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Properties of Hierarchical Archimedean Copulas

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Abstract: In this paper we analyse the properties of hierarchical Archimedean copulas. This class is a generalisation of the Archimedean copulas and allows for general non-exchangeable dependency structures. We show that the structure of the copula can be uniquely recovered from all bivariate margins. We derive the distribution of the copula value, which is particularly useful for tests and constructing confidence intervals. Furthermore, we analyse dependence orderings, multivariate dependence measures and extreme value copulas. Special attention we pay to the tail dependencies and derive several tail dependence indices for general hierarchical Archimedean copulas.

Keywords: copula; multivariate distribution; Archimedean copula; stochastic ordering; hierarchical copula.
JEL Classification: C16, C46.

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1 Introduction

Copulas play an increasingly important role in econometrics. For an arbitrary multivariate distribution they allow to separate the marginal distributions and the dependency model. As a result we obtain a convenient tool to analyse the complex relationship between variables. In particular, all common measures of dependence can be given in terms of the copula function. Modeling using copulas offers wide flexibility in terms of the form of dependence and is often encountered in applications from financial econometrics, hydrology, medicine, etc.

The copulas were first introduced in the seminal paper of Sklar (1959). Here we restate the Sklar's theorem.

Theorem 1. *Let F be an arbitrary k-dimensional continuous distribution function. Then the associated copula is unique and defined as a continuous function $C : [0,1]^k \to [0,1]$ which satisfies the equality*

$$F(x_1, \ldots, x_k) = C\{F_1(x_1), \ldots, F_k(x_k)\}, \quad x_1, \ldots, x_k \in \mathbb{R},$$

where $F_1(x_1), \ldots, F_k(x_k)$ are the respective marginal distributions.

Alternatively the copula can be defined as an arbitrary distribution function on $[0,1]^k$ with all margins being uniform. As it follows form the theorem, the copula function captures the dependency between variables, with the impact of the marginal distributions being eliminated. The Sklar's Theorem allows to express the copula function directly by

$$C(u_1, \ldots, u_k) = F\{F_1^{-1}(u_1), \ldots, F_k^{-1}(u_k)\}, \quad u_1, \ldots, u_k \in [0,1],$$

where $F_1^{-1}(\cdot), \ldots, F_k^{-1}(\cdot)$ are the corresponding quantile functions.

If the cdf F belongs to the class of elliptical distributions, for example, the Normal distribution, then this results in an elliptical copula. Note, however, that this copula cannot be given explicitly, because F and the inverse marginal distributions F_i have only integral representations. This depreciates the usefulness of the elliptical copulas. As a result, an important class of Archimedean copulas has evolved. The k-dimensional Archimedean copula function $C : [0,1]^k \to [0,1]$ is defined as

$$C(u_1, \ldots, u_k) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_k)\}, \quad u_1, \ldots, u_k \in [0,1], \tag{1}$$

where ϕ with $\phi(0) = 1$ and $\phi(\infty) = 0$ is called the generator of the copula. McNeil and Nešlehová (2008) provide necessary and sufficient conditions for ϕ to generate a feasible Archimedean copula. The generator ϕ is required to be *k-monotone*, i.e. differentiable up to the order $k-2$, with $(-1)^i \phi^{(i)}(x) \geq 0$, $i = 0, \ldots, k-2$ for any $x \in [0, \infty)$ and with $(-1)^{k-2}\phi^{(k-2)}(x)$ being nondecreasing and convex on $[0, \infty)$. We consider a stronger assumption that ϕ is a *completely monotone* function, i.e. $(-1)^i \phi^{(i)}(x) \geq 0$ for all $i \geq 0$. The class of feasible generator functions we define by (see Kimberling (1974), Theorem 1 and Theorem 2)

$$\mathcal{L} = \{\phi : [0; \infty) \to [0,1] \mid \phi(0) = 1,\ \phi(\infty) = 0;\ (-1)^i \phi^{(i)} \geq 0;\ i = 1, \ldots, \infty\}.$$

A detailed review of the properties of Archimedean copulas can be found in McNeil and Nešlehová (2008). Table 4.1 of Nelsen (2006) contains a list of common one-parameter generator functions. Throughout the paper we also consider only the generator functions with a single parameter, however, most of the theory can be easily extended to the case of several parameters.

From the Bernstein's Theorem (Bernstein (1928)) it follows that each $\phi \in \mathcal{L}$ is a Laplace transform of some distribution function. This allows us to relate the Archimedean copulas to the Laplace transforms (see Joe (1996)). Let M be the cdf of a positive random variable and ϕ denotes its Laplace transform, i.e. $\phi(t) = \int_0^\infty e^{-tw} dM(w)$. For an arbitrary cdf F there exists a unique cdf G, such that

$$F(x) = \int_0^\infty G^\alpha(x) dM(\alpha) = \phi\{-\ln G(x)\}.$$

Now consider a k-variate cumulative distribution function F with margins $F_1, \ldots, F_k$. Then it holds that

$$F(x_1, \ldots, x_k) = \int_0^\infty G_1^\alpha(x_1) \cdot \cdots \cdot G_k^\alpha(x_k) dM(\alpha) = \phi\left\{-\sum_{i=1}^k \ln G_i(x_i)\right\} = \phi\left[\sum_{i=1}^k \phi^{-1}\{F_i(x_i)\}\right].$$

This implies that the copula of F is given by (1). The representation of the copula in terms of the Laplace transforms is very useful for simulation purposes (see Whelan (2004), McNeil (2008), Hofert (2008), Marshall and Olkin (1988)).

Note that the Archimedean copula is symmetric with respect to the permutation of variables, i.e. the distribution is exchangeable. Furthermore, the multivariate dependency structure depends on a single parameter of the generator function ϕ. This is very restrictive and we can use Laplace transforms to derive flexible extensions. First, note that $G_1^\alpha \cdot \cdots \cdot G_k^\alpha$ can be seen as a product copula of the cumulative distribution functions $G_1^\alpha, \ldots, G_k^\alpha$. Second, note that the whole model depends on a single cumulative distribution function M. Replacing the product copula $G_1^\alpha \cdot \cdots \cdot G_k^\alpha$ with an arbitrary multivariate copula $K(G_1^\alpha, \ldots, G_k^\alpha)$ and replacing $M(\alpha)$ with some k-variate distribution M_k, such that the jth univariate margin has Laplace transform $\phi_j,\ j = 1, \ldots, k$, we obtain a more general type of dependency (Joe (1997)). This implies, for example, the following copula

$$C(u_1, \ldots, u_k) = \tag{2}$$
$$\int_0^\infty \ldots \int_0^\infty G_1^{\alpha_1}(u_1) G_2^{\alpha_1}(u_2) dM_1(\alpha_1, \alpha_2)\ G_3^{\alpha_2}(u_3) dM_2(\alpha_2, \alpha_3) \ldots G_k^{\alpha_{k-1}}(u_k) dM_{k-1}(\alpha_{k-1}).$$

This generalisation of the multivariate Archimedean copulas leads to the class of *hierarchical Archimedean copulas* (HAC). Other orders of integration and combinations of G_i functions lead to different dependencies. For example, the fully nested (2) HAC $C(u_1, \ldots, u_k)$ can be rewritten in terms of the generator functions arising from the cumulative distribution functions $M_1, \ldots, M_{k-1}$ as

$$\begin{aligned}
&C(u_1, \ldots, u_k) = \\
&= \phi_1[\phi_1^{-1} \circ \phi_2\{\ldots[\phi_{k-2}^{-1} \circ \phi_{k-1}\{\phi_{k-1}^{-1}(u_1) + \\
&+ \phi_{k-1}^{-1}(u_2)\} + \phi_{k-2}^{-1}(u_3)] \cdots + \phi_2^{-1}(u_{k-1})\} + \phi_1^{-1}(u_k)] \\
&= \phi_1\{\phi_1^{-1} \circ C_2(u_1, \ldots, u_{k-1}) + \phi_1^{-1}(u_k)\} = C_1\{C_2(u_1, \ldots, u_{k-1}), u_k\}.
\end{aligned}$$

The sufficient conditions on the generator functions which guarantee that C is a copula are given in Theorem 4.4 McNeil (2008). Let $\mathcal{L}^*$ denote the class of functions with a completely monotone first derivative

$$\mathcal{L}^* = \{\omega : [0; \infty) \to [0, \infty) \,|\, \omega(0) = 0,\ \omega(\infty) = \infty;\ (-1)^{i-1}\omega^{(i)} \geq 0;\ i = 1, \ldots, \infty\}.$$

Table 1: Sufficient conditions on the parameters of generator function of Nelsen (2006), Table 4.1 to guarantee the existence of HAC.

family	ϕ	ϕ^{-1}	$\phi_{\theta_1}^{-1} \circ \phi_{\theta_2}$	conditions
Gumbel	$\exp\{-x^{1/\theta}\}$	$(-\ln t)^{\theta}$	x^{θ_1/θ_2}	$\theta_1 \leq \theta_2,\ \theta \in [1,\infty)$
Clayton	$(\theta x+1)^{-1/\theta}$	$\frac{1}{\theta}(x^{-\theta}-1)$	$\frac{1}{\theta_1}\{(\theta_2 x+1)^{\theta_1/\theta_2}-1\}$	$\theta_1 \leq \theta_2,\ \theta \in (0,\infty)$
Nel. 4.2.2	$1-x^{1/\theta}$	$(1-x)^{\theta}$	x^{θ_1/θ_2}	$\theta_1 \leq \theta_2,\ \theta \in [1,\infty)$
Nel. 4.2.3	$\frac{1-\theta}{e^x-\theta}$	$\ln\frac{1-\theta(1-x)}{x}$	$\ln\frac{e^x(\theta_1-1)+\theta_2-\theta_1}{\theta_2-1}$	$\theta_1 \leq \theta_2,\ \theta \in [0,1)$
Frank	$-\frac{1}{\theta}\ln\{e^{-x}(e^{-\theta}-1)+1\}$	$-\ln\frac{e^{-\theta t}-1}{e^{-\theta}-1}$	$-\ln\frac{\{1+e^{-t}(e^{-\theta_2}-1)\}^{\theta_1/\theta_2}-1}{e^{-\theta_1}-1}$	$\theta_1 \leq \theta_2,\ \theta \in (0,\infty)$

It holds that if $\phi_i \in \mathcal{L}$ for $i = 1,\ldots,k-1$ and $\phi_i \circ \phi_{i+1} \in \mathcal{L}^*$ has a completely monotone derivative for $i = 1,\ldots,k-2$ then C is a copula. As noted by Lemma 4.1 in McNeil (2008), the fact that $\phi_i \circ \phi_{i+1} \in \mathcal{L}^*$ for $i = 1,\ldots,k-2$ also implies that $\phi_i \circ \phi_{i+h} \in \mathcal{L}^*$ for $i = 1,\ldots,k-2$.

Note that generators ϕ_i within a HAC can come either from a single generator family or from different generator families. If ϕ_i's belong to the same family, then the complete monotonicity of $\phi_i \circ \phi_{i+1}$ imposes some constraints on the parameters $\theta_1,\ldots,\theta_{k-1}$. Table 1 provides these constrains for different generators from Nelsen (2006), Table 4.1. For the majority of the copulas the parameters should decrease from the lowest to the highest level, to guarantee a feasible HAC. However, if we consider the generators from different families within a single HAC, the condition of complete monotonicity is not always fulfilled and each particular case should be analysed separately.

The aim of this paper is to provide distributional properties of HACs. First we show that if the true distribution is based on HAC then we can completely recover the true structure of HAC from all bivariate marginal distributions. This property is helpful in applications, when we estimate the HAC from data. For Normal distribution, for example, the form of the dependency is fixed and only the correlation coefficients must be estimated. For HAC both the structure and the parameters of the generators function are unknown. The established result implies that we can first estimate all bivariate copulas and then recover the tree of the HAC. Alternatively, we are forced to enumerate all possible trees, estimate the corresponding multivariate copulas and apply goodness-of-fit tests to determine the HAC with the best fit. This approach is computationally much more demanding compared with the aggregation of bivariate copulas.

Further we derive the distribution of the value of the HAC. This generalises the results of Genest and Rivest (1993) to the HAC. We take explicitly into account the hierarchical structure of the HAC and provide recursive formulas for the cdf by different types of aggregation. The results given in Section 3 can be used for developing of confidence intervals and goodness-of-fit tests. Section 4 summarises the multivariate dependence measures used in the multivariate setup and argues which of them are most convenient to be used with HAC. Section 5 contains results on the dependence orderings of HAC-based distributions. It is shown under which conditions on the generator functions one HAC is more concordant than another one. Finally Section 5 discussed the properties of HAC from the perspective of extreme value theory and provides a detailed analysis of tail dependence. In this section we establish the form of the extreme value copula and provide explicit formulas for two upper and lower tail dependence measures. All proofs are given in the appendix.

2 Determining the structure

In contrary to other distributional models, in HAC both the structure and the parameters of the copula must be specified or estimated. Okhrin, Okhrin and Schmid (2009) consider empirical methods for determining and estimation of the structure. If the structure is fixed, we can apply the maximum-likelihood approach to estimate the parameters. However, the choice of the structure itself is not obvious. One possible approach is to enumerate all structures, estimate the parameters and apply a goodness-of-fit test to determine the best one. This method is, however, unrealistic in higher dimensions. The results established in this section help to overcome this problem. In particular we show that if the true distribution is based on HAC, then we can completely recover the true distribution from all bivariate margins. This implies that instead of estimating all multivariate structures it suffices to estimate all bivariate copulas and use then to recover the full distribution. This makes the estimation of HAC particularly attractive in terms of computational efforts. The next proposition summarises the result.

Proposition 1. *Let F be an arbitrary multivariate distribution function based on HAC. Then F can be uniquely recovered from the marginal distribution functions and all bivariate copula functions.*

Assuming that marginal distributions are continuous, from the Sklar Theorem we know that the multivariate distribution function F can be split into margins and the copula function. Therefore, to recover the distribution we need to recover the structure of the HAC. The proof of the proposition consists of three parts. First, we show that any bivariate margin is a copula with the generator function which is equal to one of the generators of the full structure. Second, we show that the for any bivariate copula with a generator function from the full structure, there exists a couple of variables with the same joint bivariate distribution. Third, we suggest an aggregation procedure and show that the recovered HAC is unique.

Let

$$\mathcal{F}_{k1} = \{C_{k1} : [0;1]^k \to [0;1] : C_{k1} = \phi_\theta[\phi_\theta^{-1}(u_1)+\ldots+\phi_\theta^{-1}(u_k)], \phi \in \mathcal{L}, \theta \in \Theta, u_1, \ldots, u_k \in [0;1]\}$$

be the family of simple k-dimensional Archimedean copulas, where Θ is the set of allowable parameters of θ. The elements of Θ could be of any dimension, but in general they are scalars. Based on this class we introduce the family of k-dimensional HACs with r nodes

$$\begin{aligned}\mathcal{F}_{kr} \quad = \quad & \big\{C_{kr} : [0;1]^k \to [0;1] : \\ & C_{kr} = C\{C_{k_1 r_1}(u_{k_0=1}, \ldots, u_{k_1}), \ldots, C_{k_m - k_{m-1}, r_m}(u_{k_{m-1}+1}, \ldots, u_{k_m=k})\}, \\ & C \in \mathcal{F}_{k1},\ C_{k_i - k_{i-1}, r_i} \in F_{k_i - k_{i-1}, r_i}, \forall i = 1, \ldots, m,\ \sum_{i=1}^{m} r_i = r - 1\big\},\end{aligned}$$

where r_i denotes the number of nodes in the i-th subcopula and the variables are reordered without loss of generality. If $k_i - k_{i-1} = 1$ then $r_i = 1$ and $C_{11}(u_i) = u_i$. For example, $C = C_1\{C_2(u_1, u_2), u_3\} \in \mathcal{F}_{3,2}$, where $C_1, C_2 \in \mathcal{F}_{2,1}$ are nodes, which are also copulas. Let $\mathcal{N}(C)$ denote the set of the generator functions used in the HAC C. Let also $\mathfrak{C}_n$ denote the operator which returns a k-dimensional copula given a generator functions

$$\mathfrak{C}_n(f)(u_1, \ldots, u_k) = f\{f^{-1}(u_1) + \ldots f^{-1}(u_k)\}.$$

Based on this notation, $\mathfrak{C}_2\{\mathcal{N}(C)\} \subset \mathcal{F}_{2,1}$ is the set of all bivariate Archimedean copulas used in the structure of $C \in \mathcal{F}_{kr}$.

Let now a k-dimensional HAC $C \in \mathcal{F}_{kr}$ be fixed. The next remark shows that for any bivariate copula with generator from $\mathcal{N}(C)$ there exists a pair of variables with the same bivariate distribution.

Remark 1. $\forall i, j = 1, \ldots, k,\ i \neq j,\ \exists! C_{ij} \in \mathfrak{C}_2\{\mathcal{N}(C)\} \subset \mathcal{F}_{2,1} : (X_i, X_j) \sim C_{ij}$.

As an example we consider the following 4-dimensional case with

$$C(u_1, \ldots, u_4) \quad = \quad C_1\{C_2(u_1, u_2), C_3(u_3, u_4)\} \text{ with } \mathfrak{C}_2\{\mathcal{N}(C)\} = \{C_1, C_2, C_3\}.$$

For an arbitrary pair of variables u_i and u_j from $u_1, \ldots, u_4$, there exists a copula C_{ij} from $\{C_1, C_2, C_3\}$ such that $(u_i, u_j) \sim C_{ij}$. For example $(u_1, u_3) \sim C_1\{C_2(u_1, 1), C_3(u_3, 1)\} = C_1(u_1, u_3)$. This implies that the bivariate margins use the same generators as the generators in the nodes of the HAC.

The second step of the proof of proposition shows the inverse relationship between the bivariate margins and the set of all bivariate copulas with the generator function from $\mathcal{N}(C)$. In particular it shows that for a generator on any node, there exists a pair of variables with the bivariate distribution given by an Archimedean copula with the same generator.

Remark 2. $\forall C_{i,j} \in \mathfrak{C}_2\{\mathcal{N}(C)\} \subset \mathcal{F}_{2,1},\ \exists i^*, j^* = 1, \ldots, k :\ (X_{i^*}, X_{j^*}) \sim C_{ij}$.

Next we describe the algorithm of recovering the structure from the bivariate margins. Let C_1 denote such bivariate copula that each variable belongs to at least one bivariate margin given by C_1. This copula is the top-level copula. From the Remark 1 if the copula

$$C = C_1\{C_2(u_1, \ldots, u_{k_1}), \ldots, C_m(u_{k_{m-1}+1}, \ldots, u_k)\}$$

then $(u_i, u_j) \sim C_1$, where $i \in [i_1, i_2] \cap \mathbb{N}, j \in ([1, k] \backslash [i_1, i_2]) \cap \mathbb{N}, (i_1, i_2) \in \{(1, k_1), \ldots, (k_{m-1} + 1, k)\}$.

At the next step we drop all bivariate margins given by C_1 and identify the sets of pairs of variables with the bivariate distributions given by C_2 to C_m. For the subtrees we proceed in the same way as for C_1. To show that the structure, that we recovered is equal to the true one, one needs to explore all bivariate margins. A difference at one of the nodes would imply a change in one or several bivariate margins. But the bivariate marginal distribution coincide by construction.

For simplicity let us consider an example:

$$C(u_1, \ldots, u_6) = C_1[C_2(u_1, u_2), C_3\{u_3, C_4(u_4, u_5), u_6\}].$$

The bivariate marginal distributions are then given by

$$\begin{array}{lll}
(u_1, u_2) \sim C_2(u_1, u_2), & (u_2, u_3) \sim C_1(u_2, u_3), & (u_3, u_5) \sim C_3(u_3, u_5), \\
(u_1, u_3) \sim C_1(u_1, u_3), & (u_2, u_4) \sim C_1(u_2, u_4), & (u_3, u_6) \sim C_3(u_3, u_6), \\
(u_1, u_4) \sim C_1(u_1, u_4), & (u_2, u_5) \sim C_1(u_2, u_5), & (u_4, u_5) \sim C_4(u_4, u_5), \\
(u_1, u_5) \sim C_1(u_1, u_5), & (u_2, u_6) \sim C_1(u_2, u_6), & (u_4, u_6) \sim C_3(u_4, u_6), \\
(u_1, u_6) \sim C_1(u_1, u_6), & (u_3, u_4) \sim C_3(u_3, u_4), & (u_5, u_7) \sim C_3(u_5, u_6).
\end{array}$$

In line with Remarks 1 and 2 the set of bivariate margins is equal to

$$\mathfrak{C}_2\{\mathcal{N}(C)\} = \{C_1(\cdot,\cdot), C_2(\cdot,\cdot), C_3(\cdot,\cdot), C_4(\cdot,\cdot)\}.$$

We observe that each variable belongs to at least one bivariate margin given by C_1. This implies that the distribution of $u_1, \ldots, u_6$ has C_1 at the top level. Next we drop all margins given by C_1. Further we proceed similarly with the rest of the margins, in particular with C_3 since it covers the largest set of variables u_3, u_4, u_5, u_6. This implies that C_3 is at the top level of the subcopula containing u_3, u_4, u_5, u_6. Having information only for the copulas C_1 and C_3

$$u_1, \ldots, u_6 \sim C_1\{u_1, u_2, C_3(u_3, u_4, u_5, u_6)\}.$$

The remaining copula functions are C_2 and C_4 and they join u_1, u_2 and u_4, u_5 respectively. Summarising we obtain

$$(u_1, \ldots, u_6) \sim C_1[C_2(u_1, u_2), C_3\{u_3, C_4(u_4, u_5), u_6\}]$$

This results in the correct structure. Similarly we can apply inverse procedure by joining variables into pseudo-random variables, using low-level copulas. This problem is related to the multidimensional scaling problem, where having all paired distances between the cities, one has to recover the whole map, see Härdle and Simar (2007).

3 Distribution of HAC

For testing purposes and construction of confidence intervals we are interested in the distributions of the empirical and the true copula. Let $V = C\{F_1(X_1), \ldots, F_k(X_k)\}$ and let $K(t)$ denote the distribution function (K-distribution) of the random variable V. Genest and Rivest (1993) introduced a nonparametric estimator of K in the case $k = 2$. It is based on the concept of Kendall's process. Suppose that an independent random sample $\boldsymbol{X}_1 = (X_{11}, \ldots, X_{1k})'$, $\ldots$, $\boldsymbol{X}_n = (X_{n1}, \ldots, X_{nk})'$ of the vector $\mathbf{X} = (X_1, .., X_k)'$ is given. Let

$$V_{i,n} = \frac{1}{n+1} \sum_{j=1, j\neq i}^{n} \mathbf{I}\{\mathbf{X}_j \leq \mathbf{X}_i\}$$

and K_n denote the empirical distribution function of the $V_{i,n}$'s. Here the inequality $\mathbf{a} \leq \mathbf{b}$ means that all components of the vector $\mathbf{a}$ are less or equal than those of the vector $\mathbf{b}$. Then the Kendall process is given by

$$\alpha_n(t) = \sqrt{n}\{K_n(t) - K(t)\}.$$

Barbe, Genest, Ghoudi and Rémillard (1996) examine the limiting behavior of the empirical process $\alpha_n(t)$ for $k \geq 2$ and derived explicit formulas of its density $\kappa(t)$ and its distribution function $K(t)$ for general multivariate copulas. The authors provide explicit results for product and multivariate exchangeable Archimedean copulas. The paper of Wang and Wells (2000) used Kendall's process to determine the copula for failure data. In this section we adopt and extend the results of Barbe et al. (1996) to find the K-distribution of a HAC.

At the first step we exploit the hierarchical structure of the HAC. We consider a HAC of the form $C_1\{u_1, C_2(u_2, \ldots, u_k)\}$. Let $U_i \sim U[0,1]$ and let $V_2 = C_2(U_2, \ldots, U_k) \sim K_2$. In the next theorem we propose a recursive procedure for calculating the distribution function of $V_1 = C_1(U_1, V_2)$ which is based on the knowledge of the distribution function of V_2. This approach is particularly useful when applied to fully nested HACs.

Theorem 2. *Let $U_1 \sim U[0,1]$, $V_2 \sim K_2$ and let $P(U_1 \leq x, V_2 \leq y) = C_1\{x, K_2(y)\}$ with $C_1(a,b) = \phi\left\{\phi^{-1}(a) + \phi^{-1}(b)\right\}$ for $a, b \in [0,1]$. Assume that $\phi : [0,\infty) \to [0,1]$ is strictly decreasing with $\phi(0) = 1$ and $\phi(\infty) = 0$ and that ϕ' is strictly increasing and continuous. Moreover, suppose that K_2 is continuous. Suppose that the random variable V_2 takes values in $[0,1]$. Then the distribution function K_1 of the random variable $V_1 = C_1(U_1, V_2)$ is given by*

$$K_1(t) \;=\; t - \int_0^{\phi^{-1}(t)} \phi'\big(\phi^{-1}(t) + \phi^{-1}\left[K_2\{\phi(u)\} - u\right]\big)du \quad \textit{for} \quad t \in [0,1]. \tag{3}$$

In Theorem 2 V_2 is an arbitrary random variable on $[0,1]$ and not necessarily a copula. In the special case that V_2 is uniformly distributed on $[0,1]$ formula (3) reduces to Theorem 4.3.4 of Nelsen (2006) or to the result of Genest and Rivest (1993).

Next we consider a copula of the type $V_3 = C_3(V_4, V_5)$ with $V_4 = C_4(U_1, \ldots, U_\ell)$ and $V_5 = C_5(U_{\ell+1}, \ldots, U_k)$. Making use of the distribution functions of V_4 and V_5 a representation of the distribution function of V_3 is given in the next theorem.

Theorem 3. *Let $V_4 \sim K_4$ and $V_5 \sim K_5$ and $P(V_4 \leq x, V_5 \leq y) = C_3\{K_4(x), K_5(y)\}$ with $C_3(a,b) = \phi\left\{\phi^{-1}(a) + \phi^{-1}(b)\right\}$ for $a, b \in [0,1]$. Assume that $\phi : [0,\infty) \to [0,1]$ is strictly decreasing with $\phi(0) = 1$ and $\phi(\infty) = 0$ and that ϕ' is strictly increasing and continuous. Moreover, suppose that K_4 and K_5 are continuous and that $\phi^{-1} \circ K_4 \circ \phi$ and $\phi^{-1} \circ K_5 \circ \phi$ are of bounded variation on $[0, \phi^{-1}(t)]$. Suppose that the random variables V_4 and V_5 take values in $[0,1]$ then the distribution function K_3 of the random variable $V_3 = C_3(V_4, V_5)$ is given by*

$$K_3(t) = K_4(t) - \int_0^{\phi^{-1}(t)} \phi'\left\{\phi^{-1}[K_5\{\phi(u)\}] + \phi^{-1}\big(K_4[\phi\{\phi^{-1}(t) - u\}]\big)\right\} d\phi^{-1}\left[K_4\{\phi(u)\}\right] \tag{4}$$

for $t \in [0,1]$.

If $\phi^{-1}[K_4\{\phi(x)\}]$ has a continuous derivative then (4) can be written as

$$K_3(t) = K_4(t) - \int_0^{\phi^{-1}(t)} \frac{\phi'\left\{\phi^{-1}\left[K_5\{\phi(u)\}\right] + \phi^{-1}\big(K_4[\phi\{\phi^{-1}(t) - u\}]\big)\right\}}{\phi'\{(\phi^{-1} \circ K_4 \circ \phi)^{-1}(u)\}} \, K_4'\{\phi(u)\}\phi'(u)du$$

and similarly for the second representation. Theorem 3 reduces to Theorem 2 if V_4 or V_5 are uniformly distributed on $[0,1]$. Moreover, by taking the derivative of the generator function it can be shown that the expression in (4) is symmetric with respect to K_4 and K_5.

Note that using these two results we can establish the distribution function for an arbitrary grouping of the variables at the top level. For example, consider the copula $C_1\{u_1, u_2, C_2(u_3, \ldots, u_k)\}$. From the properties of Archimedean copulas, this copula is equivalent to $C_1[u_1, C_1\{u_2, C_2(u_3, \ldots, u_k)\}]$ and thus the result of Theorem 2 can be applied.

Theorem 2 and Theorem 3 provide recursive presentations for certain copula structures. In the next theorem we provide a direct formula for the distribution function of a copula of the form $C\{u_1, C_{k-1}(u_2, \ldots, u_k)\}$. It is an extension of the result of Barbe et al. (1996). Here we assume that u_k lies on the top level of the copula. Other cases could be derived for every single form of the copula, but it is difficult to present a general result.

Theorem 4. *Consider a HAC of the form*

$$C(u_1,\ldots,u_k) = C_1\{u_1, C_2(u_2,\ldots,u_k)\} = \phi_1\left[\phi_k^{-1}(u_k) + \phi_1^{-1}\{C_2(u_2,..,u_k)\}\right].$$

Assume that $\phi_1 : [0,\infty) \to [0,1]$ *is strictly decreasing and continuously differentiable with* $\phi_1(0) = 1$. *Then the distribution function* K_1 *of* $C(u_1,..,u_k)$ *is equal to*

$$K_1(t) = \int_0^t k(x)\,dx = \int_0^t \int_{(0,1)^{k-1}} \cdots \int h_k\{x,u_2,\ldots,u_k\}\,du_2\ldots du_k\,dx \quad for \quad t\in[0,1],$$

where

$$\begin{aligned} h_k(t,u_2,\ldots,u_k) &= \frac{\phi_1'\left\{\phi_1^{-1}(t) - \phi_1^{-1}\circ C_2(u_2,\ldots,u_k)\right\}}{\phi_1'\{\phi_1^{-1}(t)\}} \\ &\times\ c\left[\phi_1\{\phi_1^{-1}(t) - \phi_1^{-1}\circ C_2(u_2,\ldots,u_k)\}, u_2,\ldots,u_k\right] \\ &\times\ \mathbf{I}\big\{C_2(u_2,\ldots,u_k) > t\big\} \ \ for\ (u_2,\ldots,u_k)\in[0,1]^{k-1}. \end{aligned}$$

$c(u_1,..,u_k)$ *denotes the copula density of* C.

The practical calculation of K_1 using Theorem 4 seems to be quite difficult because of multivariate integration. As an example we consider the Clayton family.

Example 1. *Here we consider the simplest three-dimensional fully nested Archimedean copula with Clayton generator functions*

$$\phi_\theta(t) = (\theta t+1)^{-1/\theta}.$$

The copula function is given by

$$C(u_1,u_2,u_3;\theta_1,\theta_2) = C_{\theta_2}\{C_{\theta_1}(u_1,u_2),u_3\} = \left\{\left({u_1}^{-\theta_1} + {u_2}^{-\theta_1} - 1\right)^{-\frac{\theta_2}{\theta_1}} + {u_3}^{-\theta_2} - 1\right\}^{-\frac{1}{\theta_2}}$$

and

$$\begin{aligned} h_3(u_1,u_2,t;\theta_1,\theta_2) &= \left\{{u_2}^{\theta_1} - {u_1}^{\theta_1}\left({u_2}^{\theta_1}-1\right)\right\}^{-2}(u_1u_2)^{\theta_1-1}p_1 r^{\frac{1+\theta_2}{\theta_1}}\left(p_1 + r^{\frac{\theta_2}{\theta_1}} - 1\right)^{-3-\frac{1}{\theta_2}} \\ &\times\ \left\{1 - t^{\theta_1}(p_2-1)\right\}^{-1-\frac{1}{\theta_1}}\left\{p_1(1+\theta_1+\theta_2) + r^{\frac{\theta_2}{\theta_1}}(\theta_1-\theta_2) + \theta_2 - \theta_1\right\} \\ &\times\ (1+\theta_2)\,\mathbf{I}\left\{\frac{u_1^{-\theta_1} + u_2^{-\theta_2} - t^{-\theta_1} - 1}{\theta_1} < 0\right\} \end{aligned}$$

with

$$\begin{aligned} p_i &= \left({u_1}^{-\theta_i} + {u_2}^{-\theta_i} - 1\right)^{\frac{\theta_{-i+3}}{\theta_i}} \ \ for\ i=1,2 \\ r &= 1 + t^{-\theta_1} - p_2. \end{aligned}$$

4 Multivariate Dependence Measures

If we consider a multivariate random vector we are often interested in the dependency measures between the components of the vector. In case of the Gaussian distribution the whole dependence

can be uniquely characterized by the linear correlation coefficients. Since it is a measure of the linear dependency between random variables, it is not an appropriate measure for non-linear relationships. As an alternative correlation coefficients based on ranks of the ordered data can be considered. The most popular measure is Kendall's τ. For a bivariate copula it is defined as

$$\tau_2 = 4 \int_0^1 \int_0^1 C(u_1, u_2)\, dC(u_1, u_2) - 1.$$

The extension to the multivariate case is not straightforward and unique. A multivariate version of Kendall's τ and its empirical representation is proposed in Barbe et al. (1996) as an affine transformation of the expectations of the variables V and V_n respectively.

$$\begin{aligned}
\tau_k &= \frac{2^k}{2^{k-1}-1}\mathrm{E}(V) - 1 = \frac{2^k}{2^{k-1}-1}\int t\, dK(t) - 1 = \frac{2^k}{2^{k-1}-1}\int t\kappa(t)\, dt - 1, \\
\hat{\tau}_{kn} &= \frac{2^k}{2^{k-1}-1} \cdot \frac{1}{n}\sum_{i=1}^{n} V_n(X_{1i}, \ldots, X_{ki}) - 1 = \frac{2^k}{2^{k-1}-1}\int t\, dK_n(t) - 1,
\end{aligned}$$

where $\kappa(t)$ is the density function of the cdf $K(t)$. This justifies the name Kendall's process in the last section as coined by Genest and Rivest (1993).

Another popular measure of dependence is Spearman's ρ. In the bivariate case it is given by

$$\rho_2 = \frac{\int_0^1 \int_0^1 u_1 u_2\, dC(u_1, u_2) - \frac{1}{2}^2}{\frac{1}{12}} = \frac{\int_0^1 \int_0^1 C(u_1, u_2)\, du_1\, du_2 - \frac{1}{4}}{\frac{1}{3} - \frac{1}{4}} = 12 \int_0^1 \int_0^1 C(u_1, u_2)\, du_1\, du_2 - 3.$$

Two alternative multivariate extensions are given by

$$\begin{aligned}
\rho_{k_1} &= \frac{k+1}{2^k - (k+1)}\Big\{2^k \int_{[0,1]^k} C(\mathbf{u})\, d\mathbf{u} - 1\Big\}, \\
\rho_{k_2} &= \frac{k+1}{2^k - (k+1)}\Big\{2^k \int_{[0,1]^k} u_1 \ldots u_k\, dC(\mathbf{u}) - 1\Big\}.
\end{aligned}$$

ρ_{k_1} was introduced by Wolff (1980) and ρ_{k_2} in Joe (1997) and Nelsen (2006). Both measures were thoroughly investigated by Schmid and Schmidt (2006a) and Schmid and Schmidt (2006b). The explicit computation of ρ_{k_1} and ρ_{k_2} is difficult for almost all copula functions. Therefore, as a simplification a pairwise version of Spearman's ρ was proposed in Kendall (1970)

$$\rho_{kr} = 2^2 \sum_{m<l} \binom{k}{2}^{-1} \int_{[0,1]^2} C_{ml}(u, v)\, du dv - 1 = \frac{2}{3(k-1)k}\sum_{m<l}(\rho_{2,kl} + 3) - 1,$$

where C_{ml} is the bivariate copula for the variables u_m and u_l and $\rho_{2,kl}$ denotes the bivariate Spearman's ρ for u_k and u_l. The last representation of Spearman's ρ is very useful for HACs, because all bivariate sub copulas in a HAC are simple bivariate Archimedean copulas. Hence ρ_{kr} could be easily computed for a HAC by calculating all bivariate Spearman's ρ's.

Example 2. *For the simple exchangeable Archimedean copula the pairwise Spearman's ρ_{kr} is*

$$\rho_{kr} = \frac{(k-2)!}{24}(\rho_2 + 3) - 1,$$

where ρ_2 is the bivariate Spearman's ρ based on the generator function of the given copula.

As the third multivariate dependence measure we discuss Blomqvist's β. In the bivariate and in the multivariate cases it is computed as follows

$$\begin{aligned}\beta_2 &= \mathrm{P}\{(\mathbf{x}-\tilde{\mathbf{x}})(\mathbf{y}-\tilde{\mathbf{y}})>0\}-\mathrm{P}\{(\mathbf{x}-\tilde{\mathbf{x}})(\mathbf{y}-\tilde{\mathbf{y}})<0\},\\ \beta_k &= \frac{2^{k-1}}{2^{k-1}-1}\{C(1/2,\ldots,1/2)+\overline{C}(1/2,\ldots,1/2)-2^{1-k}\},\end{aligned}$$

where $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are the population medians, and $\overline{C}(\mathbf{u}) = 1+\sum_{s\in\mathcal{S}}(-1)^{|s|}C_s(u_j;\, j\in s)$ is the survival copula and $\mathcal{S}$ is the set of all subsets of $\{1,\ldots,k\}$. Schmid and Schmidt (2006c) provide a detailed discussion of these measures.

Example 3. *For the k-dimensional exchangeable Archimedean copula the Blomqvist's β in terms of the generator functions is given by*

$$\beta_k = \frac{2^{k-1}}{2^{k-1}-1}\left[\phi\{k\phi^{-1}(1/2)\}+\sum_{i=1}^{k}(-1)^i\binom{k}{i}\phi\{i\phi^{-1}(1/2)\}-2^{1-k}\right].$$

Example 4. *Blomqvist's β for the simplest two-dimensional fully nested Archimedean copula $C(u_1,u_2,u_3,\theta_1,\theta_2) = C_{\theta_2}\{C_{\theta_1}(u_1,u_2),u_3\}$ with the generator functions ϕ_{θ_1} and ϕ_{θ_2} respectively is given by*

$$\beta_3 = \frac{8}{3}\phi_{\theta_2}\{2\phi_{\theta_2}^{-1}(1/2)\}+\frac{4}{3}\phi_{\theta_1}\{2\phi_{\theta_1}^{-1}(1/2)\}-1.$$

For different generator functions this reduces to the expressions summarized in the following table.

family	β_3
Clayton	$\frac{4}{3}\big(2^{\theta_1+1}-1\big)^{-1/\theta_1}+\frac{8}{3}\big(2^{\theta_2+1}-1\big)^{-1/\theta_2}-1$
Gumbel	$-1+\frac{4}{3}\big(\exp\{-2^{1/\theta_1}\ln 2\}+2\exp\{-2^{1/\theta_2}\ln 2\}\big)$
Nelsen 4.2.2	$\frac{1}{3}\big(9-2^{1+1/\theta_1}-2^{2+1/\theta_1}\big)$

The considered measures depend on the copula function and can be used to measure the dependence of copula-based distributions. Unfortunately, there are numerous drawbacks of these measures. First, there is no unique decision on the superiority of one of the measures. There are papers which compare these measures in the bivariate framework (Chen (2004), Durrleman, Nikeghbali and Roncalli (2000), Fredricks and Nelsen (2004), etc.), however, nothing similar has been done in the multivariate case. Second, it is very restrictive to use a single scalar measure to quantify all the relationships between the components of a k-dimensional random vector. Third, Kendall's τ and Spearman's ρ are difficult to compute explicitly because of the multivariate integrals of the copula functions. Nevertheless, the estimators are readily available. Fourth, there is no unique method how to extend a bivariate dependence measure to the multivariate case. This inflates the number of candidates for dependence measures and makes the conclusions of empirical studies less transparent. Summarizing, due to their simplicity, we recommend to use the multivariate extension of Blomqvist's β or the pairwise multivariate Spearman's ρ_{kr} for the HAC models.

5 Dependence Orderings

Dependence orderings allows us to compare the strength of the dependence imposed by different copula functions. In this section we show some necessary conditions under which one HAC is more concordant than other. By definition (Joe (1997), p.37), C' is more concordant than C if

$$C \prec_c C' \Leftrightarrow C(\mathbf{x}) \leq C'(\mathbf{x}) \text{ and } \overline{C}(\mathbf{x}) \leq \overline{C'}(\mathbf{x}) \; \forall \mathbf{x} \in [0;1]^k.$$

This type of the ordering is also called positive quadrant ordering (PQD) or upper orthant ordering (see Müller and Stoyan (2002)). The case of two multivariate normal distributions gives us interesting insights into this ordering. Let $\boldsymbol{X} \sim \mathcal{N}_k(\mu, \Sigma)$ and $\boldsymbol{X}' \sim \mathcal{N}_k(\mu', \Sigma')$. If $\mu_i = \mu_i'$, $\sigma_{ii} = \sigma_{ii}'$ for $i = 1, \ldots, k$, and $\sigma_{ij} \leq \sigma_{ij}'$ for $1 \leq i < j \leq k$ then $\boldsymbol{X} \prec_c \boldsymbol{X}'$.

In the bivariate case the most concordant is the Fréchet upper bound and the most discordant copula is the Fréchet lower bound. Another peculiarity of the bivariate case is the relationship between the concordance ordering the dependence measures. It appears that if C_1 and C_2 are two copulas with Kendall's taus τ_1, τ_2, Spearman rhos ρ_1, ρ_2, tail dependence parameters λ_1, λ_2, Blomqvist's betas β_1, β_2 respectively, then $C_1 \prec_c C_2$ implies that $\tau_1 \leq \tau_2$, $\rho_1 \leq \rho_2$, $\lambda_1 \leq \lambda_2$ (Joe (1997)) and $\beta_1 \leq \beta_2$ (Schmid and Schmidt (2006c)).

Several interesting results can be derived if C is an Archimedean copula. First note that there is no sharp lower bound for the general class of copulas, however McNeil and Nešlehová (2008) construct the sharp lower bound in the class of Archimedean copulas. Thus there is an Archimedean copula C^L, such that $C^L \prec_c C$ for any Archimedean copula C. Joe (1997) considers in Theorem 4.8, 4.9 and 4.10 three and four dimensional HACs with different fixed structures. The theorems provide the conditions on the top level generator functions to guarantee the concordance of the HACs, assuming that the generators at lower levels are the same. In Joe (1997) the author also states that these theorems could be easily extended to any messy structure of the copula. Next we provide a general results for an arbitrary tree. The proof uses explicitly the hierarchical structure of the copula.

Theorem 5. *If two feasible hierarchical Archimedean copulas C^1 and C^2 differ only by the generator functions on the top level satisfying the condition $\phi_1^{-1} \circ \phi_2 \in \mathcal{L}^*$, then $C^1 \prec_c C^2$.*

In the next theorem we generalize this result to changes at an arbitrary level of the copula.

Theorem 6. *If two hierarchical Archimedean copulas $C^1 = C^1_{\boldsymbol{\phi}_1}(u_1, \ldots, u_k)$ and $C^2 = C^2_{\boldsymbol{\phi}_2}(u_1, \ldots, u_k)$ differ only by the generator functions on the level r as*

$$\boldsymbol{\phi}_1 = (\phi_1, \ldots, \phi_{r-1}, \phi, \phi_{r+1}, \ldots, \phi_p) \text{ and } \boldsymbol{\phi}_2 = (\phi_1, \ldots, \phi_{r-1}, \phi^*, \phi_{r+1}, \ldots, \phi_p)$$

with $\phi^{-1} \circ \phi^* \in \mathcal{L}^*$, then $C^1 \prec_c C^2$.

Note that the condition we impose on the generator function is the sufficient condition to construct a HAC (see Theorem 4.4 of McNeil (2008)). For example, consider two HACs with the same structure and with the same generator functions on the corresponding levels. For some fixed level, let θ_1 be the parameter of the generator function for the first copula, and θ_2 for the second. If the conditions given in the last column hold, then the first copula is more concordant than the second.

6 Extreme Value Theory and Tail Dependency

6.1 Extreme Value Copula

In the univariate case the distribution of the maxima or minima of a sample is defined as follows (see Embrechts, Klüppelberg and Mikosch (2001))

Definition 1. *For independent identically distributed random variables X_n if there exist constants $b_n > 0$, $a_n \in \mathbb{R}$, a non-degenerating function F^* with $\frac{m^+ - b_n}{a_n} \xrightarrow{d} F^*$ where $m^+ = \max\{X_1, \ldots, X_n\}$, then*

$$F^*(\gamma, m^+) = \exp\left\{-\left(1 - \tau\frac{m^+ - a}{b}\right)^{1/\tau}\right\}$$

with $\left(1 - \tau\frac{m^+ - a}{b}\right) > 0$ and $\gamma = (\tau, a, b)$.

The multivariate extreme value distribution can be characterised by the following theorem

Theorem 7 (Deheuvels (1978))**.** *Let $\{X_{1i}, \ldots, X_{ki}\}_{i=1,\ldots,n}$ be a sequence of the random vectors with the distribution function F, marginal distributions $F_1, \ldots, F_k$ and copula C. Let also $M_j^{(n)} = \max_{1 \le i \le n} X_{ji}$, $j = 1, \ldots, k$ be the componentwise maxima. Then*

$$\lim_{n\to\infty} P\left\{\frac{M_1^{(n)} - a_{1n}}{b_{1n}} \le x_1, \ldots, \frac{M_m^{(n)} - a_{kn}}{b_{kn}} \le x_k\right\} = F^*(x_1, \ldots, x_k), \quad \forall (x_1, \ldots, x_k) \in \mathbb{R}^k$$

with $b_{jn} > 0$, $j = 1, \ldots, k$, $n \ge 1$ if and only if

1. *for all $j = 1, \ldots, k$ there exist some constants a_{jn} and b_{jn} and a non-degenerating limit distribution F_j^* such that*

$$\lim_{n\to\infty} P\left\{\frac{M_j^{(n)} - a_{jn}}{b_{jn}} \le x_j\right\} = F_j^*(x_j), \quad \forall x_j \in \mathbb{R};$$

2. *there exists a copula C^* such that*

$$C^*(u_1, \ldots, u_k) = \lim_{n\to\infty} C^n(u_1^{1/n}, \ldots, u_k^{1/n}).$$

In this case we say that copula C^* is the extreme valued copula and the copula C belongs to the maximum domain of attraction of the copula C^* (written $C \in MDA(C^*)$). This implies that a multivariate distribution with all margins being extreme-valued distributions and with an extreme-valued copula is a multivariate extreme-valued distribution. Genest and Rivest (1989) show that the only Archimedean extreme-valued copula is the Gumbel copula. Thus, each bivariate Archimedean copula belongs to the domain of attraction of the Gumbel copula. Using Proposition 1 and the result of Genest and Rivest (1989) we state the next theorem.

Theorem 8. *If $C \in \mathcal{F}_{nr_1}$, $C^* \in \mathcal{F}_{nr_2}$ and $C \in MDA(C^*)$ then $r_1 = r_2$, $\forall \phi_\theta \in \mathcal{N}(C^*)$, $\phi_\theta = \exp\{-x^{1/\theta}\}$ and the structure of C is equal to the structure of C^*.*

6.2 Tail Dependency

Next we consider the tail dependence of HAC. The tail behavior characterises the tendency of random variables to take extreme values simultaneously. The upper and lower tail indices of two random variables $X_1 \sim F_1$ and $X_2 \sim F_2$ are given by

$$\begin{aligned}
\lambda_U &= \lim_{u\to 1^-} P\{X_2 > F_2^{-1}(u) \mid X_1 > F_1^{-1}(u)\} \\
&= \lim_{u\to 1^-} \frac{\overline{C}(u,u)}{1-u} = 2 - \lim_{u\to 1^-} \frac{1-C(u,u)}{1-u} \\
\lambda_L &= \lim_{u\to 0^+} P\{X_2 \le F_2^{-1}(u) \mid X_1 \le F_1^{-1}(u)\} \\
&= \lim_{u\to 0^+} \frac{C(u,u)}{u}.
\end{aligned}$$

The upper index is equal to limit of the probability that one variable exceed some predetermined limit conditional on the fact that another variable exceeded this limit. Similarly the lower limit describes the tendency to simultaneous undershooting some limit. We replicate the Corollary 5.4.3 of Nelsen (1997) which states explicitly the tail dependency indices for bivariate Archimedean copulas.

Theorem 9 (Nelsen (1997))**.** *For a bivariate Archimedean copula with the generator ϕ it holds*

$$\begin{aligned}
\lambda_U &= 2 - \lim_{u\to 1^-} \frac{1-\phi\{2\phi^{-1}(u)\}}{1-u} = 2 - \lim_{w\to 0^+} \frac{1-\phi(2w)}{1-\phi(w)}, \\
\lambda_L &= \lim_{u\to 0^+} \frac{\phi\{2\phi^{-1}(u)\}}{u} = \lim_{w\to\infty} \frac{\phi(2w)}{\phi(w)}.
\end{aligned}$$

In this section we extend the concept of the tail dependence to multivariate distributions and consider several alternative definitions of it. First we consider the straightforward extension of the bivariate definition. Let $\mathcal{K} = \{1,\dots,k\}$ and s is a subset of $\mathcal{K}$. Then we denote by C_s the marginal copula of the variables with indices in s, i.e. $C_s(u_i, i \in s) = C(u_1,\dots,u_k|u_j = 1, j \notin s)$. Let the upper and lower tail indices of k random variables $X_i \sim F_i$ for $i = 1,\dots,k$ are given by

$$\begin{aligned}
&\lambda_U^{(1)}(u_1,\dots,u_k|u_i) \\
&\quad= \lim_{u\to 0^+} P\{\dots, X_{i-1} > F_{i-1}^{-1}(1-u_{i-1}u), X_{i+1} > F_{i+1}^{-1}(1-u_{i+1}u), \dots \mid X_i > F_i^{-1}(1-u_iu)\} \\
&\quad= \lim_{u\to 0^+} \frac{\overline{C}(1-u_1u,\dots,1-u_ku)}{u_iu} \\
&\quad= \lim_{u\to 0^+} \frac{1-\sum_{s\subset\mathcal{K},s\neq\emptyset}(-1)^{|s|+1}C_s(1-u_ju, j\in s)}{u_iu} \\
&\quad= \sum_{s\subset\mathcal{K},s\neq\emptyset}(-1)^{|s|+1} \lim_{u\to 0^+} \frac{1-C_s(1-u_ju, j\in s)}{u_iu} =: \frac{1}{u_i}\sum_{s\in\mathcal{S},s\neq\emptyset}\beta_s, \\
&\lambda_L^{(1)}(u_1,\dots,u_k|u_i) \\
&\quad= \lim_{u\to 0^+} P\{\dots, X_{i-1} \le F_{i-1}^{-1}(u_{i-1}u), X_{i+1} \le F_{i+1}^{-1}(u_{i+1}u), \dots \mid X_i \le F_i^{-1}(u_iu)\} \\
&\quad= \lim_{u\to 0^+} \frac{C(u_1u,\dots,u_ku)}{u_iu} \;=:\; \frac{1}{u_i}\lambda_L^{(1)}(u_1,\dots,u_k).
\end{aligned}$$

$\lambda_U^{(1)}(u_1,\dots,u_k)$ defines the limit of the probabilities that all random variables simultaneously exceed the thresholds $1-u_ju$ conditional on the fact that one particular random variable exceeded

its limit $1 - u_i u$ as $u \to 0^+$. Similar motivation holds for $\lambda_L^{(1)}$ too. By setting $u_j = u^*$ for all $j = 1, \ldots, k$ we obtain a definition similar to the bivariate definition above. For example,

$$\begin{aligned}\lambda_L^{(1)} := \lambda_L^{(1)}(1, \ldots, 1) &= \lim_{u \to 0^+} P\{\ldots, X_{i-1} \leq F_{i-1}^{-1}(u), X_{i+1} \leq F_{i+1}^{-1}(u), \ldots \mid X_i \leq F_i^{-1}(u)\} \\ &= \lim_{u \to 0^+} \frac{C(u, \ldots, u)}{u},\end{aligned}$$

and similarly for $\lambda_L^{(2)} := \lambda_L^{(2)}(1, \ldots, 1) = \lim_{u \to 1^-} \frac{\overline{C}(u, \ldots, u)}{u}$.

From Theorem 9 it follows that the tail indices of Archimedean copulas are closely related to the regular variation of the generator functions. Here we replicate the definition of regular variation of functions at zero and and infinity following Shorack (2000).

Definition 2. *Call $V(\cdot) > 0$ regularly varying at 0 with characteristic exponent (tail index) $\lambda > 0$ (written $V \in \mathcal{R}_\lambda$) if $\lim_{t \to 0^+} V(tx)/V(t) = x^{-\lambda}$.*

From the monotone density theorem (see Shorack (2000), Theorem 9.1) it follows that

$$V \in \mathcal{R}_\lambda \quad \text{iff} \quad \lim_{t \to 0^+} tV'(t)/V(t) = \lambda.$$

Definition 3. *Call $V(\cdot) > 0$ on $(0, \infty)$ regularly varying at infinity with characteristic exponent $\lambda > 0$ (written $V \in \mathcal{U}_\lambda$) if $\lim_{t \to \infty} V(tx)/V(t) = x^{-\lambda}$.*

The limits exist for a very wide family of functions and all common generator functions of Archimedean copulas. Since the dependency of the copula is uniquely determined by its structure and generator functions, the multivariate tail indices can be given in terms of the tails indices of generator functions. For further exposition we find useful the following proposition which relates the variability of a function with the variability of the inverse function. Here restrict the discussion only to generator functions.

Proposition 2. *Let for the non-increasing generator function $\phi \in \mathcal{L}$ it holds that*

$$\lim_{t \to \infty} \frac{\phi(xt)}{\phi(t)} = \Phi_1(x), \quad \lim_{t \to 0^+} \frac{1 - \phi(xt)}{1 - \phi(t)} = \Phi_2(x),$$

where $\Phi_i(x) \not\equiv const$ and $\Phi_i \not\equiv \infty$ for $i = 1, 2$. Then for the inverse generator function ϕ^{-1}, i.e. $\phi^{-1}(y) = \inf\{x : \phi(x) \geq y\}$ it holds that

$$\lim_{t \to 0^+} \frac{\phi^{-1}(xt)}{\phi^{-1}(t)} = \Phi_1^{-1}(x), \quad \lim_{t \to 0^+} \frac{\phi^{-1}(1 - xt)}{\phi^{-1}(1 - t)} = \Phi_2^{-1}(x).$$

Relying on these results, the next theorem extends Theorem 9 to HAC. We derive both measures for a HAC of an arbitrary structure and with arbitrary generator functions. The theorem provides a recursive method for determining the lower and upper tail dependencies depending on the functions Φ_i for $i = 1, 2$. We discuss the case when the generator function is regularly varying at zero and/or infinity in remarks below.

Theorem 10. *Let $X_1, \ldots, X_k \sim C$ such that*

$$C(u_1, \ldots, u_k) = C_0\{C_1(u_1, \ldots, u_{k_1}), \ldots, C_m(u_{k_{m-1}+1}, \ldots, u_{k_m}), u_{k_m+1}, \ldots, u_k\},$$

where C_0 is an Archimedean copula with the generator $\phi_0 \in \mathcal{L}$ and C_i for $i = 1, \ldots, m$ are Archimedean or hierarchical Archimedean copulas with the upper and lower tail dependence indices $\lambda_{U,i}^{(1)}$ and $\lambda_{L,i}^{(1)}$ respectively. Further we assume that for the generator ϕ_0 it holds that $\Phi_i(x) \not\equiv const$ and $\Phi_i(x) \not\equiv \infty$ for $i = 1, 2$. Then the upper and lower tail dependency indices for C are given by

$$\lambda_L^{(1)}(u_1, \ldots, u_k|u_i) = \frac{1}{u_i}\Phi_1\Big[\sum_{i=1}^{m}\Phi_1^{-1}\{\lambda_{L,i}^{(1)}(u_{k_{i-1}+1}, \ldots, u_{k_i})\} + \sum_{j=k_m+1}^{k}\Phi_1^{-1}(u_j)\Big]$$

$$\lambda_U^{(1)}(u_1, \ldots, u_k|u_i) = \frac{1}{u_i}\sum_{s\in\mathcal{K}, s\neq\emptyset}(-1)^{|s|+1}\Phi_2\Big\{\sum_{i=1}^{m_s}\Phi_2^{-1}(\beta_{si}) + \sum_{i\in s\setminus\cup_{j=1}^{m_s}s_j}\Phi_2^{-1}(u_j)\Big\},$$

where β_{si}'s denote the limits $\lim_{u\to 0^+}\frac{1-C_{s_i}(1-u_j u, j\in s_i\subset s)}{u}$ for the first-level subcopulas of

$$C_s\{C_{s_1}(u_i, i \in s_1), \ldots, C_{s_{m_s}}(u_i, i \in s_{m_s}), u_{|s|_{m_s}+1}, \ldots, u_{|s|}\}$$

with $|s|_{m_s} = \sum_{i=1}^{m_s}|s_i|$.

The last theorem gives a recursive tool for determining the tail index of a HAC with arbitrary structure. Next we consider several special simplifying examples. First, assume that $u_1 = \cdots = u_k = 1$. Thus we consider the same threshold for each variable. In this case the upper and lower tail indices are given by

$$\begin{aligned}
\lambda_L^{(1)}(1, \ldots, 1) &= \lim_{u\to 0^+} u^{-1}P(X_i < F_i^{-1}(u) \text{ for all } i = 1, \ldots, k) \\
&= \Phi_1\left\{\sum_{i=1}^{m}\Phi_1^{-1}(\lambda_{L,i}^{(1)}) + k - k_m\right\} \\
\lambda_U^{(1)}(1, \ldots, 1) &= \lim_{u\to 0^+} u^{-1}P(X_i > F_i^{-1}(1-u) \text{ for all } i = 1, \ldots, k) \\
&= \sum_{s\in\mathcal{S}, s\neq\emptyset}(-1)^{|s|+1}\Phi_2\Big\{\sum_{i=1}^{m_s}\Phi_2^{-1}(\beta_{si}) + |s| - |s|_{m_s}\Big\},
\end{aligned}$$

Second, let C be a k-dimensional Archimedean copula with the generator function ϕ_0. Let λ_1 denote the tail index of ϕ_0 at infinity and λ_2 denotes the tail index of $1 - \phi_0$ at zero. Then

$$\begin{aligned}
\lambda_L^{(1)} &= \lim_{u\to 0^+}\frac{C(u, \ldots, u)}{u} = \lim_{u\to 0^+}\frac{\phi\{k\phi^{-1}(u)\}}{u} = \lim_{w\to\infty}\frac{\phi(kw)}{\phi(w)} = \Phi_1(k) = k^{-\lambda_1} \\
\lambda_U^{(1)} &= \lim_{u\to 0^+}\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\frac{1-\phi\{i\phi^{-1}(1-u)\}}{u} = \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\Phi_2(i) \\
&= \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}i^{-\lambda_2}.
\end{aligned}$$

Next we discuss the lower tail dependency index in more details. Let the generator function ϕ_0 be regularly varying at infinity, i.e. $\phi_0 \in \mathcal{U}_{\lambda_\infty}$ and $\Phi_1(x) = \lim_{t\to\infty}\frac{\phi_0(tx)}{\phi_0(t)} = x^{-\lambda_\infty}$. Note that $\Phi_1^{-1}(u_j) = u_j^{-1/\lambda_\infty} > 1$ for $u_j \in [0, 1]$. This implies that the argument of Φ_1 in the expression for $\lambda_L^{(1)}(u_1, \ldots, u_k|u_i)$ is larger than one. It follows that $\Phi_1(x) = x^{-\lambda_\infty} \in [0, 1]$ for $x > 1$ and,

therefore, $\lambda_L^{(1)}(u_1,\dots,u_k) \in [0,1]$. If $u_1 = \dots = u_k = 1$ the strength of the dependence increases as $\lambda_L^{(1)}(1,\dots,1)$ increases from zero to one.

The following two special cases deserve additional discussion. The weakest dependency (*lower tail independence*) is achieved if the tail index λ_∞ of the generator ϕ_0 at the top level is zero. Thus the generator function is a slowly varying function at infinity. This leads to $\Phi_1(x) \equiv 0$, $\Phi_1^{-1}(x) \equiv 1$ and $\lambda_L^{(1)}(1,\dots,1) = 0$. In the general case the tail independency implies that the probability of exceeding the threshold by all variables is independent on the crossing the threshold by the benchmark variable. The strongest dependence (*perfect dependence*) is obtained if λ_∞ tends to infinity and implies that the generator function is a strongly varying function with $\Phi_1(x) \equiv 1$ and $\Phi_1^{-1}(x) \equiv \infty$. In this case the tail index of the HAC is u_i and equals the probability that the i-th variable exceeds the threshold. This implies that the variables take extremely small values always simultaneously. Note that in these extreme cases the tail dependency index is independent on the generator functions and tails indices at lower level. It is completely characterised by the behaviour of the top-level generator.

For the upper tail dependency index the situation is slightly different. Let the generator function $1-\phi_0$ be regularly varying at zero. From the monotonicity properties of the generator function it follows that $\Phi_2(x) = \lim_{t\to 0^+} \frac{1-\phi_0(tx)}{1-\phi_0(t)} = x^{\lambda_0}$. This implies that $\Phi_2^{-1}(x) = x^{1/\lambda_0}$. Similarly as for the lower tail, the weakest dependence (*upper tail independence*) is achieved for $\lambda_0 = 1$, while the strongest (*perfect upper tail dependence*) is attained if λ_0 tends to zero.

Remark on non-strict generators
A generator ϕ_0 is *strict* if $\phi_0(\infty) = 0$. In this case we have a correctly specified inverse generator. If the generator is not strict, then there exists a constant c_1 such that $\phi_0(t) = 0$ for all $t > c_1$. Additionally there exists another constant c_2, such that $\phi_0^{-1}(0) = c_2$. This implies that $\Phi_1^{-1} \equiv 1$ and Φ_1 is not specified uniquely. However, if we recall that regularly varying function $\Phi_1^{-1}(x) = x^{-1/\lambda_\infty}$, then for non-strict generators the tail index equals $\lambda_\infty = 0$. This implies $\Phi_1 \equiv 0$. Thus a non-strict generator at the top level of the copula implies lower tail index equal to zero. This result is independent on the tail indices or strictness of the generators at lower levels. Note that it must hold that $\phi_0(0) = 1$ to guarantee the properties of the distribution function.

Now we consider a generalisation of the above definition of the multivariate tail dependency. $\lambda_U^{(2)}$ defines the probabilities that all random variables simultaneously exceed the thresholds u_i conditional on the fact that a subset S of the random variable exceeded them and similarly for $\lambda_L^{(2)}$.

$$
\begin{aligned}
&\lambda_U^{(2)}(u_1,\dots,u_k|u_i, i\in\mathcal{S}) \\
&= \lim_{u\to 0^+} P\{X_i > F_i^{-1}(1-u_i u) \text{ for } i \notin \mathcal{S} \subset \mathcal{K} = \{1,\dots,k\} = \mid X_j > F_j^{-1}(1-u_j u) \text{ for } j \in \mathcal{S}\} \\
&= \lim_{u\to 0^+} \frac{\overline{C}(\overbrace{1-u_1 u,\dots,1-u_k u}^{|\mathcal{K}|})}{\overline{C}(\underbrace{1-u_{j_1} u,\dots,1-u_{j_{|\mathcal{S}|}} u}_{|\mathcal{S}|})} = \lim_{u\to 0^+} \frac{\sum_{s_1\in\mathcal{K}} (-1)^{|s_1|+1}\{1-C_{s_1}(1-u_j u, j\in s_1)\}}{\sum_{s_2\in\mathcal{S}} (-1)^{|s_2|+1}\{1-C_{s_2}(1-u_j u, j\in s_2)\}}
\end{aligned}
$$

Table 2: Functions Φ_1 and Φ_2 for all strict generator functions from (Nelsen, 2006), Table 4.1.

ϕ	$\Phi_1(x)$	$\Phi_2(x)$	ϕ	$\Phi_1(x)$	$\Phi_2(x)$
1	$x^{-1/\theta}$	x	12	$x^{-1/\theta}$	$x^{1/\theta}$
3	0	x	13	0	x
4	0	$x^{1/\theta}$	14	x^{-1}	$x^{2/\theta}$
5	0	x	16	x^{-1}	x
6	0	$x^{1/\theta}$	17	0	x
9	0	x	19	1	x
10	0	x	20	1	x

$$\begin{aligned}
&\lambda_L^{(2)}(u_1,\dots,u_k|u_i, i\in\mathcal{S})\\
&= \lim_{u\to 0^+} P\{X_i \le F_i^{-1}(u_i u) \text{ for } i\notin\mathcal{S}\subset\mathcal{K}=\{1,\dots,k\} \mid X_j \le F_j^{-1}(u_j u) \text{ for } j\in\mathcal{S}\}\\
&= \lim_{u\to 0^+} \frac{C(\overbrace{1-u_1u,\dots,1-u_ku}^{|\mathcal{K}|})}{C(\underbrace{1-u_{j_1}u,\dots,1-u_{j_{|\mathcal{S}|}}u}_{|\mathcal{S}|})}.
\end{aligned}$$

The next theorem readily follows from Theorem 10.

Theorem 11. *Let $X_1,\dots,X_k\sim C$ such that*

$$C(u_1,\dots,u_k)=C_0\{C_1(u_1,\dots,u_{k_1}),\dots,C_m(u_{k_{m-1}+1},\dots,u_{k_m}),u_{k_m+1},\dots,u_k\},$$

where C_0 is an Archimedean copula with the generator $\phi_0\in\mathcal{L}$ and C_i for $i=1,\dots,m$ are Archimedean or hierarchical Archimedean copulas. Let $\mathcal{S}$ be an arbitrary subset of the variables, such that $\lambda_U^{(1)}(u_j, j\in\mathcal{S})\not\equiv 0$ and $\lambda_L^{(1)}(u_j, j\in\mathcal{S})\not\equiv 0$. Then the upper and lower tail dependence indices for C are given by

$$\lambda_L^{(2)}(u_1,\dots,u_k|u_i,i\in\mathcal{S})=\frac{\lambda_L^{(1)}(u_1,\dots,u_k)}{\lambda_L^{(1)}(u_i,i\in\mathcal{S})} \text{ and } \lambda_U^{(2)}(u_1,\dots,u_k|u_i,i\in\mathcal{S})=\frac{\lambda_U^{(1)}(u_1,\dots,u_k)}{\lambda_U^{(1)}(u_i,i\in\mathcal{S})}.$$

Example 5. *Next we provide the expressions for the function Φ_1 and Φ_2 for different strict generator functions (see Nelsen (2006), Table 4.1). Note, that the generators provided in Nelsen (2006) correspond to ϕ^{-1} in our notation. By setting $k=2$ we obtain the results in Nelsen (2006), p. 215. The possible values of the functions Φ_1 and Φ_2 provide interesting insights into the tail dependencies of copulas. If $\Phi_i(x)$ is independent on the parameter θ then the tail index depends only on the structure and the parameters at different levels do not influence the strength of the dependency in the tails. Note that most of the generators imply lower and upper tail independency. This observation is particularly important for applications and shows the need for new generator families, which allow for tail dependencies.*

7 Appendix

Proof of Remark 1. We consider here two cases. First let u_i and u_j be on different subnodes of the first level of the copula

$$\begin{aligned}
C(u_1,\dots,u_i,\dots,u_j,\dots,u_k) &= C_{kr}\{\dots,C_{k_{\ell'}-k_{\ell'-1},r_{\ell'}}(u_{k_{\ell'-1}+1},\dots,u_i,\dots,u_{k_{\ell'}}),\dots,\\
&\quad \dots,C_{k_{\ell''}-k_{\ell''-1},r_{\ell''}}(u_{k_{\ell''-1}+1},\dots,u_j,\dots,u_{k_{\ell''}}),\dots\},
\end{aligned}$$

where r is the number of all nodes, $C_{k_1,r_1} \in \mathcal{F}_{k_1,r_1}, \ldots, C_{k_m-k_{m-1},r_{k_m}} \in \mathcal{F}_{k_m,r_m}$ the subcopulas on the first level and the root $C_{kr} \in \mathcal{F}_{m,1}$. From the properties of multivariate distributions the bivariate margin of u_i and u_j is given by

$$\begin{aligned}(u_i,u_j) &\sim C(1,\ldots,u_i,\ldots,u_j,\ldots,1)\\ &= C_{kr}\{\ldots,C_{k_{\ell'}-k_{\ell'-1},r_{\ell'}}(1,\ldots,u_i,\ldots,1),\ldots,C_{k_{\ell''}-k_{\ell''-1},r_{\ell''}}(1,\ldots,u_j,\ldots,1),\ldots\}.\end{aligned}$$

Since $C(1,\ldots,1,u,1,\ldots,1)=u$ and $C(1,\ldots,1)=1$ it follows that

$$\begin{aligned}(u_i,u_j) &\sim C(1,\ldots,u_i,\ldots,u_j,\ldots,1) = C_{kr}(1,\ldots,u_i,\ldots,u_j,\ldots,1)\\ &= \phi\{\phi^{-1}(1)+\ldots+\phi^{-1}(u_i)+\ldots+\phi^{-1}(u_j)+\ldots+\phi^{-1}(1)\}\\ &= \phi\{\phi_n^{-1}(u_i)+\phi^{-1}(u_j)\} = C_{n,2}(u_i,u_j)\end{aligned}$$

where $C_{k2} \in \mathcal{F}_{2,1}$ and $C_{k2} \in \mathfrak{C}\{\mathcal{N}(C)\}$ by construction. Thus we showed, that if two variables lie in different subcopulas of the top level, their bivariate distribution is given by the top level copula. On the other case if u_i and u_j be on the different subnodes of the second level in the copula then

$$\begin{aligned}C(u_1,\ldots,u_i,\ldots,u_j,\ldots,u_k) &= C_{kr}\{\ldots,C_{k_{\ell'}-k_{\ell'-1},r_{\ell'}}(u_{k_{\ell'-1}+1},\ldots,u_i,\ldots,u_j,\ldots,u_{k_{\ell'}}),\ldots,\\ &\quad \ldots,C_{k_{\ell''}-k_{\ell''-1},r_{\ell''}}(u_{k_{\ell''-1}+1},\ldots,\ldots,u_{k_{\ell''}}),\ldots\}.\end{aligned}$$

Proceeding with the copula $C_{k_{\ell'}-k_{\ell'-1},r_{\ell'}}$ with generator ϕ_2 in the same way as with the original copula C in the first part of Remark, we obtain that $(u_i,u_j) \sim \phi_2\{\phi_2^{-1}(u_i)+\phi_2^{-1}(u_j)\}$. Continuing the recursion we complete the proof. □

Proof of Remark 2. The proof is similar to the proof of Remark 1. Let us fix the bivariate copula $C_2^* \in \mathfrak{C}_2\{\mathcal{N}(C)\}$. Without loss of generality assume that the generator $\phi = \mathcal{N}(C_2^*)$ is used to construct the subcopula at the second level of the original copula C. We reorder the variables for simplicity. Then

$$\begin{aligned}C(u_1,\ldots,u_k) &= C_{kr}\{\ldots,C_2^*(u_{1^*},\ldots,u_{m^*}),\ldots\}\\ &= C_{kr}[\ldots,\phi_1\{\phi^{-1}\circ\widetilde{C}_1(\cdot),\ldots,\phi^{-1}\circ\widetilde{C}_p(\cdot)\},\ldots].\end{aligned}$$

Now we proceed as in Remark 1 by taking two variables from different subcopulas of the second level of C_2^*. Without loss of generality we take one variable u from copula $\widetilde{C}_2$ and another v from $\widetilde{C}_p$. This shows that there exist a pair of random variables (u,v) with the joint bivariate distribution function given by $C_2^*(u,v)$. □

Proof of Theorem 2. Let $t \in [0,1]$ be fixed.
a) Then it holds that $\{V_1 \le t\} \cap \{U_1 \le t\} = \{U_1 \le t\}$ since $U_1 \le t$ is a subset of $V_1 \le t$. Moreover,

$$\{V_1 \le t\} \cap \{U_1 > t\} = \{V_2 \le g_t(U_1)\} \cap \{U_1 > t\}$$

with $g_t(x) = \phi(\phi^{-1}(t) - \phi^{-1}(x))$ for $x \in [t,1]$. The function $g_t(.) : [t,1] \to [0,1]$ is strictly decreasing with $g_t(t) = 1$.
Consequently it follows that

$$\begin{aligned} P(V_1 \le t) &= P(V_1 \le t \cap U_1 \le t) + P(V_1 \le t \cap U_1 > t) \quad (5) \\ &= t + P\{V_2 \le g_t(U_1) \cap U_1 > t\} \quad (6) \end{aligned}$$

because $P(U_1 \le t) = t$.
b) In order to calculate the second quantity of (6) we consider a partition $t = t_0 < t_1 < .. < t_N = 1$ of the interval $[t,1]$. Then it holds that

$$P\{V_2 \le g_t(U_1) \cap U_1 > t\} = \sum_{i=1}^{N} P\{V_2 \le g_t(U_1) \cap t_{i-1} < U_1 \le t_i\}$$

$$\begin{cases} \le & \sum_{i=1}^{N} P(V_2 \le g_t(t_{i-1}) \cap t_{i-1} < U_1 \le t_i) \\ \ge & \sum_{i=1}^{N} P(V_2 \le g_t(t_i) \cap t_{i-1} < U_1 \le t_i) \end{cases}$$

since $g_t(t_i) \le g_t(U_1) \le g_t(t_{i-1})$ if $t_{i-1} < U_1 \le t_i$.
c) First we consider the upper bound. We get that

$$\begin{aligned} P\{V_2 \le g_t(t_{i-1}) \cap t_{i-1} < U_1 \le t_i\} &= P\{U_1 \le t_i \cap V_2 \le g_t(t_{i-1})\} - \\ &- P\{U_1 \le t_{i-1} \cap V_2 \le g_t(t_{i-1})\} \\ &= C_1[t_i, K_2\{g_t(t_{i-1})\}] - C_1[t_{i-1}, K_2\{g_t(t_{i-1})\}] \\ &= \phi\big(\phi^{-1}(t_i) + \phi^{-1}[K_2\{g_t(t_{i-1})\}]\big) - \\ &- \phi\big(\phi^{-1}(t_{i-1}) + \phi^{-1}[K_2\{g_t(t_{i-1})\}]\big). \end{aligned}$$

Now we determine the partition by choosing $t_{N-i} = \phi(iw/N)$ for $i = 0, .., N$ with $w = \phi^{-1}(t)$. This choice fulfills the requirement that $t = t_0 < t_1 < .. < t_N \le 1$. Moreover, $g_t(t_i) = t_{N-i}$. With the notation $\zeta(x) = \phi^{-1}[K_2\{\phi(x)\}] - x$ there exists such $\xi_{N,i}$ that

$$\begin{aligned} P\{V_2 \le g_t(t_{i-1}) \cap t_{i-1} < U_1 \le t_i\} &= \phi\left\{w - \frac{w}{N} + \zeta\left(\frac{i-1}{N}w\right)\right\} - \phi\left\{w + \zeta\left(\frac{i-1}{N}w\right)\right\} \\ &= -\frac{w}{N}\,\phi'\left\{w + \zeta\left(\frac{i-1}{N}w\right) - \xi_{N,i}\right\} \end{aligned}$$

with $0 \le \xi_{N,i} \le w/N$. Now let $\delta > 0$. Since ϕ is strictly decreasing it holds for $N \ge N_0$ that

$$P\{V_2 \le g_t(t_{i-1}) \cap t_{i-1} < U_1 \le t_i\} \le -\frac{w}{N}\,\phi'\left\{w - \delta + \zeta\left(\frac{i-1}{N}w\right)\right\}.$$

Because

$$\lim_{N\to\infty} \sum_{i=1}^{N} -\frac{w}{N}\,\phi'\left\{w - \delta + \zeta\left(\frac{i-1}{N}w\right)\right\} = -\int_0^w \phi'\{w - \delta + \zeta(x)\}dx$$

it follows that

$$P(V_1 \le t \cap U_1 > t) \le \liminf_{\delta\to 0}\left[-\int_0^w \phi'\{w - \delta + \zeta(x)\}dx\right] = -\int_0^w \phi'\{w + \zeta(x)\}dx.$$

d) Next we consider the lower bound. We obtain by analogy to c) and with $\xi_{N,i}$ as above that

$$\begin{aligned} P\{V_2 \le g_t(t_i) \cap t_{i-1} < U_1 \le t_i\} &= -\frac{w}{N}\,\phi'\left\{w + \zeta\left(\frac{i}{N}w\right) - \xi_{N,i}\right\} \\ &\ge -\frac{w}{N}\,\phi'\left\{w + \zeta\left(\frac{i}{N}w\right)\right\}. \end{aligned}$$

Consequently

$$P(V_1 \le t \cap U_1 > t) \ge -\int_0^w \phi'\{w + \zeta(x)\}dx.$$

Because the upper and the lower bound are the same this completes the proof. □

Proof of Theorem 3. The proof is based on a similar argumentation as the proof of Theorem 2. Using the above arguments we get that

$$P(V_3 \le t) = K_4(t) + P(V_3 \le t \cap V_4 > t)$$

and

$$P(V_3 \le t \cap V_4 > t) \begin{cases} \le & \sum_{i=1}^N P\{V_5 \le g_t(t_{i-1}) \cap t_{i-1} < V_4 \le t_i\} \\ \ge & \sum_{i=1}^N P\{V_5 \le g_t(t_i) \cap t_{i-1} < V_4 \le t_i\} \end{cases}.$$

a) Moreover,

$$\begin{aligned} P\{V_5 \le g_t(t_{i-1}) \cap t_{i-1} < V_4 \le t_i\} &= P\{V_4 \le t_i \cap V_5 \le g_t(t_{i-1})\} - \\ &- P\{V_4 \le t_{i-1} \cap V_5 \le g_t(t_{i-1})\} \\ &= C_3[K_4(t_i), K_5\{g_t(t_{i-1})\}] - C_3[K_4(t_{i-1}), K_5\{g_t(t_{i-1})\}] \\ &= \phi\big(\phi^{-1}\{K_4(t_i)\} + \phi^{-1}[K_5\{g_t(t_{i-1})\}]\big) - \\ &- \phi\big(\phi^{-1}\{K_4(t_{i-1})\} + \phi^{-1}[K_5\{g_t(t_{i-1})\}]\big) \\ &= \phi\left\{\zeta_4\left(w - \frac{i}{N}w\right) + \zeta_5\left(\frac{i-1}{N}w\right)\right\} - \\ &- \phi\left\{\zeta_4\left(w - \frac{i-1}{N}w\right) + \zeta_5\left(\frac{i-1}{N}w\right)\right\} \end{aligned}$$

with $\zeta_4(x) = \phi^{-1}[K_4\{\phi(x)\}]$ and $\zeta_5(x) = \phi^{-1}[K_5\{\phi(x)\}]$. Then

$$\begin{aligned} P\{V_5 \le g_t(t_{i-1}) \cap t_{i-1} < V_4 \le t_i\} &= \left\{\zeta_4\left(w - \frac{i}{N}w\right) - \zeta_4\left(w - \frac{i-1}{N}w\right)\right\} \\ &\times \phi'\left\{\zeta_5\left(\frac{i-1}{N}w\right) + \zeta_4\left(w - \tilde{\xi}_{N,i}w\right)\right\} \end{aligned}$$

with $(i-1)/N \le \tilde{\xi}_{N,i} \le i/N$. Because $\zeta_5\{w(i-1)/N\} \le \zeta_5(w\tilde{\xi}_{N,i})$ it follows that

$$\begin{aligned} \sum_{i=1}^N P\{V_5 \le g_t(t_{i-1}) \cap t_{i-1} < V_4 \le t_i\} &\le \sum_{i=1}^N \left\{\zeta_4\left(w - \frac{i}{N}w\right) - \zeta_4\left(w - \frac{i-1}{N}w\right)\right\} \\ &\times \phi'\left\{\zeta_5(w\tilde{\xi}_{N,i}) + \zeta_4(w - w\tilde{\xi}_{N,i})\right\} \end{aligned}$$

$$\to -\int_0^1 \phi'\{\zeta_5(tw)+\zeta_4(w-tw)\}\, d\zeta_4(w-tw) = -\int_0^w \phi'\{\zeta_5(u)+\zeta_4(w-u)\}\, d\zeta_4(u).$$

b) For the lower bound we get

$$\begin{aligned} P\{V_5 \le g_t(t_i) \cap t_{i-1} < V_4 \le t_i\} &= \phi\left\{\zeta_4\left(w-\frac{i}{N}w\right)+\zeta_5\left(\frac{i}{N}w\right)\right\} - \\ &- \phi\left\{\zeta_4\left(w-\frac{i-1}{N}w\right)+\zeta_5\left(\frac{i}{N}w\right)\right\} \\ &= \left\{\zeta_4\left(w-\frac{i}{N}w\right)-\zeta_4\left(w-\frac{i-1}{N}w\right)\right\}\times \\ &\times \phi'\left\{\zeta_5\left(\frac{i}{N}w\right)+\zeta_4\left(w-w\xi^*_{N,i}\right)\right\} \end{aligned}$$

with $(i-1)/N \le \xi^*_{N,i} \le i/N$. Since $\zeta_5(w\xi^*_{i,N}) \le \zeta_5(wi/N)$ we obtain that

$$\begin{aligned} \sum_{i=1}^N P\{V_5 \le g_t(t_{i-1}) \cap t_{i-1} < V_4 \le t_i\} &\ge \sum_{i=1}^N \left\{\zeta_4\left(w-\frac{i}{N}w\right)-\zeta_4\left(w-\frac{i-1}{N}w\right)\right\}\times \\ &\times \phi'\left\{\zeta_5(w\xi^*_{N,i})+\zeta_4(w-w\xi^*_{N,i})\right\} \end{aligned}$$

and thus the result follows as in a). □

Proof of Theorem 4. We follow the idea of the proof of Theorem 2 of (Barbe et al., 1996). The copula is given by

$$\begin{aligned} C(u_1,\ldots,u_k) &= \phi_1\{\phi_k^{-1}(u_k)+\phi_1^{-1}\circ C_{k-1}(u_2,\ldots,u_k)\} \\ &= P\{F_1(X_1)\le u_1,\ldots,F_k(X_k)\le u_k\}. \end{aligned}$$

Since $\phi_1^{-1}(1)=0$ it holds that $C(1,u_2,\ldots,u_k)=C_2(u_2,\ldots,u_k)$. Differentiating C with respect to u_1 we get that

$$\frac{\partial C(u_1,\ldots,u_k)}{\partial u_1} = \frac{\phi_1'[\phi_1^{-1}(u_1)+\phi_1^{-1}\{C_2(u_2,\ldots,u_k)\}]}{\phi_1'\{\phi_1^{-1}(u_1)\}} > 0.$$

Next consider conditional copula $P\{F_1(X_1)\le u_1|F_2(X_2)\le u_2,\ldots,F_k(X_k)\le u_k\}$ which we denote as $C(u_1|u_2,\ldots,u_k)=\frac{C_k(u_1,\ldots,u_k)}{C_{k-1}(u_1,\ldots,u_{k-1})}$. Thus it follows that

$$\begin{aligned} P\{C(u_1,\ldots,u_k)\le t\} &= P\{C(u_1|u_2,\ldots,u_k)\cdot C_2(u_2,\ldots,u_k)\le t\} \\ &= P\left\{C(u_1|u_2,\ldots,u_k)\le \frac{t}{C_2(u_2,\ldots,u_k)}\right\}. \end{aligned}$$

Let us consider the following function

$$Q(t)=\inf\{u_1\in[0,1]: C(u_1|u_2,\ldots,u_k)\ge t\}$$

for $t \in [0,1]$. It follows as in the proof of Theorem 2 of Barbe et al. (1996) that

$$\begin{aligned} K_1(t) &= \int\limits_{(0,1)^{k-1}: C_2(u_2,\ldots,u_k)\geq t} \frac{\partial}{\partial t} Q\left\{\frac{t}{C_2(u_2,\ldots,u_k)}\right\} \\ &\times \; c_k\left[Q\left\{\frac{t}{C_2(u_2,\ldots,u_k)}\right\}, u_2,\ldots,u_k\right] du_2 \ldots du_k. \end{aligned}$$

Next we compute $\frac{\partial}{\partial t} Q\{\frac{t}{C_2(u_2,\ldots,u_k)}\}$. Note that by definition of Q it holds that

$$C\left[Q\left\{\frac{t}{C_2(u_2,\ldots,u_k)}\right\}, u_2,\ldots,u_k\right] = t.$$

Differentiation with respect to t leads to

$$1 = \left.\frac{\partial C(u_1,\ldots,u_k)}{\partial u_1}\right|_{u_1 = Q\{\frac{t}{C_2(u_2,\ldots,u_k)}\}} \cdot \frac{\partial}{\partial t} Q\left\{\frac{t}{C_2(u_2,\ldots,u_k)}\right\}.$$

Thus using the fact that C is a HAC with the generator ϕ at the highest level we obtain

$$\begin{aligned} \frac{\partial}{\partial t} Q\left\{\frac{t}{C_2(u_2,\ldots,u_k)}\right\} &= \left[\left.\frac{\partial C(u_1,\ldots,u_k)}{\partial u_1}\right|_{u_1 = Q\{\frac{t}{C_2(u_2,\ldots,u_k)}\}}\right]^{-1} \\ &= \left[\left.\frac{\phi_1'\{\phi_1^{-1}\circ C_2(u_2,\ldots,u_k) + \phi_1^{-1}(u_1)\}}{\phi_1'\{\phi_1^{-1}(u_1)\}}\right|_{u_1 = Q\{\frac{t}{C_2(u_2,\ldots,u_k)}\}}\right]^{-1} \\ &= \frac{\phi_1'\{\phi_1^{-1}(t)\}}{\phi_1'\left[\phi_1^{-1}\circ Q\left\{\frac{t}{C_2(u_2,\ldots,u_k)}\right\} + \phi_1^{-1}\circ C_2(u_2,\ldots,u_k)\right]} \\ &= \frac{\phi_1'\{\phi_1^{-1}(t)\}}{\phi_1'\circ\phi_1^{-1}\circ C\left[Q\left\{\frac{t}{C_{k-1}(u_1,\ldots,u_{k-1})}\right\}, u_2,\ldots,u_k\right]}. \end{aligned}$$

Using the following algebraic and probabilistic transformations

$$\begin{aligned} &Q_{u_2,\ldots,u_k}\left\{\frac{t}{C(1,u_2,\ldots,u_k)}\right\} = \\ &= \inf\left\{u_1\in[0,1] : C(u_1|u_2,\ldots,u_k) \geq \frac{t}{C(1,u_2,\ldots,u_k)}\right\} \\ &= \inf\{u_1\in[0,1] : P\{F_1(X_1)\leq u_1 \,|\, F_2(X_2)\leq u_2;\ldots;F_k(X_k)\leq u_k\} \\ &\times\; P\{F_2(X_2)\leq u_2;\ldots;F_k(X_k)\leq u_k\} \geq t\} \\ &= \inf\{u_1\in[0,1] : P\{F_1(X_1)\leq u_1;\ldots;F_k(X_k)\leq u_k\} \geq t\} \\ &= \inf\{u_1\in[0,1] : C(u_1,\ldots,u_k)\geq t\} \\ &= \inf\left\{u_1\in[0,1] : \phi_1\{\phi_1^{-1}(u_1) + \phi_1^{-1}\circ C_2(u_2,\ldots,u_k)\}\geq t\right\} \\ &= \inf\left\{u_1\in[0,1] : u_1 \geq \phi_1\{\phi_1^{-1}(t) - \phi_1^{-1}\circ C_2(u_2,\ldots,u_k)\}\right\} \\ &= \phi_1\{\phi_1^{-1}(t) - \phi_1^{-1}\circ C_2(u_2,\ldots,u_k)\} \end{aligned}$$

we get the final form of h_t

$$\begin{aligned} h_k(t,u_2,\ldots,u_k) &= \frac{\phi_1'\left\{\phi_1^{-1}(t) - \phi_1^{-1}\circ C_2(u_2,\ldots,u_k)\right\}}{\phi_1'\{\phi^{-1}(t)\}} \\ &\times\; c\left[\phi_1(\phi_1^{-1}(t) - \phi_1^{-1}\circ C_2(u_2,\ldots,u_k)), u_2,\ldots,u_k\right] \\ &\times\; \mathbf{I}\{C_2(u_2,\ldots,u_u) > t\} \quad \text{for } (u_1,\ldots,u_k)\in[0,1]^k. \end{aligned}$$

Further simplification of the previous formula is unfortunately too difficult because of unknown recursive formula for the HAC density, which is difficult to derive in general form. □

Proof of Theorem 5. Let $\mathbf{X} \sim C^1$ and $\mathbf{X}' \sim C^2$. To show the concordance property it is necessary to prove that

$$P\{X_i \leq x_i, i = 1, \ldots, k\} \leq P\{X_i' \leq x_i, i = 1, \ldots, k\} \tag{7}$$

and

$$P\{X_i > x_i, i = 1, \ldots, k\} \leq P\{X_i' > x_i, i = 1, \ldots, k\}. \tag{8}$$

for all $\mathbf{x} \in (0,1)^k$. The first inequality is identical to that $C^1(\mathbf{x}) \leq C^2(\mathbf{x})$, for all $\mathbf{x} \in (0,1)^k$, while the second one is equivalent to $\overline{C}^1(\mathbf{x}) \leq \overline{C}^2(\mathbf{x})$, for all $\mathbf{x} \in (0,1)^k$, where $\overline{C}$ is the survival copula of C.

As mentioned in Chapter 1, Archimedean copulas arise from the Laplace transform

$$\phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_k)\} = \int_0^\infty \prod_{i=1}^k G^\alpha_{\phi^{-1}}(u)\, dM_\phi(\alpha),$$

where the generator function $\phi(s) = \int_0^\infty e^{-sw}\, dM_\phi(w),\ s \geq 0$ is a Laplace transform of the some univariate cumulative distribution function $M_\phi(\cdot)$ of a positive random variable and $G_{\phi^{-1}}(u) = \exp\{-\phi^{-1}(u)\}$. From the statement of the theorem, C^1 and C^2 are proper HACs and they differ only by the generator function on the highest level such as $\phi_1^{-1} \circ \phi_2 \in \mathcal{L}^*$. If we denote the second level copulas as the $z_i,\ i = 1, \ldots, m$ then

$$\begin{aligned}
C^1(\mathbf{u}) &= C^1\{C_{k_1 r_1}(u_1, \ldots, u_{k_1}), \ldots, C_{k_m - k_{m-1}, r_m}(u_{k_{m-1}+1}, \ldots, u_{k_m = k})\} \\
&= \phi_1[\phi_1^{-1}\{C_{k_1 r_1}(u_1, \ldots, u_{k_1})\} + \cdots + \phi_1^{-1}\{C_{k_m - k_{m-1}, r_m}(u_{k_{m-1}+1}, \ldots, u_{k_m = k})\}] \\
&= \phi_1\{\phi_1^{-1}(z_1) + \cdots + \phi_1^{-1}(z_m)\} \\
C^2(\mathbf{u}) &= C^2\{C_{k_1 r_1}(u_1, \ldots, u_{k_1}), \ldots, C_{k_m - k_{m-1}, r_m}(u_{k_{m-1}+1}, \ldots, u_{k_m = k})\} \\
&= \phi_2[\phi_2^{-1}\{C_{k_1 r_1}(u_1, \ldots, u_{k_1})\} + \cdots + \phi_2^{-1}\{C_{k_m - k_{m-1}, r_m}(u_{k_{m-1}+1}, \ldots, u_{k_m = k})\}] \\
&= \phi_2\{\phi_2^{-1}(z_1) + \cdots + \phi_2^{-1}(z_m)\}
\end{aligned}$$

Let $\nu = \phi_1^{-1} \circ \phi_2 \in \mathcal{L}^*$, then from Theorem A.2 ((Joe, 1997)) $\chi_\alpha(u) = \exp\{-\alpha\nu(u)\}$ is the Laplace transform of some $M_\nu(\cdot; \alpha)$. This means that

$$\chi_\alpha(u) = \exp\{\alpha\phi_1^{-1} \circ \phi_2(u)\} = \int_0^\infty e^{-u\xi} dM_\nu(\xi, \alpha).$$

Similarly to the case of Archimedean copulas C^1 and C^2 ca be then transformed as follows

$$\begin{aligned}
C^1 &= \phi_1\{\phi_1^{-1}(z_1)+\cdots+\phi_1^{-1}(z_m)\} = \phi_1\{\nu\circ\phi_2^{-1}(z_1)+\cdots+\nu\circ\phi_2^{-1}(z_m)\} \\
&= \int_0^\infty e^{-\alpha\sum_{i=1}^m \nu\{\phi_2^{-1}(z_i)\}} dM_{\phi_1}(\alpha) = \int_0^\infty \prod_{i=1}^m \left[e^{-\alpha\nu\{\phi_2^{-1}(z_i)\}}\right] dM_{\phi_1}(\alpha) \\
&= \int_0^\infty \prod_{i=1}^m \left[\chi_\alpha\{\phi_2^{-1}(z_i)\}\right] dM_{\phi_1}(\alpha) = \int_0^\infty \prod_{i=1}^m \left[\int_0^\infty e^{-\gamma\phi_2^{-1}(z_i)}\, dM_\nu(\gamma,\alpha)\right] dM_{\phi_1}(\alpha) \\
&= \int_0^\infty \prod_{i=1}^m \left[\int_0^\infty G^\gamma_{\phi_2^{-1}}(z_i)\, dM_\nu(\gamma,\alpha)\right] dM_{\phi_1}(\alpha) = \phi_1\left\{-\frac{1}{\alpha}\log\prod_{i=1}^m\int_0^\infty G^\gamma_{\phi_2^{-1}}(z_i)\, dM_\nu(\gamma,\alpha)\right\} \\
C^2 &= \phi_2\{\phi_2^{-1}(z_1)+\cdots+\phi_2^{-1}(z_m)\} = \phi_1\circ\nu\{\phi_2^{-1}(z_1)+\cdots+\phi_2^{-1}(z_m)\} \\
&= \int_0^\infty \exp[-\alpha\nu\{\phi_2^{-1}(z_i)+\cdots+\phi_2^{-1}(z_m)\}] dM_{\phi_1}(\alpha) = \int_0^\infty \chi_\alpha\left\{\phi_2^{-1}(z_i)+\cdots+\phi_2^{-1}(z_m)\right\} dM_{\phi_1}(\alpha) \\
&= \int_0^\infty\int_0^\infty e^{-\gamma\sum_{i=1}^m \phi_2^{-1}(z_i)}\, dM_\nu(\gamma,\alpha) dM_{\phi_1}(\alpha) = \int_0^\infty\int_0^\infty \prod_{i=1}^m e^{-\gamma\phi_2^{-1}(z_i)}\, dM_\nu(\gamma,\alpha) dM_{\phi_1}(\alpha) \\
&= \int_0^\infty\int_0^\infty \prod_{i=1}^m G^\gamma_{\phi_2^{-1}}(z_i)\, dM_\nu(\gamma,\alpha) dM_{\phi_1}(\alpha) = \phi_1\left[-\frac{1}{\alpha}\log\int_0^\infty\prod_{i=1}^m G^\gamma_{\phi_2^{-1}}(z_i)\, dM_\nu(\gamma,\alpha)\right].
\end{aligned}$$

Note that $\phi_1\{-a\log(\cdot)\}$ is decreasing as a composition of continuous monotone decreasing functions. Since the concordance order is invariant under monotone transformations, to prove (7) it is sufficient to show that

$$\prod_{i=1}^n \int_0^\infty G^\gamma_{\phi_2^{-1}}(z_i)\, dM_\nu(\gamma,\alpha) \le \int_0^\infty \prod_{i=1}^n G^\gamma_{\phi_2^{-1}}(z_i)\, dM_\nu(\gamma,\alpha).$$

For simplicity and to emphasise the argument with respect to which we integrate, we write

$$\prod_{i=1}^n \int_0^\infty g_i(\gamma)\, dM_\nu(\gamma,\alpha) \le \int_0^\infty \prod_{i=1}^n g_i(\gamma)\, dM_\nu(\gamma,\alpha),$$

where $g_i(\gamma) = G^\gamma_{\phi_2^{-1}}(z_i)$ are bounded and decreasing functions in $\gamma \ge 0$ from the properties of $\exp\{\cdot\}$, while $G^\gamma_{\phi_2^{-1}}(z_i) = \exp\{-\gamma\phi_2^{-1}(z_i)\}$. To prove the inequalities we can use the same approach as in Joe (1997). Each bounded decreasing function can be represented as a limit of an infinite sum of a piecewise constant functions $\sum_j c_j I_{[0,b_j]}$ for positive constants c_j and b_j. As the both sides of the inequality are linear in each $g_i(\gamma)$ it is sufficient to prove the inequality for $g_i(\gamma) = I_{[0,y_j]}(\gamma)$, $j = 1,\ldots,k$. Let $B_1,\ldots,B_n \sim M_\nu(\cdot,\alpha)$ for some fixed α and are iid. By the Fréchet upper bound inequality holds that $P\{B_j \le y_j,\ j = 1,\ldots,n\} \le P\{B_1 \le \min_j y_j\} = \min_j P\{B_j \le y_j\}$ this proves the inequality (7). This means, that the copula C^1 is less positively lower orphant dependent than the copula C^2. To show the whole concordance order we have to prove that the copula C^1 is less positively upper orphant dependent than the copula C^2. Thus we need to prove the same inequality but for the survival functions.

The usual representation of the survival copula is given by

$$\overline{C}(\mathbf{u}) = 1 + \sum_{s\in\mathcal{S}} (-1)^{|s|} C_s(u_j;\, j\in s).$$

In terms of the Laplace transforms the survival copula differs from the copula by taking $H_{\phi^{-1}}(u) = 1 - G_{\phi^{-1}}(u) = 1 - \exp\{-\phi^{-1}(u)\}$ instead of the function $G_{\phi^{-1}}(u)$. Let us denote $H_{\phi^{-1},\gamma}(u) = 1 - G^{\gamma}_{\phi^{-1}}(u)$. Moreover, all $z_i,\ i = 1,\ldots,m$ are replaced by $\overline{z}_i,\ i = 1,\ldots,m$ which correspond to the respective components of the survival copula. For example if

$$\begin{aligned} z_1 &= \psi^{-1}\{C_{31}(u_1,u_2,u_3)\} \\ &= \psi^{-1}[C_{31}\{C_{21}(u_1,u_2),u_3\}] \\ &= \psi^{-1}\circ\phi[\phi^{-1}\circ\xi\{\xi^{-1}(u_1)+\xi^{-1}(u_2)\}+\phi^{-1}(u_3)] \\ G^{\eta}_{\phi^{-1}}(z_1) &= \int_0^{\infty}\int_0^{\infty} G^{\gamma}_{\xi^{-1}}(u_1)G^{\gamma}_{\xi^{-1}}(u_2)dM_{\phi^{-1}\circ\xi}(\gamma,\beta)G^{\beta}_{\phi^{-1}}(u_3)dM_{\psi^{-1}\circ\phi}(\beta,\eta), \end{aligned}$$

then the corresponding $\overline{z}_1$ is

$$\begin{aligned} H_{\phi^{-1},\eta}(\overline{z}_1) &= 1 - G^{\eta}_{\phi^{-1}}(\overline{z}_1) \\ &= \int_0^{\infty}\int_0^{\infty}\{1-G^{\gamma}_{\xi^{-1}}(u_1)\}\{1-G^{\gamma}_{\xi^{-1}}(u_2)\}dM_{\phi^{-1}\circ\xi}(\gamma,\beta)\{1-G^{\beta}_{\phi^{-1}}(u_3)\}dM_{\psi^{-1}\circ\phi}(\beta,\eta) \\ &= \int_0^{\infty}\int_0^{\infty} H_{\xi^{-1},\gamma}(u_1)H_{\xi^{-1},\gamma}(u_2)\}dM_{\phi^{-1}\circ\xi}(\gamma,\beta)H_{\phi^{-1},\beta}(u_3)\}dM_{\psi^{-1}\circ\phi}(\beta,\eta). \end{aligned}$$

Using similar transformation as in the case of positive lower orphant concordance, we have to prove the following inequality

$$\prod_{i=1}^{m}\int_0^{\infty} H_{\phi_2^{-1},\gamma}(\overline{z}_i)\,dM_{\nu}(\gamma,\alpha) \le \int_0^{\infty}\prod_{i=1}^{m} H_{\phi_2^{-1},\gamma}(\overline{z}_i)\,dM_{\nu}(\gamma,\alpha) \tag{9}$$

or

$$\prod_{i=1}^{m}\int_0^{\infty} h_i(\gamma)\,dM_{\nu}(\gamma,\alpha) \le \int_0^{\infty}\prod_{i=1}^{m} h_i(\gamma)\,dM_{\nu}(\gamma,\alpha),$$

where $h_i(\gamma) = H_{\phi_2^{-1},\gamma}(\overline{z}_i) = 1 - G_{\phi_2^{-1},\gamma}(\overline{z}_i)$ are bounded and increasing functions in $\gamma \ge 0$. Similarly as in case presented above, any increasing and bounded function can be approximated by the series $\sum_j c_j I_{[b_j,\infty)}$. It is sufficient to consider only one component of the sum. Similarly taking $B_1,\ldots,B_m \sim M_{\nu}(\cdot,\alpha)$ for some fixed α, it holds by the Fréchet upper bound inequality that $P\{B_j > y_j,\ j = 1,\ldots,m\} \le P\{B_1 > \max_j y_j\} = \min_j P\{B_j > y_j\}$. This proves the inequality (8) and completes the proof of $C^1 \prec C^2$. □

Proof of Theorem 8. Consider $\boldsymbol{X}_1 = (X_{11},\ldots,X_{1k})',\ldots,\boldsymbol{X}_n = (x_{n1},\ldots,x_{nk})'$ as a random sample from $C(u_1,\ldots,u_k)$, and then $M_1 = \max\{X_{i1}\},\ldots,M_k = \max\{X_{ik}\},$ follow the distribution

$$P\{M_1 \le x_1,\ldots,M_k \le x_k\} = C_n(x_1,\ldots,x_k),$$

where the copula C_n have the same structure as copula C but is based on the generator functions $\phi_{\ell n} = [\phi_{\ell}(t)]^n$, $\ell = 1,\ldots,r$, and inverse generator functions $\phi^{-1}_{\ell n} = \phi^{-1}_{\ell}(t^{1/n})$, $\ell = 1,\ldots,r$. For example in a three dimensional example with the copula function

$$C(u_1,u_2,u_3) \;=\; C_1\{C_2(u_1,u_2),u_3\} = \phi_1[\phi_1^{-1}\circ\phi_2\{\phi_2^{-1}(u_1)+\phi_2^{-1}(u_2)\}+\phi_1^{-1}(u_3)]$$

the extreme value copula is given by

$$\begin{aligned} P\{M_1 \le x_1,\, M_2 \le x_2,\, M_3 \le x_3\} &= \big(\phi_1[\phi_1^{-1} \circ \phi_2\{\phi_2^{-1}(x_1) + \phi_2^{-1}(x_2)\} + \phi_1^{-1}(x_3)]\big)^n \\ &= \phi_{1n}\big\{\phi_{1n}^{-1}\big([\phi_2\{\phi_2^{-1}(x_1) + \phi_2^{-1}(x_2)\}]^n\big) + \phi_{1n}^{-1}(x_3^n)\big\} \\ &= \phi_{1n}[\phi_{1n}^{-1} \circ \phi_{2n}\{\phi_{2n}^{-1}(x_1^n) + \phi_{2n}^{-1}(x_2^n)\} + \phi_{1n}^{-1}(x_3^n)] \\ &= C_n(x_1, x_2, x_3). \end{aligned}$$

Next step we have to prove the existence of the limit of $C_n(x_1, \ldots, x_k)$ when n tends to infinity. By mimicking Genest and Rivest (1989) this limit exist if and only if exists

$$\left.\frac{\partial[\phi_\ell^{-1}(t)/(\phi_\ell^{-1})'(t)]}{\partial t}\right|_{t=1}, \text{ where } \ell = 1, \ldots, k.$$

Taking into account that extreme-value distribution belong to its own domain of attraction we have

$$C_n(x_1, \ldots, x_k) = C(x_1, \ldots, x_k), \text{ for } 0 < x_1, \ldots, x_k < 1. \tag{10}$$

Let us fix some numbers $1 \le j_1 < j_2 \le k$ then, for $x_j = 1,\ j \in \{1, \ldots, k\} \setminus \{j_1, j_2\}$ copula $C(1, \ldots, 1, x_{j_1}, 1, \ldots, 1, x_{j_2}, 1, \ldots, 1)$ is a simple bivariate Archimedean copula with some generator function $\phi_{\ell_{12}}$, where $\ell_{12} \in \{1, \ldots, k\}$

$$C(1, \ldots, 1, x_{j_1}, 1, \ldots, 1, x_{j_2}, 1, \ldots, 1) = C_1(x_{j_1}, x_{j_2}) = \phi_{\ell_{12}}^{-1}\{\phi_{\ell_{12}}(x_{j_1}) + \phi_{\ell_{12}}(x_{j_2})\}$$

with the property $C_n(1, \ldots, 1, x_{j_1}, 1, \ldots, 1, x_{j_2}, 1, \ldots, 1) = C(1, \ldots, 1, x_{j_1}, 1, \ldots, 1, x_{j_2}, 1, \ldots, 1)$, from this implying Genest and Rivest (1989) we get that ϕ_1 is the Gumbel generator. Similarly by taking all other possible pairs $1 \le j_1 < j_2 \le k$ could be proved that $\phi_\ell,\ \ell \in \{1, \ldots, k\}$ are Gumbel generators. As given in the statement of the theorem that the extreme-value copula belong also to HAC family than by Proposition 1 we finish the proof, because all the bivariate margins are uniquely determined by the previous steps of the proof. □

Lemma 1. *If $f(x)$ is the continuous monotone function for $x \in \mathbb{R}$ and $f(x, n)$ is continuous with respect to both arguments and monotone with respect to the first argument function for $x, n \in \mathbb{R}$ such that*

$$f(x, n) \underset{n\to\infty}{\longrightarrow} f(x),\ \forall x \in \mathbb{R}$$

then the inverse of the function $f(x),\ \forall x \in \mathbb{R}$ exists and exists the inverse with respect to the second argument of the functions $f(x, n), \forall x, n \in \mathbb{R}$ and holds that

$$f^{-1}(x, n) \underset{n\to\infty}{\longrightarrow} f^{-1}(x),\ \forall x \in \mathbb{R}.$$

Proof. For the proof of Lemma see (Billingsley, 1995), (Resnick, 1998). □

Proof of Proposition 2. a. Let $G_t(x) = \frac{\phi(xt)}{\phi(t)}$, while $\phi(t)$ is decreasing then $G_t(x)$ is increasing for all t. Since $\lim_{t\to\infty} G_t(x) = \Phi_1(x)$ it follows from Lemma 1, that $\lim_{t\to\infty} G_t^{-1}(y) = \Phi_1^{-1}(y)$. Note that $G_t(x) = \phi(xt)/\phi(t) = y$ and $x = \phi^{-1}\{y\phi(t)\}/t = G_t^{-1}(y)$. Since $\lim_{t\to\infty} \phi(t) = 1$ it follows that

$$\Phi_1^{-1}(y) = \lim_{t\to\infty} G_t^{-1}(y) = \lim_{t\to\infty} \frac{\phi^{-1}\{y\phi(t)\}}{t} \overset{t:=\phi^{-1}(t)}{=} \lim_{t\to 0^+} \frac{\phi^{-1}(yt)}{\phi^{-1}(t)}.$$

b. Let $H_t(x) = \frac{1-\phi(xt)}{1-\phi(t)}$, while $\phi(t)$ is decreasing then $H_t(x)$ is also decreasing for all t. Since $\lim_{t\to 0^+} H_t(x) = \Phi_2(x)$ it follows from Lemma 1 (see Appendix), that $\lim_{t\to 0^+} H_t^{-1}(y) = \Phi_2^{-1}(y)$. Note that $H_t(x) = \{1-\phi(xt)\}/\{1-\phi(t)\} = y$ and $x = \phi^{-1}[1-y\{1-\phi(t)\}]/t = H_t^{-1}(y)$. Similar as above it follows that

$$\begin{aligned}\Phi_2^{-1}(y) &= \lim_{t\to 0^+} H_t^{-1}(y) = \lim_{t\to 0^+} \frac{\phi^{-1}[1-y\{1-\phi(t)\}]}{t} \\ &\overset{t:=\phi^{-1}(t)}{=} \lim_{t\to 1^-} \frac{\phi^{-1}\{1-y(1-t)\}}{\phi^{-1}(t)} \overset{t:=1-t}{=} \lim_{t\to 0^+} \frac{\phi^{-1}(1-yt)}{\phi^{-1}(1-t)}.\end{aligned}$$

□

Proof of Theorem 10. From the definition of $\lambda_L^{(1)}$ it follows that

$$\begin{aligned}\lambda_L^{(1)}(u_1,\ldots,u_k) &= \lim_{u\to 0^+} \frac{1}{u} C(u_1 u,\ldots,u_k u) \\ &= \lim_{u\to 0^+} \frac{1}{u} \phi_0\big[\phi_0^{-1}\{C_1(u_1 u,\ldots,u_{k_1} u)\} + \cdots + \phi_0^{-1}\{C_m(u_{k_{m-1}+1}u,\ldots,u_{k_m}u)\} \\ &\quad + \sum_{j=k_m+1}^{k} \phi_0^{-1}(u_j u)\big],\end{aligned}$$

We write $\phi_0^{-1}(C_j)$ for $\phi_0^{-1}\{C_j(u_{k_{j-1}+1}u,\ldots,u_{k_j}u)\}$ with $k_0 = 0$ to simplify the notation. Since $\lim_{u\to 0^+} C_j/u = \lambda_{L,j}^{(1)}(u_{k_{j-1}+1},\ldots,u_{k_j})$ we obtain

$$\begin{aligned}\lambda_L^{(1)}(u_1,\ldots,u_k) &= \lim_{u\to 0^+} \frac{1}{u} \phi_0\big\{\phi_0^{-1}(u\cdot C_1/u) + \cdots + \phi_0^{-1}(u\cdot C_m/u) + \sum_{j=k_m+1}^{k} \phi_0^{-1}(u_j u)\big\} \\ &= \lim_{u\to 0^+} \frac{1}{u} \phi_0\big\{\phi_0^{-1}(u\cdot \lambda_{L,1}^{(1)}) + \cdots + \phi_0^{-1}(u\cdot \lambda_{L,m}^{(1)}) + \sum_{j=k_m+1}^{k} \phi_0^{-1}(u_j u)\big\} \\ &= \lim_{u\to 0^+} \frac{1}{u} \phi_0\Big\{\sum_{i=1}^{m} \phi_0^{-1}(\lambda_{L,i}^{(1)}\cdot u) + \sum_{j=k_m+1}^{k} \phi_0^{-1}(u_j u)\Big\}.\end{aligned}$$

From Proposition 2 it follows that

$$\begin{aligned}\lambda_L^{(1)}(u_1,\ldots,u_k) &= \lim_{u\to 0^+} \frac{1}{u} \phi_0\Big\{\sum_{i=1}^{m} \frac{\phi_0^{-1}(\lambda_{L,i}^{(1)}\cdot u)}{\phi_0^{-1}(u)}\cdot \phi_0^{-1}(u) + \sum_{j=k_m+1}^{k} \frac{\phi_0^{-1}(u_j u)}{\phi^{-1}(u)}\cdot \phi^{-1}(u)\Big\} \\ &= \lim_{u\to 0^+} \frac{1}{u} \phi_0\Big[\Big\{\sum_{i=1}^{m} \Phi_1^{-1}(\lambda_{L,i}^{(1)}) + \sum_{j=k_m+1}^{k} \Phi_1^{-1}(u_j)\Big\}\phi_0^{-1}(u)\Big] \\ &\overset{u:=\phi_0^{-1}(u)}{=} \lim_{u\to\infty} \frac{1}{\phi_0(u)} \phi_0\Big[\Big\{\sum_{i=1}^{m} \Phi_1^{-1}(\lambda_{L,i}^{(1)}) + \sum_{j=k_m+1}^{k} \Phi_1^{-1}(u_j)\Big\}u\Big] \\ &= \Phi_1\Big[\sum_{i=1}^{m} \Phi_1^{-1}\{\lambda_{L,i}^{(1)}(u_{k_{i-1}+1},\ldots,u_{k_i})\} + \sum_{j=k_m+1}^{k} \Phi_1^{-1}(u_j)\Big].\end{aligned}$$

To prove the statement for $\lambda_U^{(1)}$ it is sufficient to derive the expression for a single β_s. Without

the loss of generality, we consider

$$\begin{aligned}
\beta &= \lim_{u\to 0^+}\frac{1}{u}\big[1-C_0\{C_1(1-u_1u,\ldots,1-u_{k_1}u),\ldots,C_m(1-u_{k_{m-1}+1}u,\ldots,1-u_{k_m}u),\\
&\qquad 1-u_{k_m+1}u,\ldots,1-u_ku\}\big]\\
&= \lim_{u\to 0^+}\frac{1}{u}\Big[1-\phi_0\Big\{\sum_{i=1}^{m}\phi_0^{-1}(C_i)+\sum_{j=k_m+1}^{k}\phi_0^{-1}(1-u_ju)\Big\}\Big]\\
&= \lim_{u\to 0^+}\frac{1}{u}\Big[1-\phi_0\Big\{\sum_{i=1}^{m}\phi_0^{-1}\left(1-\frac{1-C_i}{u}\cdot u\right)+\sum_{j=k_m+1}^{k}\phi_0^{-1}(1-u_ju)\Big\}\Big]\\
&= \lim_{u\to 0^+}\frac{1}{u}\Big[1-\phi_0\Big\{\sum_{i=1}^{m}\phi_0^{-1}(1-\beta_iu)+\sum_{j=k_m+1}^{k}\phi_0^{-1}(1-u_ju)\Big\}\Big]\\
&= \lim_{u\to 0^+}\frac{1}{u}\Big[1-\phi_0\Big\{\sum_{i=1}^{m}\frac{\phi_0^{-1}(1-\beta_iu)}{\phi_0^{-1}(1-u)}\phi_0^{-1}(1-u)+\sum_{j=k_m+1}^{k}\frac{\phi_0^{-1}(1-u_ju)}{\phi_0^{-1}(1-u)}\cdot\phi_0^{-1}(1-u)\Big\}\Big]\\
&= \lim_{u\to 0^+}\frac{1}{u}\Big(1-\phi_0\Big[\Big\{\sum_{i=1}^{m}\Phi_2^{-1}(\beta_i)+\sum_{j=k_m+1}^{k}\Phi_2^{-1}(u_j)\Big\}\phi_0^{-1}(1-u)\Big]\Big)\\
&\overset{u:=1-\phi_0(u)}{=} \Phi_2\Big[\sum_{i=1}^{m}\Phi_2^{-1}(\beta_i)+\sum_{j=k_m+1}^{k}\Phi_2^{-1}(u_j)\Big]
\end{aligned}$$

The expressions for β_i's are then derived in a recursive way. □

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