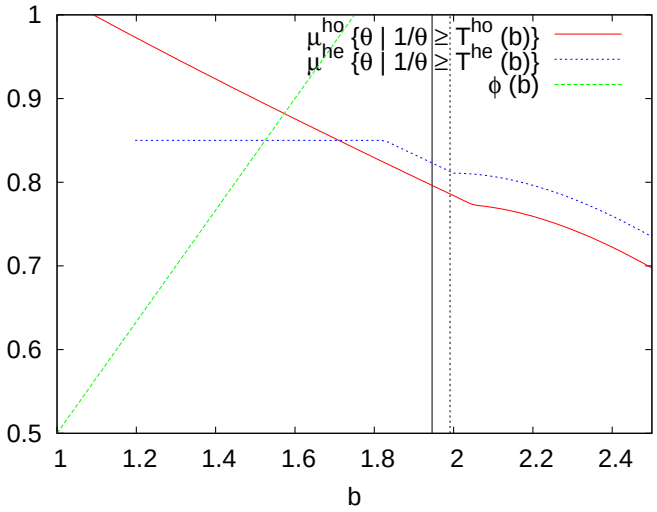

Figure 11: Fractions of acceptance $\mu\{\theta \mid 1/\theta \geq T(b)\}$ without and with income heterogeneity, as well as a debt-sensitive majority rule $\phi(b)$. As in Figure 10, the vertical lines indicate the resulting debt levels under the simple majority rule.

from the derivative

$$\frac{\mathrm{d}}{\mathrm{d}w}T(b) = -\beta \cdot \underbrace{\frac{\log h(a) - \log h(b)}{\big(\log(w+b-Ra) - \log(w+a-Ra)\big)^2}}_{>0} \cdot \underbrace{\Big(\frac{1}{w+b-Ra} - \frac{1}{w+a-Ra}\Big)}_{<0}$$

$$> 0.$$

Intuitively, if the tax base is lower, the tax revenue is smaller for any given tax level; hence *taxed* agents will consider taxation a less attractive instrument, and favor debt-making. We see from Equation (18) that b^* is shifted to the right, as long as we have an interior solution involving a strictly positive tax rate. For our example, the downward shift of the T-curve and the change of b^* are illustrated by Figure 10. Compared to Example 3, the existence of income heterogeneity makes the equilibrium variables change to

$$b^* = 1.99, \qquad \tau^* = 0.005, \qquad g^* = 0.945.$$

Figure 11 shows how the acceptance of debt-making, represented by $\mu\{\theta \mid T(b) \geq 1/\theta\}$, changes. The non-taxed agents, who comprise 15 % of the society, oppose debt-making most strictly. This is reflected by the fact that the blue curve progresses horizontally at a value of 0.85. The first kink indicates the debt level at which further agents begin to

oppose. If the green curve hits the blue curve before the red curve (as is the case in the Figure), income heterogeneity increases the effectiveness of the debt-sensitive majority rule. In the present example, higher majorities are needed for a wide range of debt levels under income heterogeneity than in the case of homogeneous incomes. The effect may, nevertheless, be reversed if the downward shift of the T curve (taxation becoming less favorable for the agents being taxed) is dominating.

9. Conclusion

Using a simple model of public-good provision, we have examined the effectiveness of debt-sensitive majority rules in restricting the excessive accumulation of public debts. The analysis has left out many issues that deserve further scrutiny in future research. In Section 8, we have already mentioned a variety of useful extensions of our model, which strengthen the effectiveness of debt-sensitive majority rules. It will be useful to calibrate debt-sensitive majority rules to business-cycle movements and to develop debt-sensitive majority rules for practical applications. Such rules may involve three or four critical thresholds at which the required majority increases. An extension of our model to an infinite (or longer-term) horizon will be worth pursuing, as this will allow to compare the long-run debt-paths under the two voting rules.

A. Proofs

Proof of Proposition 1. Consider any $\theta > 0$. The convexity of $\mathcal{P}^{\mathrm{f}}$ is obvious. On $\mathcal{P}^{\mathrm{f}}$, the function V_θ is twice continuously differentiable. The first- and second-order derivatives are given by the Equations (9) to (13) in the main text.

The assumption $(\log h)'' < 0$ from Equation (5) implies

$$\partial_\tau \partial_\tau V_\theta < 0$$

and

$$\det \operatorname{Hess} V_\theta = \frac{\theta}{(1-\tau)^2 g^2} - \beta\theta\Big(\frac{w^2\theta}{g^2} + \frac{1}{(1-\tau)^2}\Big) \cdot (\log h)''(Ra + g - \tau w) > 0;$$

hence the Hessian

$$\operatorname{Hess} V_\theta = \begin{pmatrix} \partial_\tau \partial_\tau V_\theta & \partial_g \partial_\tau V_\theta \\ \partial_g \partial_\tau V_\theta & \partial_g \partial_g V_\theta \end{pmatrix}$$

is negative definite. The function V_θ is, therefore, strictly concave on $\mathcal{P}^{\mathrm{f}}$.

Consider any closed and convex set $A \subseteq \mathcal{P}$ with $\mathcal{P}^{\mathrm{f}} \cap A \neq \emptyset$. There exists $c > -\infty$ such that the set $G := \big\{(\tau, g) \in A \;\big|\; V_\theta(\tau, g) \geq c\big\}$ is non-empty. Any point of maximum of V_θ on A lies within G. Since V_θ is continuous and A is closed, G is closed. In addition, since $\max_{\tau \in [0;1]} V_\theta(\tau, g) \to -\infty$ for both $g \to 0$ and $g \to \infty$, G is bounded, hence it is compact. Therefore, the restriction of V_θ to A attains a maximum on G, which by the definition of G, is a global maximum of the restriction of V_θ to A.

Any point of maximum of the restriction of V_θ to A must lie in $\mathcal{P}^{\mathrm{f}} \cap A$. The convexity of $\mathcal{P}^{\mathrm{f}} \cap A$ and the strict concavity of V_θ on $\mathcal{P}^{\mathrm{f}} \cap A$ imply that the point of maximum is unique. □

Proof of Remark 1. By differentiating Equation (18) totally, we achieve

$$\mathrm{d}b^0 = \frac{1}{\theta^2 \beta (\log h)'(b^0)} \mathrm{d}\theta + \Big(1 + \frac{1}{\theta}\Big) \frac{(\log h)''(b^0)}{\beta \big((\log h)'(b^0)\big)^2} \mathrm{d}b^0.$$

As $(\log h)'$ is negative and monotonically decreasing, we obtain

$$\frac{\mathrm{d}b^0}{\mathrm{d}\theta} < 0.$$

□

Proof of Lemma 1. Solving the budget constraint (3) with respect to g and inserting into (7) yields

$$V_\theta(\tau, b - Ra + \tau w) = \log(1 - \tau) + \theta \log(b - Ra + \tau w) + \beta\theta \log h(b). \tag{49}$$

The derivative with respect to τ reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau} V_\theta(\tau, b - Ra + \tau w) = -\frac{1}{1 - \tau} + \theta \frac{w}{b - Ra + \tau w}\ . \tag{50}$$

This expression is decreasing in τ. Hence, the expression in equation (49), as a function of τ, takes its maximum in $\tau = 0$ if and only if

$$\frac{\mathrm{d}}{\mathrm{d}\tau} V_\theta(\tau, b - Ra + \tau w)\Big|_{\tau=0} \leq 0.$$

An interior optimum is obtained when (50) equals zero for some $\tau > 0$. Solving for τ yields

$$\tau = \frac{\theta w - b + Ra}{w(1+\theta)}. \tag{51}$$

A corner solution at $\tau = 0$ occurs if and only if

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}\tau} V_\theta(\tau, b - Ra + \tau w)\Big|_{\tau=0} = -\frac{1}{1-\tau} + \theta \frac{w}{b - Ra + \tau w}\Big|_{\tau=0} \leq 0, \\ \Leftrightarrow \quad & \theta \frac{w}{b - Ra} \leq 1, \\ \Leftrightarrow \quad & \theta w \leq b - Ra. \end{aligned}$$

Therefore, we obtain a corner solution at $\tau = 0$ if and only if (51) takes a negative value. □

Proof of Proposition 2. Due to the first-order condition, a local optimum for b for the function

$$f_1 \colon b \mapsto C + (1+\theta)\log(w + b - Ra) + \theta\beta \log h(b), \tag{52}$$

which corresponds to the first case of (24), is given by the solution of Equation (25). As the left-hand side of this equation is monotonically decreasing in b with $(1+\theta)/(w+b-Ra) \to +\infty$ for $b \downarrow Ra - w$, and the right-hand side is monotonically increasing in b and converging to $+\infty$ for $b \uparrow \bar{b}$, a unique solution exists. Since (52) is strictly concave, the solution is a maximizer. A similar argument holds for the second case of (24), i. e. the function

$$f_2 \colon b \mapsto \log(b - Ra) + \theta\beta \log h(b),$$

and the solution of Equation (26).

By inspecting (24), we observe that $b \mapsto V_\theta\big(\tau^0_{=b}(\theta), g^0_{=b}(\theta)\big)$ is continuously differentiable and strictly concave. It attains its maximum at the solution of Equation (25) if and only

if the derivative of (52) is negative (or zero) at $b = Ra + \theta w$, i. e. if and only if

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}b}\big((1+\theta)\log(w+b-Ra) + \theta\beta \log h(b)\big)\Big|_{b=Ra+\theta w} \leq 0 \\ \Leftrightarrow \quad & \frac{1+\theta}{w+b-Ra} + \theta\beta(\log h)'(b)\Big|_{b=Ra+\theta w} \leq 0 \\ \Leftrightarrow \quad & -(\log h)'(Ra+\theta w) \geq \frac{1}{\beta\theta w}. \end{aligned}$$

Similarly, the function attains its maximum at the solution of Equation (26) if and only if the derivative is positive (or zero). Therefore, the point of maximum is given by $b^E(\theta)$.

We have to show that $b^E(\theta)$ is the minimum of the maximizer of f_1 and the maximizer of f_2: The derivatives f_1' and f_2' are continuous and strictly decreasing. As we have seen above, there is exactly one point of intersection, at $Ra + \theta w$. Calculation shows that $f_1''(Ra+\theta w) < f_2''(Ra+\theta w)$. Hence, the root of f_2' is left of the root of f_1' if $f_1'(Ra+\theta w) < 0$ and right of the root of f_1' if $f_1'(Ra+\theta w) > 0$.

By concavity, the function $b \mapsto V_\theta\big(\tau^0_{=b}(\theta), g^0_{=b}(\theta)\big)$ is increasing left of the maximum; this implies (27). By inserting this into (20) and (21), we achieve (28) and (29). □

Proof of Corollary 2. If $\tau = \tau'$, $M = \emptyset$ or $M = (0;\infty)$ by Proposition 3; hence, there is nothing to prove. We consider the case $\tau > \tau'$; the case $\tau < \tau'$ is similar.

Assume that $|M| = 1/2$. By Proposition 3, $\tilde\theta$ exists such that $M = (0;\tilde\theta)$. Since $\mu(M) = 1/2$, $\mu\big([\tilde\theta;\infty)\big) = 1/2$. Hence, $\tilde\theta$ is a median. Since the median is unique, we have $\theta^* = \tilde\theta$. It follows that $\theta^* \notin M$. □

Proof of Corollary 5. Consider any $(\tau', g') \in A$ with $(\tau', g') \neq (\tau, g)$. Since (τ', g') is a Condorcet winner within A, we have

$$\big|(\tau', g') \succ_\theta (\tau, g)\big| \leq \frac{1}{2}\,.$$

In order to reach a contradiction, we assume that

$$\big|(\tau', g') \succ_\theta (\tau, g)\big| = \frac{1}{2}\,. \tag{53}$$

By Proposition 3, we know that $\big\{(\tau', g') \succ_\theta (\tau, g)\big\}$ is of the form $(0;\tilde\theta)$ or $(\tilde\theta;\infty)$. By Equation (53), $\tilde\theta$ is a median, and by the uniqueness of the median, $\tilde\theta = \theta^*$. Hence, by Remark 3, the median type θ^* is indifferent between the two proposals (τ, g) and (τ', g').

By Corollary 3, optimal policies of the median type and Condorcet winners are the same. Since, the Condorcet winner was assumed to be unique, we obtain a contradiction. □

Proof of Theorem 2. We first prove uniqueness. Consider two equilibria (τ, g) and (τ', g'). By definition of equilibrium, $B(\tau, g) = B(\tau', g')$. Since by Corollary 4, the Condorcet winner in $\mathcal{P}_{\leq B(\tau,g)} = \mathcal{P}_{\leq B(\tau',g')}$ is unique, $(\tau, g) = (\tau', g')$.

Now let us show existence: From Corollary 4, we know that for any b, a unique Condorcet winner in $\mathcal{P}_{\leq b}$ exists. It is given by the proposal $\big(\tau^*_{\leq b}, g^*_{\leq b}\big)$, which the median voter considers to be optimal among all proposals in the set $\mathcal{P}_{\leq b}$. Let

$$\mathcal{P}^* := \Big\{ \big(\tau^*_{\leq b}, g^*_{\leq b}\big) \;\Big|\; Ra - w < b \leq \bar{b} \Big\}$$

be the set of all such proposals, and let

$$\mathcal{S} := \Big\{ (\tau, g) \in \mathcal{P}^* : \alpha(\tau, g) \geq \phi\big(B(\tau, g)\big) \Big\}.$$

By construction, $\mathcal{S}$ is the set of all semi-equilibria. Since $\alpha\big(\tau^*_{\leq a}, g^*_{\leq a}\big) = 1$, the policy $\big(\tau^*_{\leq a}, g^*_{\leq a}\big)$ is contained in $\mathcal{S}$; hence, $\mathcal{S} \neq \emptyset$. Let

$$\hat{b} := \sup \big\{ B(\tau, g) \;\big|\; (\tau, g) \in \mathcal{S} \big\},$$

and define $\hat{\tau} := \tau^*_{\leq \hat{b}}$, $\hat{g} := g^*_{\leq \hat{b}}$.

We claim that $\big(\hat{\tau}, \hat{g}\big)$ is an equilibrium under ϕ. To verify this, we only need to show that $\big(\hat{\tau}, \hat{g}\big) \in \mathcal{S}$, i. e. that it is a semi-equilibrium. Assertion (i) is clearly fulfilled. In order to prove (ii), consider a sequence of policies $(\tau_n, g_n) \in \mathcal{S}$ with $B(\tau_n, g_n) \to \hat{b}$. Since ϕ is assumed to be left-continuous and weakly increasing, it is lower semi-continuous[4]. As $B(\cdot)$, $b \mapsto \tau^*_{\leq b}$ and $b \mapsto g^*_{\leq b}$ are continuous functions, the function

$$b \mapsto \phi\Big(B\big(\tau^*_{\leq b}, g^*_{\leq b}\big)\Big)$$

is lower semi-continuous as well. If we already knew that the function

$$b \mapsto \alpha\big(\tau^*_{\leq b}, g^*_{\leq b}\big) \tag{54}$$

[4] A function f is called lower semi-continuous if for all x and all sequences (x_n) with $x_n \to x$, $\lim_{x_n \to x} f(x_n) \geq f(x)$. A function f is called upper semi-continuous if for all x and all sequences (x_n) with $x_n \to x$, $\lim_{x_n \to x} f(x_n) \leq f(x)$.

is upper semi-continuous, we could conclude that

$$\begin{aligned}\alpha\big(\hat\tau,\hat g\big) &= \alpha\big(\tau^*_{\le\hat b}, g^*_{\le\hat b}\big)\\ &\ge \lim_{n\to\infty}\alpha\big(\tau^*_{\le b_n}, g^*_{\le b_n}\big)\\ &\ge \lim_{n\to\infty}\phi\Big(B\big(\tau^*_{\le b_n}, g^*_{\le b_n}\big)\Big)\\ &\ge \phi\Big(B\big(\tau^*_{\le\hat b}, g^*_{\le\hat b}\big)\Big)\\ &= \phi\big(B(\hat\tau,\hat g)\big),\end{aligned}$$

and the proof of the theorem would be complete.

We are left with the task of proving that the function given in (54) is upper semi-continuous: For any b, let

$$M(b) := \Big\{\theta \;\Big|\; V_\theta\big(\tau^*_{\le b}, g^*_{\le b}\big) - V_\theta\big(\tau^*_{\le a}, g^*_{\le a}\big) < 0\Big\}$$

be the set of types which strictly prefer the status quo over the policy $(\tau^*_{\le b}, g^*_{\le b})$. Then,

$$\alpha\big(\tau^*_{\le b}, g^*_{\le b}\big) = 1 - \big|M(b)\big|. \tag{55}$$

Consider any b_0 and a sequence (b_n) with $b_n \to b_0$. As for each θ, the function

$$b \mapsto V_\theta\big(\tau^*_{\le b}, g^*_{\le b}\big) - V_\theta\big(\tau^*_{\le a}, g^*_{\le a}\big)$$

is continuous, we have

$$\begin{aligned}M(b_0) &= \Big\{\theta \;\Big|\; V_\theta\big(\tau^*_{\le b_0}, g^*_{\le b_0}\big) - V_\theta\big(\tau^*_{\le a}, g^*_{\le a}\big) < 0\Big\}\\ &\subseteq \Big\{\theta \;\Big|\; \exists n\;\forall m \ge n \;\Big|\; V_\theta\big(\tau^*_{\le b_m}, g^*_{\le b_m}\big) - V_\theta\big(\tau^*_{\le a}, g^*_{\le a}\big) < 0\Big\}\\ &= \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} M(b_m).\end{aligned} \tag{56}$$

This means: If a type θ prefers the status quo $\big(\tau^*_{\le a}, g^*_{\le a}\big)$ to $\big(\tau^*_{\le b_0}, g^*_{\le b_0}\big)$, he will prefer the status quo to the approximating proposals $\big(\tau^*_{\le b_m}, g^*_{\le b_m}\big)$ for all m sufficiently large.

Since μ is a probability measure, it is σ-additive and hence, σ-continuous, which means the following: If the A_n form a sequence of sets with $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and $A := \bigcup_{n=1}^{\infty} A_n$, then

$$\mu(A) = \lim_{n\to\infty}\mu(A_n).$$

Equation (56) shows that by choosing

$$A_n := \bigcap_{m=n}^{\infty} M(b_m),$$

we attain

$$\big|M(b_0)\big| \le \left|\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} M(b_m)\right| = \lim_{n\to\infty}\left|\bigcap_{m=n}^{\infty} M(b_m)\right|.$$

By using the fact that $\bigcap_{m=n}^{\infty} M(b_m) \subseteq M(b_n)$ for all n, we achieve

$$\big|M(b_0)\big| \le \lim_{n\to\infty}\big|M(b_n)\big|.$$

With (55), this yields

$$\alpha\big(\tau^*_{\le b_0}, g^*_{\le b_0}\big) \ge \lim_{n\to\infty}\alpha\big(\tau^*_{\le b_n}, g^*_{\le b_n}\big).$$

Since b_0 and the sequence b_n are arbitrary, this shows the upper semi-continuity of $b \mapsto \alpha\big(\tau^*_{\le b}, g^*_{\le b}\big)$. □

Proof of Proposition 6. Part (i) follows from the definition of T and the Equations (20) and (21).

Part (ii): To prove that $T(b)$ is strictly increasing in b for $a < b \le Ra + \theta^* w$, observe that its derivative with respect to b is given by

$$T'(b) = \beta \cdot \big(\log(w+b-Ra) - \log(w+a-Ra)\big)^{-2} \cdot z(b),$$

where

$$z(b) := -(\log h)'(b) \cdot \big(\log(w+b-Ra) - \log(w+a-Ra)\big) - \frac{\log h(a) - \log h(b)}{w+b-Ra}.$$

It is sufficient to show that $z(b) > 0$ for $b > a$. This is immediate: Since

$$\begin{aligned} z'(b) = &- (\log h)''(b) \cdot \big(\log(w+b-Ra) - \log(w+a-Ra)\big) \\ &- (\log h)'(b) \cdot \frac{1}{w+b-Ra} + (\log h)'(b) \cdot \frac{1}{w+b-Ra} \\ &+ \big(\log h(a) - \log h(b)\big) \cdot \frac{1}{(w+b-Ra)^2} \\ &> 0, \end{aligned}$$

we achieve $z(b) > z(a) = 0$ for $b > a$.

Part (iii): An agent of type θ prefers $(\tau^*_{\leq b}, g^*_{\leq b})$ to $(\tau^*_{\leq a}, g^*_{\leq a})$ if and only if the inequality

$$V_\theta(\tau^*_{\leq b}, g^*_{\leq b}) - V_\theta(\tau^*_{\leq a}, g^*_{\leq a}) \geq 0 \tag{57}$$

is satisfied. By Equation (7),

$$V_\theta(\tau^*_{\leq b}, g^*_{\leq b}) = \log(1 - \tau^*_{\leq b}) + \theta \log g^*_{\leq b} + \beta\theta \log h(b^*_{\leq b}).$$

Thus, (57) is equivalent to

$$0 \leq \log(1-\tau^*_{\leq b}) - \log(1-\tau^*_{\leq a}) + \theta\big(\log g^*_{\leq b} - \log g^*_{\leq a}\big) + \beta\theta\Big(\log h\big(b^*_{\leq b}\big) - \log h\big(b^*_{\leq a}\big)\Big). \tag{58}$$

For $b \leq b^*$, we have

$$b^*_{\leq b} = b, \quad \tau^*_{\leq b} = \tau^*_{=b}, \quad g^*_{\leq b} = g^*_{=b},$$

and similarly, for $a \leq b^*$

$$b^*_{\leq a} = a, \quad \tau^*_{\leq a} = \tau^*_{=a}, \quad g^*_{\leq a} = g^*_{=a}.$$

From this, one observes that for $a < b \leq b^*$, Condition (58) is equivalent to

$$0 \leq \log(1 - \tau^*_{=b}) - \log(1 - \tau^*_{=a}) + \theta\big(\log g^*_{=b} - \log g^*_{=a}\big) + \beta\theta\Big(\log h\big(b^*_{=b}\big) - \log h\big(b^*_{=a}\big)\Big).$$

Solving for $1/\theta$ yields

$$\frac{1}{\theta} \geq T(b).$$

□

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