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A note on Arbitrage under Transaction Costs

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Abstract

Guasoni (2006) introduced a simple condition for the absence of arbitrage opportunities. In this note we show that his results remain valid under a weaker notion of arbitrage which arises by excluding liquidation costs from the value process of a portfolio.

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1 Introduction

Most models in Mathematical Finance build on the concept of the absence of arbitrage which is an inevitable assumption in the usual theories of option pricing as well as portfolio optimization. An important result of Delbaen and Schachermayer (1994) (Theorem 7.2) states that every frictionless financial market in which the price process of a risky asset is locally bounded and not a semimartingale allows a certain kind of arbitrage. However, some effects that show up in real market data suggest asset price models which are not semimartingales. For example the long-range-dependence of asset prices which is observed in various empirical studies (see f.e. Willinger, Taqqu and Teverovsky (1999), Mandelbrot (1997)) can be captured by fractional Brownian motion which is not a semimartingale (see f.e. Liptser and Shiryaev (1989)). The long-range dependence of asset prices can also be justified economically, as it captures the effect that new information arriving in a market needs some time to be processed by all participants, see f.e. Klüppelberg and Kühn (2004).
So it is a natural question, how the usual model of a frictionless market can be extended in a way that processes which are not semimartingales become compatible with the absence of arbitrage. A promising approach is to include transaction costs in the model which obviously reduces the wealth of an investor resulting from a given trading strategy.
Indeed, Guasoni (2006) shows that pure proportional and arbitrary small transaction costs are sufficient to eliminate arbitrage opportunities for a large class of asset price processes. This class includes fractional Brownian motion and all Markov processes with regular points.
In section 2 we show that Guasoni's results remain valid under a weaker notion of arbitrage which arises by excluding liquidation costs from the value process of a portfolio. The main result is stated as Theorem 2.4.

2 No Arbitrage under proportional transaction costs

We consider a financial market model consisting of one bond (where for convenience we assume an interest rate $r = 0$) and one risky asset X. The price process $(X_t)_{t\in[0,\infty)}$ of the risky asset is assumed to satisfy the following assumption (which is the same as Assumption 2.1 of Guasoni (2006)).

2.1 Assumption
$(X_t)_{t\in[0,\infty)}$ is a process with cadlag (right-continuous with left-limits) paths,

strictly positive almost surely, adapted to a right-continuous and saturated filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, and quasi-left-continuous with respect to this filtration.

Before we formalize the trading in our model, we provide some heuristic motivation. We consider only self-financing trading strategies (with zero initial capital) which are uniquely determined by the process $(\theta_t)_{t\in[0,\infty)}$ of units of the risky asset held by an investor over time. Our model includes proportional transaction costs of size $k \in (0,1)$, i.e. for every transaction of (monetary) value $X_t|\Delta\theta_t|$ a fee of $kX_t|\Delta\theta_t|$ is charged. As strategies $(\theta_t)_{t\in[0,\infty)}$ of infinite variation on any time interval would lead to immediate ruin, it is sufficient to consider strategies of locally bounded variation. We can then introduce the value process of a trading strategy as

$$V_t = \int_{[0,t]} \theta_s dX_s \; - \int_{[0,t]} kX_s d|D\theta|_s, \tag{1}$$

where $|D\theta|_s$ denotes the total variation of $(\theta_r)_{r\in[0,s]}$ (in a pathwise sense) and the second integral is defined as the usual Stieltjes integral. However, as we have left the class of semimartingales as price processes, the first integral can not be defined as the usual stochastic integral. But as we are only interested in integrands of locally bounded variation, the conditions of Assumption 2.1 for the integrator $(X_t)_{t\in[0,\infty)}$ are sufficient to give a sensible formal definition of the first integral which can even be done in a pathwise sense. It should be mentioned that this integral which is introduced below coincides with the usual stochastic integral, if the integrator is a semimartingale (see f.e. Dellacherie and Meyer (1982), Chapter VIII), and it coincides with the usual Stieltjes integral, if the integrator is of locally bounded variation (see f.e. Hewitt and Stromberg (1965), Theorem 21.67).

2.2 Definition
Let $g : [0,\infty) \mapsto [0,\infty)$ be a cadlag function and $h : [0,\infty) \mapsto \mathbb{R}$ be left-continuous function with locally bounded variation. Then $s \mapsto h(s+) = \lim_{t\downarrow s} h(t)$ is a right-continuous function of locally bounded variation.
For $-\infty < a < b < \infty$ we define

$$\int_{[a,b]} h(s)dg(s) \; = \; h(b+)g(b) - h(a)g(a-) - \int_{[a,b]} g(s)dh(s+), \tag{2}$$

where the integral on the right hand side is defined as the ordinary Stieltjes integral.

The integral defined above can be extended to arbitrary integrands of locally bounded variation. However, Guasoni(2002, Proposition 2.5) shows that for any previsible process $(\theta_t)_{t\in[0,\infty)}$, any process $(X_t)_{t\in[0,\infty)}$ satisfying the conditions

above and any $t > 0$ we have (using the convention $\theta_{0-} = 0$)

$$\int_{[0,t]} \theta_s dX_s = \int_{[0,t]} \theta_{s-} dX_s. \tag{3}$$

We can therefore restrict ourselves to left-continuous trading strategies and the integral defined in Definition 2.2. We are now in the position to formalize the trading in our model.

2.3 Definition
A trading strategy is an adapted, left-continuous process of locally bounded variation. Let $k \in (0,1)$ be the transaction cost factor. The value process $(V_t)_{t\in[0,\infty)}$ for a trading strategy $(\theta_t)_{t\in[0,\infty)}$ is defined by

$$V_t = \int_{[0,t]} \theta_s dX_s - \int_{[0,t]} kX_s d|D\theta|_s. \tag{4}$$

A trading strategy is called admissible, if there exists $M \in \mathbb{R}$ with $V_t > M$ for all $t \geq 0$. An admissible trading strategy is called arbitrage on $[0,T]$ if

$$V_T \geq 0 \ P\text{-a.s.}, \ P(V_T > 0) > 0. \tag{5}$$

Finally, a model is called arbitrage free on $[0,T]$, if there exists no arbitrage on $[0,T]$.

The value process introduced here is different from the one considered by Guasoni (2006), as the latter one additionally includes liquidation costs $kX_t|\theta_t|$ which leads to the value process

$$V_t = \int_{[0,t]} \theta_s dX_s - \int_{[0,t]} kX_s d|D\theta|_s - kX_t|\theta_t|. \tag{6}$$

Excluding these liquidation costs can be sensible, as an investor may not want to sell the risky asset at time T, for example for speculative reasons. Another possible explanation is that the units of the risky asset held at time T may be used to equalize a short position in the risky asset resulting from other contracts. Of course, excluding liquidation costs leads to a weaker notion of arbitrage: Additionally in our model, those trading strategies are an arbitrage, which lead to a negative amount $-x$ in the bond account and a larger positive amount $y \in (x, \frac{1}{1-k}x)$ in the risky asset account. In the presence of liquidation costs, this holding in the stock is not large enough to equalize the deficit in the bond account, so in the setting of Guasoni (2006) such a strategy would not provide an arbitrage.

The following theorem (which corresponds to Proposition 2.1 of Guasoni (2006)) shows that we obtain an arbitrage free model for a large class of price processes even under this weaker notion of arbitrage.

2.4 Theorem
Let $T > 0$ and $c = \frac{k}{3}$.
Assume

$$P(\sup_{t\in[\tau,T]} |\frac{X_\tau}{X_t} - 1| < c, \tau < T) > 0 \tag{7}$$

for all stopping times τ with $P(\tau < T) > 0$.
Then our financial market model is arbitrage free on $[0, T]$.

Proof:
Let $(\theta_t)_{t\in[0,\infty)}$ be a trading strategy with $V_T \geq 0$. We define

$$\begin{aligned}
\tau &= T \wedge \inf\{t : \theta_t \neq 0\}, && (8)\\
\varepsilon_t &= X_t - X_{\tau\wedge t} \text{ for all } t \in [0, T], && (9)\\
X_{min} &= \inf_{t\in[\tau,T]} X_t, && (10)\\
A &= \{\sup_{t\in[\tau,T]} |\frac{X_\tau}{X_t} - 1| < c\} \cap \{\tau < T\}. && (11)
\end{aligned}$$

If $P(\tau < T) = 0$, it follows that

$$\theta_t = 0 \text{ for all } t \in [0, T] \ P - \text{a.s.}, \tag{12}$$

hence $(\theta_t)_{t\in[0,\infty)}$ is no arbitrage on [0,T].
Now let $P(\tau < T) > 0$. On A we have $|\varepsilon_t| < cX_t$ for all $t \in [0, T]$ and hence

$$\begin{aligned}
V_T &= \int_{[\tau,T]} \theta_s dX_{\tau\wedge s} + \int_{[\tau,T]} \theta_s d\varepsilon_s - \int_{[\tau,T]} kX_s d|D\theta|_s\\
&= \theta_T\varepsilon_T - \int_{[\tau,T]} \varepsilon_s dD\theta_s - \int_{[\tau,T]} kX_s d|D\theta|_s\\
&< cX_T|\theta_T| + \int_{[\tau,T]} cX_s d|D\theta|_s - \int_{[\tau,T]} kX_s d|D\theta|_s\\
&\leq cX_T|D\theta|_T + (c-k)X_{min}|D\theta|_T\\
&= \frac{1}{3}k\,|D\theta|_T(X_T - 2X_{min})\\
&< 0. && (13)
\end{aligned}$$

Note for the last inequality that on A it holds that $|D\theta|_T > 0$ and $(1-c)X_T < X_\tau < (1+c)X_{min}$ which implies

$$\frac{X_T}{X_{min}} < \frac{1+c}{1-c} < 2 \tag{14}$$

As we have $P(A) > 0$ by assumption, it follows that $(\theta_t)_{t\in[0,\infty)}$ is no arbitrage on $[0, T]$.

It should be mentioned that Proposition 2.1 of Guasoni (2006) which corresponds to the result above under a stronger notion of arbitrage (including liquidation costs in the value process) builds on the slightly weaker condition

$$P(\sup_{t\in[\tau,T]} |\frac{X_\tau}{X_t} - 1| < k, \tau < T) > 0. \tag{15}$$

In that paper it is shown that this condition is fulfilled, when the return process of the risky asset belongs to a certain class of stochastic processes which the author calls sticky processes. However, every sticky process even fulfils (7) as can be seen just by replacing k by $\frac{k}{3}$ in the corresponding proof (Corollary 2.1 of Guasoni (2006)). Consequently, all results of that paper remain valid under our weaker notion of arbitrage. Especially, arbitrage disappears when the return process is a strong Markov process with regular points or a fractional Brownian motion with arbitrary continuous deterministic drift (see Guasoni (2006), Propositions 3.1 and 5.1). For the convenience of the reader, the definition of a sticky process and the mentioned results are stated below.

2.5 Definition
A progressively measurable process Y is sticky with respect to the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, if for all $\varepsilon, T > 0$ and all stopping times such that $P(\tau < T) > 0$ we have that

$$P(\sup_{t\in[\tau,T]} |Y_\tau - Y_t| < \varepsilon, \tau < T) > 0. \tag{16}$$

2.6 Proposition
If $(\log X_t)_{t\in[0,\infty)}$ is sticky, then for any $T > 0$ our model is arbitrage free on the interval [0,T].

2.7 Proposition
Let $(Y_t)_{t\in[0,\infty)}$ be a stochastic process such that $(\exp(Y_t))_{t\in[0,\infty)}$ fulfils Assumption 2.1.
(i) If $(Y_t)_{t\in[0,\infty)}$ is a strong Markov process and for all finite stopping times τ and all $y \in \mathbb{R}$ we have

$$P(\inf\{t > 0 : Y_{\tau+t} = y\} = 0 | Y_\tau = y) = 1, \tag{17}$$

then $(Y_t)_{t\in[0,\infty)}$ is sticky.
(ii) Let $f : [0, \infty) \mapsto \mathbb{R}$ be continuous, $\sigma > 0$, $(B_t^H)_{t\in[0,\infty)}$ a fractional Brownian motion with Hurst-parameter $H \in (0, 1)$, and

$$Y_t = f(t) + \sigma B_t^H \tag{18}$$

for all $t \in [0, \infty)$. Then $(Y_t)_{t\in[0,\infty)}$ is sticky.

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