

Gapeev, Pavel V.

**Working Paper**

## Discounted optimal stopping for maxima in diffusion models with finite horizon

SFB 649 Discussion Paper, No. 2006,057

**Provided in Cooperation with:**

Collaborative Research Center 649: Economic Risk, Humboldt University Berlin

*Suggested Citation:* Gapeev, Pavel V. (2006) : Discounted optimal stopping for maxima in diffusion models with finite horizon, SFB 649 Discussion Paper, No. 2006,057, Humboldt University of Berlin, Collaborative Research Center 649 - Economic Risk, Berlin

This Version is available at:

<https://hdl.handle.net/10419/25140>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

# Discounted Optimal Stopping for Maxima in Diffusion Models with Finite Horizon

Pavel V. Gapeev\*



\* Weierstrass Institute for Applied Analysis and Stochastics,  
Berlin, Germany  
and  
Russian Academy of Sciences, Institute of Control Sciences,  
Moscow, Russia

This research was supported by the Deutsche  
Forschungsgemeinschaft through the SFB 649 "Economic Risk".

<http://sfb649.wiwi.hu-berlin.de>  
ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin  
Spandauer Straße 1, D-10178 Berlin



# Discounted optimal stopping for maxima in diffusion models with finite horizon<sup>\*</sup>

Pavel V. Gapeev

We present a solution to some discounted optimal stopping problem for the maximum of a geometric Brownian motion on a finite time interval. The method of proof is based on reducing the initial optimal stopping problem with the continuation region determined by an increasing continuous boundary surface to a parabolic free-boundary problem. Using the change-of-variable formula with local time on surfaces we show that the optimal boundary can be characterized as a unique solution of a nonlinear integral equation. The result can be interpreted as pricing American fixed-strike lookback option in a diffusion model with finite time horizon.

## 1. Introduction

The main aim of this paper is to develop the method proposed by Peskir [18]-[19] and apply its extension for solving optimal stopping problems for the maximum processes in diffusion models with finite time horizon. In order to demonstrate the action of this extension we consider the discounted optimal stopping problem (2.3) for the maximum associated with the geometric Brownian motion  $X$  defined in (2.1)-(2.2). This problem is related to the option pricing theory in mathematical finance, where the process  $X$  describes the price of a risky asset (e.g., a stock) on a financial market. In that case the value (2.3) can be formally interpreted as a fair price of an American fixed-strike lookback option in the Black-Scholes model with finite horizon. In the infinite horizon case the problem (2.3) was solved by Pedersen [14] and Guo and Shepp [9].

Observe that when  $K = 0$  and  $T = \infty$  the problem (2.3) turns into the Russian option problem with infinite horizon introduced and explicitly solved by Shepp and Shiryaev [22] by means of reducing the initial problem to an optimal stopping problem for a two-dimensional Markov process and solving the latter problem using the smooth-fit and normal-reflection conditions. It was further observed in [23] that the change-of-measure theorem allows to reduce the Russian option problem to a one-dimensional optimal stopping problem that explained the simplicity of the solution in [22]. Building on the optimal stopping analysis of Shepp and Shiryaev

---

<sup>\*</sup> This research was supported by Deutsche Forschungsgemeinschaft through the SFB 649 Economic Risk.

*Mathematics Subject Classification 2000:* Primary 60G40, 35R35, 45G10. Secondary 60J60, 91B28.

*Journal of Economic Literature Classification:* G13.

*Key words and phrases:* Discounted optimal stopping problem, finite horizon, geometric Brownian motion, maximum process, parabolic free-boundary problem, smooth fit, normal reflection, a nonlinear Volterra integral equation of the second kind, boundary surface, a change-of-variable formula with local time on surfaces, American lookback option problem.

[22]-[23], Duffie and Harrison [2] derived a rational economic value for the Russian option and then extended their arbitrage arguments to perpetual lookback options. More recently, Shepp, Shiryaev and Sulem [24] proposed a barrier version of the Russian option where the decision about stopping should be taken before the price process reaches a 'dangerous' positive level. Peskir [19] presented a solution to the Russian option problem in the finite horizon case (see also [3] for a numeric algorithm for solving the corresponding free-boundary problem and [5] for a study of asymptotic behavior of the optimal stopping boundary near expiration).

It is known that optimal stopping problems for Markov processes with finite horizon are inherently two-dimensional and thus analytically more difficult than those with infinite horizon. A standard approach for handling such a problem is to formulate a free-boundary problem for the (parabolic) operator associated with the (continuous) Markov process (see, e.g., [12], [8], [27], [10], [13]). Since solutions to such free-boundary problems are rarely known explicitly, the question often reduces to prove the existence and uniqueness of a solution to the free-boundary problem, which then leads to the optimal stopping boundary and the value function of the optimal stopping problem. In some cases, the optimal stopping boundary has been characterized as a unique solution of the system of (at least) countably many nonlinear integral equations (see, e.g., [10; Theorem 4.3]). Peskir [18] rigorously proved that only one equation from such a system may be sufficient to characterize the optimal stopping boundary uniquely (see also [15], [19], [7]-[6], [20] for more complicated two-dimensional optimal stopping problems).

In contrast to the finite horizon Russian option problem [19], the problem (2.3) is necessarily *three-dimensional* in the sense that it cannot be reduced to a two-dimensional optimal stopping problem. The main feature of the present paper is that we develop the method of proof proposed in [18]-[19] in order to apply its extension for the derivation of a solution to some *three-dimensional* optimal stopping problem. The proposed extension of the method should correspondingly also work for other optimal stopping problems for the maximum processes and in more general diffusion models with finite time horizon. The key argument in the proof will be the application of the change-of-variable formula with local time on surfaces, which was recently derived in [17].

The paper is organized as follows. In Section 2, for the initial problem (2.3) we construct an equivalent optimal stopping problem for a three-dimensional Markov process and show that the continuation region for the price process is determined by a continuous increasing boundary surface depending on the running maximum process. In order to find analytic expressions for the boundary, we formulate an equivalent parabolic free-boundary problem. In Section 3, we derive a nonlinear Volterra integral equation of the second kind, which also leads to the explicit formula for the value function in terms of the optimal stopping boundary. Using the change-of-variable formula from [17], we show that this equation is sufficient to determine the optimal boundary uniquely. The main result of the paper is stated in Theorem 3.1.

## 2. Preliminaries

In this section, we introduce the setting and notation of the three-dimensional optimal stopping problem, which is related to the American fixed-strike lookback option problem with finite time horizon, describe the structure of the continuation and stopping regions, and formulate the corresponding free-boundary problem. For this, we follow the schema of arguments from

[19] and [7]-[6].

2.1. For a precise formulation of the problem, let us consider a probability space  $(\Omega, \mathcal{F}, P)$  with a standard Brownian motion  $B = (B_t)_{0 \leq t \leq T}$  started at zero. Suppose that there exist a process  $X = (X_t)_{0 \leq t \leq T}$  given by:

$$X_t = x \exp \left( (r - \sigma^2/2) t + \sigma B_t \right) \quad (2.1)$$

and hence solving the stochastic differential equation:

$$dX_t = rX_t dt + \sigma X_t dB_t \quad (X_0 = x) \quad (2.2)$$

where  $x > 0$  is given and fixed. It can be assumed that the process  $X$  describes a stock price on a financial market, where  $r > 0$  is the interest rate and  $\sigma > 0$  is the volatility coefficient. The main purpose of the present paper is to derive a solution to the optimal stopping problem:

$$V = \sup_{0 \leq \tau \leq T} E \left[ e^{-\lambda \tau} \left( \max_{0 \leq t \leq \tau} X_t - K \right)^+ \right] \quad (2.3)$$

where the supremum is taken over all stopping times of the process  $X$  whose natural filtration coincides with the same of the Brownian motion  $B$ . The value (2.3) coincides with the *arbitrage-free price* of the *fixed-strike lookback option* of American type with the strike (exercise) price  $K > 0$  and  $\lambda = r + \nu \geq r > 0$  being a sum of the interest rate  $r > 0$  and the discounting rate  $\nu \geq 0$  (see, e.g., [11] or [26]).

2.2. In order to solve the problem (2.3), let us consider the extended optimal stopping problem for the Markov process  $(t, X_t, S_t)_{0 \leq t \leq T}$  given by:

$$V(t, x, s) = \sup_{0 \leq \tau \leq T-t} E_{t,x,s} [e^{-\lambda \tau} G(S_{t+\tau})] \quad (2.4)$$

where  $S = (S_t)_{0 \leq t \leq T}$  is the associated with  $X$  *maximum* process defined by:

$$S_t = s \vee \left( \max_{0 \leq u \leq t} X_u \right) \quad (2.5)$$

and  $P_{t,x,s}$  is a probability measure under which the (two-dimensional) process  $(X_{t+u}, S_{t+u})_{0 \leq u \leq T-t}$  defined in (2.1)-(2.2) and (2.5) starts at  $(x, s) \in E$ , the supremum in (2.4) is taken over all stopping times  $\tau$  of  $(X_{t+u})_{0 \leq u \leq T-t}$ , and we set  $G(s) = (s - K)^+$  for  $s > 0$ . Here by  $E = \{(x, s) \in \mathbb{R}^2 \mid 0 < x \leq s\}$  we denote the state space of the Markov process  $(X_{t+u}, S_{t+u})_{0 \leq u \leq T-t}$ . Since  $G$  is continuous on  $[K, \infty)$  and  $E_{t,x,s}[S_T]$  is finite, it is possible to apply a version of Theorem 3 in [25; page 127] for a finite time horizon and by statement (2) of that theorem conclude that an optimal stopping time exists in (2.4).

2.3. Let us first determine the structure of the optimal stopping time in the problem (2.4).

(i) Applying the arguments from [1; Subsection 3.2] and [16; Proposition 2.1] to the optimal stopping problem (2.4), we see that it is never optimal to stop when  $X_{t+u} = S_{t+u}$  for  $0 \leq u < T-t$ . It follows directly from the structure of (2.4) that it is never optimal to stop when  $S_{t+u} \leq K$  for  $0 \leq u < T-t$ . In other words, this shows that all points  $(t, x, s)$  from the set:

$$C_1 = \{(t, x, s) \in [0, T] \times E \mid 0 < x \leq s \leq K\} \quad (2.6)$$

and from the diagonal  $\{(t, x, s) \in [0, T] \times E \mid x = s\}$  belong to the continuation region:

$$C = \{(t, x, s) \in [0, T] \times E \mid V(t, x, s) > G(s)\}. \quad (2.7)$$

(Below we will show that  $V$  is continuous, so that  $C$  is open.)

(ii) Let us fix  $(t, x, s) \in C$  and let  $\tau_* = \tau_*(t, x, s)$  denote the optimal stopping time in (2.4). Then, taking some point  $(t, y, s)$  such that  $0 < x < y \leq s$ , by virtue of the structure of the optimal stopping problem (2.4) and (2.5) with (2.1), we get:

$$V(t, y, s) \geq E_{t,y,s}[e^{-\lambda\tau_*}G(S_{t+\tau_*})] \geq E_{t,x,s}[e^{-\lambda\tau_*}G(S_{t+\tau_*})] = V(t, x, s) > G(s). \quad (2.8)$$

Moreover, we recall that in the case of infinite horizon the stopping time  $\tau_* = \{t \geq 0 \mid X_t \leq g_*(S_t)\}$  is optimal in the problem (2.3) and  $0 < g_*(s) < s$  for  $s > K$  is uniquely determined from the equation (2.19) in [14], and thus, we see that all points  $(t, x, s)$  for  $0 \leq t \leq T$  and  $s > K$  with  $0 < x \leq g_*(s)$  belong to the stopping region. These arguments together with the comments in [1; Subsection 3.3] and [16; Subsection 3.3] as well as the fact that  $x \mapsto V(t, x, s)$  is convex on  $\langle 0, s \rangle$  for  $s > 0$  (see Subsection 2.4 below) show that there exists a function  $g$  satisfying  $g_*(s) \leq g(t, s) \leq s$  for  $0 \leq t \leq T$  with  $s > K$  such that the continuation region (2.7) is an open set consisting of (2.6) and of the set:

$$C_2 = \{(t, x, s) \in [0, T] \times E \mid g(t, s) < x \leq s, s > K\} \quad (2.9)$$

while the stopping region is the closure of the set:

$$D = \{(t, x, s) \in [0, T] \times E \mid 0 < x < g(t, s), s > K\} \quad (2.10)$$

with all points  $(T, x, s)$  for  $(x, s) \in E$ .

(iii) Since the problem (2.4) is time-homogeneous, in the sense that the function  $G$  does not depend on time, it follows that  $t \mapsto V(t, x, s)$  is decreasing on  $[0, T]$ . Thus, if for given  $(t, x, s) \in C_2$  we take  $t'$  such that  $0 \leq t' < t \leq T$ , then  $V(t', x, s) \geq V(t, x, s) > G(s)$ , so that  $(t', x, s) \in C_2$ . From this we may conclude in (2.9)-(2.10) that  $t \mapsto g(t, s)$  is increasing on  $[0, T]$  for each  $s > K$  fixed.

(iv) Since in (2.4) the function  $G(s) = (s - K)^+$  is linear in  $s$  for  $s > K$ , by means of standard arguments it is shown that  $s \mapsto V(t, x, s) - G(s)$  is decreasing on  $\langle K, \infty \rangle$ . Thus, if for given  $(t, x, s) \in C_2$  we take  $s'$  such that  $K < s' < s$ , then  $V(t, x, s') - G(s') \geq V(t, x, s) - G(s) > 0$ , so that  $(t, x, s') \in C_2$ . From this we may conclude in (2.9)-(2.10) that  $s \mapsto g(t, s)$  is increasing on  $\langle K, \infty \rangle$  for each  $t \in [0, T]$  fixed.

(v) Let us denote by  $W$  and  $a$  the value function and the boundary of the optimal stopping problem related to the corresponding Russian option problem with finite horizon. It is easily seen that under  $K = 0$  the functions  $W$  and  $a$  coincide with  $V$  and  $g$  from (2.4) and (2.9)-(2.10), respectively. It easily follows that  $a(t, s) = s/b(t)$  for all  $(t, s) \in [0, T] \times \langle K, \infty \rangle$ , where  $b$  is characterized as a unique solution of the nonlinear integral equation (3.4) in [19]. Suppose that  $g(t, s) > s/b(t)$  for some  $(t, s) \in [0, T] \times \langle K, \infty \rangle$ . Then for any  $x \in \langle s/b(t), g(t, s) \rangle$  given and fixed we have  $W(t, x, s) - K > s - K = V(t, x, s)$  contradicting the obvious fact that  $W(t, x, s) - K \leq V(t, x, s)$  for all  $(t, x, s) \in [0, T] \times E$  with  $s > K$ , as it is clearly seen from (2.4). Thus, we may conclude that  $g(t, s) \leq s/b(t) < s$  for all  $(t, s) \in [0, T] \times \langle K, \infty \rangle$ .

(vi) Let us finally observe that the value function  $V$  from (2.4) and the boundary  $g$  from (2.9)-(2.10) also depend on  $T$ , and let them denote here by  $V^T$  and  $g^T$ , respectively. Using the fact that  $T \mapsto V^T(t, x, s)$  is an increasing function on  $[t, \infty)$  and  $V^T(t, x, s) = G(s)$  for all  $(x, s) \in E$  such that  $0 < x \leq g(t, s)$ , we conclude that if  $T < T'$  then  $g^{T'}(t, s) \leq g^T(t, s) \leq s$  for all  $(t, x, s) \in [0, T] \times E$ . Letting in the last expression  $T'$  go to  $\infty$ , we get that  $g_*(s) \leq g^T(t, s) \leq s$ , where  $g_*(s) \equiv \lim_{T \rightarrow \infty} g^T(t, s)$  for all  $t \geq 0$ , and  $0 < g_*(s) < s$  for  $s > K$  is uniquely determined from the equation (2.19) in [14].

2.4. Let us now show that the value function  $(t, x, s) \mapsto V(t, x, s)$  is continuous on  $[0, T] \times E$ . For this it is enough to prove that:

$$s \mapsto V(t_0, x_0, s) \text{ is continuous at } s_0 \quad (2.11)$$

$$x \mapsto V(t_0, x, s) \text{ is continuous at } x_0 \text{ uniformly over } [s_0 - \delta, s_0 + \delta] \quad (2.12)$$

$$t \mapsto V(t, x, s) \text{ is continuous at } t_0 \text{ uniformly over } [x_0 - \varepsilon, x_0 + \varepsilon] \times [s_0 - \delta, s_0 + \delta] \quad (2.13)$$

for each  $(t_0, x_0, s_0) \in [0, T] \times E$  with some  $\varepsilon > 0$  and  $\delta > 0$  small enough (they may depend on  $(x_0, s_0)$ ). Since (2.11) follows by the fact that  $s \mapsto V(t, x, s)$  is convex on  $\langle 0, \infty \rangle$ , we only need to establish (2.12) and (2.13).

Observe that from the structure of (2.4) and (2.5) with (2.1) it immediately follows that:

$$x \mapsto V(t, x, s) \text{ is increasing and convex on } \langle 0, s \rangle \quad (2.14)$$

for each  $s > 0$  and  $0 \leq t \leq T$  fixed. Using the fact that  $\sup(f) - \sup(g) \leq \sup(f - g)$  and  $(y - K)^+ - (x - K)^+ \leq (y - x)^+$ , we get:

$$\begin{aligned} 0 &\leq V(t, y, s) - V(t, x, s) \\ &\leq \sup_{0 \leq \tau \leq T-t} E \left[ e^{-\lambda \tau} \left( s \vee \left( y \max_{0 \leq u \leq \tau} \frac{X_u}{x} \right) - s \vee \left( x \max_{0 \leq u \leq \tau} \frac{X_u}{x} \right) \right) \right]^+ \\ &\leq (y - x) \sup_{0 \leq \tau \leq T-t} E \left[ \frac{s}{x} \vee \left( \max_{0 \leq u \leq \tau} \frac{X_u}{x} \right) \right] \leq (y - x) E \left[ \frac{s}{x} \vee \left( \max_{0 \leq u \leq T} \frac{X_u}{x} \right) \right] \end{aligned} \quad (2.15)$$

for  $0 < x < y \leq s$  and all  $0 \leq t \leq T$ . Combining (2.15) with (2.14), we see that (2.12) follows.

It remains to establish (2.13). For this, let us fix arbitrary  $0 \leq t_1 < t_2 \leq T$  and  $(x, s) \in E$ , and let  $\tau_1 = \tau_*(t_1, x, s)$  denote the optimal stopping time for  $V(t_1, x, s)$ . Set  $\tau_2 = \tau_1 \wedge (T - t_2)$  and note since  $t \mapsto V(t, x, s)$  is decreasing on  $[0, T]$  and  $\tau_2 \leq \tau_1$  that we have:

$$\begin{aligned} 0 &\leq V(t_1, x, s) - V(t_2, x, s) \leq E[e^{-\lambda \tau_1} (S_{t_1+\tau_1} - K)^+] - E[e^{-\lambda \tau_2} (S_{t_2+\tau_2} - K)^+] \\ &\leq E[(S_{t_1+\tau_1} - K)^+ - (S_{t_2+\tau_2} - K)^+] \leq E[S_{t_1+\tau_1} - S_{t_2+\tau_2}]^+ = E[S_{\tau_1} - S_{\tau_2}]. \end{aligned} \quad (2.16)$$

Observe further that the explicit expression (2.5) yields:

$$S_{\tau_1} - S_{\tau_2} = s \vee \left( \max_{0 \leq u \leq \tau_1} X_u \right) - s \vee \left( \max_{0 \leq u \leq \tau_2} X_u \right) \quad (2.17)$$

and thus, the strong Markov property together with the fact that  $\tau_1 - \tau_2 \leq t_2 - t_1$  implies:

$$E \left[ s \vee \left( \max_{0 \leq u \leq \tau_1} X_u \right) - s \vee \left( \max_{0 \leq u \leq \tau_2} X_u \right) \right] \leq E \left[ \max_{\tau_2 \leq u \leq \tau_1} X_u \right] \leq E \left[ \frac{X_{\tau_2}}{x} \right] E \left[ \max_{0 \leq u \leq t_2-t_1} X_u \right]. \quad (2.18)$$

Hence, from (2.16)-(2.18) we get:

$$0 \leq V(t_1, x, s) - V(t_2, x, s) \leq E[S_{\tau_1} - S_{\tau_2}] \leq e^{rT} L(t_2 - t_1) \quad (2.19)$$

where the function  $L$  is defined by:

$$L(t_2 - t_1) = E \left[ \max_{0 \leq u \leq t_2 - t_1} X_u \right]. \quad (2.20)$$

Therefore, by virtue of the fact that  $L(t_2 - t_1) \rightarrow 0$  in (2.20) as  $t_2 - t_1 \downarrow 0$ , we easily conclude that (2.13) holds. In particular, this shows that the instantaneous-stopping condition (2.39) is satisfied.

2.5. In order to prove that the smooth-fit condition (2.40) holds, or equivalently,  $x \mapsto V(t, x, s)$  is  $C^1$  at  $g(t, s)$ , let us fix a point  $(t, x, s) \in [0, T] \times E$  with  $s > K$ , lying on the boundary  $g$  so that  $x = g(t, s)$ . By virtue of convexity of  $x \mapsto V(t, x, s)$  on  $\langle 0, s \rangle$  for  $s > 0$  fixed, the right-hand derivative  $V_x^+(t, x, s)$  exists, and since it is also increasing, we have:

$$V_x^+(t, x, s) \geq 0. \quad (2.21)$$

In order to prove the converse inequality, let us fix some  $\varepsilon > 0$  such that  $x < x + \varepsilon < s$  and consider the stopping time  $\tau_\varepsilon = \tau_*(t, x + \varepsilon, s)$  being optimal for  $V(t, x + \varepsilon, s)$ . Note that  $\tau_\varepsilon$  is the first exit time of the process  $(X_{t+u}^{x+\varepsilon})_{0 \leq u \leq T-t}$  from the set  $C$  in (2.9). Then (2.4) implies:

$$\begin{aligned} & \frac{V(t, x + \varepsilon, s) - V(t, x, s)}{\varepsilon} \\ & \leq \frac{1}{\varepsilon} E \left[ e^{-\lambda \tau_\varepsilon} \left( s \vee \left( \max_{0 \leq u \leq \tau_\varepsilon} X_{t+u}^{x+\varepsilon} \right) - K \right)^+ - e^{-\lambda \tau_\varepsilon} \left( s \vee \left( \max_{0 \leq u \leq \tau_\varepsilon} X_{t+u}^x \right) - K \right)^+ \right] \\ & \leq \frac{1}{\varepsilon} E \left[ e^{-\lambda \tau_\varepsilon} \left( s \vee \left( (x + \varepsilon) \max_{0 \leq u \leq \tau_\varepsilon} \frac{X_{t+u}^{x+\varepsilon}}{x + \varepsilon} \right) - s \vee \left( x \max_{0 \leq u \leq \tau_\varepsilon} \frac{X_{t+u}^x}{x} \right) \right)^+ \right] \end{aligned} \quad (2.22)$$

where we write  $X_t^x$  and  $X_t^{x+\varepsilon}$  instead of  $X_t$  in order to indicate the dependence of the process  $X$  on the starting points  $x$  and  $x + \varepsilon$ , respectively. Since the boundary  $g$  is increasing in both variables, it follows that  $\tau_\varepsilon \rightarrow 0$  ( $P$ -a.s.), so that  $\max_{0 \leq u \leq \tau_\varepsilon} X_{t+u}^x / x \rightarrow 1$  ( $P$ -a.s.) as  $\varepsilon \downarrow 0$  for  $x < x + \varepsilon < s$ . Thus, letting  $\varepsilon \downarrow 0$  in (2.22), we get:

$$V_x^+(t, x, s) \leq 0 \quad (2.23)$$

by the bounded convergence theorem. This combined with (2.21) above proves that  $V_x^+(t, x, s)$  equals zero.

2.6. Let us now show that the normal reflection condition (2.41) holds. For this, we first note that since  $s \mapsto V(t, x, s)$  is increasing (and convex) on  $\langle 0, \infty \rangle$ , it follows that  $V_s(t, s-, s) \geq 0$  for all  $(t, s) \in [0, T] \times \langle 0, \infty \rangle$ . Suppose that there exists  $(t, s) \in [0, T] \times \langle 0, \infty \rangle$  such that  $V_s(t, s-, s) > 0$ . Recalling that  $V$  is  $C^{1,2,1}$  in  $C$ , so that  $t \mapsto V_s(t, s-, s)$  is continuous on  $[0, T]$ , we see that there exist  $\varepsilon > 0$  and  $\delta > 0$  such that  $V_s(u, s-, s) \geq \varepsilon > 0$  for all  $u \in [t, t + \delta]$



with  $t + \delta < T$ . Setting  $\tau_\delta = \tau_* \wedge (t + \delta)$ , it follows by Itô's formula, using (2.36)-(2.37) and the optional sampling theorem (see, e.g., [21; Chapter II, Theorem 3.2]), that:

$$\begin{aligned} E_{t,s,s} [e^{-\lambda\tau_\delta} V(t + \tau_\delta, X_{t+\tau_\delta}, S_{t+\tau_\delta})] &= V(t, s, s) \\ &+ E_{t,s,s} \left[ \int_0^{\tau_\delta} e^{-\lambda u} (\mathbb{L}V - \lambda V)(t + u, X_{t+u}, S_{t+u}) du \right] \\ &+ E_{t,s,s} \left[ \int_0^{\tau_\delta} e^{-\lambda u} V_s(t + u, X_{t+u}, S_{t+u}) dS_{t+u} \right]. \end{aligned} \quad (2.24)$$

Since, by virtue of (2.38), the process  $(V(t + (u \wedge \tau_\delta), X_{t+(u \wedge \tau_\delta)}, S_{t+(u \wedge \tau_\delta)}))_{0 \leq u \leq T-t}$  is a martingale under  $P_{t,s,s}$ , from (2.24) it directly follows that:

$$E_{t,s,s} \left[ \int_0^{\tau_\delta} e^{-\lambda u} V_s(t + u, X_{t+u}, S_{t+u}) dS_{t+u} \right] = 0. \quad (2.25)$$

On the other hand, since it is easily seen that  $V_s(t+u, X_{t+u}, S_{t+u}) dS_{t+u} = V_s(t+u, S_{t+u}, S_{t+u}) dS_{t+u}$  and  $V_s(t+u, S_{t+u}, S_{t+u}) \geq \varepsilon > 0$  for all  $0 \leq u \leq \tau_\delta$ , we get that (2.25) implies:

$$E_{t,s,s} \left[ \int_0^{\tau_\delta} e^{-\lambda u} dS_{t+u} \right] = 0. \quad (2.26)$$

Since  $(S_{t+u})_{0 \leq u \leq T-t}$  is an increasing process and it is strictly increasing with a positive  $P_{t,s,s}$ -probability, from (2.26) we may conclude that  $\tau_\delta = 0$  ( $P_{t,s,s}$ -a.s.). As clearly this is impossible, we see that  $V_s(t, s-, s) = 0$  for all  $(t, s) \in [0, T] \times \langle 0, \infty \rangle$  as claimed in (2.41).

2.7. We proceed by proving that the boundary  $g$  is continuous on  $[0, T] \times \langle K, \infty \rangle$  and  $g(T, s) = s$  for all  $s > K$ . For this, we fix some  $(t, s) \in [0, T] \times \langle K, \infty \rangle$  and observe that for each sequence  $(t_n, s_n)$  converging to  $(t, s)$  we have  $\bar{g}(t, s) \leq g(t, s)$ , where  $\bar{g}(t, s) \equiv \lim_n g(t_n, s_n)$ . The latter inequality follows directly by the structure of the set  $D$  from (2.10) and the fact that  $(t_n, g(t_n, s_n), s_n) \in \bar{D}$  for all  $n \in \mathbb{N}$ , and thus  $(t, \bar{g}(t, s), s) \in \bar{D}$ , since  $\bar{D}$  is closed.

Suppose that at some point  $(t_*, s_*) \in \langle 0, T \rangle \times \langle K, \infty \rangle$  the function  $g$  is not continuous, so that there is a sequence  $(t'_n, s'_n)$  converging to  $(t_*, s_*)$  such that  $g'(t_*, s_*) < g(t_*, s_*)$ , where  $g'(t_*, s_*) \equiv \lim_n g(t'_n, s'_n)$ . Let us then fix a point  $(t'_n, s'_n)$  close to  $(t_*, s_*)$  and consider the half-open region  $R_n \subset C_2$  being a curved trapezoid formed by the vertexes  $(t'_n, g(t'_n, s'_n), s'_n)$ ,  $(t_*, g'(t_*, s_*), s_*)$ ,  $(t_*, x', s_*)$  and  $(t'_n, x', s'_n)$  with any  $x'$  fixed arbitrarily in the interval  $\langle g'(t_*, s_*), g(t_*, s_*) \rangle$ . Observe that the strong Markov property implies that the value function  $V$  from (2.4) is  $C^{1,2,1}$  on  $C_2$ . So that, by straightforward calculations, using (2.39) and (2.40), and taking into account the fact that  $G_{xx} = 0$ , it follows that:

$$V(t, x, s) - G(s) = \int_{g(t,s)}^x \int_{g(t,s)}^y V_{xx}(t, z, s) dz dy \quad (2.27)$$

for all  $(t, x, s) \in R_n$  and each  $n \in \mathbb{N}$  fixed. Since  $t \mapsto V(t, x, s)$  is decreasing, we have:

$$V_t(t, x, s) \leq 0 \quad (2.28)$$

for each  $(t, x, s) \in C_2$ . Finally, since the strong Markov property implies that the value function  $V$  from (2.4) solves the equation (2.38), using (2.28) and (2.36) as well as the fact that  $\lambda \geq r$ ,

we obtain:

$$V_{xx}(t, x, s) = \frac{2}{\sigma^2 x^2} \left( \lambda V - V_t - rxV_x \right)(t, x, s) \geq \frac{2r}{\sigma^2} \frac{\partial}{\partial x} \left( -\frac{V(t, x, s)}{x} \right) \quad (2.29)$$

for all  $(t, x, s) \in R_n$  and each  $n \in \mathbb{N}$  fixed. Hence, by (2.27) we get:

$$\begin{aligned} V(t'_n, x', s'_n) - G(s'_n) &\geq \frac{2r}{\sigma^2} \int_{g(t'_n, s'_n)}^{x'} \int_{g(t'_n, s'_n)}^y \frac{\partial}{\partial x} \left( -\frac{V(t'_n, z, s'_n)}{z} \right) dz dy \\ &= \frac{2r}{\sigma^2} \int_{g(t'_n, s'_n)}^{x'} \left( \frac{V(t'_n, g(t'_n, s'_n), s'_n)}{g(t'_n, s'_n)} - \frac{V(t'_n, y, s'_n)}{y} \right) dy \\ &\rightarrow \frac{2r}{\sigma^2} V(t_*, g'(t_*, s_*), s_*) \left( \frac{x'}{g'(t_*, s_*)} - \log \frac{x'}{g'(t_*, s_*)} - 1 \right) > 0 \end{aligned} \quad (2.30)$$

as  $n \rightarrow \infty$ . This implies that  $V(t_*, x', s_*) > G(s_*)$ , which contradicts the fact that  $(t_*, x', s_*)$  belongs to the stopping region  $\overline{D}$ . Thus  $g'(t_*, s_*) = g(t_*, s_*)$  showing that  $g$  is continuous at  $(t_*, s_*)$  and thus on  $[0, T] \times \langle K, \infty \rangle$  as well. We also note that the same arguments with  $t = T$  show that  $g(T-, s) = s$  for all  $s > K$ .

2.8. Summarizing the facts proved in Subsections 2.3-2.7 above, we may conclude that the following exit time is optimal in the extended problem (2.4):

$$\tau_* = \inf \{0 \leq u \leq T - t \mid X_{t+u} \leq g(t+u, S_{t+u})\} \quad (2.31)$$

(the infimum of an empty set being equal  $T - t$ ) where the boundary  $g$  satisfies the following properties (see *Figures 1 and 2* below):

$$g : [0, T] \times \langle K, \infty \rangle \rightarrow \mathbb{R} \quad \text{is continuous and increasing on } [0, T] \quad \text{and on } \langle K, \infty \rangle \quad (2.32)$$

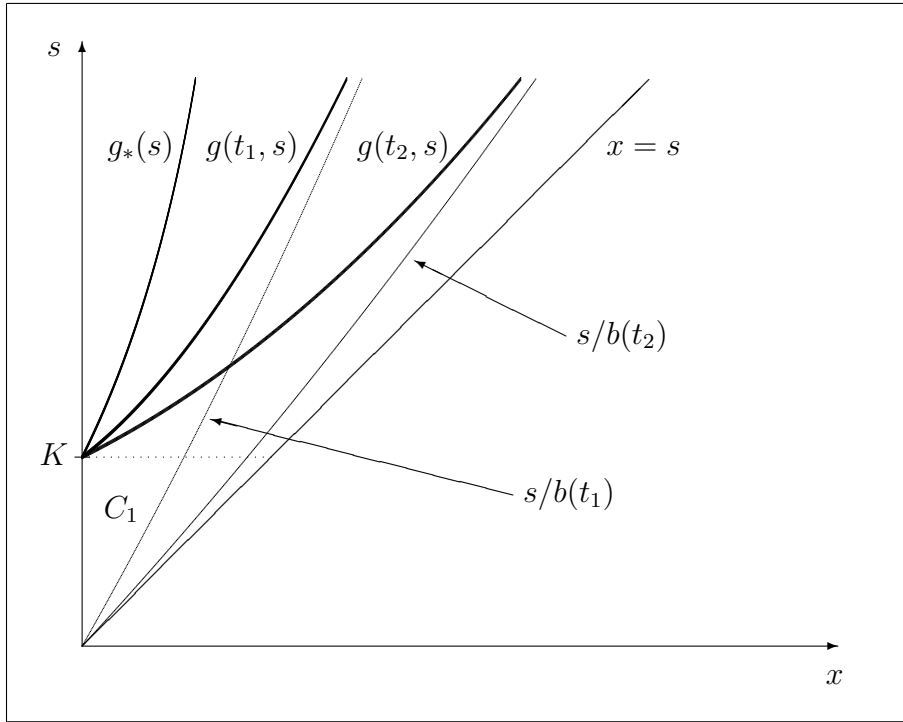
$$g_*(s) \leq g(t, s) \leq s/b(t) < s \quad \text{for all } 0 \leq t < T \quad \text{and } s > K \quad (2.33)$$

$$g(t, K+) = 0 \quad \text{for all } 0 \leq t < T \quad (2.34)$$

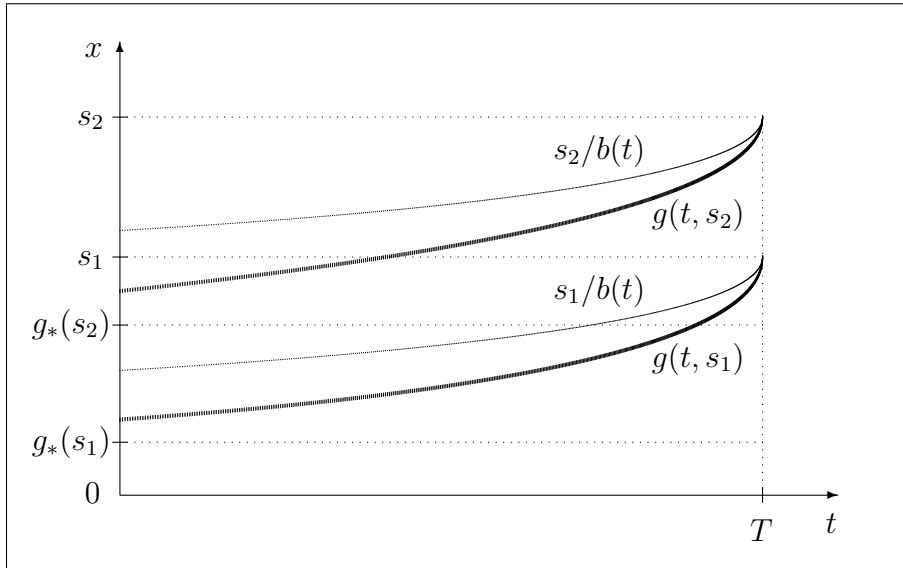
$$g(T, s) = s \quad \text{for all } s > K. \quad (2.35)$$

Here  $g_*$  satisfying  $0 < g_*(s) < s$  for all  $s > K$  is the optimal stopping boundary for the corresponding infinite horizon problem uniquely determined from the first-order nonlinear differential equation (2.19) in [14], and  $b$  is the optimal stopping boundary for the finite horizon Russian option problem uniquely characterized as a unique solution of the nonlinear integral equation (3.4) in [19]. We also note that (2.34) follows from the right continuity of the boundary  $g$  at  $s = K$ , that can be proved by the arguments in Subsection 2.7 above together with the fact that the set  $C_1$  defined in (2.6) belongs to the continuation region  $C$  given by (2.7).

We now recall that the Markov process  $(t, X_t, S_t)_{0 \leq t \leq T}$  is a three-dimensional Markov process with the state space  $[0, T] \times E$  and can change (increase) in the third coordinate only after hitting the diagonal  $\{(t, x, s) \in [0, T] \times E \mid x = s\}$ . Outside the diagonal the process  $(t, X_t, S_t)_{0 \leq t \leq T}$  changes only in the first and second coordinates and may be identified with  $(t, X_t)_{0 \leq t \leq T}$ .



**Figure 1.** A computer drawing of the optimal stopping boundaries  $s \mapsto g_*(s)$ ,  $s \mapsto g(t_1, s)$  and  $s \mapsto g(t_2, s)$  for  $0 < t_1 < t_2 < T$ .



**Figure 2.** A computer drawing of the optimal stopping boundaries  $t \mapsto g(t, s_1)$  and  $t \mapsto g(t, s_2)$  for  $K < s_1 < s_2$ .

Standard arguments then imply that the infinitesimal operator of  $(t, X_t, S_t)_{0 \leq t \leq T}$  acts on a function  $F \in C^{1,2,1}([0, T] \times E)$  according to the rule:

$$(\mathbb{L}F)(t, x, s) = \left( F_t + rxF_x + \frac{\sigma^2 x^2}{2} F_{xx} \right)(t, x, s) \quad \text{in } 0 < x < s \quad (2.36)$$

$$F_s(t, x, s) = 0 \quad \text{at } x = s \quad (2.37)$$

for all  $(t, x, s) \in [0, T] \times E$  (the latter can be shown by the same arguments as in [1; pages 238-239] or [16; pages 1619-1620]). In view of the facts proved above, we are thus naturally led to formulate the following *free-boundary problem* for the unknown value function  $V$  from (2.4) and the unknown boundary  $g$  from (2.9)-(2.10):

$$(\mathbb{L}V - \lambda V)(t, x, s) = 0 \quad \text{for } (t, x, s) \in C_2 \quad (2.38)$$

$$V(t, x, s) \Big|_{x=g(t,s)+} = G(s) \quad (\text{instantaneous stopping}) \quad (2.39)$$

$$V_x(t, x, s) \Big|_{x=g(t,s)+} = 0 \quad (\text{smooth fit}) \quad (2.40)$$

$$V_s(t, x, s) \Big|_{x=s-} = 0 \quad (\text{normal reflection}) \quad (2.41)$$

$$V(t, x, s) > G(s) \quad \text{for } (t, x, s) \in C_2 \quad (2.42)$$

$$V(t, x, s) = G(s) \quad \text{for } (t, x, s) \in D \quad (2.43)$$

where  $C_2$  and  $D$  are given by (2.9) and (2.10), the condition (2.39) is satisfied for all  $(t, s) \in [0, T] \times \langle K, \infty \rangle$ , and the conditions (2.40)-(2.41) are satisfied for all  $(t, s) \in [0, T] \times \langle K, \infty \rangle$ .

Note that the superharmonic characterization of the value function (see [4] and [25]) implies that  $V$  from (2.4) is the smallest function satisfying (2.38)-(2.39) and (2.42)-(2.43).

2.9. Observe that the arguments above show that if we start at the point  $(t, x, s) \in C_1$ , then we can stop optimally only after the process  $(X_t, S_t)_{0 \leq t \leq T}$  pass through the point  $(K, K)$ . Thus, using the strong Markov property, we obtain:

$$V(t, x, s) = E_{t,x,s} [e^{-\lambda \sigma_*} V(t + \sigma_*, K, K) I(\sigma_* \leq T - t)] \quad (2.44)$$

for all  $(t, x, s) \in C_1$ , where we set:

$$\sigma_* = \inf\{t \geq 0 \mid X_t \geq K\} \quad (2.45)$$

and  $V(t, K, K) = \lim_{s \downarrow K} V(t, K, s)$ .

By means of standard arguments (see, e.g., [21; Chapter II, Proposition 3.7]) it can be shown that (2.44) admits the representation:

$$V(t, x, s) = \int_0^{T-t} e^{-\lambda u} V(t + u, K, K) \frac{\log(K/x)}{\sigma \sqrt{2\pi} u^3} \exp \left( -\frac{1}{2u} \left( \frac{\log(K/x)}{\sigma} - \frac{2r - \sigma^2}{2\sigma^2} u \right)^2 \right) du \quad (2.46)$$

for all  $(t, x, s) \in C_1$ . Therefore, it remains us to find the function  $V$  in the region  $C_2$  and to determine the optimal stopping boundary  $g$ .

### 3. Main result and proof

In this section, using the facts proved above, we formulate and prove the main result of the paper. For proof we apply the change-of-variable formula from [17].

**Theorem 3.1.** *In the problem (2.3) the optimal stopping time  $\tau_*$  is explicitly given by:*

$$\tau_* = \inf \left\{ 0 \leq t \leq T \mid X_t \leq g\left(t, \max_{0 \leq u \leq t} X_u\right) \right\} \quad (3.1)$$

where the process  $X$  is defined in (2.1)-(2.2) with  $X_0 = x \geq K$ , and the boundary  $g$  can be characterized as a unique solution of the nonlinear integral equation:

$$\begin{aligned} s - K &= e^{-\lambda(T-t)} E_{t,g(t,s),s}[S_T - K] \\ &+ \lambda \int_0^{T-t} e^{-\lambda v} E_{t,g(t,s),s}[(S_{t+v} - K)I(X_{t+v} < g(t+v, S_{t+v}))] dv \end{aligned} \quad (3.2)$$

for  $0 \leq t \leq T$  and  $s > K$  satisfying (2.32)-(2.35) [see Figures 1 and 2 above].

More explicitly, the two terms in the equation (3.2) read as follows:

$$E_{t,g(t,s),s}[S_T - K] = \int_1^\infty \int_0^z \left( g(t,s)z \vee s - K \right) p(T-t, y, z) dy dz \quad (3.3)$$

$$\begin{aligned} &E_{t,g(t,s),s}[(S_{t+v} - K)I(X_{t+v} < g(t+v, S_{t+v}))] \\ &= \int_1^\infty \int_0^z \left( g(t,s)z \vee s - K \right) I\left(g(t,s)y < g(t+v, g(t,s)z \vee s)\right) p(v, y, z) dy dz \end{aligned} \quad (3.4)$$

for  $s > K$  and  $0 \leq v \leq T-t$  with  $0 \leq t \leq T$ . The transition density function of the process  $(X_t, S_t)_{t \geq 0}$  with  $X_0 = S_0 = 1$  under  $P$  is given by:

$$p(t, x, s) = \frac{2}{\sigma^3 \sqrt{2\pi t^3}} \frac{\log(s^2/x)}{xs} \exp\left(-\frac{\log^2(s^2/x)}{2\sigma^2 t} + \frac{\beta}{\sigma} \log x - \frac{\beta^2}{2} t\right) \quad (3.5)$$

for  $0 < x \leq s$  and  $s \geq 1$  with  $\beta = r/\sigma + \sigma/2$ , and equals zero otherwise.

**Proof.** (i) The existence of the boundary  $g$  satisfying (2.32)-(2.35) such that  $\tau_*$  from (3.1) is optimal in (2.4) was proved in Subsections 2.3-2.8 above. Since the boundary  $g$  is continuous and monotone, by the change-of-variable formula from [17], it follows that the boundary  $g$  solves the equation (3.2) (cf. (3.12)-(3.14) below). Thus, it remains us to show that the equation (3.2) has no other solution in the class of functions  $h$  satisfying (2.32)-(2.35).

Let us thus assume that a function  $h$  satisfying (2.32)-(2.35) solves the equation (3.2), and let us show that this function  $h$  must then coincide with the optimal boundary  $g$ . For this, let us introduce the function:

$$V^h(t, x, s) = \begin{cases} U^h(t, x, s), & \text{if } h(t, s) < x \leq s \\ G(s), & \text{if } 0 < x \leq h(t, s) \end{cases} \quad (3.6)$$

where the function  $U^h$  is defined by:

$$\begin{aligned} U^h(t, x, s) &= e^{-\lambda(T-t)} E_{t,x,s}[G(S_T)] \\ &+ \lambda \int_0^{T-t} e^{-\lambda v} E_{t,x,s}[(S_{t+v} - K)I(X_{t+v} < h(t+v, S_{t+v}))] dv \end{aligned} \quad (3.7)$$

for all  $(t, x, s) \in [0, T] \times E$  with  $s > K$ . Note that (3.7) with  $s - K$  instead of  $U^h(t, x, s)$  on the left-hand side coincides with (3.2) when  $x = g(t, s)$  and  $h = g$ . Since  $h$  solves (3.2) this shows that  $V^h$  is continuous on  $[0, T] \times E \setminus C_1$ , where  $C_1$  is given by (2.6). We need to verify that  $V^h$  coincides with the value function  $V$  from (2.4) and that  $h$  equals  $g$ . For this, we show that the change-of-variable formula from [17] can be applied for  $V^h$  and  $h$ , and then present the rest of verification arguments following the lines of [18]-[19] and [7]-[6] for completeness.

(ii) Using standard arguments based on the strong Markov property (or verifying directly), it follows that  $V^h$ , that is  $U^h$ , is  $C^{1,2,1}$  on  $C_h$  and that:

$$(\mathbb{L}V^h - \lambda V^h)(t, x, s) = 0 \quad \text{for } (t, x, s) \in C_h \quad (3.8)$$

where  $C_h$  is defined as  $C_2$  in (2.9) with  $h$  instead of  $g$ . It is also clear that  $V^h$ , that is  $G$ , is  $C^{1,2,1}$  on  $D_h$  and that:

$$(\mathbb{L}V^h - \lambda V^h)(t, x, s) = -\lambda G(s) \quad \text{for } (t, x, s) \in D_h \quad (3.9)$$

where  $D_h$  is defined as in (2.10) with  $h$  instead of  $g$ . Then from (3.8) and (3.9) it follows that  $\mathbb{L}V^h$  is locally bounded on  $C_h$  and  $D_h$ . Moreover, since  $U_x^h$  is continuous on  $[0, T] \times E \setminus C_1$  (which is readily verified using the explicit expressions (3.3)-(3.4) above with  $x$  instead of  $g(t, s)$  and  $h$  instead of  $g$ ) and so is  $h(t, s)$  by assumption, we see that  $V_x^h$  is continuous on  $\overline{C}_h$ , so that  $t \mapsto V_x^h(t, h(t, s)+, s)$  is continuous on  $[0, T]$ . Finally, we observe that (3.8) together with (2.36) directly imply:

$$V_{xx}^h(t, x, s) = \left( \frac{2(\lambda - r)V^h}{\sigma^2 x^2} - \frac{2V_t^h}{\sigma^2 x^2} - \frac{2r}{\sigma^2} \frac{\partial V^h}{\partial x} \frac{1}{x} \right) (t, x, s) \quad \text{for } (t, x, s) \in C_h \quad (3.10)$$

and

$$V_{xx}^h(t, x, s) = 0 \quad \text{for } (t, x, s) \in D_h. \quad (3.11)$$

Then from the arguments above it follows that for each  $s > K$  given and fixed, on the union of the sets  $\{(t, x) \mid (t, x, s) \in C_h\}$  and  $\{(t, x) \mid (t, x, s) \in D_h\}$ , the function  $(t, x) \mapsto V_{xx}^h(t, x, s)$  can be represented as a sum of two functions, where the first one is nonnegative and the second one is continuous on the closures of these sets. Therefore, by the obvious continuity of  $t \mapsto V_x^h(t, h(t, s)-, s)$  on  $[0, T]$ , the change-of-variable formula from Section 4 in [17] can be applied, and in this way we get:

$$\begin{aligned} e^{-\lambda u} V^h(t+u, X_{t+u}, S_{t+u}) &= V^h(t, x, s) \\ &+ \int_0^u e^{-\lambda v} (\mathbb{L}V^h - \lambda V^h)(t+v, X_{t+v}, S_{t+v}) I(X_{t+v} \neq h(t+v, S_{t+v})) dv \\ &+ \int_0^u e^{-\lambda v} V_s^h(t+v, X_{t+v}, S_{t+v}) I(X_{t+v} \neq h(t+v, S_{t+v}), X_{t+v} = S_{t+v}) dS_{t+v} \\ &+ M_u^h + \frac{1}{2} \int_0^u e^{-\lambda v} \Delta_x V_x^h(t+v, X_{t+v}, S_{t+v}) I(X_{t+v} = h(t+v, S_{t+v})) d\ell_u^h \end{aligned} \quad (3.12)$$

for  $0 \leq u \leq T - t$  where  $\Delta_x V_x^h(t+v, h(t+v, s), s) = V_x^h(t+v, h(t+v, s)+, s) - V_x^h(t+v, h(t+v, s)-, s)$ , the process  $(\ell_u^h)_{0 \leq u \leq T-t}$  is the local time of  $(X_{t+u})_{0 \leq u \leq T-t}$  at the (surface) boundary  $h$  (which is increasing in both variables) given by:

$$\ell_u^h = P_{t,x,s} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^u I(h(t+v, S_{t+v}) - \varepsilon < X_{t+v} < h(t+v, S_{t+v}) + \varepsilon) \sigma^2 X_{t+v}^2 dv \quad (3.13)$$

and  $(M_u^h)_{0 \leq u \leq T-t}$  defined by  $M_u^h = \int_0^u e^{-\lambda v} V_x^h(t+v, X_{t+v}, S_{t+v}) I(X_{t+v} \neq h(t+v, S_{t+v})) \sigma X_{t+v} dB_{t+v}$  is a continuous martingale under  $P_{t,x,s}$ . We also note that in (3.12) the integral with respect to  $dS_{t+v}$  is equal to zero, since the increment  $\Delta S_{t+v}$  outside the diagonal  $\{(t, x, s) \in [0, T] \times E \mid x = s\}$  equals zero, while at the diagonal we have (2.41).

Setting  $u = T - t$  in (3.12) and taking the  $P_{t,x,s}$ -expectation, using that  $V^h$  satisfies (2.38) in  $C_h$  and (2.43) in  $D_h$ , where the set  $D_h$  is defined as in (2.10) with  $h$  in place of  $g$ , we get:

$$e^{-\lambda(T-t)} E_{t,x,s}[G(S_T)] = V^h(t, x, s) - \lambda \int_0^{T-t} e^{-\lambda v} E_{t,x,s}[(S_{t+v} - K) I(X_{t+v} < h(t+v, S_{t+v}))] dv + \frac{1}{2} F(t, x, s) \quad (3.14)$$

where (by the continuity of the integrand) the function  $F$  is given by:

$$F(t, x, s) = E_{t,x,s} \left[ \int_0^{T-t} e^{-\lambda v} \Delta_x V_x^h(t+v, h(t+v, S_{t+v}), S_{t+v}) d\ell_v^h \right] \quad (3.15)$$

for all  $(t, x, s) \in [0, T] \times E$  with  $s > K$ . Thus, from (3.14) and (3.6) we see that:

$$F(t, x, s) = \begin{cases} 0, & \text{if } h(t, s) < x \leq s \\ 2(U^h(t, x, s) - G(s)), & \text{if } 0 < x \leq h(t, s) \end{cases} \quad (3.16)$$

where the function  $U^h$  is given by (3.7).

(iii) From (3.16) we see that if we are to prove that:

$$x \mapsto V^h(t, x, s) \quad \text{is } C^1 \quad \text{at } h(t, s) \quad (3.17)$$

for each  $(t, s) \in [0, T] \times \langle K, \infty \rangle$  given and fixed, then it will follow that:

$$U^h(t, x, s) = G(s) \quad \text{for all } x \leq h(t, s). \quad (3.18)$$

On the other hand, if we know that (3.18) holds, then using the general fact obtained directly from the definition (3.6) above:

$$\begin{aligned} \frac{\partial}{\partial x} (U^h(t, x, s) - G(s)) \Big|_{x=h(t,s)} &= V_x^h(t, h(t, s)+, s) - V_x^h(t, h(t, s)-, s) \\ &= \Delta_x V_x^h(t, h(t, s), s) \end{aligned} \quad (3.19)$$

for all  $(t, s) \in [0, T] \times \langle K, \infty \rangle$ , we see that (3.17) holds too. The equivalence of (3.17) and (3.18) suggests that instead of dealing with the equation (3.16) in order to derive (3.17) above we may rather concentrate on establishing (3.18) directly.

In order to derive (3.18), let us first note that using standard arguments based on the strong Markov property (or verifying directly) it follows that  $U^h$  is  $C^{1,2,1}$  in  $D_h$  and that:

$$(\mathbb{L}U^h - \lambda U^h)(t, x, s) = -\lambda G(s) \quad \text{for } (t, x, s) \in D_h. \quad (3.20)$$

It follows that (3.12) can be applied with  $U^h$  instead of  $V^h$ , and this yields:

$$\begin{aligned} e^{-\lambda u} U^h(t+u, X_{t+u}, S_{t+u}) &= U^h(t, x, s) \\ &\quad - \lambda \int_0^u e^{-\lambda v} (S_{t+v} - K) I(X_{t+v} < h(t+v, S_{t+v})) dv + N_u^h \end{aligned} \quad (3.21)$$

using (3.8) and (3.20) as well as that  $\Delta_x U_x^h(t+v, h(t+v, s), s) = 0$  for all  $0 \leq v \leq u$  since  $U_x^h$  is continuous. In (3.21) we have  $N_u^h = \int_0^u e^{-\lambda v} U_x^h(t+v, X_{t+v}, S_{t+v}) I(X_{t+v} \neq h(t+v, S_{t+v})) \sigma X_{t+v} dB_{t+v}$  and  $(N_u^h)_{0 \leq u \leq T-t}$  is a continuous martingale under  $P_{t,x,s}$ .

Next, note that (3.12) applied to  $G$  instead of  $V^h$  yields:

$$e^{-\lambda u} G(S_{t+u}) = G(s) - \lambda \int_0^u e^{-\lambda v} (S_{t+v} - K) dv + \int_0^u e^{-\lambda v} I(X_{t+v} = S_{t+v}) dS_{t+v} \quad (3.22)$$

for  $s > K$  using (3.9) and the fact that the process  $S$  may increase only at the diagonal  $\{(t, x, s) \in [0, T] \times E \mid x = s\}$ .

For  $0 < x \leq h(t, s)$  with  $0 < t < T$  and  $s > K$  let us consider the stopping time:

$$\sigma_h = \inf\{0 \leq u \leq T - t \mid X_{t+u} \geq h(t+u, S_{t+u})\}. \quad (3.23)$$

Then, using that  $U^h(t, h(t, s), s) = G(s)$  for all  $(t, s) \in [0, T] \times \langle K, \infty \rangle$  since  $h$  solves (3.2), and that  $U^h(T, x, s) = G(s)$  for all  $(x, s) \in E$  with  $s > K$ , we see that  $U^h(t + \sigma_h, X_{t+\sigma_h}, S_{t+\sigma_h}) = G(S_{t+\sigma_h})$ . Hence, from (3.21) and (3.22), using the optional sampling theorem, we find:

$$\begin{aligned} U^h(t, x, s) &= E_{t,x,s} [e^{-\lambda \sigma_h} U^h(t + \sigma_h, X_{t+\sigma_h}, S_{t+\sigma_h})] \\ &\quad + \lambda E_{t,x,s} \left[ \int_0^{\sigma_h} e^{-\lambda v} (S_{t+v} - K) I(X_{t+v} < h(t+v, S_{t+v})) dv \right] \\ &= E_{t,x,s} [e^{-\lambda \sigma_h} G(S_{t+\sigma_h})] + \lambda E_{t,x,s} \left[ \int_0^{\sigma_h} e^{-\lambda v} (S_{t+v} - K) I(X_{t+v} < h(t+v, S_{t+v})) dv \right] \\ &= G(s) - \lambda E_{t,x,s} \left[ \int_0^{\sigma_h} e^{-\lambda v} (S_{t+v} - K) dv \right] + E_{t,x,s} \left[ \int_0^{\sigma_h} e^{-\lambda v} I(X_{t+v} = S_{t+v}) dS_{t+v} \right] \\ &\quad + \lambda E_{t,x,s} \left[ \int_0^{\sigma_h} e^{-\lambda v} (S_{t+v} - K) I(X_{t+v} < h(t+v, S_{t+v})) dv \right] = G(s) \end{aligned} \quad (3.24)$$

since  $X_{t+v} < h(t+v, S_{t+v}) < S_{t+v}$  for all  $0 \leq v < \sigma_h$ . This establishes (3.18), so that (3.17) also holds.

(iv) Let us consider the stopping time:

$$\tau_h = \inf\{0 \leq u \leq T - t \mid X_{t+u} \leq h(t+u, S_{t+u})\}. \quad (3.25)$$

Observe that, by virtue of (3.17), the identity (3.12) can be written as:

$$\begin{aligned} e^{-\lambda u} V^h(t+u, X_{t+u}, S_{t+u}) &= V^h(t, x, s) \\ &\quad - \lambda \int_0^u e^{-\lambda v} (S_{t+v} - K) I(X_{t+v} < h(t+v, S_{t+v})) dv + M_u^h \end{aligned} \quad (3.26)$$

with  $(M_u^h)_{0 \leq u \leq T-t}$  being a martingale under  $P_{t,x,s}$ . Thus, inserting  $\tau_h$  into (3.26) in place of  $u$  and taking the  $P_{t,x,s}$ -expectation, by means of the optional sampling theorem, we get:

$$V^h(t, x, s) = E_{t,x,s} [e^{-\lambda \tau_h} G(S_{t+\tau_h})] \quad (3.27)$$

for all  $(t, x, s) \in [0, T] \times E$  with  $s > K$ . Then, comparing (3.27) with (2.4), we see that:

$$V^h(t, x, s) \leq V(t, x, s) \quad (3.28)$$



for all  $(t, x, s) \in [0, T] \times E$  with  $s > K$ .

(v) Let us now show that  $h \geq g$  on  $[0, T] \times \langle K, \infty \rangle$ . Using the same arguments as for  $V^h$ , we have:

$$\begin{aligned} e^{-\lambda u} V(t+u, X_{t+u}, S_{t+u}) &= V(t, x, s) \\ &\quad - \lambda \int_0^u e^{-\lambda v} (S_{t+v} - K) I(X_{t+v} < g(t+v, S_{t+v})) dv + M_u^g \end{aligned} \quad (3.29)$$

where  $(M_u^g)_{0 \leq u \leq T-t}$  is a martingale under  $P_{t,x,s}$ . Fix some  $(t, x, s)$  such that  $x < g(t, s) \wedge h(t, s)$  and consider the stopping time:

$$\sigma_g = \inf\{0 \leq u \leq T-t \mid X_{t+u} \geq g(t+u, S_{t+u})\}. \quad (3.30)$$

Inserting  $\sigma_g$  into (3.26) and (3.29) in place of  $u$  and taking the  $P_{t,x,s}$ -expectation, by means of the optional sampling theorem, we get:

$$\begin{aligned} E_{t,x,s} [e^{-\lambda \sigma_g} V^h(t + \sigma_g, X_{t+\sigma_g}, S_{t+\sigma_g})] &= G(s) \\ &\quad - \lambda E_{t,x,s} \left[ \int_0^{\sigma_g} e^{-\lambda v} (S_{t+v} - K) I(X_{t+v} < h(t+v, S_{t+v})) dv \right] \end{aligned} \quad (3.31)$$

$$\begin{aligned} E_{t,x,s} [e^{-\lambda \sigma_g} V(t + \sigma_g, X_{t+\sigma_g}, S_{t+\sigma_g})] &= G(s) \\ &\quad - \lambda E_{t,x,s} \left[ \int_0^{\sigma_g} e^{-\lambda v} (S_{t+v} - K) dv \right]. \end{aligned} \quad (3.32)$$

Hence, by means of (3.28), we see that:

$$\begin{aligned} E_{t,x,s} \left[ \int_0^{\sigma_g} e^{-\lambda v} (S_{t+v} - K) I(X_{t+v} < h(t+v, S_{t+v})) dv \right] \\ \geq E_{t,x,s} \left[ \int_0^{\sigma_g} e^{-\lambda v} (S_{t+v} - K) dv \right] \end{aligned} \quad (3.33)$$

from where, by virtue of the continuity of  $h$  and  $g$  on  $\langle 0, T \rangle \times \langle K, \infty \rangle$ , it readily follows that  $h(t, s) \geq g(t, s)$  for all  $(t, s) \in [0, T] \times \langle K, \infty \rangle$ .

(vi) Finally, we show that  $h$  coincides with  $g$ . For this, let us assume that there exists some  $(t, s) \in \langle 0, T \rangle \times \langle K, \infty \rangle$  such that  $h(t, s) > g(t, s)$  and take an arbitrary  $x$  from  $\langle g(t, s), h(t, s) \rangle$ . Then, inserting  $\tau_* = \tau_*(t, x, s)$  from (2.31) into (3.26) and (3.29) in place of  $u$  and taking the  $P_{t,x,s}$ -expectation, by means of the optional sampling theorem, we get:

$$\begin{aligned} E_{t,x,s} [e^{-\lambda \tau_*} G(S_{t+\tau_*})] &= V^h(t, x, s) \\ &\quad - \lambda E_{t,x,s} \left[ \int_0^{\tau_*} e^{-\lambda v} (S_{t+v} - K) I(X_{t+v} < h(t+v, S_{t+v})) dv \right] \end{aligned} \quad (3.34)$$

$$E_{t,x,s} [e^{-\lambda \tau_*} G(S_{t+\tau_*})] = V(t, x, s). \quad (3.35)$$

Hence, by means of (3.28), we see that:

$$E_{t,x,s} \left[ \int_0^{\tau_*} e^{-\lambda v} (S_{t+v} - K) I(X_{t+v} < h(t+v, S_{t+v})) dv \right] \leq 0 \quad (3.36)$$

which is clearly impossible by the continuity of  $h$  and  $g$ . We may therefore conclude that  $V^h$  defined in (3.6) coincides with  $V$  from (2.4) and  $h$  is equal to  $g$ . This completes the proof of the theorem.  $\square$

**Acknowledgments.** The author is gratefully thankful to Goran Peskir for many fruitful discussions concerning finite horizon optimal stopping problems.

## References

- [1] DUBINS, L., SHEPP, L. A. and SHIRYAEV, A. N. (1993). Optimal stopping rules and maximal inequalities for Bessel processes. *Theory Probab. Appl.* **38**(2) (226–261).
- [2] DUFFIE, J. D. and HARRISON, J. M. (1993). Arbitrage pricing of Russian options and perpetual lookback options. *Ann. Appl. Probab.* **3**(3) (641–651).
- [3] DUISTERMAAT, J. J., KYPRIANOU, A. E. and VAN SCHAIK, K. (2003). Finite expiry Russian options. *Stoch. Proc. Appl.* **115**(4) (609–638).
- [4] DYNKIN, E. B. (1963). The optimum choice of the instant for stopping a Markov process. *Soviet Math. Dokl.* **4** (627–629).
- [5] EKSTRÖM, E. (2004). Russian options with a finite time horizon. *J. Appl. Probab.* **41**(2) (313–326).
- [6] GAPEEV, P. V. and PESKIR, G. (2004). The Wiener sequential testing problem with finite horizon. *Stoch. Stoch. Rep.* **76** (59–75).
- [7] GAPEEV, P. V. and PESKIR, G. (2006). The Wiener disorder problem with finite horizon. *Research Report* No. 435, Dept. Theoret. Statist. Aarhus (22 pp). To appear in *Stoch. Proc. Appl.*
- [8] GRIGELIONIS, B. I. and SHIRYAEV, A. N. (1966). On Stefan’s problem and optimal stopping rules for Markov processes. *Theory Probab. Appl.* **11** (541–558).
- [9] GUO, X. and SHEPP, L. A. (2001). Some optimal stopping problems with nontrivial boundaries for pricing exotic options. *J. Appl. Probab.* **38**(3) (647–658).
- [10] JACKA, S. D. (1991). Optimal stopping and the American put. *Math. Finance* **1** (1–14).
- [11] KARATZAS, I. and SHREVE, S. E. (1998). *Methods of Mathematical Finance*. Springer, New York.

- [12] MCKEAN, H. P. JR. (1965). Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Ind. Management Rev.* **6** (32–39).
- [13] MYNENI, R. (1992). The pricing of the American option. *Ann. Appl. Probab.* **2**(1) (1–23).
- [14] PEDERSEN, J. L. (2000) Discounted optimal stopping problems for the maximum process. *J. Appl. Probab.* **37**(4) (972–983).
- [15] PEDERSEN, J. L. and PESKIR, G. (2002). On nonlinear integral equations arising in problems of optimal stopping. *Proc. Functional Anal. VII* (Dubrovnik 2001), *Various Publ. Ser.* **46** (159–175).
- [16] PESKIR, G. (1998). Optimal stopping of the maximum process: The maximality principle. *Ann. Probab.* **26**(4) (1614–1640).
- [17] PESKIR, G. (2004). A change-of-variable formula with local time on surfaces. *Research Report* No. 437, Dept. Theoret. Statist. Aarhus (23 pp). To appear in *Sém. de Probab.* (Lecture Notes in Mathematics) Springer.
- [18] PESKIR, G. (2005). On the American option problem. *Math. Finance* **15**(1) (169–181).
- [19] PESKIR, G. (2005). The Russian option: finite horizon. *Finance Stochast.* **9** (251–267).
- [20] PESKIR, G. and UYS, N. (2005). On Asian options of American type. In the Volume *Exotic Options and Advanced Levy Models*. Wiley, Chichester (217–235).
- [21] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*. Springer, Berlin.
- [22] SHEPP, L. A. and SHIRYAEV, A. N. (1993). The Russian option: reduced regret. *Ann. Appl. Probab.* **3**(3) (631–640).
- [23] SHEPP, L. A. and SHIRYAEV, A. N. (1994). A new look at the pricing of Russian options. *Theory Probab. Appl.* **39**(1) (103–119).
- [24] SHEPP, L. A., SHIRYAEV A. N. and SULEM, A. (2002). A barrier version of the Russian option. *Advances in Finance and Stochastics*. Essays in Honour of Dieter Sondermann. Sandmann, K. and Schönbucher, P. eds. Springer. (271–284).
- [25] SHIRYAEV, A. N. (1978). *Optimal Stopping Rules*. Springer, Berlin.
- [26] SHIRYAEV, A. N. (1999). *Essentials of Stochastic Finance*. World Scientific, Singapore.
- [27] VAN MOERBEKE, P. (1976). On optimal stopping and free-boundary problems. *Arch. Rational Mech. Anal.* **60** (101–148).

Pavel V. Gapeev  
Weierstraß Institute  
for Applied Analysis and Stochastics (WIAS)  
Mohrenstr. 39, D-10117 Berlin, Germany  
e-mail: gapeev@wias-berlin.de

(Russian Academy of Sciences  
Institute of Control Sciences  
Profsoyuznaya Str. 65  
117997 Moscow, Russia)

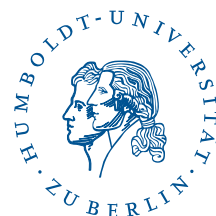
## SFB 649 Discussion Paper Series 2006

For a complete list of Discussion Papers published by the SFB 649, please visit <http://sfb649.wiwi.hu-berlin.de>.

- 001 "Calibration Risk for Exotic Options" by Kai Detlefsen and Wolfgang K. Härdle, January 2006.
- 002 "Calibration Design of Implied Volatility Surfaces" by Kai Detlefsen and Wolfgang K. Härdle, January 2006.
- 003 "On the Appropriateness of Inappropriate VaR Models" by Wolfgang Härdle, Zdeněk Hlávka and Gerhard Stahl, January 2006.
- 004 "Regional Labor Markets, Network Externalities and Migration: The Case of German Reunification" by Harald Uhlig, January/February 2006.
- 005 "British Interest Rate Convergence between the US and Europe: A Recursive Cointegration Analysis" by Enzo Weber, January 2006.
- 006 "A Combined Approach for Segment-Specific Analysis of Market Basket Data" by Yasemin Boztuğ and Thomas Reutterer, January 2006.
- 007 "Robust utility maximization in a stochastic factor model" by Daniel Hernández-Hernández and Alexander Schied, January 2006.
- 008 "Economic Growth of Agglomerations and Geographic Concentration of Industries - Evidence for Germany" by Kurt Geppert, Martin Gornig and Axel Werwatz, January 2006.
- 009 "Institutions, Bargaining Power and Labor Shares" by Benjamin Bental and Dominique Demougin, January 2006.
- 010 "Common Functional Principal Components" by Michal Benko, Wolfgang Härdle and Alois Kneip, January 2006.
- 011 "VAR Modeling for Dynamic Semiparametric Factors of Volatility Strings" by Ralf Brüggemann, Wolfgang Härdle, Julius Mungo and Carsten Trenkler, February 2006.
- 012 "Bootstrapping Systems Cointegration Tests with a Prior Adjustment for Deterministic Terms" by Carsten Trenkler, February 2006.
- 013 "Penalties and Optimality in Financial Contracts: Taking Stock" by Michel A. Robe, Eva-Maria Steiger and Pierre-Armand Michel, February 2006.
- 014 "Core Labour Standards and FDI: Friends or Foes? The Case of Child Labour" by Sebastian Braun, February 2006.
- 015 "Graphical Data Representation in Bankruptcy Analysis" by Wolfgang Härdle, Rouslan Moro and Dorothea Schäfer, February 2006.
- 016 "Fiscal Policy Effects in the European Union" by Andreas Thams, February 2006.
- 017 "Estimation with the Nested Logit Model: Specifications and Software Particularities" by Nadja Silberhorn, Yasemin Boztuğ and Lutz Hildebrandt, March 2006.
- 018 "The Bologna Process: How student mobility affects multi-cultural skills and educational quality" by Lydia Mechtenberg and Roland Strausz, March 2006.
- 019 "Cheap Talk in the Classroom" by Lydia Mechtenberg, March 2006.
- 020 "Time Dependent Relative Risk Aversion" by Enzo Giacomini, Michael Handel and Wolfgang Härdle, March 2006.
- 021 "Finite Sample Properties of Impulse Response Intervals in SVECMs with Long-Run Identifying Restrictions" by Ralf Brüggemann, March 2006.
- 022 "Barrier Option Hedging under Constraints: A Viscosity Approach" by Imen Bentahar and Bruno Bouchard, March 2006.

**SFB 649, Spandauer Straße 1, D-10178 Berlin**  
**<http://sfb649.wiwi.hu-berlin.de>**

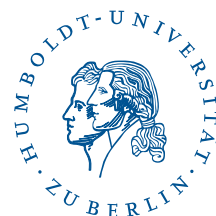
This research was supported by the Deutsche  
Forschungsgemeinschaft through the SFB 649 "Economic Risk".



- 023 "How Far Are We From The Slippery Slope? The Laffer Curve Revisited" by Mathias Trabandt and Harald Uhlig, April 2006.
- 024 "e-Learning Statistics – A Selective Review" by Wolfgang Härdle, Sigbert Klinke and Uwe Ziegenhagen, April 2006.
- 025 "Macroeconomic Regime Switches and Speculative Attacks" by Bartosz Maćkowiak, April 2006.
- 026 "External Shocks, U.S. Monetary Policy and Macroeconomic Fluctuations in Emerging Markets" by Bartosz Maćkowiak, April 2006.
- 027 "Institutional Competition, Political Process and Holdup" by Bruno Deffains and Dominique Demougin, April 2006.
- 028 "Technological Choice under Organizational Diseconomies of Scale" by Dominique Demougin and Anja Schöttner, April 2006.
- 029 "Tail Conditional Expectation for vector-valued Risks" by Imen Bentahar, April 2006.
- 030 "Approximate Solutions to Dynamic Models – Linear Methods" by Harald Uhlig, April 2006.
- 031 "Exploratory Graphics of a Financial Dataset" by Antony Unwin, Martin Theus and Wolfgang Härdle, April 2006.
- 032 "When did the 2001 recession *really* start?" by Jörg Polzehl, Vladimir Spokoiny and Cătălin Stărică, April 2006.
- 033 "Varying coefficient GARCH versus local constant volatility modeling. Comparison of the predictive power" by Jörg Polzehl and Vladimir Spokoiny, April 2006.
- 034 "Spectral calibration of exponential Lévy Models [1]" by Denis Belomestny and Markus Reiß, April 2006.
- 035 "Spectral calibration of exponential Lévy Models [2]" by Denis Belomestny and Markus Reiß, April 2006.
- 036 "Spatial aggregation of local likelihood estimates with applications to classification" by Denis Belomestny and Vladimir Spokoiny, April 2006.
- 037 "A jump-diffusion Libor model and its robust calibration" by Denis Belomestny and John Schoenmakers, April 2006.
- 038 "Adaptive Simulation Algorithms for Pricing American and Bermudan Options by Local Analysis of Financial Market" by Denis Belomestny and Grigori N. Milstein, April 2006.
- 039 "Macroeconomic Integration in Asia Pacific: Common Stochastic Trends and Business Cycle Coherence" by Enzo Weber, May 2006.
- 040 "In Search of Non-Gaussian Components of a High-Dimensional Distribution" by Gilles Blanchard, Motoaki Kawanabe, Masashi Sugiyama, Vladimir Spokoiny and Klaus-Robert Müller, May 2006.
- 041 "Forward and reverse representations for Markov chains" by Grigori N. Milstein, John G. M. Schoenmakers and Vladimir Spokoiny, May 2006.
- 042 "Discussion of 'The Source of Historical Economic Fluctuations: An Analysis using Long-Run Restrictions' by Neville Francis and Valerie A. Ramey" by Harald Uhlig, May 2006.
- 043 "An Iteration Procedure for Solving Integral Equations Related to Optimal Stopping Problems" by Denis Belomestny and Pavel V. Gapeev, May 2006.
- 044 "East Germany's Wage Gap: A non-parametric decomposition based on establishment characteristics" by Bernd Görzig, Martin Gornig and Axel Werwatz, May 2006.
- 045 "Firm Specific Wage Spread in Germany - Decomposition of regional differences in inter firm wage dispersion" by Bernd Görzig, Martin Gornig and Axel Werwatz, May 2006.

**SFB 649, Spandauer Straße 1, D-10178 Berlin**  
<http://sfb649.wiwi.hu-berlin.de>

This research was supported by the Deutsche  
Forschungsgemeinschaft through the SFB 649 "Economic Risk".



- 046 "Produktdiversifizierung: Haben die ostdeutschen Unternehmen den Anschluss an den Westen geschafft? – Eine vergleichende Analyse mit Mikrodaten der amtlichen Statistik" by Bernd Görzig, Martin Gornig and Axel Werwatz, May 2006.
- 047 "The Division of Ownership in New Ventures" by Dominique Demougin and Oliver Fabel, June 2006.
- 048 "The Anglo-German Industrial Productivity Paradox, 1895-1938: A Restatement and a Possible Resolution" by Albrecht Ritschl, May 2006.
- 049 "The Influence of Information Costs on the Integration of Financial Markets: Northern Europe, 1350-1560" by Oliver Volckart, May 2006.
- 050 "Robust Econometrics" by Pavel Čížek and Wolfgang Härdle, June 2006.
- 051 "Regression methods in pricing American and Bermudan options using consumption processes" by Denis Belomestny, Grigori N. Milstein and Vladimir Spokoiny, July 2006.
- 052 "Forecasting the Term Structure of Variance Swaps" by Kai Detlefsen and Wolfgang Härdle, July 2006.
- 053 "Governance: Who Controls Matters" by Bruno Deffains and Dominique Demougin, July 2006.
- 054 "On the Coexistence of Banks and Markets" by Hans Gersbach and Harald Uhlig, August 2006.
- 055 "Reassessing Intergenerational Mobility in Germany and the United States: The Impact of Differences in Lifecycle Earnings Patterns" by Thorsten Vogel, September 2006.
- 056 "The Euro and the Transatlantic Capital Market Leadership: A Recursive Cointegration Analysis" by Enzo Weber, September 2006.
- 057 "Discounted Optimal Stopping for Maxima in Diffusion Models with Finite Horizon" by Pavel V. Gapeev, September 2006.

**SFB 649, Spandauer Straße 1, D-10178 Berlin**  
**<http://sfb649.wiwi.hu-berlin.de>**

This research was supported by the Deutsche  
Forschungsgemeinschaft through the SFB 649 "Economic Risk".

