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A Dynamic Model of Altruistically-Motivated Transfers

Daniel Barczyk and Matthias Kredler

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Abstract

This paper studies an environment with two infinitely-lived altruistic agents whose only sources of income are a risk-free return on savings and voluntary transfers from the other player. We study the Markov-perfect equilibrium of a continuous-game (differential game). On the technical side, we address shortcomings of Markov-perfect equilibrium and standard control theory by introducing a refinement we term *limit-consistency*, in which we study a sequence of discrete-time games converging to the differential game. The concept allows us to deal with discontinuities in policies and regions where measure-valued controls are used. We find that as in the well-known two-period model of altruism savings decisions are not Pareto-efficient. Agents do not want to induce the other person to overconsume on their expense, a phenomenon known as the Samaritan's dilemma (Buchanan (1975)). Our model exhibits what we call the *dynamic Samaritan's dilemma*: distortions are present long before the "last" period. A stark difference to the two-period model is that the donor's savings decision is also distorted. Transfers only flow when the recipient's borrowing constraint binds, which is in line with stylized facts from the empirical literature.

1 Introduction

In the literature on altruism, many static models for transfers in the tradition of Becker (1974) have been analyzed¹. However, there is a conspicuous lack of models in which agents overlap for more than two periods. Yet, dynamic macroeconomic models often have to take a stance on how agents within a family are connected. Two standard models highlight this well: a representative household is often justified by the assumption that altruistic concerns connect subsequent generations, whereas in pure life-cycle overlapping-generations (OLG) models households act in complete isolation.

While these two extremes often are convenient representations, there is a substantial literature which deems it important to employ a model which lies somewhere in-between. Laitner (1988) studies the impact of social-security system on capital accumulation in an overlapping generations economy in which children and parents care for each other. However, while generations are allowed to act strategically they overlap for only one period. Altig & Davis (1988) study an array of inter- and intra-generational redistributive policies in an economy with altruistic agents. They point out that this setting has an array of implications that differ from those of the two standard workhorses of macroeconomics, the OLG economy and the infinite-horizon framework. Nonetheless, while acknowledging the importance of strategic interaction, they choose to circumvent it by assuming commitment. Lindbeck & Weibull (1988) carefully demonstrate theoretical implications of strategic interactions in a model of transfers with altruistic agents. While their discussion highlights important features of the inter-temporal aspects of altruism and transfers, the restriction to a two-period world and the fact that they allow transfers to flow in the second period only limits the usefulness of their analysis to modern dynamic macroeconomics.

The contribution of our paper is three-fold: (1) We extend the well-known two-period model with altruism in a game-theoretic setting and provide a theoretical characterization of its implications; (2) We introduce the refinement of (*discrete-time-*) *limit consistency* to Markov-perfect equilibrium in differential games, which allows us to deal with discontinuities in policies and optimality of measure-valued transfers; (3) We provide a building block for dynamic macroeconomic models which put an emphasis on redistributive policies and family behavior. The model we study is inhabited by two infinitely-lived agents who are linked to each other by mutual altruism. One-sided altruism, perfect altruism (representative household) and egoistic preferences (pure life-cycle) are all nested in this general setting.² Agents decide about consumption and savings

¹See the short literature review below.

²Following Laitner (1997) survey, throughout the paper we will refer to *one-sided altruism* as the following concept: Person A's ranking of allocations is influenced by person B's consumption – in a static setting, A may rank allocations according to a utility function $u(c_A) + \alpha u(c_B)$. Person B's ranking of allocations, however, is not affected by person A's consumption – B's preferences may be represented by utility $u(c_B)$ in this example. For *two-sided altruism*, this dependence is mutual; B's utility function would be $u(c_B) + \alpha' u(c_A)$, for example. We will refer to *perfect altruism* as the concept that person A is just as concerned about his own consumption as about person B's consumption, for example having a utility function

in a riskless asset and can make a non-negative transfer to the other agent in each instant. We characterize the Markov-perfect equilibrium; agents can only condition their strategy on current states, but not on past actions.

We find that as in the static model of altruism, transfers are positively linked to the donor's resources and negatively linked to the recipient's resources. One feature of our model is that transfers are especially likely to occur when one agent is borrowing-constrained, which is in line with empirical results (see Cox & Jappelli (1990) and Cox (1990)). When altruism is imperfect, agents' savings decisions are distorted with respect to the efficient allocation that a benevolent family ruler would choose. Both donors and recipients are led to overconsume due to strategic-altruistic considerations. Soon-to-be recipients of transfers fail to fully internalize the burden of transfers on donors; potential donors don't want to induce poorer family members to act imprudently by building up too many assets and thus creating expectations of large transfers. Almost ironically, efficiency is restored when one agent becomes dependent on the other's transfers forever – then, an effective family ruler is in charge in the person of the donor and his/her preferred allocation is implemented.

On the technical side, our paper makes the following contributions: In our setting –and in general in differential games–, some cases arise where classical control theory and the viscosity concept for Hamilton-Jacobi-Bellman (HJB) equations do not give us satisfactory answers about optimality of players' strategies. We introduce (*discrete-time-limit consistency*) as a refinement of Markov-perfect equilibrium for differential games (DGs). We consider a sequence of discrete-time games in which the time steps between the decisions go to zero in the limit. The candidate value functions and policies for the DG, if applied to the discrete games, have to induce policies that are optimal — at least in the limit.

The remainder of the paper is structured as follows: section 2 outlines the set-up of the model. It describes the set of Pareto-efficient allocations and exploits the homogeneous structure of the problem in order to reduce the state-space. The Hamilton-Jacobi-Bellman equations follow, which characterize optimal consumption-savings decision as well as transfers in smooth regions. Section 3 provides a theoretical characterization of different regions, introduces the concept of limit consistency and uses it to study boundaries between smooth regions. Subgame perfection will rule out some common-sense equilibria and we briefly describe why. Computational results are then presented in section 4. Finally, section 5 concludes.

$u(c_A) + u(c_B)$. *Imperfect* altruism is the obvious counterpart, represented for example by $u(c_A) + \alpha u(c_B)$ for $0 < \alpha < 1$.

2 Model

2.1 Setting

Time t is continuous. There are two agents in the economy who are infinitely-lived. We will denote variables for the first agent, whom we will refer to as “she”, as plain lower-case letters, e.g. c_t . Variables referring to the second agent, whom we will call “he”, are denoted with prime-superscripts, e.g. c'_t . Both agents can hold a non-negative amount k_t in a riskless asset that pays a time-invariant rate of interest r .

In each instant of time, agents choose a consumption rate $c_t \geq 0$ and a non-negative transfer rate g_t to the other agent (g stands for “gift”), so that their assets evolve according to

$$\dot{k}_t = rk_t - c_t - g_t + g'_t \quad (1)$$

$$\dot{k}'_t = rk'_t - c'_t - g'_t + g_t, \quad (2)$$

where dots denote the time-derivative of a variable. There is a no-borrowing constraint for both agents; when $k_t = 0$, we must have that she does not spend more than she receives, i.e. $c_t + g_t \leq g'_t$ (and equivalently for him, of course).

She ranks allocations using the criterion

$$V_0 = \int_0^\infty e^{-\rho t} [u(c_t) + \alpha u(c'_t)] dt,$$

where $\rho > 0$ is the discount rate and $0 \leq \alpha \leq 1$ the parameter which measures the intensity of altruism.³ He is a mirror-symmetric copy of her, but might have a different altruism parameter $0 \leq \alpha' \leq 1$ from hers. His criterion for ranking allocations is

$$W_0 = \int_0^\infty e^{-\rho t} [u(c'_t) + \alpha' u(c_t)] dt.$$

We assume that the agents do not differ in their discount rates and the form of the “primitive” utility function $u(\cdot)$. For the sake of simplicity and intuition we choose logarithmic utility as the functional form.

2.2 Pareto-optimal allocations

To find the set of Pareto-optimal allocations, consider the allocations a benevolent planner would choose placing a weight η on her life-time value. Given k_0 and k'_0 , the planner chooses optimal savings policies $k(t), k'(t)$ and consumption policies $c(t), c'(t)$ for $0 \leq t < \infty$ to maximize

$$J = \eta \int_0^\infty e^{-\rho t} \underbrace{[u(c_t) + \alpha u(c'_t)]}_{\equiv U_t} dt + (1 - \eta) \int_0^\infty e^{-\rho t} \underbrace{[u(c'_t) + \alpha' u(c_t)]}_{\equiv U'_t} dt, \quad (3)$$

³With this linearly separable formulation of altruistic preferences we are in line with the bulk of the literature.

This approach will yield all allocations on the Pareto frontier when varying $\eta \in [0, 1]$.

Obviously, we can pool the single capital stocks together and define $K = k + k'$ as total resources of the family. It is also straightforward how the planner must allocate resources intratemporally in any instant: To equalize the two margins, the following must hold:

$$[\eta + \alpha'(1 - \eta)]u_c(c_t) = [(1 - \eta) + \alpha\eta]u_c(c_t) \quad \forall t,$$

Solving for $u_c(c_t)$ yields

$$u_c(c_t) = \frac{(1 - \eta) + \alpha\eta}{\eta + \alpha'(1 - \eta)}u_c(c'_t) \quad \forall t. \quad (4)$$

This gives us c'_t as a function of c_t , and we can see that the problem has collapsed to an autarkic savings problem with a modified objective function — to see this, substitute c'_t into the objective (3). When $\eta = 1$ and $\alpha = 0$, equation (4) does not give us c'_t as a function of c_t any more – in this case, however, it is obviously optimal for the planner to set $c'_t = 0$ for all t . The same is true for the symmetric case $\eta = 0$ and $\alpha' = 0$, where $c_t = 0$ for all t is optimal.

Intertemporally, we can see that the agent's autarkic Euler equations must hold. If that was not the case, the family planner should definitely re-allocate resources in a way that yielded higher value to the agent without requiring more life-time wealth. This means that we must have

$$\begin{aligned} \frac{d}{dt}u_c(c) &= (\rho - r)u_c(c) \\ \frac{d}{dt}u_c(c') &= (\rho - r)u_c(c'), \end{aligned}$$

where the second equation is already implied by the first when we invoke intratemporal optimality from (4).

2.3 (P,K)-Space: Homogeneity

In the setting presented above her and his capital stock (k, k') are the natural state variables when we look for a stationary Markov-Perfect equilibrium. However, since the environment is homogeneous our analysis will be simplified by re-defining the state-variables (k, k') as follows

$$P = \frac{k}{K}, \quad P \in [0, 1], \quad K = k + k' \quad \forall k, k' \geq 0, \quad k + k' \neq 0.$$

Thus, P is the fraction of wealth she owns out of the combined wealth K of both players. The maximum value of P is 1 due to the no-borrowing constraints the agents face. The advantage of the (P, K) variables is that it allows us to reduce the dimensionality of the state-space to 1, in particular the only state variable will be P . Thinking about the nature of the problem it seems apparent

that a measure of relative wealth as captured by P is important for the agents' decision making. For example, when $P = 1$ then only she has wealth. Since the other player has neither wealth nor income and per-period utility has an Inada condition it has to be the case that she transfers resources when $\alpha > 0$. On the other hand, with $P = 1/2$ both players own the same fraction of total wealth and we do not expect transfers to flow.

Since our economy is homogeneous in P we conjecture that the equilibrium consumption and transfer policies are⁴,

$$\begin{aligned} c(k, k') &= \bar{c}(P)k, & g(k, k') &= \bar{g}(P)k \\ c'(k, k') &= \bar{c}'(P)k', & g'(k, k') &= \bar{g}'(P)k' \end{aligned}$$

Furthermore, by multiplying the right-hand side of the consumption and transfer functions by K/K , the conjecture of the policies in (P, K) variables is given by

$$\begin{aligned} \bar{c}(P)PK &\equiv \bar{C}(P)K = C(P, K), & \bar{g}(P)PK &\equiv \bar{G}(P)K = G(P, K) \\ \bar{c}'(P)(1-P)K &\equiv \bar{C}'(P)K = C'(P, K), & \bar{g}'(P)(1-P)K &\equiv \bar{G}'(P)K = G'(P, K) \end{aligned}$$

Thus, the key equilibrium objects are the factors of proportionality which are functions of P only:

$$\bar{C}, \bar{C}', \bar{G}, \bar{G}' : [0, 1] \rightarrow \bar{\mathbb{R}}_+, \quad \bar{c}, \bar{c}', \bar{g}, \bar{g}' : [0, 1] \rightarrow \bar{\mathbb{R}}_+.$$

The $\bar{}$ emphasizes that the function represents a proportion. If the proportion is written using a capital letter, e.g. \bar{C} , it indicates that it is related to the sum of players' wealth K , whereas proportions written in lower-case letters, e.g. \bar{c} , refer to the private wealth of the agent k or k' . In order to minimize cluttering in the notation, arguments of functions will be suppressed.

2.3.1 Her Problem in (P, K) Space

Using the guess of the strategies, her problem given initial values (P_0, K_0) and his strategy $\{C'(P, K), G'(P, K)\}$ is

$$V(P, K) \equiv \max_{\bar{C}, \bar{G}} \left\{ \int_0^\infty e^{-\rho t} [\ln(\bar{C}K) + \alpha \ln(\bar{C}'K)] dt \right\} \quad (5)$$

subject to

$$\begin{aligned} \dot{K} &= [r - \bar{C} - \bar{C}'] K \\ \dot{P} &= -(1-P)\bar{C} + P\bar{C}' + [\bar{G}' - \bar{G}] \\ P &\geq 0, \quad \bar{C} \geq 0, \quad \bar{G} \geq 0, \quad \forall P. \end{aligned}$$

We can simplify (5) by integration and defining the following function:

$$\bar{V}(P) \equiv \int_0^\infty e^{-\rho t} \left\{ \ln(\bar{C}_t) + \alpha \ln(\bar{C}'_t) - (1 + \alpha) \int_0^t [\bar{C}(P_\tau) + \bar{C}'(P_\tau)] d\tau \right\} dt \quad (6)$$

⁴Recall that in the homogeneous standard consumption-savings problem with logarithmic utility the consumption policy is simply to consume a constant fraction ρ out of wealth i.e. $c(k) = \rho k$.

The interpretation of this complicated integral will become clear in a moment. We can now write (5) as

$$\begin{aligned}
V(P, K) &\equiv \max_{\bar{C}, \bar{G}} \left\{ \frac{1 + \alpha}{\rho} \left[\frac{r}{\rho} + \ln(K) \right] + \bar{V}(P) \right\} & (7) \\
&\text{subject to} \\
\dot{K} &= [r - \bar{C} - \bar{C}'] K \\
\dot{P} &= -(1 - P)\bar{C} + P\bar{C}' + [\bar{G}' - \bar{G}] \\
P &\geq 0, \quad \bar{C} \geq 0, \quad \bar{G} \geq 0, \quad \forall P.
\end{aligned}$$

From (7), we note that the value function in terms of K is known so that only \bar{V} remains to be determined (we use the *bar* to indicate that this part of the value function $V(P, K)$ only depends on proportions).

Note that we may also see her problem (7) as follows: She maximizes (6) subject to the law of motion for P and the constraints on policies. We can see easily that since r only influence a part of $V(P_0, K_0)$ that is fixed, best responses will be independent of r . As for ρ , it is not hard to show that the maximizer must be homogeneous in this parameter; we have:

Proposition 1 (Equilibrium independent of r and homogeneous in ρ)

If $\{C_1(\cdot), C_1'(\cdot), G_1(\cdot), G_1'(\cdot)\}$ are an equilibrium for $(\rho, r) = (1, 1)$, then $\{\rho C_1(\cdot), \rho C_1'(\cdot), \rho G_1(\cdot), \rho G_1'(\cdot)\}$ are an equilibrium for any vector (ρ, r) and vice versa.

Proof: It is obvious that r does not enter the problem of maximizing (6) given the law of motion for P and the no-borrowing constraint.

As for ρ , consider the following argument: Denote the time path of P for given P_0 under the policies $\{C_1, C_1', G_1, G_1'\}$ by $P_t^{(1)}$. Then we can easily see that $P_t^{(\rho)} = P_t^{(1)\rho t}$ gives us the time path of P for the suggested policies $\{C_\rho, C_\rho', G_\rho, G_\rho'\} = \{\rho C_1, \rho C_1', \rho G_1, \rho G_1'\}$ for $P_0^{(\rho)} = P_0^{(1)}$ since the ODE for the law of motion for P is fulfilled.

Now apply the change of variable $g(t) = \rho t$ to the integral inside the brackets in (6):

$$\int_0^t \rho C_1(\rho\tau) + \rho C_1'(\rho\tau) d\tau = \int_0^{\rho t} C_1(x) + C_1'(x) dx.$$

Applying again the change of variable $g(t) = \rho t$, now to the entire integral in (6), we obtain

$$\bar{V}_\rho(P; \{\rho C, \rho G\}) = \frac{(1 + \alpha) \ln \rho}{\rho} + \frac{1}{\rho} \bar{V}_1(P; \{C, G\}),$$

where $\bar{V}_\rho(\cdot; C, G)$ denotes the value function for a given value of the parameter ρ given a policy $\{C, G\}$. So if V_1 attained by the policy $\{C_1, G_1\}$ dominates the value \tilde{V}_1 attained by any other feasible $\{\tilde{C}_1, \tilde{G}_1\}$, then V_ρ attained by $\{\rho C_1, \rho G_1\}$ must dominate the value \tilde{V}_ρ attained by $\{\rho \tilde{C}_1, \rho \tilde{G}_1\}$. Since any feasible policy under the discount rate ρ corresponds to one such policy $\{\rho C_1, \rho G_1\}$ and $V_\rho(\rho C, \rho G)$ is a monotone transform of $V_1(C, G)$, $\{\rho C_1, \rho G_1\}$ must be an

optimal policy. Since the same holds for him, the statement in the proposition follows. ■

This proposition tells us that we are essentially left with the two parameters (α, α') ; we can easily map equilibria between different values for (ρ, r) . The intuition is the following: The interest rate r does not matter since the income and substitution effect cancel with log-utility. As for ρ , consider the following parable: If we changed the unit of time from a year to a month, then we should divide the discount rate by 12. When also dividing all consumption (and transfer) rates by 12, we obtain the exact same allocation as before, which is of course also an equilibrium – this is at least true when utility is homothetic and the nature of the physical environment is homogeneous.

With (5) and (7) we can use standard tools of dynamic programming to obtain her⁵ Hamiltonian-Jacobi-Bellman (HJB) equation⁶

$$\begin{aligned} \rho \bar{V} = & \alpha \ln \bar{C}' - \bar{C}' \left[\frac{1 + \alpha}{\rho} \right] + P \bar{C}' \bar{V}_P + \bar{G}' \bar{V}_P + \\ & + \max_{\bar{C}, \bar{G}} \left\{ \ln \bar{C} - \bar{C} \left[\frac{1 + \alpha}{\rho} \right] - (1 - P) \bar{C} \bar{V}_P - \bar{G} \bar{V}_P \right\}. \end{aligned} \quad (8)$$

Since (8) is an equation in P only it is an ordinary differential equation which is an additional simplification with regard to (k, k') state variables, in which the HJB is a partial differential equation (PDE). When she is unconstrained her first-order condition with respect to her consumption rate is given by:

$$\frac{1}{\bar{C}} = \frac{1 + \alpha}{\rho} + (1 - P) \bar{V}_P, \quad \text{for } P \in (0, 1]. \quad (9)$$

When $\bar{V}_P = 0$ we will refer to the resulting consumption rate $\rho/(1 + \alpha)$ as the *wealth-pooling rate*. The reason is that when $\bar{V}_P = 0$ she is indifferent towards the asset distribution. If $P = 1$ he is broke and she implements her first-best which in this case is the wealth-pooling rate, i.e. in the planner's objective (3) she obtains a weight of $\eta = 1$ (also see Lemma (12) in the appendix). Besides these special cases we can think of the consumption rate as the sum of the valuation of common resources and a value on the distribution of resources between the players. In other words, $V_K = \rho/(1 + \alpha)$ measures the value obtained if an additional \$1 rains on the both players while leaving the distribution unchanged, whereas, $V_P = \bar{V}_P$ measures the value to her when 1% of total assets K are transferred from him to her while K remains unchanged. It follows that as long as $\bar{V}_P \geq 0$ she will not make any transfer to him. When $\bar{V}_P < 0$ she wants to make a transfer – vice versa, if \bar{V}_P' he wants to make a transfer – and we will refer to \bar{V}_P and \bar{V}_P' as her and his *transfer motive*, respectively.

From (8) we can quickly see that the maximization part instructs to transfer

⁵Whenever it is obvious that the other player is a mirror-symmetric copy we provide characterizations and analysis in terms of “her” problem.

⁶See appendix A.1 for the derivation of the HJB from (7) and (6).

resources precisely when $\bar{V}_P < 0$ i.e.

$$\max_{\bar{C}} \left\{ \ln \bar{C} - \bar{C} \left[\frac{1+\alpha}{\rho} \right] - (1-P)\bar{C}\bar{V}_P \right\} + \max_{\bar{G}} \{-\bar{G}\bar{V}_P\}$$

We see that if $\bar{V}_P > 0$ the maximizer is $\bar{G}^* = 0$, whereas, if $\bar{V}_P < 0$ the maximizer is a mass-point $\bar{G}^* = \infty$. When $\bar{V}_P = 0$, any transfer is optimal.

A great simplification in the continuous-time setting is that computing optimal consumption at a given point P in the state-space does not require to solve a fixed-point problem as is the case in discrete time. The technical reason for this is that second-order effects become unimportant and therefore only the current value of the state matters⁷. The advantage of this becomes particularly clear when we compute solutions to the model. In discrete-time one would have to solve the Nash equilibrium for every node in the state-space. While this requires an enormous amount of computational effort an additional problem may arise because multiple Nash equilibria exist at a particular node.

As for the mass-point transfer we can think of this as a discrete jump in the state-space of the agents. The donor provides instantaneously an amount of transfers in order to ensure $\bar{V}_P = 0$. This, however, does not exclude the possibility that transfers may also occur as a flow. In particular, if one of the agents is liquidity constrained then the optimal amount of transfers the donor provides is, as in the static model, given by

$$u_c(c_t) = \alpha u_c(c'_t)$$

assuming she is the donor. But even within the state-space it may be plausible to have a flow of transfers as would be the case when $\bar{V}_P = 0$.

Finally we obtain the Euler equations⁸

$$\begin{aligned} \frac{d}{dt} \bar{V}_P(t) &= [P\bar{C}' - (1-P)\bar{C} - \bar{G} + \bar{G}'] \bar{V}_{PP} = \\ &= [\rho - \bar{C} - \bar{C}'] \bar{V}_P + \left[\frac{1}{\bar{C}} - \frac{\alpha}{\bar{C}'} - \bar{V}_P \right] \bar{C}'_P - \bar{G}'_P \bar{V}_P \quad (10) \\ \frac{d}{dt} \bar{V}'_P(T) &= [P\bar{C}' - (1-P)\bar{C} - \bar{G} + \bar{G}'] \bar{V}'_{PP} = \\ &= [\rho - \bar{C} - \bar{C}'] \bar{V}'_P + \left[\frac{1}{\bar{C}'} - \frac{\alpha'}{\bar{C}} + \bar{V}'_P \right] \bar{C}_P + \bar{G}_P \bar{V}'_P. \end{aligned}$$

⁷Recall that in discrete-time the FOC would be given by the Euler equation $u_c(c_t) = R\beta V'(k_{t+1})$. Solving this equation requires knowledge of k_{t+1} , which depends on current strategies of the other player.

⁸The term $G_P(P)\bar{V}_P(P)$ vanishes in regions where $G = 0$ since $G_P = 0$ and in regions where $G > 0$ since $\bar{V}_P = 0$; a similar argument shows that the term $G(P)\bar{V}_{PP}(P)$ vanishes in both regions. Of course, the same arguments apply for the respective terms in the HJB for $\bar{V}'(\cdot)$.

3 Characterization of equilibrium

We will restrict our attention to equilibria that consist of finitely many *smooth regions* inside which players' policies are continuous and satisfy certain regularity conditions. Inside smooth regions we can use standard control theory to analyze equilibria. On the boundaries between the smooth regions⁹ and inside mass-transfer regions, however, standard control theory does not provide us with answers on how to find best responses. To address this issue, we introduce the concept of *(discrete-time) limit consistency* which formalizes what we mean by optimality of a policy at these special points. In a nutshell, the concept says that a player's policy must be optimal to a first order over a short interval Δt for any point in the state space.

Formally, our assumptions are as follows:

Assumption 1 (Finitely many smooth regions) *The state space $\mathcal{P} = [0, 1]$ is divided into finitely many regions (intervals) $\mathcal{P}_i = (P_{i-1}, P_i)$, $i = 1, \dots, n$, where $0 = P_0 < P_1 < \dots < P_{n-1} < P_n = 1$. Inside each region, the value functions $\bar{V}(\cdot)$ and $\bar{V}'(\cdot)$ are continuously differentiable and the policy functions $\bar{C}(\cdot)$, $\bar{C}'(\cdot)$, $\bar{G}(\cdot)$ and $\bar{G}'(\cdot)$ are continuous.*

Note that policies are allowed to be discontinuous at the boundaries P_0, \dots, P_n , which may be associated with non-differentiabilities (*kinks*) of the value functions.

Assumption 2 (Consumption lower-bounded under altruism) *If $\alpha > 0$, then $\bar{C}'(P) > \epsilon$ for all $P \in [0, 1]$ for some $\epsilon > 0$. If $\alpha' > 0$, then $\bar{C}(P) > \epsilon$ for all $P \in [0, 1]$ for some $\epsilon > 0$.*

We introduce this assumption since it is very hard to rule out equilibria where one player's consumption is zero for some $P \in \mathcal{P}$. For example, if we had $\bar{C}(0) = 0$, then it would be a best response for him to set $\bar{G}(0) = 0$ – giving her transfers would not help, since she would consume nothing anyway and both players would obtain utility of minus infinity. For her, since he is not giving any transfer, setting $\bar{C}(0) > 0$ does not pay since she would still be left with zero consumption, so this pair of policies is an equilibrium. However, this is clearly not in the spirit of the altruism framework because both players have strong incentives to avoid such situations. We thus exclude this case from our analysis.

Note that we do *not* restrict her consumption to be lower-bounded if he is not altruistic towards her. To see why, consider the case $\alpha = \alpha' = 0$: Here, autarky with her consuming $\bar{C}(P) = \rho P$ is clearly an equilibrium where $\bar{C}(P) \rightarrow 0$, which is reasonable since he has no incentives to help her out when $\alpha' = 0$.

Finally, we make the following technical assumption:

Assumption 3 (Limit-consumption exists) *For each region \mathcal{P}_i , the limits of consumption on the boundaries of the region exist: $\bar{C}(P_{i-1})_{lim} \equiv \lim_{P \rightarrow P_{i-1}^+} \bar{C}(P)$ and analogously for $\bar{C}'(P_{i-1})_{lim}$, $\bar{C}(P_i)_{lim}$ and $\bar{C}'(P_i)_{lim}$.*

⁹including the boundaries of the state space $P = 0$ and $P = 1$

Given these three assumptions, it is not hard to show that both players' value functions are continuous also on the boundaries (see Lemma 11 in the appendix). If both players are altruistic, the value functions are also bounded (see Lemma 10 in the appendix).

3.1 Refinement of equilibrium: (Discrete-time-) Limit consistency

In our setting – and in general in differential games – cases may arise where classical control theory and the viscosity concept¹⁰ for HJBs do not give us satisfactory answers about optimality of players' strategies. The two most important ones that lead to difficulties in our setting are mass-transfer regions and boundaries between regions.¹¹

We will introduce (*discrete-time-*) *limit-consistency* as a refinement of Markov-perfect equilibrium for differential games. We consider a sequence of discrete-time games in which the time steps between the decisions go to zero in the limit. The candidate value functions and policies for the differential game, if applied to the discrete games, have to induce policies that are optimal — at least in the limit.

Consider a general differential game. Each decision $x_j^{(i)}$ of player $i = 1, \dots, n$ is made only once over a typical time interval $[t_0, t_0 + \Delta t)$, where $j \in 1, \dots, J_i$ indexes player i 's decisions (in our example: consumption and transfers, $j = 1, 2$). The sequencing of the decisions is governed by a *timing protocol*, which consists of a number of stages of the interval game $K \leq \sum_{i=1}^N J_i$ with associated numbers $s_k \in [0, 1)$, $s_1 = 0$ and $s_{k+1} \geq s_k$ for $k = 2, \dots, K$, where a subvector $\tilde{x}_k^{(i)}$ of $x^{(i)}$ is chosen by player i at $t_i + s_k \Delta t$ with knowledge of all previous decisions at stages $\tilde{k} < k$.

For our game, we adopt the following timing protocol with $K = 2$ stages for the interval game:

1. g and g' are chosen simultaneously at t_0 , i.e. $s_1 = 0$.
2. As soon as transfer flows have been realized and observed, c and c' are chosen simultaneously $t = t_0$, i.e. $s_2 = 0$.

¹⁰The viscosity concept is the agreed-upon sense in which the HJB has to be fulfilled as a PDE; unlike classical control theory, the viscosity concept is able to deal with non-differentiabilities of the value functions, see the seminal paper by Crandall & Lions (1983) and for differential games Bressan (2009)

¹¹As for mass transfers, note that in a region where she receives a mass transfer, setting $\bar{C} = 0$ would be a best response for her since no time is spent in the mass-transfer region in equilibrium thus her choice does not matter. However, taking $\bar{C} = 0$ as given, he is forced to give a mass transfer if $\alpha' > 0$ since any short interval of time spent with $\bar{C} = 0$ leads to utility of minus infinity for him. But clearly the “suicide threat” of $\bar{C} = 0$ is not what one would consider a credible threat in the spirit of subgame perfection. As for boundaries between regions, consider the case where his policy changes in a discontinuous fashion at a boundary. This leads to a discontinuous Hamiltonian in her control problem, which is a case that neither classical control theory nor the viscosity concept of the HJB is able to deal with. CITATION FOR VISCOSITY NEEDED HERE!

We will consider the following problem for the player:

$$V(P_{t_0+s_k\Delta t}) = \max_{x_k \in X_k(P)} \left\{ U(P, x, x'(P))\Delta t_k + e^{-\rho\Delta t_k} V(P_{t_0+s_{k+1}\Delta t}) \right\} \quad (11)$$

where $\Delta t_k = (s_{k+1} - s_k)\Delta t$ for $k = 1, \dots, K-1$ and $\Delta t_K = (1 - s_K)\Delta t$ denote the time interval until the next decision stage. $x_k, k = 1, \dots, K$, denote the sub-vector of decisions taken at step k of the interval game, $X_k(P)$ is the feasible set for these decisions and $U(P, x, x'(P)) = \ln c + \alpha \ln c'(P)$ is flow utility resulting from the decision taken over the previous Δt . $P_{t_0+s_{k+1}\Delta t}$ is determined by the realization of transfers after stage 1 and by the realization of consumption at stage 2.

We require (11) to hold in the limit as $\Delta t \rightarrow 0$. As for the transfer decision, the problem becomes

$$V(P_{t_0}) = \max_{G \geq 0} V(P_{t_0} + G'(P_{t_0})\Delta t - G\Delta t). \quad (12)$$

(12) implies that her value function must be weakly increasing. When policies are of the flow-type, (12) has no bite since $P_{t_0+s_{k+1}\Delta t} \rightarrow P_{t_0+s_k\Delta t}$ and $U(\cdot)\Delta t_k \rightarrow 0$.

As is standard in control theory, we will thus require that (11) holds up to terms of order Δt . For points P where V is differentiable, this yields the usual HJB. We require additionally that on boundaries P_i (where the value function may have kinks) the Hamiltonian be maximized when taking the derivative into the appropriate direction. For a given policy of the other player $x'(\cdot)$ and a given value function $\bar{V}(\cdot) \in C^0$ (and C^1 inside smooth regions), we define the *limit Hamiltonian* as

$$H_{lim}(x; P, x'(P)) = U(P, x, x'(P)) + \begin{cases} f(P, x, x'(P))V_P^+(P) & \text{if } f(P, x, x'(P)) \geq 0 \\ f(P, x, x'(P))V_P^-(P) & \text{if } f(P, x, x'(P)) < 0 \end{cases}$$

where $U(P, x, x'(P)) = \ln c + \alpha \ln c'(P)$ is the flow utility for strategy x , $V_P^+(P)$ is the upward derivative of the value function at P and $V_P^-(P)$ is the downward derivative. Observe that inside smooth regions, H_{lim} equals the usual Hamiltonian; on boundaries between smooth regions, however, the concept has bite. Also, note that it is important how we specify the opponent's strategy $x'(P)$ at P if P is a boundary between regions.

Definition 1 *We say that a pair of strategies $\{x(\cdot), x'(\cdot)\}$ and associated value functions $\{V(\cdot), V'(\cdot)\}$ constitute a discrete-time-limit-consistent (or short: limit-consistent) Markov-perfect equilibrium with respect to a certain timing protocol if the following holds for all players $i = 1, \dots, n$, for all stages $k = 1, \dots, K$ and for all $P \in \mathcal{P}$:*

1. (0th-order consistency) (11) holds in the limit as $\Delta t \rightarrow 0$.
2. (1st-order consistency) For k with $\Delta t_k > 0$

(a) strategies $\{x(\cdot), x'(\cdot)\}$ are first-order optimal in the sense that

$$x(P) = \arg \max_x H_{lim}(x; P, x'(P))$$

(b) and they are consistent with $V(\cdot)$ in the sense that

$$\rho V(P) = H_{lim}(x(P); P, x'(P)).$$

Obviously, in smooth regions without mass transfers limit consistency collapses to the simple requirement that value functions be continuous (0th-order consistency), that policies maximize the Hamiltonian and the HJB be fulfilled (1st-order consistency) and thus do not impose any additional constraint to ordinary control theory. However, limit consistency will be key to characterize equilibria on boundaries as well as to study credible strategies inside mass-transfer regions and in bankruptcy.

3.2 Characterization of regions

We will now proceed to characterize the equilibrium on the interior of smooth regions. The following is an exhaustive listing of possible types of regions:

- No-transfer region (NT): $\bar{G}(0) = \bar{G}'(0) = 0$ for all $P \in \mathcal{P}_i$.
- Flow-transfer region (FT): $\bar{G}(P) > 0$ for all $P \in \mathcal{P}_i$ (or equivalently $\bar{G}'(P) > 0$ for him).
- Bang-bang-transfer region (BB): $\bar{G}(P) = (P_i - P)\delta$ for all $P \in \mathcal{P}_i$ (or equivalently $\bar{G}(P) = (P - P_{i-1})\delta$ for him).¹²

Furthermore, we will study the following important special types of regions: Autarkic regions (AUT), where policies equal the autarkic ones (a special kind of NT); bankruptcy (BR) at $P = 0$ and $P = 1$ as an absorbing state (a special kind of FT); and wealth-pooling regions (WP), where both players' consumption is given by the wealth-pooling policies (a special kind of FT).

3.2.1 No-transfer regions (NT)

The value functions and consumption policies satisfy her HJB (8), his HJB (not shown) and Euler equations (10), in which all terms in \bar{G} and \bar{G}' are set to zero. The Euler equations yield a system of two non-linear ordinary differential equations of first order for the consumption policies $\bar{C}(\cdot)$ and $\bar{C}'(\cdot)$ on \mathcal{P}_i . Given limiting consumption policies $\bar{C}(P_{i-1}^+)$ and $\bar{C}'(P_{i-1}^+)$, we can solve for consumption throughout \mathcal{P}_i (or equivalently for P_i from the top).

MAY WANT TO STATE Euler Equations IN C,CP HERE. ALSO: THEOREM THAT NT HAS TO BE REPELLING UNLESS AUTARKIC.

¹² δ denotes the *Dirac delta*: The path P_i would instantaneously jump from P to P_i if it reached P at any point.

We will now be interested in stationary points of the economy inside a NT-region. First, we express the law of motion for P in terms of (c, c') :

$$\dot{P} = P(1 - P)(c' - c)$$

So P increases over time if and only if he consumes at a higher rate out of his own capital stock than he does. At a stationary point P_0 , we have $c' = c = c_0$. We will now study stationary points with $c_0 \neq \rho$ (i.e. which are not autarkic regions). We can express the Euler equations (10) in terms of consumption rates (see appendix A.4 for some intermediate steps) and use the identity $c = CP$ and $c' = C'(1 - P)$ to obtain

$$\begin{aligned} C_P &= -\frac{\frac{1}{\bar{C}'} - \frac{1+\alpha'}{\rho}}{1 + \alpha'} \left(\frac{c_0 - \rho}{\frac{1}{\rho} - \frac{1}{c_0}} \right) \leq 0 & \Rightarrow & c_P = \frac{C_P - c}{P} < 0 \\ C'_P &= \frac{\frac{1}{\bar{C}} - \frac{1+\alpha}{\rho}}{1 + \alpha} \left(\frac{c_0 - \rho}{\frac{1}{\rho} - \frac{1}{c_0}} \right) \geq 0 & \Rightarrow & c'_P = \frac{C'_P + c'}{1 - P} > 0. \end{aligned}$$

The inequalities follow from the parenthesis always being positive and $\bar{C} \leq \rho/(1 + \alpha)$, $\bar{C}' \leq \rho/(1 + \alpha')$. So it must be that $\dot{P} > 0$ to the right of P_0 and $\dot{P} < 0$ to the left, so \dot{P} is not an attracting steady state.

3.2.2 Flow-transfer regions (FT)

We will consider a FT region where she gives transfers, i.e. $\bar{G} > 0$. First, observe that $\bar{G}(P) > 0$ implies $\bar{V}_P(P) = 0$ for all $P \in \mathcal{P}_i$. By her FOC (5) for consumption, this also means that $\bar{C} = \bar{C}_{wp}$ throughout \mathcal{P}_i . Then, from her HJB (8), we can see that \bar{C}' must also be constant on P since the function $\alpha \ln \bar{C}' - \bar{C}'[(1 + \alpha)/\rho - P\bar{V}_P]$ would vary if \bar{C}' did, violating her HJB (note that all other terms of her HJB are constant in P).¹³

From his Euler equation in (10), we now see that

$$\bar{V}'_P(\rho - \bar{C} - \bar{C}' + \bar{G}_P) = 0.$$

Observe that the \bar{G}_P enters the Euler equation with the same sign as ρ , which stands for impatience. So when her transfers are *increasing* in P , this is a disincentive for him to save. By the same token, she can give him incentives to save if she makes transfers *decreasing* in P , i.e. *increasing* in his wealth share. So rewarding thriftiness with more money later can induce savings, as intuition suggests.

His Euler equation now leads us to the following two relevant cases:

1. $\bar{V}'_P = 0$: In this case, we must have $\bar{C} = \bar{C}' = \bar{C}_{wp}$ on P and transfers are indeterminate – we might indeed have $\bar{G}' > 0$ now. Note that the Euler equations cease to contain information in this case. The HJBs tell us that \bar{V}' and V are pinned down at their wealth-pooling levels, so P would be a *wealth-pooling region* (see section 3.2.6).

¹³We would arrive at the same conclusion by studying her Euler equation.

2. $\bar{V}'_P < 0$: In this case, his Euler equation tells us that

$$\begin{aligned}\bar{G}_P &= \bar{C}_{WP} + \bar{C}' - \rho = \bar{C}' - \frac{\alpha}{1+\alpha}\rho = \bar{C}' - \alpha\bar{C}_{WP} \\ \bar{G}'_P &= \alpha'\bar{C}'_{WP} - \bar{C},\end{aligned}\tag{13}$$

where we have stated the ODE for his transfer in the case where he is giving transfers for future reference. The last expression in (13) involves $\alpha\bar{C}_{WP}$, which is what she would like him to consume if she could set his consumption, i.e. $\bar{C}'^*(\bar{C}_{WP})$. If he consumes more than this level then transfers have to increase at a certain rate in P in order to ensure that \bar{C}' is held constant.

Another way to think of (13) is that it tells her the \bar{G}_P that is needed to induce a desired consumption rate \bar{C}' for him. We see that the lower this desired rate is, the lower must be \bar{G}_P , i.e. the more she must make transfers increasing in *his* wealth share to provide strong-enough incentives to save.

Note that (13) is a very simple ODE for $\bar{G}(\cdot)$ – it tells us that the slope of $\bar{G}(\cdot)$ is constant on \mathcal{P}_i . Since \bar{C}' is constant on \mathcal{P}_i , this is the only ODE we have to solve when provided with boundary values $\bar{G}(P_i)$ and $\bar{C}'(P_i)$, for example.

Furthermore, we state the following result which tells us that no-transfers regimes are always transitory (except for points with measure zero on the state space):

Lemma 1 (FT not absorbing) *There cannot be any interval $I = (a, b)$ inside FT for which $P_t \in I$ implies $P_s \in I$ for all $s > t$.*

Proof: Suppose there was such an interval I contained in FT. Since both consumption rates are constant on FT, consumption rates would be the same constants for all $s > t$ for any $P_t \in I$. But this implies that both agents obtain the same continuation value starting from any $P_t \in I$, meaning that their value functions are constant on I . But this would mean that I is of WP-type and the recipient should play WP-consumption too, a contradiction to I being contained in an FT region. ■

3.2.3 Bang-bang-transfer regions (BB)

Consider a region where she gives a mass transfer, i.e. $\bar{G}(P) = (P_i - P)\delta$ for $P \in \mathcal{P}_i$. Since BB is always left immediately (as argued before), we have $\bar{V}(P) = \bar{V}(P_i)$ and $\bar{V}'(P) = \bar{V}'(P_i)$ for all $P \in \mathcal{P}_i$, which of course implies $\bar{V}_P = \bar{V}'_P = 0$ throughout \mathcal{P}_i . If she decided to defer the mass transfer by Δt , then both players should choose wealth-pooling consumption over this interval. Thus the only “credible threat” is $\bar{C}' = \bar{C}'_{WP}$ for him, which is formalized by the requirement of limit consistency. For her, equivalently $\bar{C} = \bar{C}_{WP}$ throughout the region. Her threat will be essential when considering deviations from a mass transfer at her bankruptcy, as will be discussed in 3.2.4.

3.2.4 Bankruptcy (BR)

An important special case is the point $P = 0$ where she is bankrupt (and $P = 1$ for him equivalently). First, note that for any interior point $P = \epsilon > 0$, any policy $\bar{C}(\epsilon)$ becomes feasible in the interval game for Δt small enough in the consumption stage. So she becomes unconstrained (i.e. she can choose an arbitrarily high consumption rate) with any small amount of resources when the time horizon is short enough. On the other hand, notice that whenever we have $P = 0$ after the transfer stage, she will be forced to consume zero over the next interval Δt and the value function would be minus infinity.

To escape this dilemma, we will interpret the policy $\bar{C}(0)$ as the policy when she is constrained in the stage of the game where consumption is chosen, i.e. on the interval $I_0(\Delta t) = (0, \bar{C}_{lim}(0)\Delta t]$. As the relevant derivative of the value function, only $\bar{V}_{P,lim}(0) \equiv \lim_{\epsilon \rightarrow 0} \bar{V}_P(\epsilon)$ is a sensible candidate. Under this interpretation, it is easy to see that for any $P \in I_0(\Delta t)$ the only limit-consistent policy for her is to consume all her resources. We interpret this as $\bar{C}(0) = \bar{C}_{lim}(0)$.¹⁴ So we have:

Lemma 2 (Recipient's policy continuous at BR) *In any limit-consistent equilibrium $\bar{C}(0) = \bar{C}_{lim}(0) \equiv \lim_{P \rightarrow 0} \bar{C}(P)$, and equivalently $\bar{C}'(1) = \bar{C}'_{lim}(1)$.*

As for the donor, he should not give transfers that lift her out of $I_0(\Delta t)$ if $\bar{V}'_{P,lim} < 0$. When giving a transfer $\bar{G}'(0) \leq \bar{C}(0)$, then the economy will stay at $P = 0$. Note that this implies that he can achieve his globally preferred allocation if $\bar{C}(0) \geq \alpha' \bar{C}'_{WP}$ by setting $\bar{C}'(0) = \bar{C}'_{WP}$ and $\bar{G}'(0) = \alpha' \bar{C}'_{WP}$ which implies $\bar{C}_0^* = \alpha' \bar{C}'_{WP}$ (see the Pareto problem (3) with weight $\eta = 0$). We summarize:

Lemma 3 (Donor's policies at BR) *For $P = 0$ limit-consistent implies:*

- $\bar{C}(0) \geq \alpha' \bar{C}'_{WP}$ implies $\bar{G}'(0) = \alpha' \bar{C}'_{WP}$ and that his globally preferred allocation is played at $P = 0$.
- $\lim_{P \rightarrow 0} \bar{V}'_P(P) < 0$ (\mathcal{P}_1 is NT) implies $\bar{G}'(0) \leq \bar{C}(0)$.

3.2.5 Autarkic regions (AUT)

We refer to an *autarkic region* as an interval $\mathcal{P}_i = (P_{i-1}, P_i)$ where the consumption rates equal the autarkic ones, i.e. $\bar{C}(P) = \rho P$ and $\bar{C}'(P) = \rho(1 - P)$, and where transfers are zero, i.e. $\bar{G} = \bar{G}' = 0$. For the laws of motion, this implies $\dot{P} = 0$ and $\dot{K} = (r - \rho)K$, which is, as explained in section 2.2, efficient. Observe that $\dot{P} = 0$ implies that any autarkic region is absorbing: If the economy is started in P or reaches \mathcal{P}_i at some time, it will stay there forever.

¹⁴Note that this does not mean that the actual *consumption* since this is given by \bar{C}_0^* , which depends on his transfer $\bar{G}'(0)$.

Value functions are given by the autarkic ones, which are easily verified to be:

$$\begin{aligned}\rho\bar{V}^{(aut)} &= (1 + \alpha)(\ln \rho - 1) + \ln P + \alpha \ln(1 - P) \\ \rho\bar{V}'^{(aut)} &= (1 + \alpha')(\ln \rho - 1) + \ln(1 - P) + \alpha' \ln P.\end{aligned}$$

Taking the derivative in P gives us

$$\begin{aligned}\rho\bar{V}_P^{(aut)} &= \frac{1}{P} - \frac{\alpha}{1 + P} \\ \rho\bar{V}'_P^{(aut)} &= \frac{-1}{1 - P} + \frac{\alpha'}{P},\end{aligned}$$

In this case, the Euler equations and the first-order conditions contain the same information.

Furthermore, since $\bar{G} = \bar{G}' = 0$ have to be optimal, we also have the inequalities $\bar{V}_P^{(aut)} \geq 0$ and $\bar{V}'_P^{(aut)} \leq 0$, from which we obtain the restriction

$$P \in \left[\frac{\alpha'}{1 + \alpha'}, \frac{1}{1 + \alpha} \right] \quad \text{for all } P \in \mathcal{P}_i$$

Any autarkic region has to be contained in this interval. The intuition for this result is very simple: If one player becomes too poor, the marginal utility of helping the other out becomes higher than the marginal utility of own consumption.

3.2.6 Wealth-pooling regions (WP)

We refer to WP-regions as those where consumption policies are of the wealth-pooling type, which is equivalent to both value functions being constant in this region. WP-regions are either BB-regions or FT-regions where both players transfer motives are zero. Her HJB tells us for such a region that

$$\rho\bar{V}^{(WP)} = \ln \bar{C}_{WP} + \alpha \ln \bar{C}'_{WP} - (\bar{C}_{WP} + \bar{C}'_{WP}) \frac{1 + \alpha}{\rho} = \text{constant}.$$

The following statement follows immediately:

Lemma 4 *All WP-regions must be connected. In a symmetric equilibrium they must form an interval $(\tilde{P}, 1 - \tilde{P})$, where $\tilde{P} \in [0, \frac{1}{2}]$.*

Proof: Suppose there were two disconnected WP-regions. Then between those regions there must be a point where the transfer motive is positive for at least one player – if not, everything in between would be a WP-region, too. Without loss of generality, say that $\bar{V}_p > 0$ at some point. But then, since her value function is \bar{V}_{WP} in both WP-regions, there also has to be a region where $\bar{V}_p < 0$ in between, which is a contradiction. ■

3.3 Characterizing boundaries

We will now turn to characterize policies on boundaries between two regions A and B , where $A = \mathcal{P}_i$ lies to the left of $B = \mathcal{P}_{i+1}$. We denote the boundary as $\tilde{P} = P_i$. It turns out that it is convenient to work with policies in terms of agents' own capital stock; we denote $\bar{c}_A = \lim_{P \rightarrow \tilde{P}^-} \bar{c}(P)$ as the limit of consumption on the A -side. The notation for $\bar{c}'_A, \bar{c}_B, \bar{g}_A$ and so forth is analogous. The policies directly on the boundary are denoted by $\bar{c}_K = \bar{c}(\tilde{P})$ etc., where K stands for "kink". When saying $H_{lim} = H_A$, we mean the Hamiltonian using \bar{V}_p^- and analogously for $H_{lim} = H_B$ using \bar{V}_p^+ .

The following lemma tells us that whenever policies on a kink lead us into region $i \in \{A, B\}$, then the policies on the kink must be like the limit policies in i (if i is not a wealth-pooling region):

Lemma 5 (Policy matching) *If an equilibrium allocation (c_K, g_K, c'_K, g'_K) on a kink is such that*

- $\dot{P} < 0$, then we have:

$$\begin{array}{ll} c_K = c_A, & c'_K = c'_A \\ \bar{V}_P^A > 0 \Rightarrow g_K = g_A = 0 & \bar{V}_P^A < 0 \Rightarrow g'_K = g'_A = 0 \end{array}$$

- Similarly, if $\dot{P} > 0$:

$$\begin{array}{ll} c_K = c_B & c'_K = c'_B \\ \bar{V}_p^B > 0 \Rightarrow g_K = g_B = 0 & \bar{V}_p^B < 0 \Rightarrow g'_K = g'_B = 0 \end{array}$$

Proof: Suppose that policies on the kink are such that $\dot{P} < 0$. Since \dot{P} is a continuous function in c , there is an interval $B_\epsilon(c_K) = (c_K - \epsilon, c_K + \epsilon)$ such that $\dot{P} < 0$ for all $c \in B_\epsilon(c_K)$. On $B_\epsilon(c_K)$, H_{lim} is given by H_A and c_K must optimize H_A on $B_\epsilon(c_K)$. Since H_A is differentiable, c_K must fulfill the FOC_A . But only c_A does this, so $c_K = c_A$. Analogously, $c'_K = c'_A$.

In the stage of the game where transfers are chosen it is obvious that $\bar{V}_P^A > 0$ implies $g_K = 0$. The argumentation for the remaining statements is entirely similar.

This tells us that on any boundary that is left the consumption policies must be as in one of the adjacent regions. As for transfers, it is unproblematic to find optimizing transfers on a boundary: If the transfer motive is zero, any policy is optimal, if it is negative, than transfers must be zero. So for this kind of boundaries it is easy to find limit-consistent kink policies; we only have to make sure that value matching is satisfied between the regions A and B .

This is not so easy, however, for *attracting boundaries*¹⁵. The above lemma tells us, for example, that neither the A -policies nor the B -policies can be optimal kink policies if both A and B are NT -regions. The following lemma tells us something about the implications of limit-consistency on the consumption policies on the kink:

¹⁵By an *attracting boundary* we mean that $\dot{P}_A > 0$ and $\dot{P}_B < 0$.

Lemma 6 (Consumption ordering on attracting kinks) *Suppose $\dot{P}_K = 0$, i.e. there are limit-consistent kink policies such that the kink is never left. This implies that*

$$\begin{aligned} c_A &\leq c_K \leq c_B \\ c'_B &\leq c'_K \leq c'_A. \end{aligned}$$

Proof: Consider first her situation: Note that $c_K = c'_K$ in equilibrium since $\dot{P}_K = 0$. When considering deviations $c > c'_K$, changes in the limit Hamiltonian are given by $FOC_A(\cdot)$ since we would have $\dot{P} < 0$. In equilibrium, such a deviation cannot be profitable, so we need $FOC_A(c_K) \leq 0$. But this is true if and only if $c_K \geq c_A$, since $FOC_A(\cdot)$ is a strictly decreasing function with root c_A . By the same token, deviations $c < c_K$ should not be profitable. For this we need $FOC'_B(c_K) \geq 0$, which is equivalent to $c_K \leq c_B$ since $FOC'_B(\cdot)$ is a strictly decreasing function with root c_B . This concludes the proof for her ordering of consumption rates; the argumentation for his ordering is entirely similar. ■

To prove this statement, we have only used the property that limit-consistent policies maximize the limit Hamiltonian; note that in practice, we still have to check if candidate policies also fulfill value matching, i.e. if the maximized limit Hamiltonian H_K takes the same value as H_A and H_B for both players.

3.3.1 Implications of value matching

Since both value functions have to be continuous at boundaries (see Lemma 11), they have to converge to the same values on the A- and B-sides (referred to as *value matching* in the literature). Using the HJBs, such as hers (8), substituting consumption rates for \bar{V}_P from the FOCs, such as hers (9), and taking limits towards the boundary, this implies the *value-matching conditions*

$$H(\bar{C}_A, \bar{C}'_A) = H(\bar{C}_B, \bar{C}'_B) \quad (14)$$

$$H'(\bar{C}'_A, \bar{C}_A) = H'(\bar{C}'_B, \bar{C}_B) \quad (15)$$

$$\begin{aligned} H(\bar{C}, \bar{C}') &= \ln \bar{C} - 1 + \alpha \ln \bar{C}' - \frac{\bar{C}'}{1-P} \left[\frac{1+\alpha}{\rho} - \frac{P}{\bar{C}} \right] + \frac{\bar{G}'}{1-P} \left[\frac{1+\alpha}{\rho} - \frac{1}{\bar{C}} \right] \\ &= \ln \bar{C} + \alpha \ln \bar{C}' - (\bar{C} + \bar{C}') \frac{1+\alpha}{\rho} + \dot{P} \bar{V}_P \end{aligned}$$

$$\begin{aligned} H'(\bar{C}, \bar{C}') &= \ln \bar{C}' - 1 + \alpha' \ln \bar{C} - \frac{\bar{C}}{P} \left[\frac{1+\alpha'}{\rho} - \frac{1-P}{\bar{C}'} \right] + \frac{\bar{G}}{P} \left[\frac{1+\alpha'}{\rho} - \frac{1}{\bar{C}'} \right] \\ &= \ln \bar{C}' + \alpha' \ln \bar{C} - (\bar{C} + \bar{C}') \frac{1+\alpha'}{\rho} + \dot{P} \bar{V}'_P \end{aligned}$$

Fixing policies in the A-region, (14) and (23) constitute a system of two non-linear equations in two unknowns from the B-side: If $\bar{V}_P = 0$, then $\bar{C} = \bar{C}_{WP}$ but \bar{G} is unknown; otherwise, $\bar{G} = 0$ and \bar{C} is unknown (and equivalently for him).

When B is of FT-type, it is easy to obtain the solution to the system. Say he is giving transfers in B. Then $\bar{C}'_B = \bar{C}'_{WP}$ and $\bar{G}_B = 0$, so we can use (23) to obtain \bar{C}_B (there are usually two solutions since the terms in \bar{C}_B constitute a concave function). We then use (14) to pin down \bar{G}'_B uniquely.

If B is of NT-type, matters are more complicated. However, it turns out that we can exploit the concavity of the utility function to characterize the solutions:

Proposition 2 (Discontinuities in policies) *Suppose that B is a NT-region. Then, for a given pair \bar{V}_A and \bar{V}'_A , there are generically at least two candidates for the consumption policies $\{c_B, c'_B\}$ on the B-side which are consistent with value matching. One of these candidates coincides with $\{c_A, c'_A\}$. The solutions are independent of P . The only exception is the case where $c_A = c'_A = \rho$, where the unique solution is $c_B = c'_B = \rho$.*

If also region A is of NT-type, then

$$\begin{aligned} c_A > c'_A &\Leftrightarrow c_B < c'_B \\ c_A < c'_A &\Leftrightarrow c_B > c'_B \\ c_A = c'_A &\Leftrightarrow c_B = c'_B. \end{aligned}$$

If either $k(c_A, c'_A) \leq k(\rho, \rho)$ or $k'(c_A, c'_A) \leq k'(\rho, \rho)$, then there are exactly two solutions for (c_B, c'_B) with the following property: If he is under-consuming in her eyes on the A-side, he must be over-consuming in her eyes on the B-side:

$$\frac{\alpha}{c'_A} > \frac{1+\alpha}{\rho} - \frac{1}{c_A} \quad \Leftrightarrow \quad \frac{\alpha}{c'_B} > \frac{1+\alpha}{\rho} - \frac{1}{c_B}.$$

The same is true reversing the inequalities, and of course for her under-/over-consumption in a symmetric fashion.

A proof is given in appendix A.3. It also provides an algorithm for finding a/the second solution to the system. In one specific case we were not able to prove that there are *exactly* two solutions. Numerical exercises, however, suggest that also in this case there exists no third solution.

In the following, we will characterize boundaries between the different types of regions. We start with the analysis of the different region types bordering bankruptcy and then proceed with the characterization of boundaries on the interior.

3.3.2 Bankruptcy-bang-bang (BR-BB) structure

It is intuitive to expect an equilibrium where the richer agent gives a mass-point transfer in a region where the other is very poor, i.e. $\mathcal{P}_0 = [0, P_1)$ is a BB-region. However, it turns out that this is impossible:

Proposition 3 (No BR-BB equilibrium) *If $\alpha\alpha' < 1$, then in equilibrium he does not give a mass transfer at $P = 0$ (and she does not give one at $P = 1$).*

Proof: If \mathcal{P}_1 is of BB-type, then $\bar{C} = \bar{C}_{WP}$ throughout \mathcal{P}_1 and thus also $\bar{C}(0) = \bar{C}_{WP} = \rho/(1 + \alpha) > \alpha'\rho/(1 + \alpha') = \alpha'\bar{C}'_{WP}$ if $\alpha\alpha' < 1$.¹⁶ Then, by Lemma 3, $\bar{G}'(0) = \alpha'\bar{C}'_{WP}$, and thus $\bar{C}_0^* < \bar{C}_{WP}$. ■

Intuitively, the poor agent has incentives to consume a lot; indeed she chooses C_{WP} in the entire *BB*-region. So at $P = 0$, the donor (he) can restrict transfers to a flow $G' = \alpha C'_{WP}$ and so obtain his preferred allocation – but this flow-transfer contradicts a mass-transfer being given when she is bankrupt.

3.3.3 Bankruptcy-flow-transfer (BR-FT) structure

Here, we clearly have the restriction $C_{FT} \leq \alpha C'_{WP}$ (i.e. the recipient's consumption must be below of what the donor would choose for the recipient if he could). If that was not the case, limit consistency would break down at $P = 0$ since he would set $G'(0) = \alpha' C'_{WP} < C_{FT}$.

BR-FT boundaries can only be of the smooth kind; we summarize this and other properties in the following proposition:

Proposition 4 (BR-FT properties) *Policies are continuous on a BR-FT boundary: $C'(0) = C'_{lim} = C'_{WP}$, $C(0) = C_{lim}(0) < \alpha C'_{WP}$ and $G'(0) = G'_{lim}(0) \geq C(0)$. Furthermore, $\dot{P} > 0$ for all $P > 0$, so she is lifted out of FT from any $P > 0$.*

Proof: Limit consistency clearly implies that $C'(0) = C'_{WP}$ so that $C'(0) = C'_{lim}(0)$. Now, his VM (23) implies that also $C(0) = C_{lim}(0)$ since the term $\dot{P}\bar{V}'_P$ is always zero. Then her VM (14) tells us that $G'(0) = G'_{lim}(0)$: If $\dot{P} > 0$ at $P = 0$, this is obviously necessary; if $\dot{P}(0) = 0$, then we also need $\dot{P}_{lim}(0) = 0$ since $\bar{V}_P > 0$, which again tells us that transfers are continuous. Feasibility at $P = 0$ requires $G'(0) \geq C(0)$. Then Lemma 3 tells us that it must be that $C(0) \leq \alpha' C'_{WP}$, if not he would restrict her consumption to his desired level at bankruptcy.

Now, we will rule out the possibility that $C(0) = \alpha' C'_{WP}$. If this was the case, then his HJB says that he obtains V'_{max} for all P in FT. The unique consumption sequence that ensures this is that his preferred consumption rates are placed forever. But this implies one of the following: (i) either FT is never left from any point P in FT – this is impossible since FT-regions cannot be absorbing, see Lemma 1, (ii) or his preferred consumption rates are played on the entire state space $[0, 1]$, which is inconsistent with her optimality at $P = 1$ (and not symmetric, either). So we can conclude that $C(0) < \alpha' C'_{WP}$.

From his ODE for transfers (13) we see that his transfers are increasing in P , i.e. $G'_P > 0$ on FT. From the law of motion for P REF??? and $G'(0) \geq C(0)$ we conclude that $\dot{P} > 0$ for all $P > 0$ inside FT, so she will be lifted out of FT from any point $P > 0$ on FT. ■

Note that if we introduced the possibility of in-kind transfers into the model (i.e. a transfer that cannot be saved but must be consumed), then this structure would break down since he would give her an in-kind transfer of $\alpha' C'_{WP}$ at

¹⁶Observe that $1/(1 + \alpha) \geq 1/2 \geq \alpha'/(1 + \alpha')$.

$P = 0$ and set monetary transfers equal to zero, which would guarantee that his preferred allocation was played forever.

3.3.4 No-transfer (NT) region and bankruptcy (BR): Parties

First, we establish the following lemma:

Lemma 7 (Donor's limit consumption) *Suppose $\alpha\alpha' > 0$. Then, under assumptions 2 and 3 in any equilibrium where a NT-region borders her BR:*

$$\lim_{P \rightarrow 0} \bar{C}'(P) = \bar{C}'_{WP} = \frac{\rho}{1 + \alpha'}$$

$$\lim_{P \rightarrow 0} \dot{P} = -\bar{C}_{lim} < 0,$$

i.e. bankruptcy is an absorbing state. The same is true in a mirror-symmetric way for $P = 1$.

Note that this is not obvious since it is not true that $\bar{V}'_P \rightarrow 0$. The intuition is that his consumption policy does not influence the proportion of asset she owns any more when her assets become very small; then, if her consumption does not approach zero, she must head into bankruptcy.

Given this result, we can show that

Proposition 5 (Party Theorem) *If $0 < \alpha = \alpha' < 1$ and assumptions 1, 2, 3 are satisfied, then any equilibrium where a NT-region borders her bankruptcy has the following properties:*

1. $\dot{P}_{lim} < 0$ and $\dot{P}_0 = 0$: *Bankruptcy is an absorbing state.*
2. $\bar{C}(0) = \alpha' \bar{C}'_{WP} = \frac{\alpha' \rho}{1 + \alpha'}$: *In her bankruptcy, his preferred allocation is played.*
3. $\bar{C}_{lim} = \exp\left(\frac{1 - \alpha\alpha'}{1 + \alpha'}\right) \bar{C}(0) > \bar{C}(0)$: *(Party) On reaching bankruptcy, her consumption path has a downward jump.*
4. $\bar{V}'_P(0) > 0$: *He strictly prefers her bankruptcy to her returning to be unconstrained.*

Proof: First, it is convenient to rule out that $\bar{V}_P(0) = \lim_{P \rightarrow 0} \bar{V}_P(P) = 0$.¹⁷ By way of contradiction, suppose that $\bar{V}_P(0) = 0$. Then, since $\bar{C}_{lim} = \bar{C}_{WP}$ her value-matching condition (VM) becomes (14)

$$\ln \bar{C}_0^* - \bar{C}_0^* \frac{1 + \alpha}{\rho} = \ln \bar{C}_{lim}(0) - \bar{C}_{lim} \frac{1 + \alpha}{\rho},$$

which implies $\bar{C}_0^* = \bar{C}_{WP}$ (note that the left- and right-hand sides are uniquely maximized by \bar{C}_{WP}). Since $\alpha\alpha' < 1$, by Lemma 3 the donor should then set

¹⁷Note that although NT is not a transfer region, it is possible that the transfer motive is negative throughout NT but tends to zero in the limit of the region.

$\bar{G}'(0) = \alpha' \bar{C}'_{WP}$ to obtain his globally preferred allocation. We would thus have $\bar{C}_0^* < \bar{C}_{WP}$, a contradiction. So from now on we may assume $\bar{V}_P(0) > 0$.

Now we will consider separately the two possibilities for the law of motion at her bankruptcy: $\dot{P}_0 = 0$ and $\dot{P}_0 > 0$.

(i) Suppose first that $\dot{P}_0 > 0$, i.e. $\bar{C}(0) = \bar{C}_0^* < \bar{G}'(0)$. By Lemma 5, it must be that $\bar{C}(0) = \bar{C}_{lim}$. For him, $\bar{C}'(0) = \bar{C}'_{lim} = \bar{C}'_{WP}$ by Lemma 7. Now, since both agents' policies are the same at $P = 0$ and $P \rightarrow 0^+$, her VM (14) collapses to:

$$\dot{P}_0 \bar{V}_P(0) = \dot{P}_{lim} \bar{V}_P(0),$$

from which we conclude that $\dot{P}_0 = \dot{P}_{lim} < 0$ since $\bar{V}_P(0) > 0$. But this is a contradiction to $\dot{P}_0 > 0$.

(ii) So it can only be that $\dot{P}_0 = 0$, which –together with Lemma 7– proves point 1 of the proposition. Now, define the function $H_{WP}(\bar{C}) = \ln \bar{C} - \bar{C} \frac{1+\alpha}{\rho}$ and re-write her VM (14) as

$$H_{WP}(\bar{C}_0^*) = H_{WP}(\bar{C}_{lim}) + \dot{P}_{lim} \bar{V}_P(0).$$

Since $\bar{V}_P(0) > 0$ and $\dot{P}_{lim} < 0$ (see Lemma 2), the fact that $H_{WP}(\cdot)$ is a strictly increasing function on $[0, \frac{\rho}{1+\alpha}]$ implies that $\bar{C}_0^* < \bar{C}_{lim}$, which is what we term a *party* before bankruptcy.

Now define $G_{WP}(\bar{C}) = \alpha' \ln \bar{C} - \bar{C} \frac{1+\alpha'}{\rho}$ and re-write his VM (23) as

$$G_{WP}(\bar{C}_0^*) = G_{WP}(\bar{C}_{lim}) + \dot{P}_{lim} \bar{V}'_P(0).$$

Since $\dot{P}_{lim} < 0$ and $\bar{V}'_P(0) \leq 0$, it is clear that $\dot{P}_{lim} \bar{V}'_P(0) \geq 0$ and thus $G_{WP}(\bar{C}_0^*) \geq G_{WP}(\bar{C}_{lim})$. Observe that $G_{WP}(\cdot)$ is a strictly increasing function on $(0, \alpha' \bar{C}'_{WP}]$ and a strictly decreasing function on $[\alpha' \bar{C}'_{WP}, \infty)$. By the ordering $\bar{C}_{lim} > \bar{C}_0^*$ from before, it can therefore be ruled out that $\bar{C}_{lim} \leq \alpha' \bar{C}'_{WP}$.¹⁸

Furthermore, by Lemma 2 we know that $\bar{C}(0) = \bar{C}_{lim}$, so he must set $\bar{G}'(0) = \alpha' \bar{C}'_{WP}$ to obtain his globally preferred allocation. Thus, $\bar{C}_0^* = \min\{\bar{C}(0), \bar{G}'(0)\} = \alpha' \bar{C}'_{WP}$, which is point 2 of the proposition.

Now, using again her VM (14) and the FOC for \bar{C}_{lim} in (9), can find the closed-form expression for $\bar{C}_{lim}(0)$ given in point 3 of the proposition. Finally, using his VM (23) again, we see that $\bar{V}'_P(0) < 0$ since $G_{WP}(\bar{C}_0^*) > G_{WP}(\bar{C}_{lim}(0))$, which proves point 4. ■

Note that the discontinuity of the recipient's consumption path is a gross violation of efficiency; a Pareto planner would lower consumption before bankruptcy and increase it in bankruptcy. This *party* is equivalent to the Samaritan's dilemma in a 2-period model: The recipient of transfers is not fully internalizing the negative effects that her consumption has on the donor when choosing consumption in the period/instant before bankruptcy. As the Euler equations (10) make clear, this inefficiency will propagate further back in time and so affect the entire span of time before bankruptcy – an effect that 2-period models are silent on.

¹⁸In intuitive terms, he must dislike the party and prefer the bankruptcy consumption rates to it, otherwise his VM cannot be fulfilled.

3.3.5 Boundaries between two no-transfer regions (NT-NT)

Lemma 6 and Proposition 2 together imply:

Corollary 1 (No attracting kink between NT-regions) *There cannot be an attracting boundary if both A and B are NT-regions.*

Repelling boundaries (i.e. $\dot{P}_A < 0$ and $\dot{P}_B > 0$) are possible if they satisfy value matching. The policies on the two sides are characterized by Proposition 2. Both the A- and the B-side policies are then possible limit-consistent policies on the boundary.

3.3.6 Boundary between autarkic and no-transfer region (NT-AUT)

Proposition 2 tells us that any NT-AUT boundary must be smooth, i.e. policies in the NT-region must converge to the autarkic levels.

3.3.7 Flow-transfer and no-transfer region (FT-NT)

We distinguish between *smooth* and *non-smooth* boundaries between FT- and NT-regions. As a smooth boundary we regard one where value functions are differentiable and thus policies are continuous.

For example, let A be a region where he is giving transfers, so $\bar{C}'_A = \bar{C}'_{WP} = \bar{C}'_B$. From her value matching it follows that $G'_A = 0$. It is obvious that this fulfills VM for both players and that $\bar{C}_K = \bar{C}_A = \bar{C}_B$ and $\bar{C}'_K = \bar{C}'_A = \bar{C}'_B$ are limit-consistent policies on the boundary. From the ODE for transfers (13) and the fact that transfers cannot be negative the following characterization then follows easily:

Lemma 8 (Smooth FT-NT/NT-FT boundary) *Let \tilde{P} be the boundary between a region A on the left and a region B on the right, where one of the two is NT and the other FT. Then*

1. $C'_A = C'_B = C'_K = C'_{WP}$ and $C_A = C_B = C_K$.
2. $G'_A = G'_B = G'_K = 0$.
3. *If A is of NT-type, then $C \geq \alpha' C'_{WP}$.*
4. *If B is of NT-type, then $C \leq \alpha' C'_{WP}$.*

This result tells us that when we reach a point \tilde{P} where V'_P becomes positive in an NT-region, then it is only possible to start an FT-region at \tilde{P} if she is under-consuming in his eyes at \tilde{P} .

Appendix A.2.3 provides a result on non-smooth FT-NT boundaries.

3.3.8 FT-FT structure

When looking for a symmetric equilibrium, it will prove important to rule out the case that an FT-region (in which he is giving transfers) extends to $P = 1/2$ from the left. The following result formalizes this:

Lemma 9 (No FT-FT at $1/2$ in symmetric equilibrium) *Let $(P_i, P_{i+1}) \in [0, 1/2]$ with $G' > 0$ be an FT region and suppose $\alpha = \alpha'$. In any symmetric limit-consistent equilibrium it must be that $P_{i+1} < 1/2$.*

Proof: Let c_{WP} and c_{FT} denote the donor's and recipient's consumption levels in terms of their own capital stock. We are looking for a symmetric consumption level c_{symm} at $P = 1/2$ that is consistent with both value matching and limit consistency. Note that by symmetry, we also have $g(1/2) = g'(1/2)$ and $\dot{P}(1/2) = 0$.

For c_{symm} to be limit consistent, we need $c_{FT} \leq c_{symm} \leq c_{WP}$, as inspection of the relevant limit Hamiltonian reveals. Value matching requires that $k(c_{WP}, c_{NT}) \leq k(\rho, \rho) = \max_c k(c, c)$. Also, observe that $c_{WP} > \rho$. In Figure 19, which plots the loci where $k = k(\rho, \rho)$, we must thus be in one of the following two regions (recall that $k(c, \cdot)$ is concave for all c):

1. $c_{WP} > \rho, c_{NT} \leq \gamma'_-(c)$: But then, for him we have $k'(c_{WP}, c_{NT}) > k(\rho, \rho)$ by the properties of k' . This rules out value matching for him:

$$k(c_{symm}, c_{symm}) = k'(c_{WP}, c_{NT}) - g_{NT} V'_{P, NT} \geq k'(c_{WP}, c_{NT}) > k(\rho, \rho) \geq k(c_{symm}, c_{symm}),$$

a contradiction.

2. $c_{WP} > \rho, c_{NT} \geq \gamma'_+(c)$: Then $c_{NT} > \rho$, since $\gamma'_+(c) > \rho$ for $c > \rho$. But now, observe that

$$k_{symm}(c_{symm}) \leq k_{symm}(c_{NT}) < k(c_{WP}, c_{NT})$$

where the weak inequality follows from $c_{symm} \geq c_{NT}$ and k_{symm} being strictly decreasing on $[\rho, \infty)$ and the second inequality follows from $c_{WP} > c_{NT}$ and k being increasing in its first argument below the diagonal (i.e. for $c > c'$). The inequality makes value matching for her impossible.

The statement in the lemma follows, since these are the only possible cases. ■

3.3.9 FT-WP structure

First, notice that at boundary between an FT- and a WP-region, the recipient's consumption in FT is pinned down exactly. To see this, assume that he is the donor in the FT-region. Then by his VM (23), her consumption C_{FT} – recall that the recipient's consumption is constant inside FT – must solve

$$f(C_{FT}) \equiv \alpha' \ln C_{FT} - C_{FT} \frac{1 + \alpha'}{\rho} = \alpha' \ln C_{WP} - C_{WP} \frac{1 + \alpha'}{\rho} = f(C_{WP}) \quad (16)$$

One solution to this equation is obviously $C_{FT} = C_{WP}$. However, this would mean that FT was also a WP-region, which contradicts our assumption. Since $f(\cdot)$ is concave, uniquely maximized at $\alpha' C_{WP}$ and $\lim_{C \rightarrow 0} f(C) = -\infty$, there must exist exactly one further solution $C_{FT} \in (0, \alpha' C_{WP})$, where she under-consumes in his eyes. Interestingly, this solution is homogeneous in ρ , as is easily verified: If $C_{FT}(1, \alpha, \alpha')$ solves (16) for $\rho = 1$, then $C_{FT}(\rho, \alpha, \alpha') = \rho C_{FT}(1, \alpha, \alpha')$ solves (16) for any $\rho > 0$.¹⁹

From her VM (22), we get

$$\underbrace{\ln C_{FT} - C_{FT} \frac{1+\alpha}{\rho} + \tilde{P}_A \bar{V}_P}_{\equiv \tilde{h}(C_{FT})} = \ln C_{WP} - C_{WP} \frac{1+\alpha}{\rho} = \tilde{h}(C_{WP})$$

Since $C_{FT} < \frac{\alpha'}{1+\alpha'} \rho < \frac{\rho}{2} < \frac{1}{1+\alpha} = C_{WP}$ (if $\alpha\alpha' < 1$) and \tilde{h} is strictly increasing for $C \leq C_{WP}$, it must be that $\tilde{h}(C_{FT}) < \tilde{h}(C_{WP})$. Now, since $\bar{V}_P^{(A)} > 0$, there is a unique $G'_A > 0$ that solves the above VM. It is obtained as

$$G'_A = (1-\tilde{P})\rho \underbrace{\frac{(1+\alpha)C_{FT}(1, \alpha, \alpha') - \ln C_{FT}(1, \alpha, \alpha') - 1 - \ln(1+\alpha)}{\frac{1}{C_{FT}(1, \alpha, \alpha')} - (1+\alpha)}}_{\equiv \tilde{H}(\alpha, \alpha')} + (1-\tilde{P})C_{FT} - \tilde{P}C'_{WP}. \quad (17)$$

Note that for $\tilde{P} = 0$, we have $G'_A > C_{FT}$. By continuity, there must thus be some $\tilde{P} > 0$ such that $G'(0) \geq C_{FT}$ (note that $G'(0) < G'(\tilde{P})$ since $G'_P > 0$, see Proposition 4).

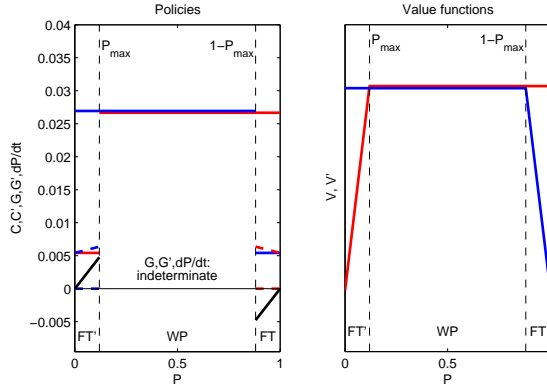


Figure 1: FT-WP-equilibrium: $\alpha = \alpha' = 0.5$, $\tilde{P} = P_{max}$

Note that the right-hand side of (17) is linearly decreasing in \tilde{P} ; we can back out the maximally-possible \tilde{P} that can sustain a FT-WP equilibrium from the

¹⁹Note that it is impossible to establish a similar result for α and α' , even in the special case $\alpha = \alpha'$.

equation

$$G'(0) = G'_A(\tilde{P}_{max}) - G'_P\tilde{P}_{max} = C_{FT}$$

Using (17) and (13), we find that this value is independent of ρ and given by

$$\tilde{P}_{max}(\rho, \alpha, \alpha') = \frac{\tilde{H}(\alpha, \alpha')}{\tilde{H}(\alpha, \alpha') + 1}, \quad (18)$$

where $\tilde{H}(\alpha, \alpha')$ is defined in (17). Figure 2 shows \tilde{P}_{max} as a function of (α, α') .

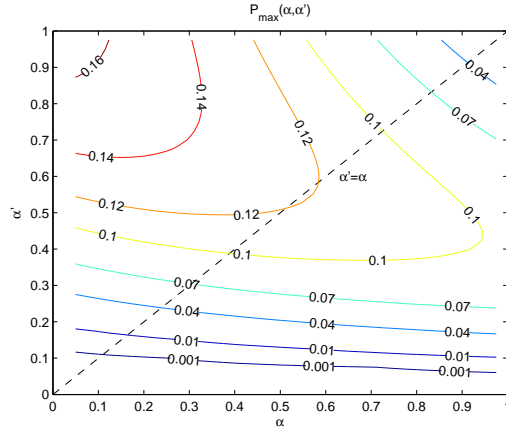


Figure 2: Maximal boundary in FT-WP-equilibrium

The following proposition collects the previous results:

Proposition 6 (Continuum of FT-WP equilibria) *If $\alpha\alpha' \in (0, 1)$, then there exists a continuum of equilibria of the following type: He gives transfers on $[0, P_1)$; there is a WP-region $[P_1, P_2]$ and she gives transfers on $(P_2, 1]$, where $P_1 \in (0, \tilde{P}'_{max})$ and $P_2 \in [1 - \tilde{P}_{max}, 1)$ and \tilde{P}_{max} is given in (18). Transfers on $[0, P_1)$ are given by (17) at P_1 and follow from (13) for $P \in (0, P_1)$. Transfers inside WP are indeterminate, but are such that WP is never left from any point: $P_0 \in [P_1, P_2] \Rightarrow P_t \in [P_1, P_2]$ for all $t > 0$.*

Note that limit consistency at P_1 is unproblematic: Since his value function is flat in both directions, $C'(P_1) = C'_{WP}$ and any $G'(P_1) \geq 0$ are clearly limit-consistent. For her, $C(P_1) = C_{WP}$ is fine since it is the global maximum of the limit Hamiltonian for any V_P , see 12; decreasing C below this value and heading into FT must decrease H_{lim} .

A noteworthy feature of this equilibrium is the following: Since transfers G' are linearly increasing in P on FT, we see from the law of motion for P in (7) that \dot{P} linearly increases in P . This means that the economy is moving out of FT at increasing speed as she becomes richer. If $P_1 < P_{max}$, then the equilibrium is

such that FT is left in finite time for any starting value P_0 , even when $P_0 = 0$. For $P_1 = P_{max}$, however, the initial time spent in FT increases without bound as $P_0 \rightarrow 0$ and is indeed infinite when $P_0 = 0$: Then $\dot{P} = 0$ and the economy is stuck at $P = 0$ forever.

Note that type of equilibrium described in Proposition 6 breaks down if we allow agents to give in-kind transfers, as was already noticed in section 4: He would give an in-kind transfer of $\alpha' C'_{WP}$ instead of monetary transfers inside NT, which she would be “forced” to consume instantaneously. This would lead to his globally-preferred allocation being played, so it would be a profitable deviation for him.

4 Computational Results

In this section we present the results of computing a stationary Markov-Perfect equilibrium using value function iteration. We find that *there is an equilibrium where transfers flow only when one of the players is broke*. In other words, for $P \in (0, 1)$ there are neither bang-bang transfers nor flow transfers. Only when $P = 0$ (he is broke) or $P = 1$ (she is broke) transfers flow. While we have not been able to prove that this equilibrium is unique we have reason to believe that it is. In order to check the robustness of this equilibrium we consider a finite-horizon version of this model. In the last period T the sharing rule is based upon the static altruism game. Using backward induction, including time as a state-variable, yields the same equilibrium as the one found by value function iteration. We also use a host of other sharing rules for the final period and consistently find convergence to the same equilibrium. Thus, this equilibrium is the unique limit of the sequence of equilibria in (equivalent) finite-horizon games.

In this equilibrium, there is a *party* of the recipient just before bankruptcy. For example, consider the situation where he is about to become broke. The relationship between his limit-consumption and consumption in bankruptcy is given by

$$\lim_{P \rightarrow 1} \bar{C}'(P) = \exp\left(\frac{1 - \alpha\alpha'}{1 - \alpha}\right) \frac{\alpha\rho}{1 + \alpha} > \frac{\alpha\rho}{1 + \alpha} = \bar{C}'(1) = \bar{G}(1) \quad (19)$$

For $\alpha' < 1$ his limit-consumption is strictly larger than his consumption when he is broke. Thus, his consumption path exhibits a discontinuity in the form of a downward jump of size which we can read off equation (19).

4.1 Asset Evolution

In this section we present the results of computing the stationary Markov-Perfect equilibrium. In order to build intuition for the results of the model it is helpful to simulate the distribution of wealth P over time. Figures 3 to 6 show the evolution of P over a time horizon of 30 years. Initial values of P range between

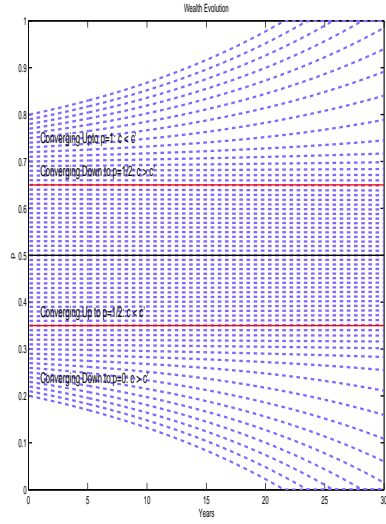


Figure 3: $\alpha = \alpha' = 0.2$

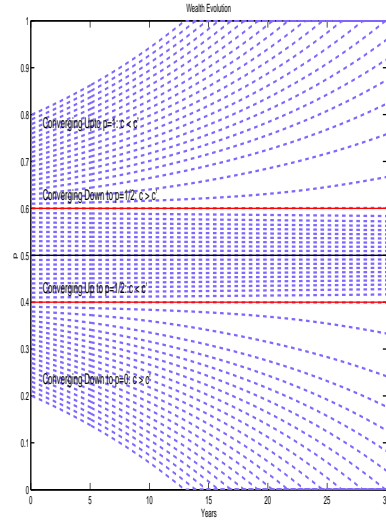


Figure 4: $\alpha = \alpha' = 0.4$

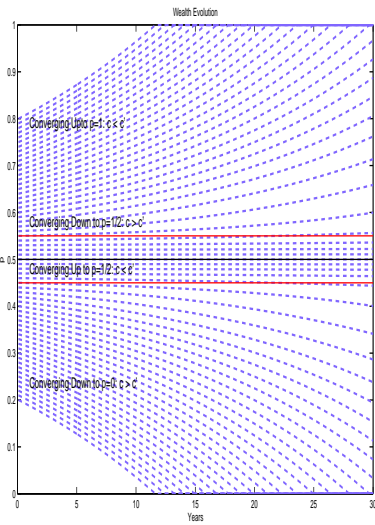


Figure 5: $\alpha = \alpha' = 0.6$

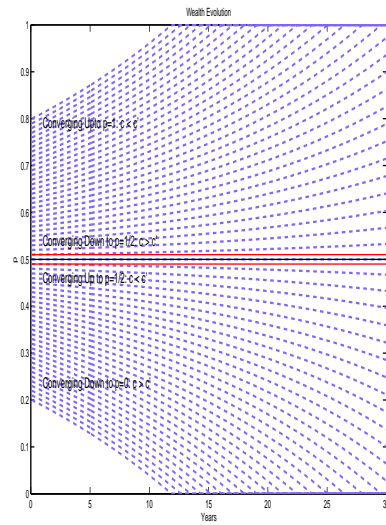


Figure 6: $\alpha = \alpha' = 0.8$

0.2 and 0.8. The altruism parameters are assumed to be symmetric and given by $\alpha = \alpha' = \{0.2, 0.4, 0.6, 0.8\}$.

In all 4 figures, which correspond to the various rates of altruism, three regions become apparent: (1) Starting with a value of P closer to one leads eventually to his bankruptcy which is an absorbing state; (2) With starting values of P closer to $1/2$ neither of the players become bankrupt but rather converge to $P = 1/2$; (3) For values of P closer to zero she reaches bankruptcy in a matter of time. Additionally, comparing Figure 3 in which $\alpha = \alpha' = 0.2$ with Figure 6 in which $\alpha = \alpha' = 0.8$ demonstrates that the range of P for which bankruptcy results increases in the altruism parameter. This is not surprising since the “soon-to-be” bankrupt can count on higher transfers from the donor. Another way of saying this is that the optimal time of bankruptcy is sooner the higher the degree of altruism. For example, in Figure 3 if we start the players off with $P_0 = 0.2$ and follow them through time she becomes bankrupt after approximately 22 years. On the other hand, Figure 4 shows that she reaches bankruptcy already after 13 years.

The horizontal lines are meant to visually approximate intervals for values of P when one of the players’ consumption rate is larger or smaller than the others. This can be easily seen when we write the law of motion for P in terms of \bar{c}, \bar{c}' , the consumption proportions out of the private capital stock,

$$\dot{P} = -(1 - P)P(\bar{c}' - \bar{c})$$

Clearly, when he consumes at a faster rate than she does $\dot{P} > 0$ and if he consumes at a slower rate $\dot{P} < 0$. In Figure 4, for example, below the horizontal line $P = 0.4$ we see that $\dot{P} < 0$, she consumes at a faster rate than he does, and as we will see below above her autarkic rate ρ . The situation is mirror-symmetric above $P = 0.6$. When $P = 0.5$ then we trivially have that $\bar{c} = \bar{c}'$ and $\dot{P} = 0$. Thus, within the interval $[0.4, 0.6]$ they consume at approximately the same rate and we will see that this rate is close to the autarkic rate. At values of P close to the horizontal lines there are strong conflicting forces among the two players. Each player has a strong incentive to direct the evolution of P in opposing directions. This conflict of interest will become especially clear in the following section.

4.2 Consumption Policies

Figures 7 to 10 plot for $\alpha = \alpha' = \{0.2, 0.4, 0.6, 0.8\}$ consumption proportions in terms of common wealth \bar{C}, \bar{C}' (the left box in each figure) and in terms of private wealth \bar{c}, \bar{c}' (the right box in each figure) for $P \in [0, 1]$. As reference points we also depict the autarkic consumption proportion and the wealth-pooling rate. The autarkic consumption proportion out of combined capital stock are respectively given by $\bar{C}'_{aut} = \rho \cdot (1 - P)$ for him and $\bar{C}_{aut} = \rho \cdot P$ for her, and players wealth-pooling rate is $\bar{C}_{wp} = \bar{C}'_{wp} = \rho/(1 + \alpha)$. Similarly, for the consumption proportions out of private wealth her and his autarkic rate is given by $\bar{c}_{aut} = \bar{c}'_{aut} = \rho$, his wealth-pooling rate is given by $\bar{c}'_{wp} = \frac{1}{1-P} \frac{\rho}{1+\alpha}$, and $\bar{c}_{wp} = \frac{1}{P} \frac{\rho}{1+\alpha}$ for her.

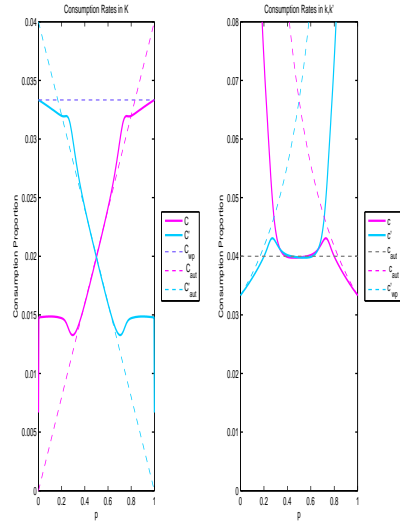


Figure 7: $\alpha = \alpha' = 0.2$

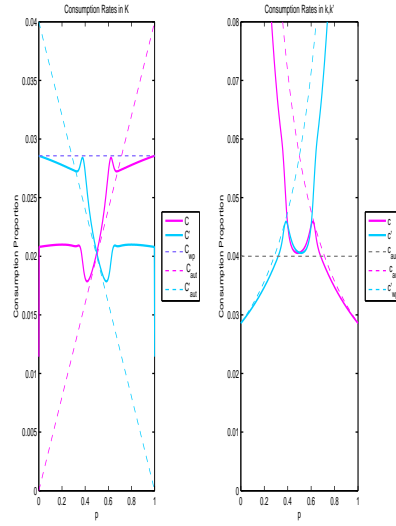


Figure 8: $\alpha = \alpha' = 0.4$

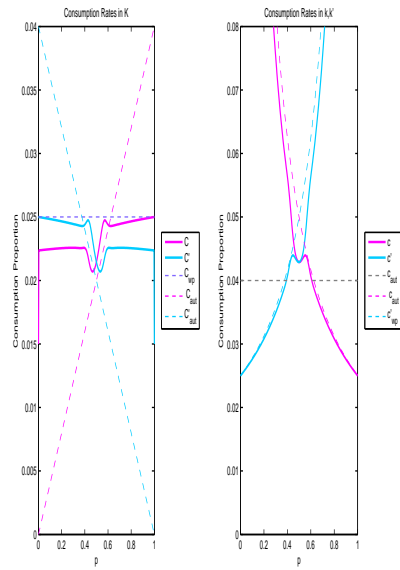


Figure 9: $\alpha = \alpha' = 0.6$

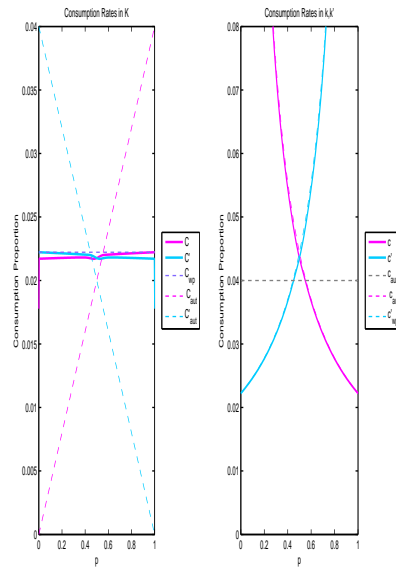


Figure 10: $\alpha = \alpha' = 0.8$

For example, Figure 8 can be read as follows. In the left box the pink line is her consumption rate \bar{C} and the thick baby-blue line is his consumption rate \bar{C}' . The wealth-pooling rate is constant and equal for both players and is shown as the horizontal line. The two discontinuous lines from the upper corner to the lower corner represent his and her autarkic consumption rates. Note that for the right box it is the autarkic rate which is equal for both agents but not the wealth-pooling rates.

Consider, for example, Figure 8 with P starting off at zero. At $P = 0$ she is bankrupt and receives transfers. His consumption rate equals the wealth-pooling rate. We can see that his consumption rate is below the autarkic rate and her consumption rate is, of course, above. He implements his first best by doing what the planner would do. Instead of consuming ρ out of $K = k'$ he consumes $\rho/(1 + \alpha)$ out of $K = k'$ and transfers $\alpha\rho/(1 + \alpha)$ to her. It is the sum of his consumption and transfer rate which equals ρ (see the left box). In terms of consumption out of private capital stock (see the right box) the story is entirely analogous.

As we increase P slightly her consumption rate exhibits an upward jump since briefly before she is about to become bankrupt she is “having a party”. From then on until approximately $P = 0.4$ her consumption rate out of K is roughly constant and much larger than it would be in the autarkic case. His consumption rate is decreasing and slightly below his autarkic rate. This is what we refer to as the *dynamic Samaritan’s dilemma*. She vastly overconsumes relative to what the planner would do since she knows that eventually she will be taken care of. His consumption rate is also distorted even though he consumes below the autarkic rate. Since it is the growth rate of marginal utility of consumption which matters for overconsumption it turns out to be the case that his rate is too fast.

At roughly $P = 0.4$ his consumption rate displays an upward hump, shooting up but not exceeding his wealth-pooling rate, and her consumption displays a downward hump, dropping down but not below her autarkic rate. It is here that the characterization of regions becomes especially important. Since the players know that to the left of $P = \underline{P}$ they are in a region in which eventually she will be receiving transfers from him but to the right no transfers will ever flow there is a conflict of interest in which direction P should evolve. Since both players have control using consumption and transfers over the law of motion for P it is important to understand on how the regions are connected. It is conceivable that the value functions exhibit convexities at these critical points which would be in line with Laitner (1988)’s suspicion. This could be interpreted as risk-loving behavior even in the absence of risk. Another possibility is that while the value functions are continuous they are not differentiable at these points of conflict in the state-space. Then the humps would be an artifact and an interpretation is difficult. Increasing P above $P = \underline{P}$ we can see that both consumption rates are getting close, but never equal, to the autarkic rates.

Some more observations from the other figures are worthwhile to point out. Comparing figures 8 and 9 we see that qualitatively the interpretation conforms to the one just outlined. Notice that in figure 7 there is a fairly large region in

which the agents consume as if they are in autarky. This is sensible since their altruism parameter is fairly small. From figure 10 we can see that the agents essentially behave as if they are wealth-pooling, at least, for $P \in (0, 1)$. Note however, that this is somewhat of an illusion since the transfer motives of him and her are not even close to zero ²⁰.

4.3 Strategic-Altruistic Distortions

4.3.1 Euler Equation in (k, k')

In order to supplement intuition about the intertemporally optimal consumption plan a slight detour into the (k, k') state-space is in order. In particular we will take a look at her Euler equation under the assumption that we are within a smooth region in which there are neither transfers nor binding borrowing constraints (e.g. a no-transfer region). Euler equations tell us how marginal utility of consumption changes over time – recall that in discrete time for a selfish agent, we have $u_c(c_t) = \beta R u_c(c_{t+1})$, which is a *difference* equation in c_t . In continuous time, we can obtain an analogous *differential* equation from the HJB: First, let us focus on the selfish case by setting $\alpha = \alpha' = 0$. We obtain the following Euler equation for a selfish agent:

$$\frac{d}{dt} V_k(k_t) = (\rho - r) V_k = \frac{d}{dt} u_c(c_t),$$

This equation tells us something familiar: If $\rho > r$, the agent is impatient compared to other market participants and her marginal utility grows at rate $(\rho - r)$ over time, which implies a decreasing consumption path. If $\rho = r$, then marginal utility is time-invariant. For $\rho < r$, the agent is more patient than other market participants and his consumption is increasing over time.

When carrying out the same steps as in the selfish problem for the case *with* altruism (i.e. $\alpha > 0$ and $\alpha' > 0$)

$$\frac{dV_k}{dt} = (V_k)_k \dot{k} + (V_k)_{k'} \dot{k}' = (\rho - r) V_k + \underbrace{[V_{k'} - \alpha u_c(c')]}_{\equiv \lambda(k_t, k'_t)} \frac{\partial c'}{\partial k}.$$

The intuition here is the following: The term $(\rho - r) V_k$ tells us that marginal utility increases at rate $(\rho - r)$ along the optimal consumption path, which is the same as in the selfish setting. The second term $\lambda(\cdot)$ is related to altruistic/strategic considerations: When she saves one unit more over some interval of time, her capital stock will increase and there will be a (usually positive²¹)

²⁰It turns out that for α 's large enough and logarithmic utility one can fairly well approximate the consumption function as a linear combination of wealth-pooling rates which is itself again a wealth-pooling rate.

²¹In the case where he is richer than she is, he will feel more confident about consuming more when she increases her wealth since it becomes less likely that he must help her out or later. If she is rich relative to him, he will know that there are more resources available that she can potentially provide to him in the form of transfers, which again tends to increase his consumption.

effect on his consumption which is captured in the term $\partial c'/\partial k$. This increase in c' gives an immediate gain $\alpha u_c(c')$ to her since she feels happy for him; this is an additional incentive to save, so it enters the equation with the same sign as r does. However, his increased consumption also means that he saves less, which has negative effects on her (through decreased future consumption and decreased future transfers) that are captured in the term $V_{k'}$. These enter with the same sign as ρ , so they act as a disincentive to save.

4.3.2 Distortion in (P,K)

In (P, K) -space the Euler equations are given by equation (10) in section 2.3 which captures the growth of her and his transfer motive. Thus it is not directly suitable in understanding over- and under-consumption. Instead, we will compute the growth rate of marginal utility which is after all the left side of the Euler equation in (k, k') -space. To this end recall that in the autarkic case, or equivalently in the planner's problem, the growth rate of marginal utility is given by:

$$\frac{\frac{d}{dt}u_c(c_t)}{u_c(c_t)} = \rho - r \quad (20)$$

In (P, K) -space we compute:

$$\begin{aligned} \frac{\frac{d}{dt}u_c(c_t)}{u_c(c_t)} &= \frac{d}{dt} \ln(u_c(c_t)) = -\frac{\dot{\bar{C}}}{\bar{C}} - \frac{\dot{K}}{K} = -\frac{d}{dt} [\ln(\bar{C}) + \ln(K)] = \\ &= -\frac{\bar{C}_P}{\bar{C}} \dot{P} - (r - \bar{C} - \bar{C}') \end{aligned}$$

Equivalently, we can express the growth rate of marginal utility in terms of private capital stocks \bar{c}, \bar{c}' rates as follows:

$$\frac{\frac{d}{dt}u_c(c_t)}{u_c(c_t)} = \bar{c} - \frac{\dot{\bar{c}}}{\bar{c}} - r = (\bar{c} - r) - \frac{\bar{c}_P}{\bar{c}} \dot{P}$$

For simplicity, let us set $r = \rho$, then from (20) we see that efficiency corresponds to a zero growth rate of marginal utility. In contrast, if $(\bar{c} - r) - \frac{\bar{c}_P}{\bar{c}} \dot{P} > 0$ there is overconsumption. Next, we substitute the law of motion in terms of private capital stock $\dot{P} = -(1 - P)P(\bar{c} - \bar{c}')$ into the distortion,

$$(\bar{c} - r) + (1 - P)P(\bar{c} - \bar{c}') \frac{\bar{c}_P}{\bar{c}} > 0 \quad (21)$$

Figures (11) to (14) show the distortion (21) for the various symmetric values of the altruism parameters as a function of P . Also shown is the zero-line which corresponds to the efficient growth rate as well as the growth rate corresponding to the wealth-pooling consumption rate (this rate is non-zero since we are working with small variables \bar{c} and \bar{c}' , the upper horizontal line).

There are two regularities which jump out immediately. First, note that over the entire state-space the distortion is practically always non-negative. Thus,

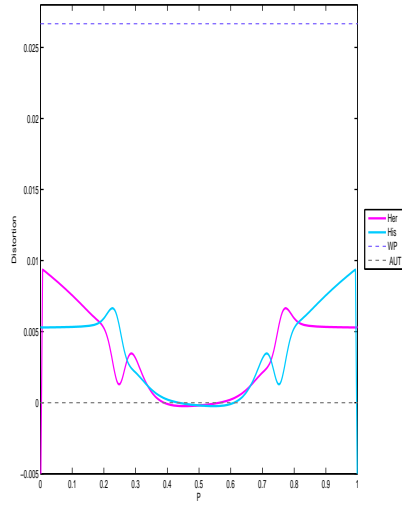


Figure 11: $\alpha = \alpha' = 0.2$

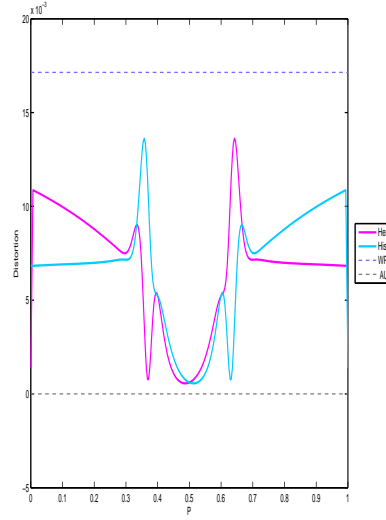


Figure 12: $\alpha = \alpha' = 0.4$

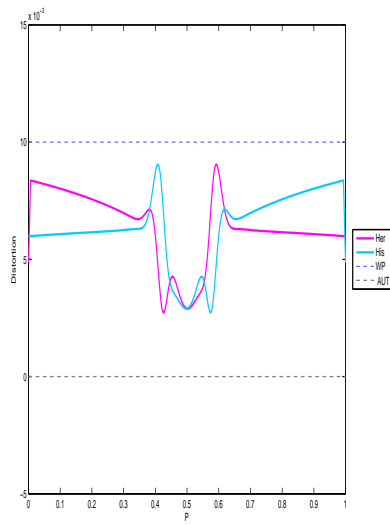


Figure 13: $\alpha = \alpha' = 0.6$

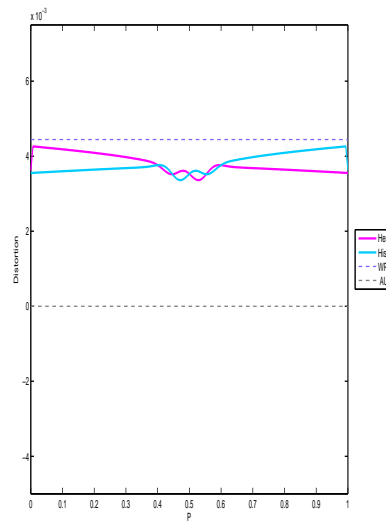


Figure 14: $\alpha = \alpha' = 0.8$

for all values of P there is overconsumption. This highlights our definition of over-consumption. We do not mean that the consumption rate is larger than the planner would choose, but that the *growth* of the consumption rate is faster than the planner would set it. As discussed above when she is bankrupt efficiency is restored, an allocation that a planner chooses, when all the weight is placed on his life-time utility. Yet, his consumption rate is below his autarkic rate! The key of efficiency is that marginal utility for both agents grows at a rate zero.

Second, the distortion due to the wealth-pooling rate is never surpassed. The intuition for this can be understood when considering *the tragedy of the commons*. In that scenario there is a resource which can be accessed and exploited by various individuals who do not care for each other. All the players have an incentive to extract more from the resource than they would as owner. Similarly, in order for both agents to consume at the wealth-pooling rate it would have to be the case that $\bar{V}_P = \bar{V}'_P = 0$. When these derivatives are both zero then both agents do not care about the distribution of wealth but only about the common capital stock K . This can be interpreted as an absence of property rights. As a result there is an *altruistic tragedy of the commons*. The degree of altruism ameliorates the underlying force in the tragedy of the commons, but, as we can see it is still less efficient than our Markov-Perfect equilibrium.

As an example of what these figures say consider figure (12). For P close to zero the growth rate of her marginal utility of consumption is somewhat above 1%. While this is significantly smaller than the distortion would be in the wealth-pooling case, about 1.7%, it is much larger than the efficient rate of zero. For the future donor the growth rate is also positive and roughly 0.7%. As P increases her distortion decreases and is smallest when $P = 0.5$. His distortion slightly increases before it decreases to its lowest point at $P = 0.5$. The interpretation of the spikes in the distortion hinge on whether the value function is convex in that region or non-differentiable.

In a two-period model it is easy to show that the savings decision of the agent who expects to be the recipient in the second period is inefficient. In our model the situation is similar but distortions occur long before actual transfers flow. A difference however is that in our model both agent's savings decision are distorted, as opposed to a two-period model in which only the recipients is. In addition, to this dynamic Samaritan's dilemma there is the standard Samaritan's dilemma which occurs before one of the agents is broke as signified by the discontinuity of the recipient's consumption function.

4.3.3 A History

In Figure (15) a history of a tuple of players who start off with $P = 0.25$ and symmetric altruism of $\alpha = \alpha' = 0.4$ is illustrated. Her optimal time of bankruptcy is in about 17 years at which point her consumption rate drops discontinuously from $\bar{C} = 2.2\%$ to 1.1% . At that time she receives transfers from him, portrayed by the discontinuous baby-blue line $\bar{G}' = \bar{g}' = \rho\alpha'/(1+\alpha')$ with his consumption rate being equal to the wealth-pooling rate $\bar{C}'_{wp} = \bar{c}'_{wp} = \rho/(1+\alpha)$. The growth rate of common resources $\dot{K}/K = -(r - \bar{C} - \bar{C}')$ is close

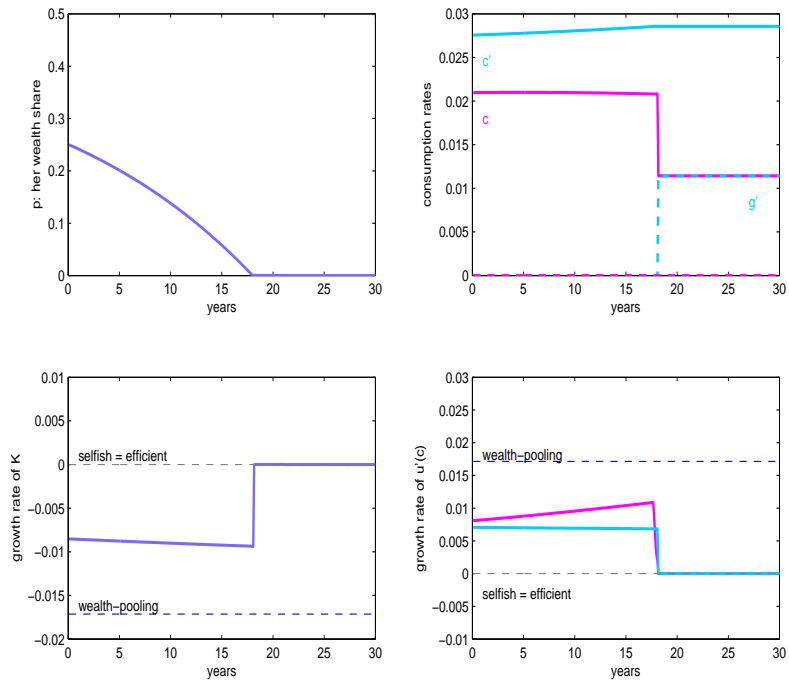


Figure 15: A history of a tuple of players with $\alpha = \alpha' = 0.4$ starting off the simulation with $P = 1/4$. Along the horizontal axis is time measured in years and along the vertical axis is P , consumption and transfer rates $\bar{C}, \bar{C}', \bar{G}, \bar{G}'$, growth of combined wealth K , and growth of her marginal utility.

to -1% over the first 17 years which is in stark contrast to the efficient growth rate of zero percent. Note that both agents contribute to this inefficiency as is evident from the last sub-figure which is in contrast to a 2-period model in which only the future recipients savings decision is distorted but not the donors.

4.3.4 Welfare

A particular way of assessing the welfare implications of the equilibrium is to ask how much the players would value a scheme in which they could commit to a redistribution of resources at time zero equal to the present value of future transfers the donor would give away. Table 1 demonstrates the percentage of common resources K the agents would require in order to make them indifferent between the actual equilibrium and a hypothetical situation in which the donor transfers a lump-sum at time zero equal to the amount of transfers in present value terms of the actual equilibrium. One would expect that the future recipient as well as the future donor attain a higher level of life-time utility if the donor could credibly commit to that scheme. In that case strategic considerations are eliminated and both players consume at their unique optimal rate ρ . Since she is able to perfectly smooth her consumption stream she has to be clearly better off. If she is better off then this contributes well-being to him since her well-being is part of his preferences. Furthermore, he can also consume at his unique optimal rate ρ .

More specifically, we first compute the optimal time of her bankruptcy T^* . As of that point we know that

$$\frac{\dot{K}}{K} = r - \frac{\alpha' \rho}{1 + \alpha'} - \frac{\rho}{1 + \alpha'} = r - \rho$$

Next, we compute the corresponding combined capital stock K_{T^*} by numerically integrating. The present value of transfers at time T^* is given by:

$$G'_{T^*} = \int_{T^*}^{\infty} e^{-r(s-T^*)} \frac{\alpha' \rho}{1 + \alpha'} e^{(r-\rho)(s-T^*)} K_{T^*} dt = \frac{\alpha'}{1 + \alpha'} K_{T^*}$$

and at time zero by:

$$G'_0 = e^{-rT^*} \frac{\alpha'}{1 + \alpha'} K_{T^*}$$

Consumption policies if the agents could commit to transfer G'_0 at time zero and then shut down transfer possibilities is given by:

$$c_{com}(t) = (G'_0 + e^{(r-\rho)t} k_0) \rho, \quad c'_{com}(t) = (-G'_0 + e^{(r-\rho)t} k'_0) \rho$$

Her value function is then given by:

$$V = \int_0^{\infty} e^{-\rho t} \ln(c_{com}(t)) dt + \alpha \int_0^{\infty} e^{-\rho t} \ln(c'_{com}(t)) dt$$

In the case that $r = \rho$ consumption is constant over time so that her value is:

$$V_{com} = \frac{\ln(c_{com}) + \alpha \ln(c'_{com})}{\rho}$$

Similarly for him:

$$V'_{com} = \frac{\ln(c'_{com}) + \alpha' \ln(c_{com})}{\rho}$$

The welfare measure we employ is to find two values of K for a given initial distribution P_0 such that:

$$V_{com} - V(K) = 0, \quad V'_{com} - V'(K') = 0$$

The following table displays various values of this welfare measure in percentages (brackets are for him):

	0	0.2	0.4	0.6	0.8	1
0.05	0 (0)	4.3 (1.6)	1.8 (1.4)	1.1 (1.1)	0.82 (0.91)	0 (0)
0.1	0 (0)	6.9 (2.2)	3.1 (1.9)	1.6 (1.4)	0.96 (0.99)	0 (0)
0.2	0 (0)	8.4 (2.5)	4.7 (2.5)	2.3 (1.8)	1.1 (1.1)	0 (0)
0.3	0 (0)	1.5 (2.3)	5.1 (2.7)	2.6 (2.1)	1.2 (1.2)	0 (0)
0.4	0 (0)	-	0.5 (3.1)	2.7 (2.1)	1.3 (1.3)	0 (0)

Table 1: Percentage compensation of K required for her and (him) in order to be indifferent between the equilibrium and an equivalent transfer of resources at time zero under commitment. Along rows the starting value of P_0 varies; along columns the symmetric value of the altruism parameter varies i.e. $\alpha \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

5 Conclusions

This paper provides a dynamic model of voluntary transfers with two-sided altruism in a Markov-perfect setting. The continuous-time setting allows us to neglect certain second-order effects and provides additional tractability.

We find that transfers are especially likely when the recipient is liquidity constrained, which is in line with results from the empirical literature. We show that the equilibrium allocation is not Pareto-efficient because agents' savings decisions are distorted by altruistic-strategic considerations. Agents have an extra disincentive to save since they fear that their relatives react by overspending in reliance on their larger resources. On the technical side we introduce the concept of "limit consistency" as a refinement of Markov-perfect equilibrium for differential games.

In future work, we plan to extend the model to allow for shocks in order to be able to tie the framework to the partial insurance literature. It is then imperative to confront the model with data to assess its empirical performance.

We plan on testing restrictions implied by the Euler equations using panel data such as the Health and Retirement Study. Another avenue for future work is to revisit Ricardian equivalence and study the effects of intergenerational altruism with its strategic considerations on the allocations of the economy.

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A Appendix

A.1 HJB in P/K -space

A.1.1 Our approach

Using the definition of agents’ preferences, we can re-write the value function recursively (to a first order):

$$V(P, K) = [\ln(CK) + \alpha \ln(C'K)]\Delta t + e^{-\rho\Delta t}V(P + \Delta P, K + \Delta K)$$

where the law of motions for the state variables are given in (7). We then substitute in our P/K -separable expression for V on both sides, take a first-order expansion of $e^{-\Delta t}V(P + \Delta P, K + \Delta K)$ in Δt and take limits as $\Delta t \rightarrow 0$. The terms in K disappear and we obtain the HJB (8).

A.1.2 Working directly on (6)

Another way of doing things is to recognize that the agent cannot influence the value stemming from K in (7) and thus to maximize \bar{V} in (6) by choosing a path for \bar{C} given his policy rule $\bar{C}'(P)$. This dynamic-programming problem is non-standard since the current policy \bar{C}_0 appears in the double integral in (6), which makes it harder to state the Bellman equation (i.e. to write the value function recursively).²²

We now want to write the double integral in (6) recursively. To a first order, we have:

$$\begin{aligned} J(0) &= \int_0^\infty e^{-\rho t} \left[\int_0^t C_s ds \right] dt = \\ &= \int_0^{\Delta t} e^{-\rho t} C_0 t dt + e^{-\rho \Delta t} \left(\int_{\Delta t}^\infty e^{-\rho(t-\Delta t)} \left[C_0 \Delta t + \int_{\Delta t}^\infty C_s ds \right] dt \right) = \\ &= 0 + \frac{C_0}{\rho} \Delta t + e^{-\rho \Delta t} \int_{\Delta t}^\infty e^{-\rho(t-\Delta t)} \left[\int_{\Delta t}^t C_s ds \right] dt = \\ &= \frac{C_0}{\rho} \Delta t + e^{-\rho \Delta t} J(\Delta t) \end{aligned}$$

From line 1 to 2, we split up the integrals in two parts at Δt and use the fact that $C_t = C_0 + o(t)$ for small t by continuity. From line 2 to 3, we see that the first term is of order Δt^2 (note that $t \leq \Delta t$ in the integrand and thus the entire term is bounded by $C_0 \Delta t^2$). Line 3 to 4 uses the definition of $J(\cdot)$ from the first line to arrive at the recursive formulation.

Using this expression, it is then easy to see that, to a first order,

$$\bar{V}(P_0) = \alpha \ln \bar{C}'(P_0) \Delta t - \bar{C}'_0 \frac{1+\alpha}{\rho} \Delta t + \max_{\bar{C}, \bar{G}} \left\{ \ln \bar{C} \Delta t - \bar{C} \frac{1+\alpha}{\rho} \Delta t + e^{-\rho \Delta t} V(P_{\Delta t}) \right\}$$

where, again to a first order,

$$e^{-\rho \Delta t} V(P_{\Delta t}) \simeq (1 - \rho \Delta t) V(P_0) + [(1 - P) \bar{C}'(0) - P \bar{C} - \bar{G} + \bar{G}'(P_0)] \Delta t \bar{V}_P(P_0).$$

From this it is then easy to obtain the HJB (8) taking limits as $\Delta t \rightarrow 0$.

Economically, the term in the double integral represent the damage that consumption today has on the size of the “pie”

²²Note the similarity of this problem to the problem of finding the HJB for habit or recursive preferences, see Fwu-Rang Chang REF???, chapter 4.4.1, for example.

A.2 Further results and proofs

Lemma 10 (Value functions bounded under altruism) *Let consumption functions satisfy assumption 2. Then, if $\alpha > 0$ and $\alpha' > 0$, then there exist numbers $\underline{M} > -\infty$ and $\bar{M} < \infty$ such that $\underline{M} < \bar{V}(P) < \bar{M}$ and $\underline{M} < \bar{V}'(P) < \bar{M}$.*

Proof: Clearly, her value function is upper-bounded by the value of the Pareto problem where the planner puts full weight on her. To find a lower bound, notice that her flow-utility is lower bounded when we invoke assumption 2:

$$FU_{min} \equiv \min_{\bar{C} \in [\epsilon, \frac{\rho}{1+\alpha}], \bar{C}' \in [\epsilon, \frac{\rho}{1+\alpha'}]} \left\{ \ln \bar{C} + \alpha \ln \bar{C}' - (\bar{C} + \bar{C}') \frac{1+\alpha}{\rho} \right\} > -\infty$$

since we are minimizing a continuous function over a compact set. Now we see that

$$\bar{V}(P) = \int_0^\infty e^{-\rho t} [\ln \bar{C}_t + \alpha \ln \bar{C}'_t - (\bar{C}_t + \bar{C}'_t) \frac{1+\alpha}{\rho}] dt \geq \frac{FU_{min}}{\rho} > -\infty$$

Since we can do the same for him, the statement follows. ■

Lemma 11 (Value functions continuous) *Let the policy functions satisfy assumptions 2 and 3. Then the value functions of both players are continuous at the boundaries $\{P_i\}$ (and thus continuous throughout \mathcal{P}).*

Proof: Suppose that the value function is discontinuous at P_i for her (without loss of generality). Since his policies \bar{C}' and \bar{G}' converge to finite positive numbers at P_i from both sides by assumptions 2 and 3, she can always choose her policy \bar{C} such that $\dot{P} = < 0$ (or $\dot{P} > 0$) in a neighborhood around P_i by the law of motion for P in ((5))²³. Now, if $V(P_i^+) > V(P_i^-)$ (these two denoting the right- and left-side limits at P_i), then the inequality

$$V(P_i - \epsilon) \geq [\ln(\bar{C}) + \alpha \ln(\bar{C}')] \Delta t + e^{-\rho \Delta t} V(P_i) + o(\Delta t)$$

can be violated for some small $\epsilon > 0$ and some \bar{C} . Δt is the amount of time it takes to reach P_i under the given policies, which vanishes as $\epsilon \rightarrow 0$. In other words, she could obtain a higher value than $V(P_i - \epsilon)$ by steering the economy to P_i . The case $V(P_i^+) < V(P_i^-)$ is entirely analogous. ■

A.2.1 Proof for Lemma 7: Donor's limit consumption

Proof: Since his FOC is given by

$$\frac{1}{\bar{C}'} = \frac{1 + \alpha'}{\rho} + P \bar{V}'_P$$

²³In the special case where $P_i = 0$ this must also be true since $\bar{G}'(0) > 0$ by our assumption that $\bar{C}(0) > 0$. Then $\dot{P} > 0$ may be achieved by setting $\bar{C} < \bar{G}'(0)$. A similar argument applies if $P_i = 1$.

and $\lim_{P \rightarrow 0} P = 0$, it is sufficient to show that \bar{V}'_P is upper-bounded in absolute value to prove the claim.

From his HJB, we obtain

$$\bar{V}'_P = \frac{1}{P} \left\{ \rho \bar{V} - \left[\ln \bar{C}' + \alpha' \ln \bar{C} - (\bar{C}' + \bar{C}) \frac{1 + \alpha'}{\rho} \right] \right\}.$$

Invoking assumptions 2 and 3,

$$\dot{P}_{lim} \equiv \lim_{P \rightarrow 0} -(1 - P)\bar{C}(P) + P\bar{C}'(P) = -\bar{C}_{lim} \leq -\epsilon.$$

since consumption policies have finite non-zero limits. Thus,

$$|\bar{V}'_P| = \left| \frac{1}{\dot{P}_{lim}} \right| |\{\cdot\}| \leq \frac{1}{\epsilon} |\{\cdot\}|$$

since $|1/\dot{P}_{lim}|$ is bounded by $1/\epsilon$. The term in curly brackets is bounded since \bar{V}' is bounded (by Lemma 10) and the remaining expression is a continuous function of consumption rates, which are confined to a compact set by assumption 2. This shows that \bar{V}'_P is bounded, which concludes the proof. ■

A.2.2 Global maximizer of H

Define the maximization problem contained in her HJB w.r.t. her consumption rate in common capital stocks as:

$$H(\bar{C}; \bar{V}_p) = \ln \bar{C} - \bar{C} \left(\frac{1 + \alpha}{\rho} + (1 - P)\bar{V}_p \right)$$

Lemma 12 $H(\bar{C}_{WP}; 0)$ is the unique global maximum of $H(\cdot; \cdot)$.

Proof: Substitute the first-order condition to obtain:

$$H^*(\bar{V}_p) = -\ln \left(\frac{1 + \alpha}{\rho} + (1 - P)\bar{V}_p \right) - \left(\frac{1 + \alpha}{\rho} + (1 - P)\bar{V}_p \right)^{-1} \left(\frac{1 + \alpha}{\rho} + (1 - P)\bar{V}_p \right)$$

$\Rightarrow H^*$ is strictly decreasing in \bar{V}_p so that it is maximized at $\bar{V}_p = 0$. Using this fact in the first-order condition shows that $(\bar{C} = \bar{C}_{WP}, \bar{V}_p = 0)$ maximizes H. ■

A.2.3 Lemma on non-smooth NT-FT boundary

We will characterize the boundary of a flow transfer region FT with an NT -region for a special case. We are assuming that he is the donor, i.e. $\bar{V}'_p = 0$.

We know that $c'_{FT} = c'_{wp}$. Her VM (14) yields:

$$\text{VM: } \rho \bar{V}_{NT} = \rho \bar{V}_{FT} \quad \Rightarrow \quad k(c_{NT}, c'_{NT}) = k(c_{FT}, c'_{FT}) + \frac{g_{FT}}{P} \bar{V}_{p,FT}$$

His VM (23) yields:

$$\text{VM': } k'(c_{NT}, c'_{NT}) = k'(c_{FT}, c'_{FT}),$$

since $\bar{V}'_{P,FT} = 0$. Given (c_{NT}, c'_{NT}) , and knowing that $c'_{FT} = c'_{wp}$ we can solve VM and VM' for (c_{FT}, g'_{FT}) (note that there are usually two candidate solutions). Then take ODE to solve forward for c, g' .

We now check limit consistency. Consider first the ordering of the two regions such that NT is to the left of FT and $\tilde{P} < 1/2$. The donor's limit Hamiltonian as a function of c' fixing (g'_K, c_K) reads as:

$$H'_{lim}(c'; g'_K, c_K) = \alpha' \ln c_K - c_K P \frac{1 + \alpha'}{\rho} + \left\{ \ln c' - c'(1 - P) \frac{1 + \alpha'}{\rho} + \dot{P} \nu'(c'; g'_K, c_K) \right\},$$

where

$$\nu'(c'; g'_K, c_K) = \begin{cases} \bar{V}'_{p,NT} & \text{if } \dot{P} < 0 \\ \bar{V}'_{p,FT} = 0 & \text{if } \dot{P} \geq 0 \end{cases}$$

$$\dot{P} = (1 - P)P \left(c' - c_K + \frac{g'_K}{P} - \frac{g_K}{1 - P} \right).$$

Define

$$\tilde{H}(c', g') = \left\{ \ln c' - c'(1 - P) \frac{1 + \alpha'}{\rho} + c'(1 - P)P \nu'(c', g'; c_K) \right\} + \left\{ (1 - P)P \frac{g'}{P} \nu'(c', g'; c_K) \right\}$$

The second curly bracket is always zero (recall that $V_P^A > 0$ implies $g_A = 0$). The first curly bracket is maximized over all c', ν' if $c' = c'_{wp}$, see Lemma 12. By choosing a rate of transfers g' high enough, he can always ensure $\dot{P} > 0$ and set $c' = c'_{wp}$ which must then be optimal. We thus have the following lemma:

Lemma 13 *Consider a kink $\tilde{P} > 0$ where $\bar{V}'_{B,P} = 0$. Then, on the kink, limit consistency requires that $c'_K = c'_{wp}$ and $\dot{P}_K \geq 0$. Analogously, if $\bar{V}'_{A,P} = 0$, then we have $c_K = c_{wp}$ and $\dot{P}_K \leq 0$.*

A.3 Restrictions from value matching

In this section we describe the condition on policies on the left (A) and right side (B) of a boundary \tilde{P} between two smooth regions and prove the statements in Proposition 2. Assume that inside A and B, respectively, the solution is smooth and the Euler equations hold. At \tilde{P} , however, we consider the possibility that policies (i.e. consumption levels) jump, and thus value functions might have kinks. We denote the left-hand limits of consumption at \tilde{P} by $\bar{\mathbf{C}}_A = (\bar{C}_A, \bar{C}'_A)$ and the right-hand limits by $\bar{\mathbf{C}}_B = (\bar{C}_B, \bar{C}'_B)$. Since both value functions must be continuous at \tilde{P} (see Lemma 11), certain conditions must be fulfilled by the pair $(\bar{\mathbf{C}}_A, \bar{\mathbf{C}}_B)$ since the two HJBs must equalize in the limit (*value matching*).

Write her HJB (8) writing \bar{V}_P in terms of \bar{C} using the FOC (9) (and doing the same for him) to obtain

$$\begin{aligned}\rho\bar{V} &= \underbrace{\alpha \ln \bar{C}' - \frac{\bar{C}'}{1-P} \left[\frac{1+\alpha}{\rho} - \frac{P}{\bar{C}} \right]}_{\equiv G(\bar{C}, \bar{C}')} + \underbrace{\ln \bar{C} - 1}_{\equiv H^*(\bar{C})} \\ \rho\bar{V}' &= \underbrace{\alpha' \ln \bar{C} - \frac{\bar{C}}{1-P} \left[\frac{1+\alpha'}{\rho} - \frac{1-P}{\bar{C}'} \right]}_{\equiv G'(\bar{C}, \bar{C}')} + \underbrace{\ln \bar{C}' - 1}_{\equiv H^*(\bar{C}')}\end{aligned}$$

Now, expressing consumption rates in terms of the agents' own wealth, i.e. using the identities $\bar{c} = \bar{C}/P$ and $\bar{c}' = \bar{C}'/(1-P)$, we obtain

$$\begin{aligned}\rho\bar{V} &= \underbrace{\ln P + \alpha \ln(1-P)}_{\equiv f(P)} + \underbrace{\alpha \ln \bar{c}' - \bar{c}' \left[\frac{1+\alpha}{\rho} - \frac{1}{\bar{c}} \right]}_{\equiv g(\bar{c}, \bar{c}')} + \underbrace{\ln \bar{c} - 1}_{\equiv h^*(\bar{c})} \\ \rho\bar{V}' &= \underbrace{\ln(1-P) + \alpha' \ln P}_{\equiv f'(P)} + \underbrace{\alpha' \ln \bar{c} - \bar{c} \left[\frac{1+\alpha'}{\rho} - \frac{1}{\bar{c}'} \right]}_{\equiv g'(\bar{c}, \bar{c}')} + \underbrace{\ln \bar{c}' - 1}_{\equiv h^*(\bar{c}')}\end{aligned}$$

We see that two regimes (\bar{c}_A, \bar{c}'_A) and (\bar{c}_B, \bar{c}'_B) can only be consistent with value matching if the following two value-matching (VM) conditions hold:

$$k(\bar{c}_A, \bar{c}'_A) \equiv g(\bar{c}_A, \bar{c}'_A) + h^*(\bar{c}_A) = g(\bar{c}_B, \bar{c}'_B) + h^*(\bar{c}_B) = k(\bar{c}_B, \bar{c}'_B) \quad (22)$$

$$k'(\bar{c}_A, \bar{c}'_A) \equiv g'(\bar{c}_A, \bar{c}'_A) + h^*(\bar{c}'_A) = g'(\bar{c}_B, \bar{c}'_B) + h^*(\bar{c}'_B) = k'(\bar{c}_B, \bar{c}'_B) \quad (23)$$

We see that this value-matching condition is independent of \tilde{P} – this is not surprising, since the HJBs in (k, k') are independent of \tilde{P} , too, and value matching has to hold in (k, k') -space as well.

We now want to determine what the relevant range for $\bar{\mathbf{c}}_B \equiv (\bar{c}_B, \bar{c}'_B)$ is. The FOCs for consumption together with $\bar{V}_P \geq 0$ and non-negativity of consumption give us the following bounds:

$$0 \leq \bar{C} \leq \frac{\rho}{1+\alpha} \quad 0 \leq \bar{c} \leq \frac{\rho}{\tilde{P}(1+\alpha)} \quad (24)$$

$$0 \leq \bar{C}' \leq \frac{\rho}{1+\alpha'} \quad 0 \leq \bar{c}' \leq \frac{\rho}{(1-\tilde{P})(1+\alpha')}. \quad (25)$$

Note that these bounds *do* depend on \tilde{P} , so we cannot neglect \tilde{P} altogether when trying to determine $\bar{\mathbf{c}}_B$ for a given $\bar{\mathbf{c}}_A$.

In order to characterize the solution, it will be crucial to study the derivatives of k . We start with the derivative in an agent's own consumption:

$$\frac{\partial k(c, c')}{\partial c} = \frac{1}{c} \left(1 - \frac{c'}{c} \right), \quad \frac{\partial k'(c, c')}{\partial c'} = \frac{1}{c'} \left(1 - \frac{c}{c'} \right). \quad (26)$$

Fixing c' , k is decreasing in c for $c < c'$ (i.e. above the diagonal of the (c, c') -plane) and increasing in c for $c > c'$ (below the diagonal). For given c , k is minimized by $c' = c$.

The derivative in the other agent's consumption is

$$\frac{\partial k(c, c')}{\partial c'} = \frac{\alpha}{c'} - \left[\frac{1 + \alpha}{\rho} - \frac{1}{\bar{c}} \right], \quad \frac{\partial k'(c, c')}{\partial c} = \frac{\alpha'}{\bar{c}} - \left[\frac{1 + \alpha'}{\rho} - \frac{1}{c'} \right]. \quad (27)$$

Note that if the bracket on the right-hand of (27) side is negative ($\bar{c} \leq \rho/(1+\alpha)$), then an increase in \bar{c}' always leads to an increase in k . In this case, her marginal value \bar{V}_P of having the wealth distribution tilted in her favor is so high that this dominates the marginal value of common funds $(1 + \alpha)/\rho$; she would set infinite consumption for him if she had the choice (this is what the expression $\bar{c}^* = \alpha/0 = \infty$ is supposed to say). If the bracket is positive, $\partial k/\partial c'$ is strictly decreasing in c' for fixed c .

It is useful to study the following related maximization problem:

$$\begin{aligned} k^*(\bar{c}) &\equiv \max_{\bar{c}'} k(\bar{c}, \bar{c}') \\ \bar{c}'^*(\bar{c}) &\equiv \arg \max_{\bar{c}'} k(\bar{c}, \bar{c}') = \frac{\alpha}{\max \left\{ \frac{1+\alpha}{\rho} - \frac{1}{\bar{c}}, 0 \right\}} \end{aligned} \quad (28)$$

Since this is a strictly concave problem, the FOC (27) is sufficient. $\bar{c}'^*(\bar{c})$ has the interpretation as the consumption rate that she would choose for him given the slope of the value function \bar{V}_P that is consistent with a given consumption rate \bar{c} . k is increasing in c' below the function $\bar{c}'^*(\bar{c})$ (again in (c, c') -space) and decreasing in c' above, as the FOC (27) shows.

A.3.1 Properties of $\bar{c}'^*(\cdot)$ and $\bar{c}^*(\cdot)$

Our goal will now be to show that (i) the unique intersection between the functions $\bar{c}'^*(\cdot)$ and $\bar{c}^*(\cdot)$ in the (\bar{c}, \bar{c}') -plane is at the point (ρ, ρ) , (ii) the graph of \bar{c}'^* lies *above* the graph of \bar{c}^* for values $\bar{c} > \rho$ and (iii) \bar{c}'^* lies *below* \bar{c}^* for values $\bar{c} < \rho$.

We first re-write $\bar{c}'^*(\cdot)$ in the area where it is bounded:

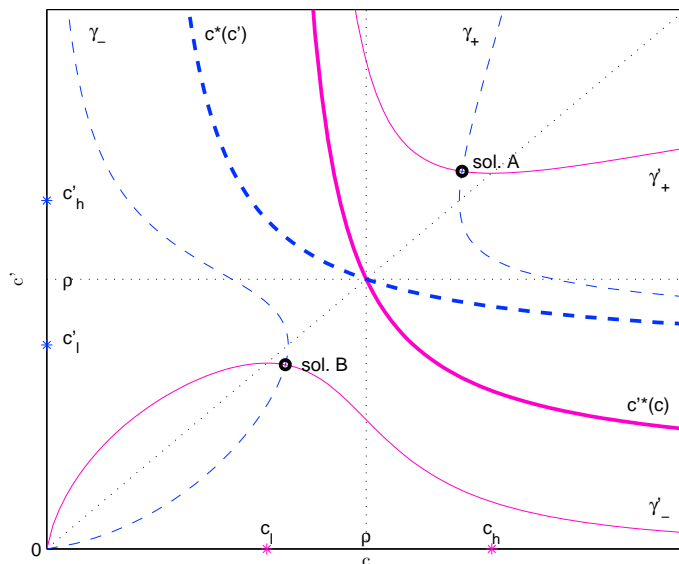
$$\bar{c}'^*(\bar{c}) = \frac{1}{\frac{1}{\rho} + \frac{1}{\alpha} \left(\frac{1}{\rho} - \frac{1}{\bar{c}} \right)}.$$

Now, we invert this function and then reverse the roles of the two players to obtain that graph of \bar{c}^* as a function of \bar{c}' in \bar{c} :

$$\bar{c}^*(\bar{c}') = \frac{1}{\frac{1}{\rho} + \alpha' \left(\frac{1}{\rho} - \frac{1}{\bar{c}'} \right)}.$$

Inspection of the two functions makes clear that the claims above are true whenever $\alpha\alpha' < 1$, see also Figure 16 for an illustration. The functions fall on top of each other in the case $\alpha = \alpha' = 1$.

Figure 16: $k_A < k_\rho$ and $k'_A < k'_\rho$ (case 1)



A.3.2 Properties of $k^*(\bar{c})$ and $k'^*(\bar{c}')$

By substituting $c'^*(\bar{c})$ into $k(\bar{c}, c')$, it can be verified that $k^*(\cdot)$ has the following properties:²⁴

1. k^* is smooth and strictly convex.
2. k^* is uniquely minimized at $\bar{c} = \rho$.
3. $\lim_{\bar{c} \rightarrow 0} k^*(\bar{c}) = \lim_{\bar{c} \rightarrow \infty} k^*(\bar{c}) = \infty$.

The same is true for the function $k'^*(\cdot)$, making the obvious adjustments. The above properties together with the derivatives (26) and (27) imply that the point (ρ, ρ) is the saddle point of both functions $k^*(\cdot)$ and $k'^*(\cdot)$: Geometrically, $\bar{c}^*(\bar{c})$ is the ridge of a mountain (the maximal point of the k -surface keeping \bar{c} fixed and varying c'). At $\bar{c} = \rho$, this ridge attains its lowest point. The equivalent statements hold for $k'^*(\cdot)$ and $c^*(\cdot)$.

²⁴It is worthwhile noting that these properties do not depend on the choice $u(c) = \ln c$ for instantaneous utility. They can be derived generally for concave utility functions $u(\cdot)$ satisfying $\lim_{c \rightarrow 0} u(c) = -\infty$ and $\lim_{c \rightarrow \infty} u(c) = \infty$ using only the convexity properties of Legendre transforms.

A.3.3 Find \bar{c}' given \bar{c} to solve her VM

We will now study the following problem: Given a fixed \bar{c} , find \bar{c}' to solve $k(\bar{c}, \bar{c}') = k_A$ (her VM). In order to find the solution(s), it will be useful to study the function $g_{\bar{c}}(\bar{c}') \equiv g(\bar{c}, \bar{c}')$. Obviously, $g_{\bar{c}}(\cdot)$ is smooth and concave, and we have $\lim_{\bar{c}' \rightarrow 0} g_{\bar{c}}(\bar{c}') = -\infty$.

There are two cases to consider:

1. $\bar{c} > \rho/(1 + \alpha)$: $g_{\bar{c}}(\cdot)$ attains its unique maximum at $\bar{c}'^*(\bar{c})$ and we have $\lim_{\bar{c}' \rightarrow \infty} g_{\bar{c}}(\bar{c}') = -\infty$. There are the following sub-cases:
 - (a) $k^*(\bar{c}) > k(\bar{c}_A, \bar{c}'_A)$: There are two solutions; we denote the upper solution by $\gamma'_+(\bar{c})$ and the lower one by $\gamma'_-(\bar{c})$. Of course, $\gamma'_+(\bar{c}) > \bar{c}'^*(\bar{c}) > \gamma'_-(\bar{c})$.
 - (b) $k^*(\bar{c}) = k(\bar{c}_A, \bar{c}'_A)$: There is exactly one solution $\gamma'(\bar{c}) = \bar{c}'^*(\bar{c})$.
 - (c) $k^*(\bar{c}) < k(\bar{c}_A, \bar{c}'_A)$: There is no solution.
2. $\bar{c} \leq \rho/(1 + \alpha)$: $g_{\bar{c}}$ is globally increasing and $\lim_{\bar{c}' \rightarrow \infty} g_{\bar{c}}(\bar{c}') = \infty$. Thus there is exactly one value for \bar{c}' that solves VM, which we denote by $\gamma'_-(\bar{c})$, since it lies below $\bar{c}'^*(\bar{c}) = \infty$.

A.3.4 Characterization of γ -functions

Since $k(\cdot, \cdot)$ is a smooth function, both $\gamma'_-(\cdot)$ and $\gamma'_+(\cdot)$ will be smooth functions (on the range where they are defined). The implicit-function theorem gives us

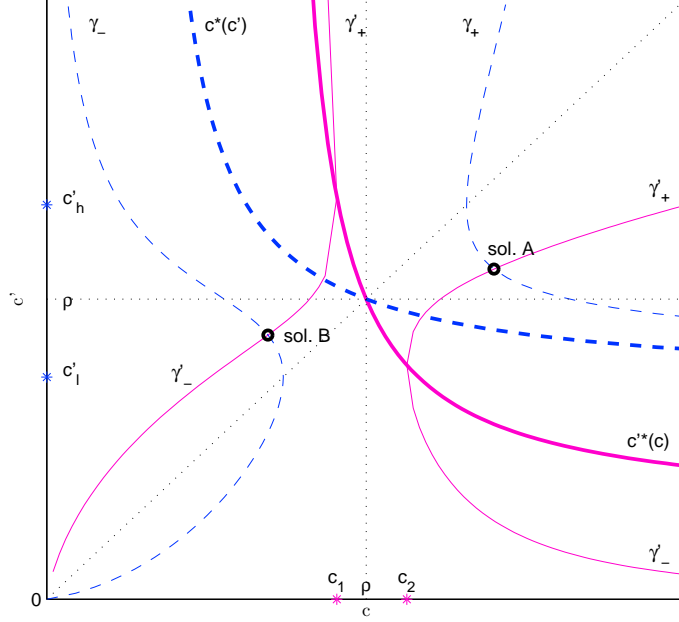
$$\frac{\partial \gamma'}{\partial \bar{c}} = \frac{d\bar{c}'}{d\bar{c}} = -\frac{\frac{1}{\bar{c}} \left(1 - \frac{\bar{c}'}{\bar{c}}\right)}{\frac{\alpha}{\bar{c}'} - \left(\frac{1+\alpha}{\rho} - \frac{1}{\bar{c}}\right)} = \frac{(c' - c)c}{cc' + \alpha c^2 - \frac{1+\alpha}{\rho} cc'} \quad (29)$$

$$\frac{\partial \gamma}{\partial \bar{c}'} = \frac{d\bar{c}}{d\bar{c}'} = -\frac{\frac{1}{\bar{c}'} \left(1 - \frac{\bar{c}}{\bar{c}'}\right)}{\frac{\alpha'}{\bar{c}} - \left(\frac{1+\alpha'}{\rho} - \frac{1}{\bar{c}'}\right)} = \frac{(c - c')c'}{c'c + \alpha' c'^2 - \frac{1+\alpha'}{\rho} c'c}, \quad (30)$$

where of course we have to set $\bar{c}' = \gamma'(\bar{c})$ along the graph of $\gamma'(\cdot)$ and $c = \gamma(c')$ along the graph of $\gamma(\cdot)$. The denominator equals $\lambda(\bar{c}, \bar{c}') = \partial g(\bar{c}, \bar{c}')/\partial \bar{c}'$ (the distortive term in her Euler equation). Along $\gamma'_+(\cdot)$, λ is always negative, which means that she would prefer him to consume less (we term this *over-consumption*). Along γ'_- , λ is always positive, which means that she would prefer him to consume more (which we term *under-consumption*).

Furthermore, we see that the numerator in (29) is zero if and only if $\bar{c} = \bar{c}'$, i.e. on the diagonal of the (\bar{c}, \bar{c}') -plane; it is positive above the diagonal and negative below. The denominator coincides with the FOC of the problem (28), so it is zero on $\bar{c}'^*(\bar{c})$, positive below and negative above. So γ'_+ (which lies above \bar{c}'^*) is decreasing above the diagonal and increasing below it. γ'_- (which lies below \bar{c}'^*) is increasing above the diagonal and decreasing below. Furthermore,

Figure 17: $k_A \leq k_\rho$ and $k'_A \geq k'_\rho$ (case 2)



γ'_+ and γ'_- have infinite slope at the point where they cross the function $c^*(\bar{c})$, which is the point where the two solutions collapse to a single one; see the right pink curve in Figure 18 for an illustration.

Since the diagonal $\bar{c} = c'$ plays a crucial role, it is useful to study the function $k(\cdot, \cdot)$ along it. Define

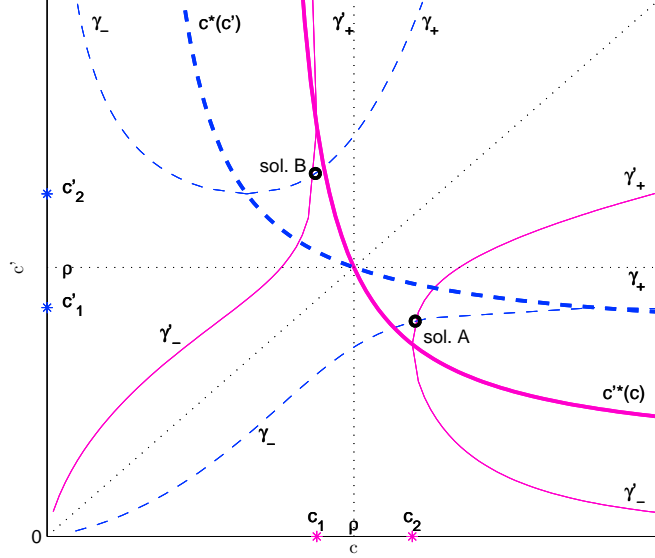
$$k_{symm}(c) \equiv k(c, c) = (1 + \alpha) \left(\ln c - \frac{c}{\rho} \right).$$

It is clear that $k_{symm}(\cdot)$ is smooth, convex, uniquely maximized at $c = \rho$ and that $\lim_{c \rightarrow 0} k_{symm}(c) = \lim_{c \rightarrow \infty} k_{symm}(c) = -\infty$.

We will now characterize the γ -functions. It turns out that the value of k on the saddle point $k_\rho \equiv k_{symm}(\rho) = k(\rho, \rho) = (1 + \alpha)(\ln \rho - 1)$ plays a crucial role in distinguishing different cases:

1. $k_A < k_\rho$: For each $\bar{c} \in (0, \infty)$, $\gamma'_+(\bar{c})$ and $\gamma'_-(\bar{c})$ exist since $k^*(\bar{c}) > k_\rho$ for all \bar{c} . By the properties of k_{symm} , there are exactly two numbers c_l (with $0 < c_l < \rho$) and c_h (with $c_h > \rho$) such that $k_{symm} = k_A$. This implies that γ'_- is a smooth function on $(0, \infty)$, is uniquely maximized at c_l , is increasing and above the diagonal for $\bar{c} < c_l$ and decreasing and below the

Figure 18: $k_A \geq k_\rho$ and $k'_A \geq k'_\rho$ (case 3)



diagonal for $\bar{c} > c_l$. γ'_+ is a smooth function on $(\rho/(1+\alpha), \infty)$, is uniquely minimized at c_h , is decreasing above the diagonal and increasing below it. The situation is illustrated in Figure 16.

2. $k_A > k_\rho$: By the properties of k^* , there are exactly two values $c_1 < \rho$ and $c_2 > \rho$ such that $k^*(c_1) = k^*(c_2) = k_A$. No solution \bar{c}' exist to $k(\bar{c}, \bar{c}') = k_A$ on the range $\bar{c} \in (c_1, c_2)$ since $k^*(\bar{c}) < k_A$, so the γ -functions are not defined on this range. Also note that the γ -functions cannot cross the diagonal since $k_{symm}(c) \leq k_\rho < k_A$. Thus γ'_- is an increasing function that stays above the diagonal for $\bar{c} \in (0, c_1)$ and a decreasing function that stays below the diagonal for $\bar{c} \in (c_2, \infty)$. γ'_+ is a decreasing function above the diagonal for $\bar{c} \in (0, c_1)$ and an increasing function below the diagonal for $\bar{c} \in (c_2, \infty)$. The situation is illustrated in Figure 18.
3. $k_A = k_\rho$: By the same argument as in the case $k_A < k_\rho$, γ'_+ and γ'_- are defined for all values $\bar{c} \in (0, \infty)$, if we allow the coincidence $gamma'_+(\rho) = \gamma'_-(\rho) = \rho$ on the saddle point (ρ, ρ) . Since $k(\cdot, \cdot)$ is differentiable, level lines must be differentiable, too. So γ'_- must have the same slope just left of ρ as γ'_+ has just right of ρ . Also, it must be that γ'_- has slope smaller than unity just right of $c = \rho$: If γ'_- came out above the diagonal, (29) tells us that its slope should be negative, which is a contradiction. So $\rho < \gamma'_+(c) < c$ for all $c > \rho$. This in turn implies that

γ'_- comes out above the diagonal just left of ρ and that $\gamma'_-(c) > c$ for all $c < \rho$ and $\rho < \gamma'_+(c) < c$. For the other two branches, we clearly have $\gamma'_+(c) > c^*(c) > \rho$ for all $c < \rho$ and $\gamma'_-(c) < c^*(c) < \rho$ for all $c > \rho$. The situation is illustrated in Figure 19.

Another important property which is independent of the different cases is the following: For \bar{c} large enough, the function $\gamma'_+(\bar{c})$ always exists, stays below the diagonal and is increasing. As \bar{c} goes to infinity, we have

$$\lim_{\bar{c} \rightarrow \infty} \gamma'_+(\bar{c}) = \infty \quad \lim_{\bar{c}' \rightarrow \infty} \gamma_+(\bar{c}') = \infty$$

Suppose this was not the case and there was a bound \bar{c}' to which γ'_+ converged. Then (29) would tell us that the slope of γ'_+ approaches unity as \bar{c} grows large, which is a contradiction to γ'_+ being bounded.

A.3.5 Find (\bar{c}, \bar{c}') to solve both VM-conditions

We can now finally look at the different types of solutions to the system (22) and (23) of *both* VM-conditions. The following is an exhaustive list of the cases that arise. (Note that the arguments also provide algorithms to find the respective solutions.)

1. $k_A \leq k_\rho$ and $k'_A \leq k'_\rho$, where one of the inequalities is strict: Without loss of generality, assume that $c_l \leq c'_l$, i.e. γ'_- crosses the diagonal closer to the origin than γ_- does.²⁵

Then, there are exactly two solutions (see Figure 16 for an illustration):

- (a) Following γ'_- to the right from c_l on (i.e. on the range $\bar{c} > c_l$), there must be a unique intersection point of γ'_- with γ_- (since γ'_- is decreasing in \bar{c} , and γ_- is increasing in \bar{c}' for $\bar{c}' < c'_l \leq \rho$, see A.3.4). This is a solution where both under-consume (we are on the γ_- -parts), $\bar{c}_B < \rho$, $\bar{c}'_B < \rho$ and $\bar{c} > \bar{c}'$.
- (b) Notice that c_l is the lower solution to $\ln c - c/\rho - k_A/(1 + \alpha) = 0$, and c'_l is the lower solution to $\ln c' - c'/\rho - k'_A/(1 + \alpha') = 0$. It is easy to see that $c_l \leq c'_l$ implies $c_h \geq c'_h$ for the upper solutions. Now, proceed similarly as in 1a: Follow γ_+ upward from c'_h (i.e. in the range $\bar{c}' \geq c'_h$). Since γ_+ is increasing in \bar{c}' and γ'_+ is decreasing in \bar{c} function on $(\rho/(1 + \alpha), c_h)$, there must be a unique intersection point (\bar{c}_B, \bar{c}'_B) . This solution is such that both over-consume, $\bar{c}_B > \rho$, $\bar{c}'_B > \rho$ and $\bar{c} < \bar{c}'$. Note that on the other side of the diagonal ($\bar{c} > \bar{c}'$), the γ -functions cannot intersect due to the properties described in A.3.4.

2. $k_A \leq k_\rho$ and $k'_A \geq k'_\rho$, where one of the inequalities is strict. See Figure 17 for an illustration.

²⁵Just reverse the roles of the two in case the crossing is the other way around.

- (a) Again, follow γ'_- to the right from c_l on (i.e. on the range $\bar{c} > c_l$). Note that γ'_- must cross $c'^*(\cdot)$ on its way to $\bar{c}^*(\cdot)$ by the ordering of c^* and c'^* described in A.3.1. Since γ'_- always stays below c'^* , there must be an intersection with γ_+ . Again, by the properties of the γ -functions (see A.3.4) this must be the only intersection that γ'_- can have with γ_- and γ_+ . The solution has the property that both under-consume, $\bar{c}_B \geq \bar{c}'_B$ and $\bar{c}'_B \leq \rho$.
- (b) Now, consider γ'_+ . By the properties of γ_+ and γ_- , it is clear that γ'_+ yield a solution to the right of the diagonal. When following γ'_+ to the left from c_h , γ'_+ grows unbounded as $\bar{c} \rightarrow \rho/(1 + \alpha)$. Since γ_+ grows unbounded in \bar{c}' , there must be a unique intersection of γ'_+ and γ_+ . At this solution, both over-consume, $\bar{c} \leq \bar{c}'$ and $\bar{c}' > \rho$.
3. $k_A \geq k_\rho$ and $k'_A \geq k'_\rho$, where one of the inequalities is strict: There are two cases to consider, depending on the value of k' at $(c_2, \gamma'(c_2))$ ²⁶; see Figure 18 for an illustration (which corresponds to sub-case).
- (a) $k'(c_2, \gamma'(c_2)) \leq k'_A$: γ_- must lie below $(c_2, \gamma'(c_2))$ at $c = c_2$. When following γ'_- letting c increase from c_2 on, at least one intersection with γ_- must take place since γ_- must cut c^* at some point, and c^* is above c'^* , which again lies above γ'_- . This intersection must be unique, because k' strictly increases when we follow γ'_- south-east by the derivatives of k' (which are analogous to (26) and (27)). This solution is such that $c > c'$ and that both under-consume.
- (b) $k'(c_2, \gamma'(c_2)) > k'_A$: γ_- must lie above $(c_2, \gamma'(c_2))$ at $c = c_2$. When following γ'_+ letting c increase from c_2 on, at least one intersection with γ_- must take place since γ'_+ grows unbounded. This intersection must be such that $c > \rho > c'$, he over-consumes and she under-consumes. In this case, we cannot rule out that another crossing happens between γ_- and γ'_+ , so there might be another solution. We could not find any such case computationally, though. Also, note that this is the *only* case in which there can be mixed solutions, i.e. where she over-consumes and he under-consumes.

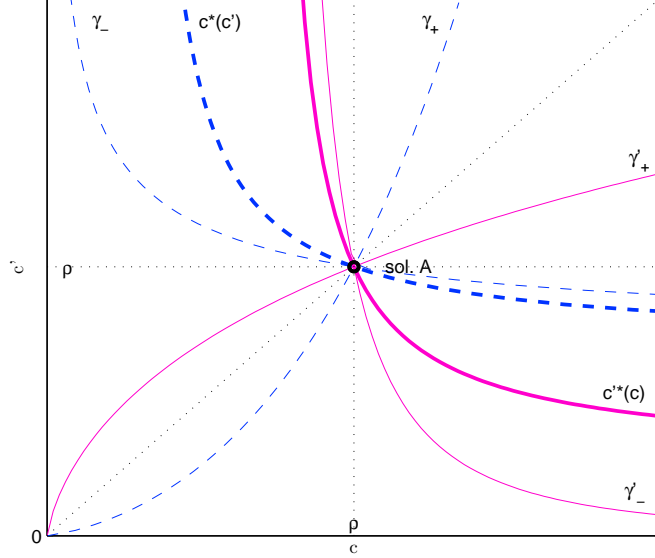
We can follow the same procedure starting at the point $(c'_2, c^*(c'_2))$ and will find (at least) one more solution there.

4. $k_A = k_\rho$ and $k'_A = k'_\rho$: All γ -functions (for both agents) must contain the point (ρ, ρ) , which is one solution to the system. These properties imply that there cannot be any other solution in the entire (c, c') -space, see Figure 19 for an illustration.

Note that having found a candidate solution (c_B, c'_B) (which is independent of P), of course we still have to check if it respects the bounds given in (24) and (25), which depend on P .

²⁶Recall that this is the locus where there is exactly one solution c' for her VM given c , so $\gamma'_-(c_2) = \gamma'_+(c_2) = c'^*(c_2)$, so we simply write $\gamma'(c_2)$

Figure 19: $k_A = k_\rho$ and $k'_A = k_\rho$ (case 4)



A.4 Euler equations: Different formulations

The Euler equations in terms of consumption rates (\bar{C}, \bar{C}') out of the total capital stock K are

$$-\dot{P} \frac{\bar{C}_P}{\bar{C}^2} = \left[\frac{1}{\bar{C}} - \frac{1+\alpha}{\rho} \right] \left\{ (\rho - \bar{C} - \bar{C}') - \frac{\dot{P}}{(1-P)} \right\} + \left[\left(\frac{1+\alpha}{\rho} - \frac{P}{\bar{C}} \right) - \alpha \frac{(1-P)}{\bar{C}'} \right] \bar{C}'_P$$

$$\dot{P} \frac{\bar{C}'_P}{\bar{C}'^2} = - \left[\frac{1}{\bar{C}'} - \frac{1+\alpha'}{\rho} \right] \left\{ (\rho - \bar{C} - \bar{C}') + \frac{\dot{P}}{P} \right\} + \left[\left(\frac{1+\alpha'}{\rho} - \frac{(1-P)}{\bar{C}'} \right) - \alpha' \frac{P}{\bar{C}} \right] \bar{C}_P.$$

It is instructive to replace most of the terms in (\bar{C}, \bar{C}') in the above equations by consumption rates (\bar{c}, \bar{c}') out of agents own capital stocks (k, k') :

$$- \underbrace{\left[\frac{1-P}{\bar{C}} \left(\frac{\bar{c}' - \bar{c}}{\bar{c}} \right) \right]}_{\equiv q_1(P, c, c')} \bar{C}_P = \underbrace{\left[\frac{1}{\bar{C}} - \frac{1+\alpha}{\rho} \right]}_{\equiv q_3(P, c, c')} [\rho - \bar{c}'] + \underbrace{\left[\left(\frac{1}{\rho} - \frac{1}{\bar{c}} \right) + \alpha \left(\frac{1}{\rho} - \frac{1}{\bar{c}'} \right) \right]}_{\equiv q_2(c, c')} \bar{C}'_P \quad (31)$$

$$- \underbrace{\left[\frac{P}{\bar{C}'} \left(\frac{\bar{c} - \bar{c}'}{\bar{c}'} \right) \right]}_{\equiv q'_1(P, c, c')} \bar{C}'_P = - \underbrace{\left[\frac{1}{\bar{C}'} - \frac{1+\alpha'}{\rho} \right]}_{\equiv q'_3(P, c, c')} [\rho - \bar{c}] + \underbrace{\left[\left(\frac{1}{\rho} - \frac{1}{\bar{c}'} \right) + \alpha' \left(\frac{1}{\rho} - \frac{1}{\bar{c}} \right) \right]}_{\equiv q'_2(c, c')} \bar{C}_P \quad (32)$$

where we note that

$$\dot{P} = P(1 - P)(\bar{c}' - \bar{c}).$$

We can write the Euler equations entirely in small consumption rates:

$$\begin{aligned} q_1 P c_P + q_2 (1 - P) c'_P &= -q_1 c + q_2 c' - q_3 \\ q'_2 P c_P + q'_1 (1 - P) c'_P &= q'_1 c - q'_2 c' + q'_3 \end{aligned}$$

Note that $q_1 = q_2 = q_3 = q'_1 = q'_2 = q'_3 = 0$ when $c = c' = \rho$. Then any pair (c_P, c'_P) is solution to the above system for given (P, c, c') . To study the solutions in the neighborhood of $(c, c', P) = (\rho, \rho, P)$, we will linearize the above equations around this point.

For this, we first calculate the partial derivatives of $(q_i, q'_i)_{i=1}^3$ in (c, c') at $(c, c') = (\rho, \rho)$ for a given P – note that all derivatives in P vanish since $c = c' = \rho$:

$$\begin{aligned} \frac{\partial q_1}{\partial c} &= -\frac{1 - P}{P\rho^2} & \frac{\partial q_1}{\partial c'} &= \frac{1 - P}{P\rho^2} & \frac{\partial q'_1}{\partial c} &= \frac{P}{(1 - P)\rho^2} & \frac{\partial q'_1}{\partial c'} &= -\frac{P}{(1 - P)\rho^2} \\ \frac{\partial q_2}{\partial c} &= \frac{1}{\rho^2} & \frac{\partial q_2}{\partial c'} &= \frac{\alpha}{\rho^2} & \frac{\partial q'_2}{\partial c} &= \frac{\alpha'}{\rho^2} & \frac{\partial q'_2}{\partial c'} &= \frac{1}{\rho^2} \\ \frac{\partial q_3}{\partial c} &= 0 & \frac{\partial q_3}{\partial c'} &= \frac{(1 + \alpha) - \frac{1}{P}}{\rho} & \frac{\partial q'_3}{\partial c} &= \frac{(1 + \alpha') - \frac{1}{1 - P}}{\rho} & \frac{\partial q'_3}{\partial c'} &= 0 \end{aligned}$$

We then use these above to linearize the system around (ρ, ρ) – note that all terms in ΔP vanish:

$$\begin{aligned} &\left[\Delta c \begin{pmatrix} \frac{\partial q_1}{\partial c} P & \frac{\partial q_2}{\partial c} (1 - P) \\ \frac{\partial q_2}{\partial c} P & \frac{\partial q_1}{\partial c} (1 - P) \end{pmatrix} + \Delta c' \begin{pmatrix} \frac{\partial q_1}{\partial c'} P & \frac{\partial q_2}{\partial c'} (1 - P) \\ \frac{\partial q_2}{\partial c'} P & \frac{\partial q_1}{\partial c'} (1 - P) \end{pmatrix} \right] \begin{pmatrix} c_P \\ c'_P \end{pmatrix} = \\ &= \Delta c \begin{pmatrix} -\frac{\partial q_1}{\partial c} \rho + \frac{\partial q_2}{\partial c} \rho - \frac{\partial q_3}{\partial c} \\ \frac{\partial q_1}{\partial c} \rho - \frac{\partial q_2}{\partial c} \rho + \frac{\partial q_3}{\partial c} \end{pmatrix} + \Delta c' \begin{pmatrix} -\frac{\partial q_1}{\partial c'} \rho + \frac{\partial q_2}{\partial c'} \rho - \frac{\partial q_3}{\partial c'} \\ \frac{\partial q_1}{\partial c'} \rho - \frac{\partial q_2}{\partial c'} \rho + \frac{\partial q_3}{\partial c'} \end{pmatrix} \end{aligned}$$

Simplifications yield

$$\begin{aligned} \frac{1}{\rho} \left[\sin \phi \begin{pmatrix} P - 1 & 1 - P \\ \alpha' P & P \end{pmatrix} + \cos \phi \begin{pmatrix} 1 - P & (1 - P)\alpha \\ P & -P \end{pmatrix} \right] \begin{pmatrix} c_P \\ c'_P \end{pmatrix} = \\ = \sin \phi \begin{pmatrix} \frac{1}{P} \\ 0 \end{pmatrix} + \cos \phi \begin{pmatrix} 0 \\ -\frac{1}{1 - P} \end{pmatrix} \end{aligned}$$

When setting $\Delta c = 0$, we see from the second row (or equation) that c_P and c'_P must be of opposite sign; the same is true for the solution when setting $\Delta c' = 0$. The solutions to these two “extreme” problems are

$$\begin{aligned} c_P(\Delta c, 0) &= -\frac{1}{P(1 - P)} \frac{\rho}{1 + \alpha'} & c'_P(\Delta c, 0) &= \frac{1}{P(1 - P)} \frac{\alpha' \rho}{1 + \alpha'} \\ c_P(0, \Delta c') &= -\frac{1}{P(1 - P)} \frac{\alpha \rho}{1 + \alpha} & c'_P(0, \Delta c') &= \frac{1}{P(1 - P)} \frac{\rho}{1 + \alpha} \end{aligned}$$

$$\det A(\phi) = -P(1 - P) [(1 + \alpha') \sin^2 \phi + (1 + \alpha) \cos^2 \phi - (1 - \alpha\alpha') \cos \phi \sin \phi]$$

The following shows that $A(\phi) < 0$ for all ϕ , so the above system has a unique solution for all ϕ :

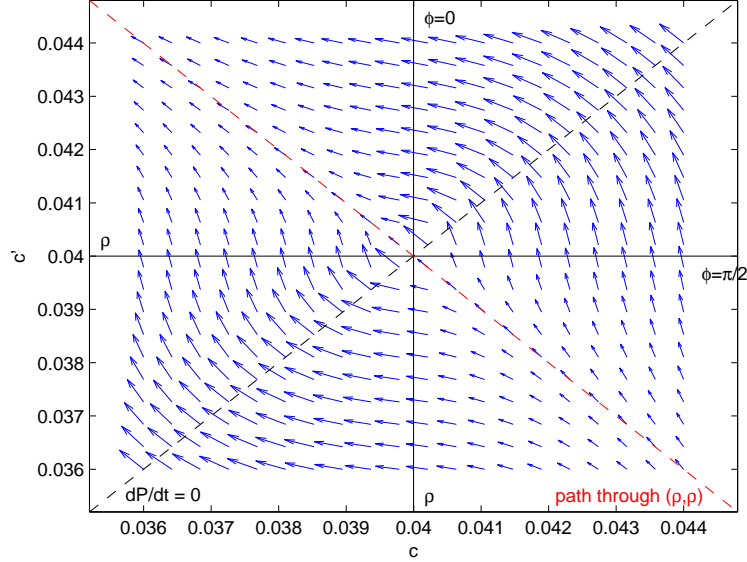
$$\begin{aligned} 0 &< \left(\sqrt{1 + \alpha'} \sin \phi - \sqrt{1 + \alpha} \cos \phi \right)^2 = \\ &= (1 + \alpha') \sin^2 \phi + (1 + \alpha) \cos^2 \phi - 2\sqrt{1 + \alpha'} \sqrt{1 + \alpha} \sin \phi \cos \phi \leq -\frac{A(\phi)}{P(1 - P)} \end{aligned}$$

where the last step follows from $2\sqrt{1 + \alpha'} \sqrt{1 + \alpha} > 1 \geq 1 - \alpha\alpha'$ for any (α, α') . As solution, we obtain

$$\begin{pmatrix} c_P \\ c'_P \end{pmatrix}(\phi) = \frac{1}{A(\phi)} \begin{pmatrix} \sin^2 \phi + \alpha \cos^2 \phi \\ -\alpha' \sin^2 \phi - \cos^2 \phi \end{pmatrix}$$

so obviously $c_P(\phi) < 0$ and $c'_P(\phi) > 0$ for all directions ϕ . Note that since $\sin(\phi + \pi) = -\sin \phi$ and $\cos(\phi + \pi) = -\cos(\phi)$, $A(\phi) = A(\phi + \pi)$ and $\mathbf{c}_P(\phi + \pi) = \mathbf{c}_P(\phi)$. So completing half a circle leads to the same vector direction, as is evident in Figure

Figure 20: \mathbf{c}_P around (ρ, ρ) (for $\alpha = \alpha' = 0.2$)



We are looking for an angle $\phi \in [0, \pi)$ such that
Since $\sin^2 \phi + \cos^2 \phi = 1$.

The angle of the vector \mathbf{c}_P is

$$\tan \xi(\phi) = -\frac{(1-\alpha)\sin^2\phi + \alpha}{1 - (1-\alpha')\sin^2\phi} \equiv -\frac{N(\phi)}{D(\phi)}$$

It is easy to see that $N'(\phi)$ has the opposite sign of $D'(\phi)$; $\tan \xi$ is maximized at $\phi = 0$ with value $-\alpha$, it is decreasing on $[0, \pi/2]$, takes its minimum at $\phi = \pi/2$ with value $-1/\alpha'$ and increases again on $[\pi/2, \pi]$.

A path $\bar{c}(P)$ going through (ρ, ρ) must be such that $\xi(\phi) = \phi$, or equivalently – since $\tan(\cdot)$ is a strictly increasing function on $[-\pi/2, \pi/2]$ –:

$$\tan \xi(\phi) = \tan \phi$$

Since $\tan \xi(\phi) \leq 0$ for all ϕ and $\tan(\phi) > 0$ for $\phi > 0$, there cannot be any solution to this fixed-point problem on the range $(0, \phi/2)$. On $[-\pi/2, 0]$, there must exist at least one fixed point since $\tan(\phi)$ goes crosses the entire range of $\tan \xi(\phi)$. BUT $\tan \xi$ LOOKS DECREASING ON THE FIGURE!